arXiv:2311.18309v1 [math.CO] 30 Nov 2023

A NOTE ON CONSTRUCTION OF THE LEECH LATTICE

ICHIRO SHIMADA

Abstract. In this paper, we present a method to construct the Leech lattice from other Niemeier lattices.

1. Introduction

The Leech lattice is an important mathematical object. Various constructions of this beautiful lattice are known. In this paper, we give a method to construct the Leech lattice as modifications of other Niemeier lattices.

Let N be a Niemeier lattice with the intersection form $\langle\ ,\ \rangle_N$. Suppose that N is not isomorphic to the Leech lattice. Let R be the set of vectors in N of square-norm 2, and $\langle R\rangle \subset N$ the sublattice generated by R. Then $N/\langle R\rangle$ is a finite abelian group, which we call the *code* of N. (In [6], it is called the *glue code*.) For each codeword of this code, we give a method to construct the Leech lattice. The main result is given in Theorem 7.8.

Considering the case where this codeword is 0, we obtain the following. Let N and R be as above. We choose a simple root system Θ of R, that is, Θ is a set of vectors in R that form a basis of $\langle R\rangle$ with the dual graph being a Dynkin diagram of an ordinary ADE-type. Then we have a vector $\rho \in N\otimes\mathbb{Q}$ such that $\langle r, \rho\rangle_N = -1$ for all $r \in \Theta$. In fact, we have $\rho \in N$. Let h be the Coxeter number of N.

Corollary 1.1. *The $\mathbb{Z}$-module*

$$\{\, u \in N \mid \langle u, \rho\rangle_N \equiv 0 \bmod 2h + 1 \,\}$$

with the quadratic form

$$u \mapsto \langle u, u\rangle_N - \frac{2}{(2h + 1)^2}\,(\langle u, \rho\rangle_N)^2$$

is isomorphic to the Leech lattice. □

Our construction is similar to the twenty-three holy constructions of Conway and Sloane [7]. See also [5]. In the proof of our construction, the classification of deep holes of the Leech lattice [4] and the determination of the fundamental domain of the Weyl group of the even unimodular Lorentzian lattice $II_{25,1}$ [3] play important roles. Some ideas in our construction have already appeared in Borcherds [1].

A novelty of our approach is that we use the geometry of $K3$ surfaces as a heuristic guide. Let $\mathbb{X}$ be a $K3$ surface whose Néron-Severi lattice $S_{\mathbb{X}}$ is an even unimodular lattice $II_{1,25}$ of signature $(1, 25)$. It is needless to say that such a $K3$ surface $\mathbb{X}$ does *not* exist. Applying the lattice theoretic tools for the study of $K3$ surfaces to this *virtual* $K3$ surface $\mathbb{X}$, however, we can rephrase the above mentioned

2020 *Mathematics Subject Classification.* 11H06.

Key words and phrases. Leech lattice, Niemeier lattice, deep hole.

This work was supported by JSPS KAKENHI Grant Number 20H01798.

results of Conway et al. [3, 4] in terms of geometry of $K3$ surfaces. Then, using an algorithm that we have developed in [14] for the study of $K3$ surfaces, we obtain our construction.

Note that, in the proof of our main result (Theorem 7.8), we do not use $\mathbb{X}$, and hence the result is rigorously correct. In fact, we can verify our result by direct computations. See Remark 7.10. Computational data relevant to this paper is presented in [15] in the format of `GAP` [10].

Notation. A lattice is a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear form that takes values in $\mathbb{Z}$. This bilinear form is called the *intersection form* of the lattice. For a lattice M with the intersection form $\langle\ ,\ \rangle_M$, let M^- denote the lattice whose underlying $\mathbb{Z}$-module is M and whose intersection form is equal to $\langle\ ,\ \rangle_M^- := -\langle\ ,\ \rangle_M$.

Let Λ be the Leech lattice. In this paper, we construct the *negative-definite* Leech lattice Λ^- from *negative-definite* Niemeier lattices N^- using the result on the lattice $II_{25,1}^- = II_{1,25}$. We make this change of sign because we want to use ideas coming from the geometry of algebraic surfaces.

2. Preliminaries

2.1. **Roots and reflections.** A lattice M is *even* if $\langle v, v\rangle_M \in 2\mathbb{Z}$ holds for all $v \in M$. Let M be an even lattice. A vector r of M is said to be a *root* if $|\langle r, r\rangle_M| = 2$. A lattice is said to be a *root lattice* if it is generated by roots. A root r of M with $\langle r, r\rangle_M = \pm 2$ defines the *reflection*

$$s_r \colon x \mapsto x \mp \langle x, r\rangle_M \cdot r,$$

which is an element of the automorphism group $\mathrm{O}(M)$ of the lattice M. Let $M^\vee$ denote the *dual lattice*

$$\{\, x \in M \otimes \mathbb{Q} \mid \langle x, v\rangle_M \in \mathbb{Z} \ \text{ for all } \ v \in M \,\}.$$

The *discriminant group* of M is the finite abelian group $M^\vee/M$. The group $\mathrm{O}(M)$ acts on $M^\vee$, and hence on $M^\vee/M$. Since $s_r(x) - x \in M$ holds for any $x \in M^\vee$, the reflections $s_r \in \mathrm{O}(M)$ act on $M^\vee/M$ trivially.

2.2. ADE**-configurations.** A (-2)*-vector* of an even lattice M is a vector $r \in M$ such that $\langle r, r\rangle_M = -2$. Let $\{r_1, \dots, r_m\}$ be a set of (-2)-vectors of an even lattice M such that $\langle r_i, r_j\rangle_M \in \{0, 1, 2\}$ holds for any i, j with $i \neq j$. The *dual graph* of $\{r_1, \dots, r_m\}$ is the graph whose set of nodes is $\{r_1, \dots, r_m\}$ and whose set of simple edges (resp. of double edges) is the set of pairs $\{r_i, r_j\}$ such that $\langle r_i, r_j\rangle_M = 1$ (resp. $\langle r_i, r_j\rangle_M = 2$). (A double edge appears only in extended ADE-configuration whose ADE-type contains A_1.) We say that $\{r_1, \dots, r_m\}$ forms an *ordinary* ADE*-configuration* (resp. an *extended* ADE*-configuration*) if the dual graph is the ordinary Dynkin diagram (resp. the extended Dynkin diagram) of type ADE, and in this case, we define the type of the configuration to be the type of the dual graph.

Let M be a negative-definite root lattice, and R the set of (-2)-vectors in M. We have $M = \langle R\rangle$. A *simple root system* of M is a subset of R that is a basis of M and that forms an ordinary ADE-configuration. A negative-definite root lattice always has a simple root system. The set of simple root systems of M is described as follows. We denote by $W(M)$ the subgroup of $\mathrm{O}(M)$ generated by all the reflections

s_r with respect to $r \in R$, and call it the *Weyl group* of M. We consider the unit sphere

$$S := \{ x \in M \otimes \mathbb{R} \mid \langle x, x \rangle_M = -1 \}.$$

For $r \in R$, we put

$$(r)^\perp := \{ x \in S \mid \langle x, r \rangle_M = 0 \}, \quad H^+(r) := \{ x \in S \mid \langle x, r \rangle_M > 0 \},$$

and set

$$S^\circ := S \ \setminus \ \bigcup_{r \in R} (r)^\perp.$$

Then $W(M)$ acts on the set of connected components of S° simple-transitively. A subset Θ of R is a simple root system of M if and only if the space

$$\Delta(\Theta) := \bigcap_{r \in \Theta} H^+(r)$$

is a connected component of S° and, for each $r \in \Theta$, the intersection $(r)^\perp \cap \overline{\Delta(\Theta)}$ contains a non-empty open subset of $(r)^\perp$, where $\overline{\Delta(\Theta)}$ is the closure of $\Delta(\Theta)$ in S. Therefore $W(M)$ acts on the set of simple root systems of M simple-transitively.

2.3. **The coefficients of the highest root.** Let $\widetilde{\Sigma}$ be a set of (-2)-vectors that form an extended ADE-configuration of type τ with the dual graph being *connected*, that is, we have $\tau = A_l$ or $\tau = D_m$ or $\tau = E_n$. We choose a vector $\theta \in \Sigma$ such that $\Sigma := \widetilde{\Sigma} \setminus \{\theta\}$ forms an ordinary ADE-configuration of type τ. Then Σ is a simple root system of the negative-definite root lattice $\langle \Sigma \rangle$. Let $\mu \in \langle \Sigma \rangle$ be the highest root with respect to Σ (see [9, Section 1.5] for the definition). We define a function $m \colon \widetilde{\Sigma} \to \mathbb{Z}_{>0}$ by

$$m(r) := \begin{cases} 1 & \text{if } r = \theta, \\ \text{the coefficient of } r \text{ in } \mu & \text{if } r \in \Sigma. \end{cases}$$

In fact, the function m does not depend on the choice of $\theta \in \widetilde{\Sigma}$. We have $m(r) = 1$ if and only if $\widetilde{\Sigma} \setminus \{r\}$ forms an ordinary ADE-configuration of type τ. The values of the function m for each connected ADE-type τ can be found in [8, Figure 23.1] or in [9, Figure 1.8].

Let $\mathbb{Z}^{\widetilde{\Sigma}}$ be the $\mathbb{Z}$-module of functions $\widetilde{\Sigma} \to \mathbb{Z}$. We can also define m as the function $\widetilde{\Sigma} \to \mathbb{Z}$ that takes values in $\mathbb{Z}_{>0}$ and that is the generator of the following $\mathbb{Z}$-submodule of of $\mathbb{Z}^{\widetilde{\Sigma}}$:

$$\{ \, (\ x(r)\ ) \in \mathbb{Z}^{\widetilde{\Sigma}} \mid \textstyle\sum_{r \in \widetilde{\Sigma}} x(r) \langle r, r' \rangle = 0 \ \text{ for all } r' \in \widetilde{\Sigma} \, \}.$$

2.4. **Hyperbolic lattices.** We say that a lattice M of rank $n > 1$ is *hyperbolic* if its signature is $(1, n-1)$.

Example 2.1. If n is a positive integer satisfying $n \equiv 2 \bmod 8$, then there exists an even unimodular hyperbolic lattice L_n of rank n, and L_n is unique up to isomorphism. The lattice $L_{26} \cong II_{1,25}$ plays a central role in this paper.

Example 2.2. The *hyperbolic plane* U is the even unimodular hyperbolic lattice L_2 of rank 2. We fix a basis u_0, u_1 of U such that the Gram matrix of U with respect to u_0, u_1 is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let M be an even hyperbolic lattice. A *positive cone* of M is one of the two connected components of the space $\{\, x \in M \otimes \mathbb{R} \mid \langle x, x \rangle_M > 0 \,\}$. Let $\mathcal{P}_M$ be a positive cone of M. We denote by $\overline{\mathcal{P}}_M$ the closure of $\mathcal{P}_M$ in $M \otimes \mathbb{R}$, and by $\partial \overline{\mathcal{P}}_M$ the boundary $\overline{\mathcal{P}}_M \setminus \mathcal{P}_M$. We put

$$\mathcal{R}_M := \{\, r \in M \mid \langle r, r \rangle_M = -2 \,\},$$

and, for $r \in \mathcal{R}_M$, let $(r)^\perp$ be the hyperplane $\{\, x \in \mathcal{P}_M \mid \langle x, r \rangle_M = 0 \,\}$ in $\mathcal{P}_M$. The *Weyl group* $W(M)$ is a subgroup of $\mathrm{O}(M)$ generated by all the reflections s_r associated with $r \in \mathcal{R}_M$. Then $W(M)$ acts on $\mathcal{P}_M$.

Definition 2.3. A *standard fundamental domain of* $W(M)$ is the closure in $\mathcal{P}_M$ of a connected component of

$$\mathcal{P}_M \setminus \bigcup_{r \in \mathcal{R}_M} (r)^\perp.$$

Let D be a standard fundamental domain of $W(M)$. We say that $r \in \mathcal{R}_M$ *defines a wall of* D if $D \cap (r)^\perp$ contains a non-empty open subset of $(r)^\perp$ and $\langle x, r \rangle_M \geq 0$ holds for all $x \in D$.

Note that, contrary to the case where M is a negative-definite root lattice, a standard fundamental domain of $W(M)$ in the positive cone of hyperbolic lattice may have infinitely many walls.

3. Niemeier lattices

A *Niemeier lattice* is an even positive-definite unimodular lattice of rank 24. Niemeier [11] classified Niemeier lattices. It was shown in [11] that, up to isomorphism, there exist exactly 24 Niemeier lattices, that one of them contains no roots, whereas each of the other 23 lattices contains a sublattice of finite index generated by roots. See also [6]. The Niemeier lattice containing no roots is called the *Leech lattice*, and is denoted by Λ.

Let N be a Niemeier lattice with roots. Let R be the set of (-2)-vectors of N^-, and $\langle R \rangle$ the sublattice of N^- generated by R. Then the *code* $N^-/\langle R \rangle$ of N^- is a subgroup of the discriminant group $\langle R \rangle^\vee / \langle R \rangle$. Let $\Theta \subset R$ be a simple root system of $\langle R \rangle$, and let $\tau(N)$ denote the ADE-type of Θ. We call $\tau(N)$ the ADE-*type* of N^-. Note that $\mathrm{O}(N^-)$ is a subgroup of $\mathrm{O}(\langle R \rangle)$. More precisely, we have

$$\mathrm{O}(N^-) = \{\, g \in \mathrm{O}(\langle R \rangle) \mid \text{the action of } g \text{ on } \langle R \rangle^\vee / \langle R \rangle \text{ preserves } N^-/\langle R \rangle \,\}.$$

Since the subgroup $W(\langle R \rangle)$ of $\mathrm{O}(\langle R \rangle)$ acts on $\langle R \rangle^\vee / \langle R \rangle$ trivially, we see that $W(\langle R \rangle)$ is contained in $\mathrm{O}(N^-)$. Since $W(\langle R \rangle)$ acts on the set of simple root systems of $\langle R \rangle$ transitively, we have the following:

Proposition 3.1. *The group* $\mathrm{O}(N^-)$ *acts on the set of simple root systems of* $\langle R \rangle$ *transitively.* □

We write

$$\tau(N) = \tau(N)_1 + \cdots + \tau(N)_K, \tag{3.1}$$

where $\tau(N)_1, \ldots, \tau(N)_K$ are the ADE-types of the connected components of the dual graph of Θ. Accordingly, we obtain the decompositions

$$\Theta = \Theta_1 \sqcup \cdots \sqcup \Theta_K, \quad R = R_1 \sqcup \cdots \sqcup R_K, \tag{3.2}$$

in such a way that $\Theta_i \subset R_i$ is a simple root system of the root lattice $\langle R_i \rangle$ of type $\tau(N)_i$. We put

$$n_i := \mathrm{Card}(\Theta_i) = \mathrm{rank}\,\langle R_i \rangle.$$

Then we have $24 = n_1 + \cdots + n_K$.

Definition 3.2. The *Coxeter number* h_i of R_i is defined by any of the following:

(a) $\mathrm{Card}(R_i) = n_i h_i$.

(b) Let $\rho_i \in \langle R_i \rangle \otimes \mathbb{Q}$ be the vector satisfying $\langle \rho_i, r \rangle_N^- = 1$ for all $r \in \Theta_i$. Then we have $\langle \rho_i, \rho_i \rangle_N^- = -n_i h_i (h_i + 1)/12$.

(c) Let $\mu_i \in R_i$ denote the highest root with respect to Θ_i. (See Section 1.5 of [9]). Then we have $h_i = \langle \mu_i, \rho_i \rangle_N^- + 1$, that is, $h_i - 1$ is the sum of the coefficients $m(r)$ of μ_i expressed as a linear combination of vectors $r \in \Theta_i$, where m is the function defined in Section 2.3.

(d) The product of all the reflections s_r with respect to $r \in \Theta_i$ (where the product is taken in arbitrary order) is of order h_i in $\mathrm{O}(\langle R_i \rangle)$.

A remarkable fact about the ADE-type $\tau(N)$ is that h_i does not depend on i. We put

$$h := h_1 = \cdots = h_K,$$

and call it the *Coxeter number* of N^-. (See Table 3.1.) We also put

$$\rho := \rho_1 + \cdots + \rho_K,$$

which is called a *Weyl vector* of N^-. By property (b) of h above, we have

$$\langle \rho, \rho \rangle_N^- = -2h(h+1). \tag{3.3}$$

Remark 3.3. The Weyl vector $\rho \in \langle R \rangle \otimes \mathbb{Q} = N^- \otimes \mathbb{Q}$ is in fact a vector of N^-. This fact can be easily confirmed by direct computation. Borcherds [1] gave a proof of this fact. See also Remark 7.7.

4. Deep holes

In this section, we review the classification of deep holes of the Leech lattice Λ due to Conway, Parker, Sloane [4], and the determination of the fundamental domain of the Weyl group $W(L_{26})$ of $L_{26} \cong II_{1,25}$ due to Conway [3].

For $x, y \in \Lambda \otimes \mathbb{R}$, we put

$$d(x,y) := \sqrt{\langle x-y, x-y \rangle_\Lambda}\,, \quad \text{and} \quad d(x,\Lambda) := \min{}_{\lambda \in \Lambda}\, d(x,\lambda).$$

The *covering radius* of Λ is defined to be the maximum of $d(x,\Lambda)$, where x runs through $\Lambda \otimes \mathbb{R}$. In [4], the following was proved:

Theorem 4.1. *The covering radius of Λ is $\sqrt{2}$.* □

Using Vinberg's algorithm [16] and Theorem 4.1, Conway [3] proved the following. Let $\langle\ ,\ \rangle_L$ denote the intersection form of $L_{26} \cong II_{1,25}$. Let $\mathcal{R}_L$ be the set of (-2)-vectors of L_{26}. We choose a positive cone $\mathcal{P}_L$ of L_{26}.

Definition 4.2. We call a standard fundamental domain of $W(L_{26})$ in $\mathcal{P}_L$ a *Conway chamber*. A non-zero primitive vector $w \in L_{26}$ is called a *Weyl vector* if

No.	τ	h	$N^-/\langle R\rangle$	$\langle R\rangle^\vee/\langle R\rangle$
1	D_{24}	46	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$
2	$3E_8$	30	0	0
3	$D_{16}+E_8$	30	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2$
4	A_{24}	25	$\mathbb{Z}/5\mathbb{Z}$	$\mathbb{Z}/25\mathbb{Z}$
5	$2D_{12}$	22	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^4$
6	$A_{17}+E_7$	18	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/18\mathbb{Z}$
7	$D_{10}+2E_7$	18	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^4$
8	$A_{15}+D_9$	16	$\mathbb{Z}/8\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/16\mathbb{Z}$
9	$3D_8$	14	$(\mathbb{Z}/2\mathbb{Z})^3$	$(\mathbb{Z}/2\mathbb{Z})^6$
10	$2A_{12}$	13	$\mathbb{Z}/13\mathbb{Z}$	$(\mathbb{Z}/13\mathbb{Z})^2$
11	$4E_6$	12	$(\mathbb{Z}/3\mathbb{Z})^2$	$(\mathbb{Z}/3\mathbb{Z})^4$
12	$A_{11}+D_7+E_6$	12	$\mathbb{Z}/12\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/12\mathbb{Z}$
13	$4D_6$	10	$(\mathbb{Z}/2\mathbb{Z})^4$	$(\mathbb{Z}/2\mathbb{Z})^8$
14	$2A_9+D_6$	10	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/10\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2\times(\mathbb{Z}/10\mathbb{Z})^2$
15	$3A_8$	9	$\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/9\mathbb{Z}$	$(\mathbb{Z}/9\mathbb{Z})^3$
16	$2A_7+2D_5$	8	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/8\mathbb{Z}$	$(\mathbb{Z}/4\mathbb{Z})^2\times(\mathbb{Z}/8\mathbb{Z})^2$
17	$4A_6$	7	$(\mathbb{Z}/7\mathbb{Z})^2$	$(\mathbb{Z}/7\mathbb{Z})^4$
18	$6D_4$	6	$(\mathbb{Z}/2\mathbb{Z})^6$	$(\mathbb{Z}/2\mathbb{Z})^{12}$
19	$4A_5+D_4$	6	$\mathbb{Z}/2\mathbb{Z}\times(\mathbb{Z}/6\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^2\times(\mathbb{Z}/6\mathbb{Z})^4$
20	$6A_4$	5	$(\mathbb{Z}/5\mathbb{Z})^3$	$(\mathbb{Z}/5\mathbb{Z})^6$
21	$8A_3$	4	$(\mathbb{Z}/4\mathbb{Z})^4$	$(\mathbb{Z}/4\mathbb{Z})^8$
22	$12A_2$	3	$(\mathbb{Z}/3\mathbb{Z})^6$	$(\mathbb{Z}/3\mathbb{Z})^{12}$
23	$24A_1$	2	$(\mathbb{Z}/2\mathbb{Z})^{12}$	$(\mathbb{Z}/2\mathbb{Z})^{24}$

TABLE 3.1. Niemeier lattices with roots

$w \in \partial\,\overline{\mathcal{P}}_L$ (in particular, we have $\langle w, w\rangle_L = 0$) and the lattice $(\mathbb{Z}w)^\perp/\mathbb{Z}w$ is isomorphic to Λ^-. For a Weyl vector w, we put

$$\begin{aligned}\mathcal{L}(w) &:= \{\, r \in \mathcal{R}_L \mid \langle w, r\rangle_L = 1\,\},\\ \mathcal{C}(w) &:= \{\, x \in \mathcal{P}_L \mid \langle x, r\rangle_L \ge 0 \ \text{ for all } \ r \in \mathcal{L}(w)\,\}.\end{aligned}$$

An element of $\mathcal{L}(w)$ is called a *Leech root* of the Weyl vector w.

Theorem 4.3 (Conway [3]). (1) *The mapping $w \mapsto \mathcal{C}(w)$ gives a bijection between the set of Weyl vectors and the set of Conway chambers.*

(2) *Let w be a Weyl vector. A (-2)-vector r of L_{26} defines a wall of the Conway chamber $\mathcal{C}(w)$ if and only if $r \in \mathcal{L}(w)$.* □

Let Co_∞ denote the group of affine isometries of the Leech lattice Λ. We have $\mathrm{Co}_\infty = \Lambda \rtimes \mathrm{O}(\Lambda)$, where Λ acts on Λ by translation. Let U_Λ be a copy of the hyperbolic plane U, and we put

$$L_\Lambda := U_\Lambda \oplus \Lambda^-,$$

which is isomorphic to L_{26}. We write elements of L_Λ as

$$(a, b, v)_\Lambda := au_0 + bu_1 + v, \quad \text{where } a, b \in \mathbb{Z} \text{ and } v \in \Lambda^-,$$

where u_0 and u_1 are the basis of U given in Example 2.2. Then the vector

$$w_\Lambda := (1, 0, 0)_\Lambda$$

is a Weyl vector of L_Λ, and the mapping

$$\lambda \;\mapsto\; r_\lambda := (\,-1-\lambda^2/2,\ 1,\ \lambda\,)_\Lambda, \quad \text{where } \lambda^2 = \langle \lambda, \lambda\rangle_\Lambda^-, \tag{4.1}$$

gives a bijection $\Lambda^- \cong \mathcal{L}(w_\Lambda)$. Then Theorem 4.3 implies the following:

Corollary 4.4. *The automorphism group*

$$\mathrm{O}(L_\Lambda, w_\Lambda) := \{\, g \in \mathrm{O}(L_\Lambda) \mid w_\Lambda^g = w_\Lambda \,\}$$

of the Conway chamber $\mathcal{C}(w_\Lambda)$ is isomorphic to Co_∞ via the bijection $r_\lambda \mapsto \lambda$. □

A point $c \in \Lambda^- \otimes \mathbb{R}$ is called a *deep hole* if c satisfies $d(c, \Lambda) = \sqrt{2}$. The group Co_∞ acts on the set of deep holes. In [4], deep holes are classified up to the action of Co_∞. For a deep hole $c \in \Lambda \otimes \mathbb{R}$, we put

$$P_0(c) := \{\, \lambda \in \Lambda^- \mid d(c, \lambda) = \sqrt{2}\,\},$$

and call it the set of *vertices* of c. We then consider the set

$$\Xi_0(c) := \{\, r_\lambda \in \mathcal{L}(w_\Lambda) \mid \lambda \in P_0(c)\,\}.$$

Theorem 4.5 (Conway, Parker, Sloane [4])**.** (1) *For each deep hole c, the set $\Xi_0(c)$ forms an extended* ADE*-configuration, and its type $\tau(c)$ is one of the* 23 *types $\tau(N)$ of Niemeier lattices N with roots.*

(2) *Conversely, for an* ADE*-type $\tau(N)$ of a Niemeier lattice N with roots, there exists a deep hole c such that $\tau(c) = \tau(N)$.*

(3) *Two deep holes c and c' are Co_∞-equivalent if and only if $\tau(c) = \tau(c')$.* □

Definition 4.6. The *Coxeter number of a deep hole c* is the Coxeter number of the Niemeier lattice N with roots such that $\tau(N) = \tau(c)$.

5. The shape of a Conway chamber

For each Co_∞-equivalence class of deep holes, we have computed a representative element c and its vertices $P_0(c)$ explicitly in [13]. The results are also presented on the web page [15]. Using this data, we can confirm the following:

Proposition 5.1. *Let $c \in \Lambda^- \otimes \mathbb{Q}$ be a deep hole, and h the Coxeter number of c. Then hc is a primitive vector of Λ^-, and $h\langle c, c\rangle_\Lambda^-/2$ is an integer.* □

Recall that $L_\Lambda = U_\Lambda \oplus \Lambda^-$. We consider the Conway chamber $\mathcal{C}(w_\Lambda)$ of L_Λ corresponding to the Weyl vector $w_\Lambda = (1, 0, 0)_\Lambda$. Let $\overline{\mathcal{C}}(w_\Lambda)$ be the closure of $\mathcal{C}(w_\Lambda)$ in $\overline{\mathcal{P}}_L$. Let $c \in \Lambda^- \otimes \mathbb{Q}$ be a deep hole with the Coxeter number h. We put

$$\bar{f}(c) := (\,-\langle c, c\rangle_\Lambda^-/2,\ 1,\ c\,)_\Lambda \;\in\; L_\Lambda \otimes \mathbb{Q}, \quad f(c) := h\bar{f}(c). \tag{5.1}$$

By Proposition 5.1, the vector $f(c)$ is a primitive vector of L_Λ with $\langle f(c), f(c)\rangle_L = 0$. Then we have the following.

Proposition 5.2. *The intersection $\overline{\mathcal{C}}(w_\Lambda) \cap \partial\,\overline{\mathcal{P}}_L$ is a union of the half-lines $\mathbb{R}_{\ge 0} w_\Lambda$ and $\mathbb{R}_{\ge 0} f(c)$, where c runs through the set of deep holes.*

Proof. It is obvious that $\mathbb{R}_{\ge 0} w_\Lambda \subset \overline{\mathcal{C}}(w_\Lambda) \cap \partial\,\overline{\mathcal{P}}_L$. Let ℓ be a point of $\partial\,\overline{\mathcal{P}}_L \setminus \mathbb{R}_{\ge 0} w_\Lambda$. Then we have $\langle \ell, w_\Lambda\rangle_L > 0$. Rescaling ℓ by a positive real number, we assume that $\langle \ell, w_\Lambda\rangle_L = 1$ so that we have

$$\ell = (\,-\langle v, v\rangle_\Lambda^-/2,\ 1,\ v\,)_\Lambda$$

for some $v \in \Lambda \otimes \mathbb{R}$. Then, for each $\lambda \in \Lambda^-$, we have

$$\langle \ell, r_\lambda \rangle_L = -1 - \frac{\langle \lambda, \lambda \rangle_\Lambda^-}{2} - \frac{\langle v, v \rangle_\Lambda^-}{2} + \langle v, \lambda \rangle_\Lambda^- = -1 + \frac{d(v,\lambda)^2}{2}. \tag{5.2}$$

Therefore ℓ belongs to $\overline{\mathcal{C}}(w_\Lambda)$ if and only if v is a deep hole c, and in this case, we have $\ell = \bar{f}(c)$. □

Let $c \in \Lambda^- \otimes \mathbb{Q}$ be a deep hole. We have $\langle f(c), r_\lambda \rangle_L \in \mathbb{Z}_{\geq 0}$ for any $\lambda \in \Lambda^-$. For $\nu \in \mathbb{Z}_{\geq 0}$, we put

$$\Xi_\nu(c) := \{\, r_\lambda \in \mathcal{L}(w_\Lambda) \mid \langle r_\lambda, f(c) \rangle_L = \nu \,\}.$$

By (5.2), we see that $\Xi_\nu(c)$ is in one-to-one correspondence with the set

$$P_\nu(c) := \{\, \lambda \in \Lambda^- \mid d(c,\lambda)^2 = 2(1 + \nu/h) \,\}$$

by the bijection $\lambda \mapsto r_\lambda$ between Λ^- and $\mathcal{L}(w_\Lambda)$. Note that these definitions are compatible with the definitions of $\Xi_0(c)$ and $P_0(c)$ in Section 4. The set $P_0(c)$ is the set of points of Λ nearest to c, and $P_1(c)$ is the set of points of Λ next nearest to c.

Remark 5.3. The intersection form $\langle\ ,\ \rangle_L$ of $L \otimes \mathbb{R}$ restricted to the affine subspace of $L_\Lambda \otimes \mathbb{R}$ defined by $\langle w_\Lambda, x \rangle_L = 1$ and $\langle f(c), x \rangle_L = \nu$ is an inhomogeneous quadratic form whose homogeneous part of degree 2 is negative-definite. Hence we can explicitly calculate the set $\Xi_\nu(c)$. Then we obtain the set $P_\nu(c)$.

We investigate the sets Ξ_0 and Ξ_1. Propositions 5.4, 5.5, and 5.6 below were observed in [7]. We can also confirm them by looking at the computational data in [15]. As will be explained in Section 6.2, they have geometric meanings in terms of the virtual $K3$ surface $\mathbb{X}$.

Recall that $\Xi_0(c)$ forms an extended ADE-configuration of type $\tau(c)$. We write $\tau(c)$ as

$$\tau(c) = \tau(c)_1 + \cdots + \tau(c)_K,$$

where $\tau(c)_i$ are the ADE-types of the connected components of the dual graph of $\Xi_0(c)$. Let

$$\Xi_0(c) = \Xi_0(c)_1 \sqcup \cdots \sqcup \Xi_0(c)_K$$

be the corresponding decomposition. Then we have a function $m\colon \Xi_0(c)_i \to \mathbb{Z}_{>0}$ defined in Section 2.3 for $i = 1, \dots, K$.

Proposition 5.4. *We have* $\displaystyle\sum_{r \in \Xi_0(c)_i} m(r) r = f(c)$ *for* $i = 1, \dots, K$. □

Next, we investigate the set $\Xi_1(c)$.

Proposition 5.5. *Let s be an element of $\Xi_1(c)$. Then, for each $i = 1, \dots, K$, there exists a unique element $\theta(i,s)$ of $\Xi_0(c)_i$ such that, for all $r \in \Xi_0(c)_i$, we have*

$$\langle r, s \rangle_L = \begin{cases} 1 & \text{if } r = \theta(i,s), \\ 0 & \text{otherwise.} \end{cases}$$

We then have $m(\theta(i,s)) = 1$, and hence

$$\Theta(c,s)_i := \Xi_0(c)_i \setminus \{\theta(i,s)\} \tag{5.3}$$

forms an ordinary ADE-*configuration of type $\tau(c)_i$.* □

We choose and fix an element

$$z \in \Xi_1(c), \tag{5.4}$$

and let $U(c,z)$ denote the hyperbolic plane in L_Λ generated by $f(c)$ and z. Its orthogonal complement $U(c,z)^\perp$ in L_Λ contains the set

$$\Theta(c,z) := \Theta(c,z)_1 \sqcup \cdots \sqcup \Theta(c,z)_K$$

of (-2)-vectors that form an ordinary ADE-configuration of type $\tau(c)$, where $\Theta(c,z)_i$ is defined by (5.3). Let N be the Niemeier lattice such that $\tau(c) = \tau(N)$. Since $U(c,z)^\perp$ is unimodular and of rank 24, the lattice

$$N^-(c,z) := U(c,z)^\perp$$

is isomorphic to N^-, and contains $\Theta(c,z)$ as a simple root system. Thus L_Λ has two orthogonal direct-sum decompositions

$$L_\Lambda = U_\Lambda \oplus \Lambda^- = U(c,z) \oplus N^-(c,z). \tag{5.5}$$

Let $\langle \Theta(c,z) \rangle$ be the sublattice of $N^-(c,z)$ generated by $\Theta(c,z)$, and we put

$$\mathit{\Gamma}(c,z) := N^-(c,z)/\langle \Theta(c,z) \rangle,$$

which is a finite abelian group isomorphic to the code of N^-. We have a natural homomorphism

$$L_\Lambda \;\to\; N^-(c,z) \;\to\; \mathit{\Gamma}(c,z), \tag{5.6}$$

where $L_\Lambda \to N^-(c,z)$ is the projection by the second decomposition (5.5).

Proposition 5.6. *The mapping* (5.6) *induces a bijection* $\Xi_1(c) \cong \mathit{\Gamma}(c,z)$. □

Remark 5.7. The (-2)-vector $\theta(i,z) \in \Xi_0(c)_i$ satisfies the following:
(a) $\langle f(c), \theta(i,z) \rangle_L = 0$, $\langle z, \theta(i,z) \rangle_L = 1$,
(b) if $j \neq i$, then $\langle r, \theta(i,z) \rangle_L = 0$ for all $r \in \Theta(c,z)_j$, and
(c) $\Theta(c,z)_i \cup \{\theta(i,z)\}$ forms an extended ADE-configuration of type $\tau(c)_i$.

Since $f(c)$, z and the 24 vectors in $\Theta(c,z)$ span $L_\Lambda \otimes \mathbb{Q}$, these properties characterize the vector $\theta(i,z) \in \Xi_0(c)_i$ uniquely.

6. The virtual $K3$ surface $\mathbb{X}$

The results in Section 5 can be interpreted as geometric results on a virtual, non-existing $K3$ surface $\mathbb{X}$.

6.1. $K3$ **surfaces.** First, we give a brief review of lattice theoretic aspects of the theory of (non-virtual) $K3$ surfaces. See the book [12, Chapter 11] for details. See also [14] for a review from a computational point of view.

For simplicity, we work over an algebraically closed field of characteristic $\neq 2, 3$. Let X be a $K3$ surface. We denote by S_X the Néron-Severi lattice of X, that is, S_X is the lattice of numerical equivalence classes of divisors on X. Let $\langle\ ,\ \rangle_S$ denote the intersection form of S_X. For a curve C on X, let $[C] \in S_X$ be the class of C. Suppose that the Picard number $\operatorname{rank} S_X$ is > 1. Then S_X is an even hyperbolic lattice. Let $\mathcal{P}_X$ be the positive cone of S_X that contains an ample class. The *nef-and-big cone* $\mathcal{N}_X$ of X is defined by

$$\mathcal{N}_X := \{\, x \in \mathcal{P}_X \mid \langle x, [C] \rangle_S \geq 0 \ \text{ for all curves } C \text{ on } X \,\}.$$

We denote by $\overline{\mathcal{N}}_X$ the closure of $\mathcal{N}_X$ in $\overline{\mathcal{P}}_X$. We put

$$\mathrm{Rats}(X) := \{\, r \in S_X \mid r \text{ is the class of a smooth rational curve on } X \,\}.$$

Then we have the following:

Proposition 6.1. (1) *The nef-and-big cone $\mathcal{N}_X$ is the standard fundamental domain of the Weyl group $W(S_X)$ containing an ample class.*

(2) *A (-2)-vector r of S_X belongs to* $\mathrm{Rats}(X)$ *if and only if r defines a wall of the standard fundamental domain $\mathcal{N}_X$.* □

Let f and z be vectors of S_X such that

$$\langle f, f\rangle_S = 0, \quad \langle f, z\rangle_S = 1, \quad \langle z, z\rangle_S = -2, \quad f \in \overline{\mathcal{N}}_X, \quad z \in \mathrm{Rats}(X).$$

Then we have an elliptic fibration $\phi\colon X \to \mathbb{P}^1$ with a section $\zeta\colon \mathbb{P}^1 \to X$ such that f is the class of a fiber of ϕ and that z is the class of the image of ζ. Let $U_{f,z}$ denote the hyperbolic plane in S_X generated by f and z, and let $U_{f,z}^{\perp}$ be the orthogonal complement of $U_{f,z}$ in S_X. Note that $U_{f,z}^{\perp}$ is negative-definite. Let $R(U_{f,z}^{\perp})$ be the set of (-2)-vectors in $U_{f,z}^{\perp}$, and $\langle R(U_{f,z}^{\perp})\rangle$ the sublattice of $U_{f,z}^{\perp}$ generated by $R(U_{f,z}^{\perp})$. We put

$$\widetilde{\Theta}_\phi := \{\, [C] \mid C \text{ is a smooth rational curve on } X \text{ mapped to a point by } \phi \,\}.$$

Then $\widetilde{\Theta}_\phi$ forms an extended ADE-configuration. Let

$$\widetilde{\Theta}_\phi = \widetilde{\Theta}_{\phi,1} \sqcup \cdots \sqcup \widetilde{\Theta}_{\phi,K}$$

be the decomposition according to the connected components of the dual graph of $\widetilde{\Theta}_\phi$. Then the fibration $\phi\colon X \to \mathbb{P}^1$ has exactly K reducible fibers $\phi^{-1}(p_1), \dots, \phi^{-1}(p_K)$, and, under an appropriate numbering of the points $p_1, \dots, p_K \in \mathbb{P}^1$, we have

$$[C] \in \widetilde{\Theta}_{\phi,i} \iff C \subset \phi^{-1}(p_i)$$

for a smooth rational curve C on X. Let $m\colon \widetilde{\Theta}_{\phi,i} \to \mathbb{Z}_{>0}$ be the function defined in Section 2.3. Then we have

$$\phi^*(p_i) = \sum_{C \subset \phi^{-1}(p_i)} m([C])C, \tag{6.1}$$

where C runs through the set of irreducible components of $\phi^{-1}(p_i)$. Let C_{i0} be the unique irreducible component of $\phi^{-1}(p_i)$ that intersects the zero section ζ, and we put

$$\Theta_{\phi,\zeta,i} := \widetilde{\Theta}_{\phi,i} \setminus \{[C_{i0}]\}.$$

Then each $\Theta_{\phi,\zeta,i}$ forms a connected ordinary ADE-configuration. We put

$$\Theta_{\phi,\zeta} := \Theta_{\phi,\zeta,1} \sqcup \cdots \sqcup \Theta_{\phi,\zeta,K}.$$

Then we have

$$\Theta_{\phi,\zeta} = R(U_{f,z}^{\perp}) \cap \mathrm{Rats}(X).$$

We can calculate the classes

$$[C_{i0}] = f - \sum_{r \in \Theta_{\phi,\zeta,i}} m(r)r$$

of smooth rational curves in fibers of ϕ that intersect ζ. Note that $\sum_{r \in \Theta_{\phi,\zeta,i}} m(r)r$ is the highest root of $\langle \Theta_{\phi,\zeta,i}\rangle$ with respect to $\Theta_{\phi,\zeta,i}$.

Let $\mathrm{MW}_{\phi,\zeta}$ be the Mordell-Weil group of the Jacobian fibration $\phi\colon X \to \mathbb{P}^1$ with the zero section $\zeta\colon \mathbb{P}^1 \to X$, that is, $\mathrm{MW}_{\phi,\zeta}$ is the group of sections $s\colon \mathbb{P}^1 \to X$ of ϕ with ζ being 0. For $s \in \mathrm{MW}_{\phi,\zeta}$, let $[s] \in \mathrm{Rats}(X)$ denote the class of the image of the section s. Then we have the following famous result. See [12].

Proposition 6.2. *The mapping $s \mapsto [s]$ induces an isomorphism*

$$\mathrm{MW}_{\phi,\zeta} \;\cong\; S_X/(U_{f,z} \oplus \langle \Theta_{\phi,\zeta} \rangle) \tag{6.2}$$

of abelian groups. □

Remark 6.3. We have an algorithm [14, Section 4] that calculates, for any $v \in S_X$, the class $[s_v] \in \mathrm{Rats}(X)$ of the section $s_v \in \mathrm{MW}_{\phi,\zeta}$ corresponding to the class of v modulo $U_{f,z} \oplus \langle \Theta_{\phi,\zeta} \rangle$ via (6.2).

6.2. **Geometry of the virtual $K3$ surface $\mathbb{X}$.** Let $\mathbb{X}$ be a *virtual* $K3$ surface with an isometry

$$S_{\mathbb{X}} \cong L_\Lambda. \tag{6.3}$$

Applying to $\mathbb{X}$ the results of $K3$ surfaces explained in Section 6.1, we obtain natural explanations to the results Propositions 5.4, 5.5, and 5.6 that were observed in [7]. Remark that such a $K3$ surface $\mathbb{X}$ does *not* exist, and the arguments below should be considered only as a heuristic guide.

By composing (6.3) with an element of the Weyl group $W(L_\Lambda)$, we can assume that (6.3) maps the nef-and-big cone $\mathcal{N}_{\mathbb{X}}$ of $\mathbb{X}$ to the Conway chamber $\mathcal{C}(w_\Lambda)$. In the following, we identify L_Λ with $S_{\mathbb{X}}$, and $\mathcal{C}(w_\Lambda)$ with $\mathcal{N}_{\mathbb{X}}$.

By Theorem 4.3 and Proposition 6.1, the set $\mathrm{Rats}(\mathbb{X})$ of the classes of smooth rational curves on $\mathbb{X}$ is equal to the set $\mathcal{L}(w_\Lambda)$ of Leech roots of w_Λ. The primitive vector $w_\Lambda \in S_{\mathbb{X}} \cap \overline{\mathcal{N}}_{\mathbb{X}}$ corresponds to the class of a fiber of an elliptic fibration

$$\phi_\Lambda\colon \mathbb{X} \to \mathbb{P}^1.$$

Since $\langle w_\Lambda, r_\lambda \rangle_L = 1$ for any $r_\lambda \in \mathcal{L}(w_\Lambda)$, every smooth rational curve on $\mathbb{X}$ is a section of ϕ_Λ. If we choose a smooth rational curve and consider it as a zero section $\zeta\colon \mathbb{P}^1 \to \mathbb{X}$ of ϕ_Λ, then, by Proposition 6.2, the Mordell-Weil group of this Jacobian fibration (ϕ_Λ, ζ) is isomorphic to $\mathbb{Z}^{24}$, because we have $\Theta_{\phi_\Lambda,\zeta} = \emptyset$.

Remark 6.4. More strongly, the Mordell-Weil lattice (see [12]) of (ϕ_Λ, ζ) is isomorphic to the Leech lattice Λ.

Let $c \in \Lambda^- \otimes \mathbb{Q}$ be a deep hole. We have defined in (5.1) a primitive vector $f(c) \in L_\Lambda$ generating a half-line in $\overline{\mathcal{C}}(w_\Lambda) \cap \partial\,\overline{\mathcal{P}}_L$. Then the vector $f(c) \in S_{\mathbb{X}} \cap \overline{\mathcal{N}}_{\mathbb{X}}$ is the class of a fiber of an elliptic fibration

$$\phi(c)\colon \mathbb{X} \to \mathbb{P}^1.$$

The set $\Xi_0(c)$ is the set of classes of smooth rational curves on $\mathbb{X}$ that are contained in fibers of $\phi(c)$. Let $\phi(c)^*(p_i)$ $(i = 1, \dots, K)$ be the reducible fibers of $\phi(c)$. Renumbering the points $p_1, \dots, p_K \in \mathbb{P}^1$ if necessary, we have that, for each $i = 1, \dots, K$, the set $\Xi_0(c)_i$ is the set of classes $[C]$ of irreducible components C of $\phi(c)^*(p_i)$ with $m([C]) \in \mathbb{Z}_{>0}$ being the multiplicity of C in the fiber $\phi(c)^*(p_i)$. The set $\Xi_1(c)$ is the set of classes of sections of $\phi(c)$. Hence Proposition 5.4 follows from (6.1), Proposition 5.5 follows from the fact that a section and a fiber intersect only at one point and with intersection multiplicity 1. We choose $z \in \Xi_1(c)$ as in (5.4). Then we have a section $\zeta\colon \mathbb{P}^1 \to \mathbb{X}$ whose class is z. We consider ζ as a zero section of $\phi(c)$. Then $\theta(i,z)$ is the class of the smooth rational curve in $\phi(c)^*(p_i)$ that intersects ζ, and hence $\Theta(c,z)$ is the set of classes of irreducible components of reducible fibers of $\phi(c)$ that are disjoint from ζ. Hence the Mordell-Weil group of the Jacobian fibration $(\phi(c),\zeta)$ is isomorphic to $\mathit{\Gamma}(c,z)=N^-(c,z)/\langle\Theta(c,z)\rangle$ by Proposition 6.2, and Proposition 5.6 also follows from Proposition 6.2.

7. Constructions of the Leech lattice

Let N be a Niemeier lattice with roots. Let

$$\Theta=\Theta_1\sqcup\cdots\sqcup\Theta_K \quad\subset\quad R \quad\subset\quad \langle R\rangle \quad\subset\quad N^-$$

and $\tau(N)=\tau(N)_1+\cdots+\tau(N)_K$ be defined as in Section 3. We give a construction of the Leech lattice Λ^- for each codeword γ of the finite abelian group $N^-/\langle R\rangle$.

Remark 7.1. By Proposition 3.1, this construction does not depend on the choice of the simple root system Θ of $\langle\Theta\rangle=\langle R\rangle$ up to the action of $\mathrm{O}(N^-)$.

First we present a lemma about the discriminant group of a negative-definite root lattice. Let $\Sigma=\{r_1,\dots,r_k\}$ be a set of (-2)-vectors that form a *connected* ordinary ADE-configuration, and let $\langle\Sigma\rangle$ denote the lattice generated by Σ. Let $m\colon\Sigma\to\mathbb{Z}_{>0}$ be the function defined in Section 2.3. We put

$$J(\Sigma):=\{\,j\mid m(r_j)=1\,\}\quad\subset\quad\{1,\dots,k\}.$$

Let $r_1^\vee,\dots,r_k^\vee$ be the basis of $\langle\Sigma\rangle^\vee$ dual to the basis $r_1,\dots,r_k$ of $\langle\Sigma\rangle$. Then the following can be checked easily for each connected ADE-configuration Σ.

Lemma 7.2. *The mapping $j\mapsto r_j^\vee \bmod \langle\Sigma\rangle$ gives rise to a bijection from $J(\Sigma)$ to the set of non-zero elements of the discriminant group $\langle\Sigma\rangle^\vee/\langle\Sigma\rangle$.* □

Definition 7.3. For a codeword $\alpha\in\langle\Sigma\rangle^\vee/\langle\Sigma\rangle$, we define its *canonical representative* $\tilde{\alpha}\in\langle\Sigma\rangle^\vee$ by the following: if $\alpha=0$, then $\tilde{\alpha}=0$, and if $\alpha\neq0$, then $\tilde{\alpha}=r_j^\vee$, where $j\in J(\Sigma)$ corresponds to α by the bijection in Lemma 7.2.

We put

$$A_i:=\langle\Theta_i\rangle^\vee/\langle\Theta_i\rangle.$$

By the natural embedding

$$N^-\hookrightarrow\langle R\rangle^\vee=\langle\Theta_1\rangle^\vee\oplus\cdots\oplus\langle\Theta_K\rangle^\vee, \tag{7.1}$$

we have an embedding

$$N^-/\langle R\rangle \quad\hookrightarrow\quad A_1\times\cdots\times A_K. \tag{7.2}$$

Let $\gamma\in N^-/\langle R\rangle$ be a codeword. For $i=1,\dots,K$, let $\gamma_i\in A_i$ be the ith component of γ by the embedding (7.2). Let $\tilde{\gamma}_i\in\langle\Theta_i\rangle^\vee$ be the canonical representative of γ_i, and we put

$$v_\gamma:=\tilde{\gamma}_1+\cdots+\tilde{\gamma}_K\quad\in\quad\langle\Theta_1\rangle^\vee\oplus\cdots\oplus\langle\Theta_K\rangle^\vee=\langle R\rangle^\vee.$$

Then we have the following:

Proposition 7.4. *We have $v_\gamma\in N^-$.*

Proof. Let U_N be a copy of the hyperbolic plane U, and we put

$$L_N := U_N \oplus N^-.$$

The intersection form on L_N is denoted by $\langle\ ,\ \rangle_L$. Let u_0, u_1 be the basis of U_N given in Example 2.2. A vector of L_N is written as

$$(a, b, v)_N = au_0 + bu_1 + v, \quad \text{where } a, b \in \mathbb{Z} \text{ and } v \in N^-. \tag{7.3}$$

By $v \mapsto (0, 0, v)_N$, we regard N^- as a sublattice of L_N. In particular, we have $R \subset L_N$ and $\Theta \subset L_N$. We then put

$$f_N := (1, 0, 0)_N \in L_N, \quad z_N := (-1, 1, 0)_N \in L_N.$$

Note that L_N is isomorphic to L_Λ. We construct an isometry $L_\Lambda \cong L_N$ explicitly by means of deep holes. Let $c \in \Lambda^- \otimes \mathbb{Q}$ be a deep hole such that $\tau(c) = \tau(N)$. Recall from Section 5 that we have defined subsets $\Xi_0(c)$ and $\Xi_1(c)$ of the set $\mathcal{L}(w_\Lambda)$ of the Leech roots of w_Λ, and for a fixed element $z \in \Xi_1(c)$, we constructed an orthogonal direct-sum decomposition

$$L_\Lambda = U(c, z) \oplus N^-(c, z).$$

Note that $N^-(c, z)$ is isomorphic to N^-. By Proposition 3.1, after renumbering the connected components $\Theta_1, \dots, \Theta_K$ of the simple root system Θ of N^- if necessary, we have an isometry $N^-(c, z) \cong N^-$ that maps $\Theta(c, z)_i$ to Θ_i for $i = 1, \dots, K$. Then we obtain an isometry

$$L_\Lambda \cong L_N \ \text{ satisfying } \ f(c) \mapsto f_N, \ \ z \mapsto z_N, \ \ \Theta(c, z)_i \xrightarrow{\sim} \Theta_i \ (i = 1, \dots, K). \tag{7.4}$$

For $i = 1, \dots, K$, let $\mu_i \in \langle \Theta_i \rangle$ be the highest root with respect to Θ_i. We put

$$\theta_i := (1, 0, -\mu_i)_N.$$

The (-2)-vector $\theta_i \in L_N$ satisfies the following:

(a) $\langle f_N, \theta_i \rangle_L = 0$, $\langle z_N, \theta_i \rangle_L = 1$,
(b) if $j \neq i$, then $\langle r, \theta_i \rangle_L = 0$ for all $r \in \Theta_j$, and
(c) $\Theta_i \cup \{\theta_i\}$ form an extended ADE-configuration of type τ_i.

By Remark 5.7, we see that $\theta_i \in L_N$ corresponds to $\theta(i, z) \in \Xi_0(c)_i \subset L_\Lambda$ via the isometry (7.4) given above. We put

$$\widetilde{\Theta}_i := \Theta_i \cup \{\theta_i\}.$$

Then the isometry (7.4) induces a bijection from $\Xi_0(c)_i$ to $\widetilde{\Theta}_i$.

The isometry (7.4) induces an isomorphism

$$\mathit{\Gamma}(c, z) = N^-(c, z)/\langle \Theta(c, z) \rangle \cong N^-/\langle R \rangle. \tag{7.5}$$

Let $s'_\gamma \in \Xi_1(c)$ denote the (-2)-vector of L_Λ that corresponds to the codeword $\gamma \in N^-/\langle R \rangle$ via the isomorphism (7.5) and the bijection $\Xi_1(c) \cong \mathit{\Gamma}(c, z)$ in Proposition 5.6. Let s_γ be the (-2)-vector of L_N that corresponds to the (-2)-vector $s'_\gamma \in L_\Lambda$ via the isomorphism (7.4). Since $\langle f_N, s_\gamma \rangle_L = \langle f(c), s'_\gamma \rangle_L = 1$ and $\langle s_\gamma, s_\gamma \rangle_L = -2$, there exists a vector $u_\gamma \in N^-$ such that

$$s_\gamma = (a, 1, u_\gamma)_N, \quad \text{where } a = -1 - \langle u_\gamma, u_\gamma \rangle_N^-/2.$$

By Proposition 5.5 transplanted to $s_\gamma \in L_N$ from $s'_\gamma \in L_\Lambda$, we see that the ith component of u_γ by the embedding (7.1) is equal to the canonical representative

$\tilde{\gamma}_i$. Indeed, Proposition 5.5 implies that there exists a unique element $r_\gamma \in \widetilde{\Theta}_i$ such that $m(r_\gamma) = 1$ and that, for any $r \in \widetilde{\Theta}_i$, we have

$$\langle s_\gamma, r\rangle_L = \begin{cases} 1 & \text{if } r = r_\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Hence s_γ intersects the elements of Θ_i with the same intersection numbers as the canonical representative $\tilde{\gamma}_i \in \langle\Theta_i\rangle^\vee$ does. (If $r_\gamma = \theta_i$, then we have $\tilde{\gamma}_i = 0$.) Therefore we obtain $v_\gamma = u_\gamma \in N^-$. □

Definition 7.5. We call $v_\gamma \in N^-$ the *canonical representative* of $\gamma \in N^-/\langle R\rangle$.

Recall from Section 3 that h is the Coxeter number of N^- and that $\rho \in N^- \otimes \mathbb{Q}$ is the Weyl vector of N^- with respect to Θ.

Proposition 7.6. *The vector*

$$w_N := (h+1, h, \rho)_N$$

corresponds to the Weyl vector $w_\Lambda \in L_\Lambda$ via the isometry (7.4). In particular, the lattice $(\mathbb{Z}w_N)^\perp/\mathbb{Z}w_N$ is isomorphic to Λ^-.

Remark 7.7. The vector w_N appeared in Borcherds [1]. Proposition 7.6 gives a proof that $\rho \in N^- \otimes \mathbb{Q}$ is in fact $\rho \in N^-$.

Proof. Note that equality (3.3) implies $\langle w_N, w_N\rangle_L = 0$. From defining property (b) of ρ_i and property (c) of $h = h_i$ in Definition 3.2, we see that the vector $w_N \in L_N \otimes \mathbb{Q}$ satisfies $\langle w_N, z_N\rangle_L = 1$ and

$$\langle w_N, r\rangle_L = 1 \quad \text{for all } r \in \Theta, \qquad \langle w_N, \theta_i\rangle_L = 1 \quad \text{for } i = 1, \dots, K.$$

Since z_N, r ($r \in \Theta$) and θ_i ($i = 1, \dots, K$) span $L_N \otimes \mathbb{Q}$ and correspond, via the isometry (7.4), to Leech roots of w_Λ, we see that w_N corresponds to w_Λ via the isometry (7.4). □

Theorem 7.8. *Let γ be a codeword of the code $N^-/\langle R\rangle$, and let $v_\gamma \in N^-$ be the canonical representative of γ. We put*

$$n_\gamma := \langle v_\gamma, v_\gamma\rangle^-_N, \quad a_\gamma := 2h + 1 + h\, n_\gamma/2.$$

We define linear forms $\alpha_0 \colon N^- \to \mathbb{Q}$ and $\alpha_1 \colon N^- \to \mathbb{Q}$ by

$$\begin{aligned} \alpha_0(u) &:= \langle h v_\gamma - \rho, u\rangle^-_N / a_\gamma, \\ \alpha_1(u) &:= (1 + n_\gamma/2)\, \alpha_0(u) - \langle v_\gamma, u\rangle^-_N, \end{aligned}$$

and put

$$\Lambda^-(\gamma) := \{\, u \in N^- \mid \alpha_0(u) \in \mathbb{Z} \,\}.$$

Then the $\mathbb{Z}$-module $\Lambda^-(\gamma)$ with the intersection form

$$\langle u, u'\rangle := \langle u, u'\rangle^-_N + \alpha_0(u)\alpha_1(u') + \alpha_1(u)\alpha_0(u') \tag{7.6}$$

is isomorphic to the negative-definite Leech lattice Λ^-.

Proof. Recall that $s_\gamma = (-1 - n_\gamma/2, 1, v_\gamma)_N \in L_N$ is the vector corresponding, via the isometry (7.4), to the Leech root $s'_\gamma \in \Xi_1(c)$ of w_Λ. Let $U(w_N, s_\gamma)$ be the hyperbolic plane in L_N generated by w_N and s_γ. Since w_N is a Weyl vector of L_N, the orthogonal complement $U(w_N, s_\gamma)^\perp \cong (\mathbb{Z}w_N)^\perp/\mathbb{Z}w_N$ is isomorphic to the negative-definite Leech lattice Λ^-. A vector $(x, y, u)_N$ of $L_N \otimes \mathbb{Q}$ is orthogonal to both of w_N and s_γ if and only if

$$x = \alpha_1(u), \quad y = \alpha_0(u).$$

Hence the image of the orthogonal projection

$$U(w_N, s_\gamma)^\perp \hookrightarrow L_N = U_N \oplus N^- \to N^-$$

is equal to the $\mathbb{Z}$-submodule $\mathit{\Lambda}^-(\gamma)$ of N^-, and the restriction of $\langle\ ,\ \rangle_L$ to $U(w_N, s_\gamma)^\perp$ gives rise to the intersection form (7.6) on $\mathit{\Lambda}^-(\gamma)$. $\square$

Remark 7.9. In terms of $\mathbb{X}$, the construction above is described as follows. The sublattice $U_N \subset L_N$ yields a Jacobian fibration of $\mathbb{X}$ by the isometry $L_N \cong L_\Lambda = S_{\mathbb{X}}$ given in (7.4), and the (-2)-vector s_γ is the class of the image of the element of the Mordell-Weil group corresponding to γ.

Considering the case where $\gamma = 0$ in Theorem 7.8, and changing signs of intersection forms of lattices, we obtain Corollary 1.1 in Introduction.

Remark 7.10. Since the Leech lattice Λ^- is characterized, up to isomorphism, as the unique even unimodular negative-definite lattice of rank 24 with no roots (see [2]), we can confirm Theorem 7.8 by direct computation, once we compute canonical representatives v_γ of codewords $\gamma \in N^-/\langle R \rangle$ explicitly.

The canonical representatives are computed as follows. The set of (-2)-vectors $s_\gamma \in L_N$, where γ runs through $N^-/\langle R \rangle$, is equal to

$$\{\, r \in L_N \mid\ \langle f_N, r \rangle_L = \langle w_N, r \rangle_L = 1,\ \langle r, r \rangle = -2\ \},$$

and, as was explained in Remark 5.3, this set can be computed easily as the set of integer solutions of a negative-definite inhomogeneous quadratic form. The set of canonical representatives v_γ is then obtained from this set by the projection $L_N \to N^-$.

We can also use the algorithm for the study of elliptic $K3$ surfaces described in [14, Section 4]. See Remark 6.3.

References

[1] R. E. Borcherds. The Leech lattice. *Proc. Roy. Soc. London Ser. A*, 398(1815):365–376, 1985.

[2] J. H. Conway. A characterisation of Leech's lattice. *Invent. Math.*, 7:137–142, 1969. Chapter 12 of [8].

[3] J. H. Conway. The automorphism group of the 26-dimensional even unimodular Lorentzian lattice. *J. Algebra*, 80(1):159–163, 1983. Chapter 27 of [8].

[4] J. H. Conway, R. A. Parker, and N. J. A. Sloane. The covering radius of the Leech lattice. *Proc. Roy. Soc. London Ser. A*, 380(1779):261–290, 1982. Chapter 23 of [8].

[5] J. H. Conway and N. J. A. Sloane. Lorentzian forms for the Leech lattice. *Bull. Amer. Math. Soc. (N.S.)*, 6(2):215–217, 1982. Chapter 26 of [8].

[6] J. H. Conway and N. J. A. Sloane. On the enumeration of lattices of determinant one. *J. Number Theory*, 15(1):83–94, 1982. Chapter 16 of [8].

[7] J. H. Conway and N. J. A. Sloane. Twenty-three constructions for the Leech lattice. *Proc. Roy. Soc. London Ser. A*, 381(1781):275–283, 1982. Chapter 24 of [8].

[8] J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups*, volume 290 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, third edition, 1999.

[9] Wolfgang Ebeling. *Lattices and codes*. Advanced Lectures in Mathematics. Springer Spektrum, Wiesbaden, third edition, 2013. A course partially based on lectures by Friedrich Hirzebruch.

[10] The GAP Group. GAP - Groups, Algorithms, and Programming. Version 4.11.0 of 29-Feb-2020 (http://www.gap-system.org).

[11] Hans-Volker Niemeier. Definite quadratische Formen der Dimension 24 und Diskriminante 1. *J. Number Theory*, 5:142–178, 1973.

[12] Matthias Schütt and Tetsuji Shioda. *Mordell-Weil lattices*, volume 70 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Singapore, 2019.

[13] Ichiro Shimada. Holes of the Leech lattice and the projective models of $K3$ surfaces. *Math. Proc. Cambridge Philos. Soc.*, 163(1):125–143, 2017.

[14] Ichiro Shimada. Mordell-Weil groups and automorphism groups of elliptic $K3$ surfaces, 2022. Preprint, arXiv: 2210.01328.

[15] Ichiro Shimada. Explanation of the computation data for the paper "A note on construction of the Leech lattice", 2023.
https://www.math.sci.hiroshima-u.ac.jp/~shimada/lattice.html.

[16] È. B. Vinberg. Some arithmetical discrete groups in Lobačevskiĭ spaces. In *Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973)*, pages 323–348. 1975.

Department of Mathematics, Graduate School of Science, Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526 JAPAN

Email address: ichiro-shimada@hiroshima-u.ac.jp