# Four infinite families of chiral 3-polytopes of type $\{4,8\}$ with solvable automorphism groups 

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#### Abstract

We construct four infinite families of chiral 3 -polytopes of type $\{4,8\}$, with $1024 m^{4}, 2048 m^{4}, 4096 m^{4}$ and $8192 m^{4}$ automorphisms for every positive integer $m$, respectively. The automorphism groups of these polytopes are solvable groups, and when $m$ is a power of 2 , they provide examples with automorphism groups of order $2^{n}$ where $n \geq 10$. (On the other hand, no chiral polytopes of type $\{4,8\}$ exist for $n \leq 9$.) In particular, our families give a partial answer to a problem proposed by Schulte and Weiss in [Problems on polytopes, their groups, and realizations, Period. Math. Hungar. 53 (2006), 231-255] and a problem proposed by Pellicer in [Developments and open problems on chiral polytopes, Ars Math. Contemp 5 (2012), 333-354].


Keywords: Chiral 3-polytope, Automorphism group, 2-group.

## 1 Introduction

In [15], Schulte and Weiss proposed the following problem:
Problem 1.1 Characterize the groups of orders $2^{n}$, with $n$ a positive integer, which are automorphism groups of regular or chiral polytopes.

Let $\mathcal{P}$ be a regular or chiral $d$-polytope, whose automorphism group has order $2^{n}$ with type $\left\{2^{k_{1}}, 2^{k_{2}}, \cdots, 2^{k_{d}}\right\}$. Here $k_{i}>1$ for $i \in\{1,2, \cdots, d\}$. The atlas [2] contains information about all regular or chiral polytopes with automorphism group of order at most 2000. The first author, Feng and Lemmans in [9, 10] shows that if $k_{1}+k_{2}+\cdots+k_{d} \leq n-1$, then there exists an regular $d$-polytope of order $2^{n}$ with type $\left\{2^{k_{1}}, 2^{k_{2}}, \cdots, 2^{k_{d}}\right\}$ for $n \geq 10$. That means all possible type can be achieved for regular polytope of order $2^{n}$. However,

[^0]there are just a few results for chiral polytopes of order $2^{n}$, see [6, 17]. Even for the case $d=3$, to the best of our knowledge, the only infinite family is the so-called tight chiralpolytopes, that is, it has type $\left\{2^{k_{1}}, 2^{k_{2}}\right\}$ and has order $2^{n}=2^{k_{1} k_{2}}$, see [8]. On the other hand, by the famous book [7, Chapter 8], one can see that $\left\{2^{k_{1}}, 2^{k_{2}}\right\} \neq\{4,4\}$. Inspired by these results listed above, we naturally consider the following problem:
Problem 1.2 what is the smallest type of chiral 3 -polytopes of order $2^{n}$ ?
Here we construct four families of chiral 3-polytopes with type $\{4,8\}$. Each family contains one example with $1024 m^{4}, 2048 m^{4}, 4096 m^{4}$ or $8192 m^{4}$ automorphisms, respectively, for every integer $m \geq 1$. In particular, if we let $m$ be an arbitrary power of 2 , say $2^{k}$ (with $k \geq 0$ ), the the automorphism group has order $2^{10+4 k}, 2^{11+4 k}, 2^{12+4 k}$ or $2^{13+4 k}$, which can be expressed as $2^{n}$ for an arbitrary integer $n \geq 10$. It means that the smallest type of chiral 3 -polytopes of order $2^{n}$ is $\{4,8\}$ for $n \geq 10$. Hence, these results givie an answer to problem 1.2 and a partial answer to problem 1.1.

On the other hand, chiral 3-polytopes are also known as chiral maps. The genus of a chiral map is the genus of the carrier surface. The genus $g$ and type $\left\{k_{1}, k_{2}\right\}$ of a chiral map are related via the Euler-Poincaré formula by $2-2 g=\chi=|G|\left(\frac{1}{k_{1}}+\frac{1}{k_{2}}-\frac{1}{2}\right)$, where $\chi$ is the Euler characteristic of the carrier surface and $G$ is the automorphism group of chiral maps. All chiral maps on the tours was described by Coxeter in [7]. In fact, no chiral maps lies on a non-orientable surface and there are no chiral maps on orientable surfaces from genus 2 to genus 6. These resultes were extended up to genus 15 by Conder and Dobcsányi, with the help of computational methods, and then more recently by Conder much further, up to genus 301, see [3, 4]. In [13], Pellicer proposed the following problem:
Problem 1.3 Determine all positive integers $g$ for which there are chiral maps(polyhedra) on orientable surfaces with genus $g$.

It was proved in [5] that orientable surfaces with genera $p+1,(\chi=-2 p)$, with $p$ prime and $p-1$ not divisible by 3,5 or 8 , do not admit chiral maps. Our chiral polytopes shows that for every $n \geq 10$, there exists a chiral polytope(map) of type $\{4,8\}$ on an orientable surfaces with genera $g=2^{n-4}+1 \geq 65,\left(\chi=-2^{n-3}\right)$, with $2^{n}$ automorphisms. Hence, these results given a partial answer to problem 1.3 .

Our main result is the following theorem:
Theorem 1.4 For every positive integer $m \geq 1$, there exist chiral 3-polytopes $\mathcal{P}_{m}^{1}, \mathcal{P}_{m}^{2}$, $\mathcal{P}^{3}$ and $\mathcal{P}_{m}^{4}$ of type $\{4,8\}$ with solvable automorphism groups of order $1024 m^{4}, 2048 m^{4}$, $4096 m^{4}$ and $8192 m^{4}$, respectively.

As a special case we have the following Corollary, which is an immediate consequence of Theorem 1.4 when $m$ is taken as a power of 2 .
Corollary 1.5 For $n \geq 10$, the smallest type of chiral 3-polytopes of order $2^{n}$ is $\{4,8\}$, and the corresponding surfaces has Euler characteristic $\chi=-2^{n-3}\left(g=2^{n-4}+1\right)$.

Together with Conder's Date [2, 4], we have the following Corollary.
Corollary 1.6 For any $l \geq 0$, let $\mathcal{S}$ be an orientable surface with Euler characteristic $\chi=-2^{l}$, then there exists a chiral map on $\mathcal{S}$ if and only if $l \neq 1,2,3$ and 4 , that is $\chi \neq-2,-4,-8$ and -16 .

## 2 Additional background

In this section we give some further background that may be helpful for the rest of the paper, see [12, 14, 16].

### 2.1 Abstract polytopes: definition, structure and properties

An abstract polytope of rank $n$ is a partially ordered set $\mathcal{P}$ endowed with a strictly monotone rank function with range $\{-1,0, \cdots, n\}$, which satisfies four conditions, to be given shortly.

The elements of $\mathcal{P}$ are called faces of $\mathcal{P}$. More specifically, the elements of $\mathcal{P}$ of rank $j$ are called $j$-faces, and a typical $j$-face is denoted by $F_{j}$. Two faces $F$ and $G$ of $\mathcal{P}$ are said to be incident with each other if $F \leq G$ or $F \geq G$ in $\mathcal{P}$. A chain of $\mathcal{P}$ is a totally ordered subset of $\mathcal{P}$, and is said to have length $i$ if it contains exactly $i+1$ faces. The maximal chains in $\mathcal{P}$ are called the flags of $\mathcal{P}$. Two flags are said to be $j$-adjacent if they differ in just one face of rank $j$, or simply adjacent (to each other) if they are $j$-adjacent for some $j$. Also if $F$ and $G$ are faces of $\mathcal{P}$ with $F \leq G$, then the set $\{H \in \mathcal{P} \mid F \leq H \leq G\}$ is called a section of $\mathcal{P}$, and is denoted by $G / F$. Such a section has rank $m-k-1$, where $m$ and $k$ are the ranks of $G$ and $F$ respectively. A section of rank $d$ is called a $d$-section.

We can now give the four conditions that are required of $\mathcal{P}$ to make it an abstract polytope. These are listed as (P1) to (P4) below:
(P1) $\mathcal{P}$ contains a least face and a greatest face, denoted by $F_{-1}$ and $F_{n}$, respectively.
(P2) Each flag of $\mathcal{P}$ has length $n+1$ (so has exactly $n+2$ faces, including $F_{-1}$ and $F_{n}$ ).
(P3) $\mathcal{P}$ is strong flag-connected, which means that any two flags $\Phi$ and $\Psi$ of $P$ can be joined by a sequence of successively adjacent flags $\Phi=\Phi_{0}, \Phi_{1}, \cdots, \Phi_{k}=\Psi$, each of which contains $\Phi \cap \Psi$.
(P4) The rank 1 sections of $\mathcal{P}$ have a certain homogeneity property known as the diamond condition, namely as follows: if $F$ and $G$ are incidence faces of $\mathcal{P}$, of ranks $i-1$ and $i+1$, respectively, where $0 \leq i \leq n-1$, then there exist precisely two $i$-faces $H$ in $\mathcal{P}$ such that $F<H<G$.

An easy case of the diamond condition occurs for polytopes of rank 3 (or polyhedra): if $v$ is a vertex of same face $f$, then there are two edges that are incident with both $v$ and $f$.

If $F_{n-1}$ is a facet ( of rank $n-1$ ), then the section $F_{n-1} / F_{-1}$ is also called a facet of $\mathcal{P}$, while if $F_{0}$ is a vertex, then the section $F_{n} / F_{0}=\left\{G \in \mathcal{P} \mid F_{0} \leq G\right\}$ is called a vertex-figure of $\mathcal{P}$ at $F_{0}$. Next, every 2 -section $G / F$ of $\mathcal{P}$ is isomorphic to the face lattice of a polygon. Now if it happens that the number of sides of every such polygon depends only on the rank of $G$, and not on $F$ or $G$ itself, then we say that the polytope $\mathcal{P}$ is equivelar. In this case, if $k_{i}$ is the number of edges of every 2 -section between an $(i-2)$-face and an $(i+1)$-face of $\mathcal{P}$, for $1 \leq i \leq n$, then the expression $\left\{k_{1}, k_{2}, \cdots, k_{n-1}\right\}$ is called the Schläfli type of $\mathcal{P}$. (For example, if $\mathcal{P}$ has rank 3 , then $k_{1}$ and $k_{2}$ are the valency of each vertex and the size of each face, respectively.)

### 2.2 Automorphisms of polytopes

An automorphism of an abstract polytope $\mathcal{P}$ is an order-preserving permutation of its elements. In particular, every automorphism preserves the set of faces of any given rank. Under permutation composition, the set of all automorphisms of $\mathcal{P}$ forms a group, called the automorphism group of $\mathcal{P}$, and denoted by $\operatorname{Aut}(\mathcal{P})$ or sometimes more simply as $\Gamma(\mathcal{P})$. Also it is not difficult to use the diamond condition and strong flag-connectedness to prove that if an automorphism preserves of flag of $\mathcal{P}$, then it fixes every flag of $\mathcal{P}$ and hence every element of $\mathcal{P}$. It follows that $\Gamma(\mathcal{P})$ acts semi-regularly on flags of $\mathcal{P}$.

A polytope $\mathcal{P}$ is said to be regular if its automorphism group $\Gamma(\mathcal{P})$ acts transitively (and hence regularly) on the set of flags of $\mathcal{P}$. In this case, the number of automorphisms of $\mathcal{P}$ is as large as possible, and equal to the number of flags of $\mathcal{P}$. In particular, $\mathcal{P}$ is equivelar, and the stabiliser in $\Gamma(\mathcal{P})$ of every 2 -section of $\mathcal{P}$ induces the full dihedral group on the corresponding polygon. Moreover, for a given flag $\Phi$ and for every $i \in\{0,1, \ldots, n-1\}$, the polytope $\mathcal{P}$ has a unique automorphism $\rho_{i}$ that takes $\Phi$ to the unique flag $(\Phi)^{i}$ that differs from $\Phi$ in precisely its $i$-face, and then the automorphisms $\rho_{0}, \rho_{1}, \ldots, \rho_{n-1}$ generate $\Gamma(\mathcal{P})$ and satisfy the defining relations for the string Coxeter group $\left[k_{1}, k_{2}, \cdots, k_{n-1}\right]$, where the $k_{i}$ are as given in the previous subsection for the Schläfli type of $\mathcal{P}$. Here, the string Coxeter group $\left[k_{1}, k_{2}, \cdots, k_{n-1}\right]$, is defined as the group with presentation

$$
\begin{gathered}
\left\langle\rho_{0}, \rho_{1}, \cdots, \rho_{n-1}\right| \rho_{i}^{2}=1 \text { for } 0 \leq i \leq n-1,\left(\rho_{i} \rho_{i+1}\right)^{k_{i+1}}=1 \text { for } 0 \leq i \leq n-2, \\
\left.\left(\rho_{i} \rho_{j}\right)^{2}=1 \text { for } 0 \leq i<j-1<n-1\right\rangle .
\end{gathered}
$$

They also satisfy a certain 'intersection condition', which follows from the diamond and strong flag-connectedness conditions. These and many more properties of regular polytopes may be found in [12].

We now turn to chiral polytopes, for which two good references are [14] and [13].
A polytope $\mathcal{P}$ said to be chiral if its automorphism group $\Gamma(\mathcal{P})$ has two orbits on flags, with every two adjacent flags lying in different orbits. (Another way of viewing this definition is to consider $\mathcal{P}$ as admitting no 'reflecting' automorphism that interchanges a flag with an adjacent flag.) Here the number of flags of $\mathcal{P}$ is $2|\Gamma(\mathcal{P})|$, and $\Gamma(\mathcal{P})$ acts regularly on each of two orbits. Again $\mathcal{P}$ is equivelar, with the stabiliser in $\Gamma(\mathcal{P})$ of every 2 -section of $\mathcal{P}$ inducing the full cyclic group on the corresponding polygon.

For a given flag $\Phi$, denote by $F_{i}$ be the $i$-face in $\Phi$ for each $0 \leq i \leq n$, and for every $j \in\{1,2, \ldots, n-1\}$, the chiral polytope $\mathcal{P}$ admits an automorphism $\sigma_{j}$ that fixes each $F_{i}$ with $i \neq j-1, j$ and cyclically permutes consecutive $j$ - and $(j-1)$-faces in the 2 -section $F_{j+1} / F_{j-2}$, that is, $\sigma_{j}$ takes $\Phi$ to the flag $(\Phi)^{j, j-1}$ which differs from $\Phi$ in precisely its $(j-1)$-and $j$-faces. This automorphism $\sigma_{j}$ is the analogue of the abstract rotation $\rho_{j-1} \rho_{j}$ in the regular case, for each $j$. These automorphisms $\sigma_{1}, \sigma_{2}, \ldots, \sigma_{n-1}$ generate $\Gamma(\mathcal{P})$, and if $\mathcal{P}$ has Schläfli type $\left\{k_{1}, k_{2}, \ldots, k_{n-1}\right\}$, then they satisfy the defining relations for the orientation-preserving subgroup of (index 2 in) the string Coxeter group [ $k_{1}, k_{2}, \cdots, k_{n-1}$ ]. Also they satisfy a 'chiral' form of the intersection condition, which is a variant of the one mentioned earlier for regular polytopes.

Chiral polytopes occur in pairs (or enantiomorphic forms), such that each member of the pair is the 'mirror image' of the other. Suppose one of them is $\mathcal{P}$, and has Schläfli type $\left\{k_{1}, k_{2}, \cdots, k_{n-1}\right\}$. Then $\Gamma(\mathcal{P})$ is isomorphic to the quotient of the orientation-preserving subgroup $\Lambda^{\circ}$ of the string Coxeter group $\Lambda=\left[k_{1}, k_{2}, \cdots, k_{n-1}\right]$ via some normal subgroup $K$. By chirality, $K$ is not normal in the full Coxeter group $\Lambda$, but is conjugated by any orientation-reversing element $c \in \Lambda$ to another normal subgroup $K^{c}$ which is the kernel of an epimorphism from $\Lambda^{\circ}$ to the automorphism group $\Gamma\left(\mathcal{P}^{c}\right)$ of the mirror image $\mathcal{P}^{c}$ of $\mathcal{P}$.

The automorphism groups of $\mathcal{P}$ and $\mathcal{P}^{c}$ are isomorphic to each other, but their canonical generating sets satisfy different defining relations. In fact, replacing the elements $\sigma_{1}$ and $\sigma_{2}$ in the canonical generating tuple $\left(\sigma_{1}, \sigma_{2}, \sigma_{3}, \ldots, \sigma_{n-1}\right)$ by $\sigma_{1}^{-1}$ and $\sigma_{1}^{2} \sigma_{2}$ gives a set of generators for $\Gamma(\mathcal{P})$ that satisfy the same defining relations as a canonical generating tuple for $\Gamma\left(\mathcal{P}^{c}\right)$, but chirality ensures that there is no automorphism of $\Gamma(\mathcal{P})$ that takes ( $\sigma_{1}, \sigma_{2}$ ) to $\left(\sigma_{1}^{-1}, \sigma_{1}^{2} \sigma_{2}\right)$ and fixes all the other $\sigma_{j}$.

Conversely, any finite group $G$ that is generated by $n-1$ elements $\sigma_{1}, \sigma_{2}, \ldots, \sigma_{n-1}$ which satisfy both the defining relations for $\Lambda^{\circ}$ and the chiral form of the intersection condition is the 'rotation subgroup' of an abstract $n$-polytope $\mathcal{P}$ that is either regular or chiral. Indeed, $\mathcal{P}$ is regular if and only if $G$ admits a group automorphism $\rho$ of order 2 that takes $\left(\sigma_{1}, \sigma_{2}, \sigma_{3}, \ldots, \sigma_{n-1}\right)$ to $\left(\sigma_{1}^{-1}, \sigma_{1}^{2} \sigma_{2}, \sigma_{3}, \ldots, \sigma_{n-1}\right)$.

We now focus our attention on the rank 3 case. Here the generators $\sigma_{1}, \sigma_{2}$ for $\Gamma(\mathcal{P})$ satisfy the canonical relations $\sigma_{1}^{k_{1}}=\sigma_{2}^{k_{2}}=\left(\sigma_{1} \sigma_{2}\right)^{2}=1$, and the chiral form of the intersection condition can be abbreviated to $\left\langle\sigma_{1}\right\rangle \cap\left\langle\sigma_{2}\right\rangle=\{1\}$.

### 2.3 Group theory

We use standard notation for group theory, as in [16] for example. We also need the following, which are elementary and so we give them without proof.

Proposition 2.1 Let $G$ be the free abelian group $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ of rank 4, generated by four elements $x_{1}, x_{2}, x_{3}$ and $x_{4}$ subject to the single defining relation $\left[x_{i}, x_{j}\right]=1, i, j \in$ $\{1,2,3,4\}$. Then for every positive integer $m$, the subgroup $G_{m}=\left\langle x_{1}^{m}, x_{2}^{m}, x_{3}^{m}, x_{4}^{m}\right\rangle$ is characteristic in $G$, with index $\left|G: G_{m}\right|=m^{4}$.

Finally, we will use some Reidemeister-Schreier theory, which produces a defining presentation for a subgroup $H$ of finite index in a finitely-presented group $G$. An easily readable reference for this is [11, Chapter IV], but in practice we use its implementation as the Rewrite command in the Magma computation system [1]. We also found the groups that we use in the next section with the help of Magma in constructing and analysing some small examples.

## 3 Proof of Theorem 1.4

Proof. We begin by defining $\mathcal{U}$ as the finitely-presented group

$$
\left\langle a, b \mid a^{4}=b^{8}=(a b)^{2}=\left[a^{2}, b^{2}\right]^{2}=\left(a b^{3} a^{2} b^{4}\right)^{2}=1\right\rangle
$$

This group $\mathcal{U}$ has four normal subgroups of index 1024, 2048, 4096 and 8192. The quotients of $\mathcal{U}$ by each of these give the initial members of our four infinite families.

Case 1: Take $N^{1}$ as the subgroup of $\mathcal{U}$ generated by $x_{1}, x_{2}, x_{3}, x_{4}$, where

$$
\begin{gathered}
x_{1}=\left(b^{-2} a\right)^{4}, x_{2}=\left(b^{4}\right)^{a b^{-1}}\left(b^{4}\right)^{a^{-1}} \\
x_{3}=\left(b^{2} a^{2}\right)^{4}, \quad x_{4}=\left(\left(b^{2} a^{2}\right)^{4}\right)^{a} .
\end{gathered}
$$

A short computation with Magma shows that $N^{1}$ is normal in $\mathcal{U}$, with index 1024. In fact, the defining relations for $\mathcal{U}$ can be used to show that

$$
\begin{array}{lllll}
x_{1}^{a}=x_{2} x_{4}^{-1} & x_{2}^{a}=x_{1}^{-1} x_{3}^{-1} & x_{3}^{a}=x_{4} & \text { and } & x_{4}^{a}=x_{1}^{-1} x_{2}^{-1}, \\
x_{1}^{b}=x_{1} x_{4} & x_{2}^{b}=x_{1}^{-1} & x_{3}^{b}=x_{1}^{-1} x_{2} x_{3}^{-1} x_{4}^{-1} & \text { and } & x_{4}^{b}=x_{3} .
\end{array}
$$

One way to prove these relations is by hand, which we leave as a challenging exercise for the interested reader. Another is by a partial enumeration of cosets of the identity subgroup in $\mathcal{U}$. For example, if this is done using the ToddCoxeter command in Magma, allowing the definition of just 110000 cosets, then multiplication by each of the words $x_{1}^{a}\left(x_{2} x_{4}^{-1}\right)^{-1}$ and $x_{1}^{b}\left(x_{1} x_{4}\right)^{-1}$ is found to fix the trivial coset, and therefore $x_{1}^{a}\left(x_{2} x_{4}^{-1}\right)^{-1}=1=x_{1}^{b}\left(x_{1} x_{4}\right)^{-1}$.

Also Magma's Rewrite command gives a defining presentation for $N^{1}$, with

$$
\left[x_{1}, x_{2}\right]=\left[x_{1}, x_{3}\right]=\left[x_{1}, x_{4}\right]=\left[x_{2}, x_{3}\right]=\left[x_{2}, x_{4}\right]=\left[x_{3}, x_{4}\right]=1 .
$$

Hence the normal subgroup $N^{1}$ is free abelian of rank 4.
The quotient $\mathcal{U} / N$ is isomorphic to the automorphism group of the chiral 3-polytope of type $\{4,8\}$ with 1024 automorphisms listed at [2].

Now for any positive integer $m$, let $N_{m}^{1}$ be the subgroup generated by $x_{1}^{m}, x_{2}^{m}, x_{3}^{m}$ and $x_{4}^{m}$. By Proposition 2.1, we know that $N_{m}$ is characteristic in $N$ and hence normal in $\mathcal{U}$, with index $\left|\mathcal{U}: N_{m}\right|=|\mathcal{U}: N|\left|N: N_{m}\right|=1024 m^{4}$. Moreover, in the quotient $G_{m}^{1}=$ $\mathcal{U} / N_{m}$, the subgroup $N / N_{m}$ is abelian and normal, with quotient $\left(\mathcal{U} / N_{m}\right) /\left(N / N_{m}\right) \cong \mathcal{U} / N$ being a 2 -group, and so $G_{m}^{1}$ is solvable.

Next, we claim that $\langle a\rangle \cap\langle b\rangle=\{1\}$ in $G_{m}^{1}$. Since $a^{4}=1$, we have $|\langle a\rangle \cap\langle b\rangle|=1,2$ or 4. If $|\langle a\rangle \cap\langle b\rangle|=4$, then $\langle a\rangle \leq\langle b\rangle$ and $G_{m}=\langle b\rangle$, contradicting $\left|G_{m}^{1}\right|=1024 m^{4}$. If $|\langle a\rangle \cap\langle b\rangle|=2$, then $\left\langle a^{2}\right\rangle=\left\langle b^{4}\right\rangle$. It follows that $\left\langle a^{2}\right\rangle \unlhd G_{m}^{1}$. Consider the quotient group $G_{m}^{1} /\left\langle a^{2}\right\rangle$. It is easy to see $\left(a\left\langle a^{2}\right\rangle\right)^{2}=\left(b\left\langle a^{2}\right\rangle\right)^{4}=\left(a b\left\langle a^{2}\right\rangle\right)^{2}=\left\langle a^{2}\right\rangle$. Then $\left|G_{m}^{1} /\left\langle a^{2}\right\rangle\right| \leq 8$ and hence $\left|G_{m}^{1}\right|=\left|G_{m}^{1} /\left\langle a^{2}\right\rangle\right| \cdot\left|\left\langle a^{2}\right\rangle\right| \leq 8 \cdot 2=16$, which is impossible because $\left|G_{m}^{1}\right|=1024 m^{4}$.

Finally, if there exists an automorphism $\rho$ of $G_{m}^{1}$ taking $(a, b)$ to $\left(a^{-1}, a^{2} b\right)$, and it follows from the relation $1=\left(a b^{3} a^{2} b^{4}\right)^{2}$ that also $1=\left(a^{-2}\left(a^{2} b\right)^{3} a^{-2}\left(a^{2} b\right)^{4}\right)^{2}$ in $G_{m}^{1}$. However, by Magma, $a^{-2}\left(a^{2} b\right)^{3} a^{-2}\left(a^{2} b\right)^{4}$ has order 4 in $G_{1}^{1}(m=1)$. On the other hand, $G_{1}^{1}$ is the quotient group of $G_{m}^{1}$. Then $a^{-1} b^{-3} a^{-2} b^{-4}$ has order at least 4 in $G_{m}^{1}$, which is impossible because $a^{-2}\left(a^{2} b\right)^{3} a^{-2}\left(a^{2} b\right)^{4}$ has order 2 in $G_{m}^{1}$.

Thus $\mathcal{P}_{m}^{1}$ is chiral, with solvable automorphism group $G_{m}^{1}$ of order $1024 m^{4}$.
Magma programs for Case 1: Here we provide some Magma code for determining $\mathcal{U} \unrhd N^{1}=\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, x_{1}^{a}=x_{2} x_{4}^{-1}$, and $x_{1}^{b}=x_{1} x_{4}$.
$\mathrm{U}<\mathrm{a}, \mathrm{b}>:=\operatorname{Group}<\mathrm{a}, \mathrm{b} \mid \mathrm{a}^{4}, \mathrm{~b}^{8},(\mathrm{a} * \mathrm{~b})^{2},\left(\mathrm{a}^{2}, \mathrm{~b}^{2}\right)^{2},\left(\mathrm{a} * \mathrm{~b}^{3} * \mathrm{a}^{2} * \mathrm{~b}^{4}\right)^{2}>;$

```
\(\mathrm{x} 1:=\left(\mathrm{b}^{-2} * \mathrm{a}\right)^{4}\);
\(\mathrm{x} 2:=\left(\mathrm{b}^{4}\right)^{\mathrm{a} * \mathrm{~b}^{-1}} *\left(\mathrm{~b}^{4}\right)^{\mathrm{a}^{-1}}\);
x3: \(=\left(b^{2} * a^{2}\right)^{4}\);
\(\mathrm{x} 4:=\left(\left(\mathrm{b}^{2} * \mathrm{a}^{2}\right)^{4}\right)^{\mathrm{a}}\);
\(N^{1}:=s u b<U \mid x 1, x 2, x 3, x 4>;\)
```

Index $\left(\mathrm{U}, \mathrm{N}^{1}\right)$; // Returns the index of $\mathrm{N}^{1}$ in U
IsNormal ( $\mathrm{U}, \mathrm{N}^{1}$ ) ; // Return true if $\mathrm{N}^{1}$ is a normal subgroup of U and false otherwise
AbelianQuotientInvariants $\left(\mathrm{N}^{1}\right)$; // This function computes the elementary divisors of the derived quotient group $\mathrm{N}^{1} /\left[\mathrm{N}^{1}, \mathrm{~N}^{1}\right]$

Rewrite ( $\mathrm{U}, \mathrm{N}^{1}$ ) ; // Compute a defining set of relations for $\mathrm{N}^{1}$ on the existing generators
nc, ctb:=ToddCoxeter (U,sub<U|>:CosetLimit:=110000); // This function attempts to build up a coset table of 1 in U using the Todd-Coxeter procedure

```
for g in [a,b] do
    for i in [-1,0,1] do
        for j in [-1,0,1] do
            for k in [-1,0,1] do
                for l in [-1,0,1] do
                    rel:=(x\mp@subsup{1}{}{g})*(x\mp@subsup{1}{}{i}*x\mp@subsup{2}{}{j}*x\mp@subsup{3}{}{\textrm{k}}*\textrm{x}\mp@subsup{4}{}{1}\mp@subsup{)}{}{-1};
                        if ctb(1,rel) eq 1 then
                    print g,":",i,j,k,l;
                    end if;
            end for;
            end for;
        end for;
    end for;
end for;
```

Case 2: Take $N^{2}$ as the subgroup of $\mathcal{U}$ generated by $y_{1}, y_{2}, y_{3}, y_{4}$, where

$$
\begin{gathered}
y_{1}=b^{2} a^{2} b a^{-1} b^{4} a^{-1} b^{2} a^{-1} b a^{-1}, y_{2}=\left(b^{-1} a\right)^{8}, \\
y_{3}=b^{3} a^{2} b^{2} a b^{-1} a b^{-1} a b^{-3} a^{-1}, y_{4}=\left(a b^{-3}\right)^{4} .
\end{gathered}
$$

A short computation with Magma shows that $N^{2}$ is normal in $\mathcal{U}$, with index 2048, and moreover, the Rewrite command tells us that $N^{2}$ is free abelian of rank 4. In this case, the defining relations for $\mathcal{U}$ give

```
dongdonghoudeMacBook-Air:~ dongdonghou$ /Applications/Magma/magma ; exit;
Magma V2.24-5 Mon Jan 23 2023 10:59:40 on dongdonghoudeMacBook-Air [Seed =
1239929370]
Type ? for help. Type <Ctrl>-D to quit.
```



```
> x1:=( (b^-2*a )^4;
> x2:=( (b^4)^(a* ^ ^ -1)*( (b^4)^(a^-1);
> x3:=( (b^2*a^2)^4;
> x4:=(( (b^2*a^2)^4)^a;
> N1:=sub<U|x1,x2,x3,x4>;
> Index(U,N1),IsNormal(U,N1),AbelianQuotientInvariants(N1),Rewrite(U,N1);
1024 true [ 0, 0, 0, 0 ]
Finitely presented group on 4 generators
Generators as words in group U
    $.1 = (b^-2* a ( ^^4
    $.2 = b^2 * a * b^-3*a* * b^-1*a* * b^-4* a^-1
    $.3 = b^2 * * a^2 * b^2 * a^2 * b * a^-1 * b b^-2* * ^^-1 * b * a^-2
    $.4 = a^-1 * b^2 * a^2 * b^2 * a * b^^1 * a^-2 * b^-2 * a^-1 * b * a^-1
Relations
    ($.1^-1, $.2) = Id($)
    ($.\mp@subsup{3}{}{\wedge}-1, $.1^-1) = Id($)
    ($.4, $.3) = Id($)
    ($.4, $.1^-1) = Id($)
    ($.\mp@subsup{2}{}{\wedge}-1, $.4) = Id($)
    ($.3, $. 2^-1) = Id($)
>
> nc,ctb:=ToddCoxeter(U,sub<U|>:CosetLimit:=110000);
> for g in [a,b] do
for> for i in [-2,-1,0,1,2] do
for|for> for }j\mathrm{ in [-2,-1,0,1,2] do
for|for|for> for }k\mathrm{ in [-2,-1,0,1,2] do
for|for|for|for> for }1\mathrm{ in [-2,-1,0,1,2] do
for|for|for|for|for> rel:=(x\mp@subsup{1}{}{\wedge}g)*(x\mp@subsup{1}{}{\wedge}i*x\mp@subsup{2}{}{\wedge}j*x\mp@subsup{3}{}{\wedge}k*x\mp@subsup{4}{}{\wedge}l\mp@subsup{)}{}{\wedge}-1;
for||or|for|for|for> if ctb(1,rel) eq 1 then
for|for|for|for|for|if> print g,":",i,j,k,l;
[for|for|for|for|for|if> end if;end for;end for;end for;end for;end for;
a : 0 1 0 -1
b : 1 0 0 1
>
```

Figure 1: Magma programs for Case 1

$$
\begin{array}{lllll}
y_{1}^{a}=y_{3} & y_{2}^{a}=y_{1} y_{3}^{-1} y_{4}^{-1} & y_{3}^{a}=y_{3} y_{4} & \text { and } & y_{4}^{a}=y_{2} y_{4}^{-1} \\
y_{1}^{b}=y_{2}^{-1} y_{3}^{-1} & y_{2}^{b}=y_{1}^{-1} y_{3}^{-1} y_{4}^{-1} & y_{3}^{b}=y_{1} & \text { and } & y_{4}^{b}=y_{1}^{-1} y_{3}^{-1}
\end{array}
$$

The quotient $\mathcal{U} / N^{2}$ is isomorphic to the automorphism group of the chiral 3-polytope of type $\{4,8\}$ with 2048 automorphisms listed at [2].

Now for any positive integer $m$, let $N_{m}^{2}$ be the subgroup generated by $y_{1}^{m}, y_{2}^{m}, y_{3}^{m}$, and $y_{4}^{m}$. Using Proposition 2.1, we find that $N_{m}^{2}$ is characteristic in $N_{2}$ and hence normal in $\mathcal{U}$, with index $\left|\mathcal{U}: N_{m}^{2}\right|=\left|\mathcal{U}: N^{2}\right|\left|N^{2}: N_{m}^{2}\right|=2048 m^{4}$. Also the quotient $G_{m}^{2}=\mathcal{U} / N_{m}^{2}$ is solvable.

Moreover, the same argument as used in case(1) shows that $\mathcal{P}_{m}^{2}$ is chiral. Thus, $\mathcal{P}_{m}^{2}$ is chiral, with automorphism group $G_{m}^{2}$ of order $2048 m^{4}$.

Case 3: Take $N^{3}$ as the subgroup of $\mathcal{U}$ generated by $z_{1}, z_{2}, z_{3}, z_{4}$, where

$$
\begin{gathered}
z_{1}=\left(b^{-1} a\right)^{8}, z_{2}=\left(b^{-3} a\right)^{4}, \\
z_{3}=\left(a b^{-1}\right)^{8}, z_{4}=\left(a b^{2} a^{-1} b a^{-1}\right)^{4} .
\end{gathered}
$$

A short computation with Magma shows that $N^{3}$ is normal in $\mathcal{U}$, with index 4096, and moreover, the Rewrite command tells us that $N^{3}$ is free abelian of rank 4. In this case, the defining relations for $\mathcal{U}$ give

$$
\begin{array}{lllll}
z_{1}^{a}=z_{3}^{-1} & z_{2}^{a}=z_{2}^{-1} z_{3}^{-1} & z_{3}^{a}=z_{1} & \text { and } & z_{4}^{a}=z_{1}^{-1} z_{4} \\
z_{1}^{b}=z_{2} z_{4}^{-1} & z_{2}^{b}=z_{1}^{-1} z_{2} & z_{3}^{b}=z_{1} & \text { and } & z_{4}^{b}=z_{3}^{-1} z_{4}^{-1}
\end{array}
$$

Now for any positive integer $m$, let $N_{m}^{3}$ be the subgroup generated by $z_{1}^{m}, z_{2}^{m}, z_{3}^{m}$, and $z_{4}^{m}$. Using Proposition 2.1, we find that $N_{m}^{3}$ is characteristic in $N^{3}$ and hence normal in $\mathcal{U}$, with index $\left|\mathcal{U}: N_{m}^{3}\right|=\left|\mathcal{U}: N^{3}\right|\left|N^{3}: N_{m}^{3}\right|=4096 m^{4}$. Also the quotient $G_{m}^{3}=\mathcal{U} / N_{m}^{3}$ is solvable.

Moreover, the same argument as used in case(1) shows that $\mathcal{P}_{m}^{3}$ is chiral. Thus, $\mathcal{P}_{m}^{3}$ is chiral, with automorphism group $G_{m}^{3}$ of order $4096 \mathrm{~m}^{4}$.

Case 4: Take $N^{4}$ as the subgroup of $\mathcal{U}$ generated by $w_{1}, w_{2}, w_{3}, w_{4}$, where

$$
\begin{gathered}
w_{1}=\left(a b^{-1}\right)^{8}, w_{2}=\left(\left(b^{-1} a\right)^{8}\right)^{b}, w_{3}=\left(\left(b^{-2} a\right)^{8}\right)^{b} \\
w_{4}=\left(\left(a^{2} b^{2}\right)^{4}\right)^{b^{-1}}\left(\left(b^{-2} a^{2}\right)^{4}\right)^{a} .
\end{gathered}
$$

A short computation with Magma shows that $N^{4}$ is normal in $\mathcal{U}$, with index 8192, and moreover, the Rewrite command tells us that $N^{4}$ is free abelian of rank 4. In this case, the defining relations for $\mathcal{U}$ give

$$
\begin{array}{lllll}
w_{1}^{a}=w_{2} w_{3}^{-1} w_{4}^{-1} & w_{2}^{a}=w_{2} w_{3}^{-1} & w_{3}^{a}=w_{3}^{-1} & \text { and } & w_{4}^{a}=w_{1} w_{2} \\
w_{1}^{b}=w_{2} w_{3}^{-1} w_{4}^{-1} & w_{2}^{b}=w_{2}^{-1} w_{3} & w_{3}^{b}=w_{1}^{-1} w_{2}^{-1} w_{3} & \text { and } & w_{4}^{b}=w_{1} w_{2}^{-1} .
\end{array}
$$

The quotient $\mathcal{U} / N^{4}$ is isomorphic to the automorphism group of the chiral 3-polytope of type $\{4,8\}$ with 8192 automorphisms.

Now for any positive integer $m$, let $N_{m}^{4}$ be the subgroup generated by $w_{1}^{m}, w_{2}^{m}, w_{3}^{m}$, and $w_{4}^{m}$. Using Proposition 2.1, we find that $N_{m}^{4}$ is characteristic in $N_{4}$ and hence normal in
$\mathcal{U}$, with index $\left|\mathcal{U}: N_{m}^{4}\right|=\left|\mathcal{U}: N^{4}\right|\left|N^{4}: N_{m}^{4}\right|=8192 m^{4}$. Also the quotient $G_{m}^{4}=\mathcal{U} / N_{m}^{4}$ is solvable.

Moreover, the same argument as used in case(1) shows that $\mathcal{P}_{m}^{4}$ is chiral (in this case, $a^{-2}\left(a^{2} b\right)^{3} a^{-2}\left(a^{2} b\right)^{4}$ has order 8 in $G_{4}^{1}(m=1)$.). Thus, $\mathcal{P}_{m}^{4}$ is chiral, with automorphism group $G_{m}^{4}$ of order $8192 m^{4}$.

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