Large Girth and Small Oriented Diameter Graphs

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Abstract

In 2015, Dankelmann and Bau proved that for every bridgeless graph G of order n and minimum degree δ there is an orientation of diameter at most $11\frac{n}{\delta+1} + 9$. In 2016, Surmacs reduced this bound to $7\frac{n}{\delta+1}$. In this paper, we consider the girth of a graph g and show that for any $\varepsilon > 0$ there is a bound of the form $(2g + \varepsilon)\frac{n}{h(\delta,g)} + O(1)$, where $h(\delta,g)$ is a polynomial. Letting g = 3 and $\varepsilon < 1$ gives an inprovement on the result by Surmacs.

Keywords: diameter, oriented diameter, orientation, oriented graph, distance, size, girth

1. Definitions

Let G = (V, E) denote a finite simple graph with vertex set V and edge set $E \subseteq \binom{V}{2}$. Given G = (V, E), a subgraph H of G, denoted $H \subseteq G$, is a graph H = (V', E') for which $V' \subseteq V$ and $E' \subseteq E \cap \binom{V'}{2}$. By |G| we mean the order of G, |V(G)|. A digraph $\overrightarrow{G} = (V, A)$ is a graph with a vertex set V and an arc set A where each arc is oriented and the orientation of the arc a with ends u and v is in the direction from u to v will be denoted as \overrightarrow{uv} . If a set of arcs A when considered to be unordered is the set E, we call \overrightarrow{G} an orientation of the graph G. A path is defined as P = (V, E), where $V = \{v_0, v_1, \ldots, v_n\}$ and $E = \{x_0x_1, x_1x_2, \ldots, x_{n-1}x_n\}$. We will denote this path $P = v_0v_1 \ldots v_n$.

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Given such a path P, a cycle is defined as a graph $G = (V(P), E(P) \cup \{v_0v_n\})$. Given an unoriented path $P = v_0v_1 \dots v_n$, we denote using \overrightarrow{P} the corresponding oriented path from v_0 to v_n , we will denote using \overleftarrow{P} the oriented path from v_n to v_0 . Denote the interior of a path $\overrightarrow{P} = v_1 \dots v_{n-1}$. Given a graph G and an edge set $E' \subseteq E$, define $G \setminus E' = (V, E \setminus E')$. Given an edge set containing a single edge, $E' = \{e\}$, we may leave off the brackets, i.e. $G \setminus \{e\} = G \setminus e = (V, E \setminus \{e\})$. We define a forest as a graph containing no cycles. A connected forest is called a tree.

For a set $B \subseteq V(G)$, the induced subgraph of G on the vertex set B is denoted by G[B]. That is, $G[B] = (B, {B \choose 2} \cap V)$. Given G a simple graph and $v \in V(G)$, the degree of v in G is the number of vertices adjacent to v, denoted deg $(v) = |\{uv \mid u \in V(G), u \neq v, uv \in E(G)\}|$. The minimum degree of a graph G is $\delta(G) = \min\{\deg(v) \mid v \in V(G)\}$. If the graph G is unambiguous, we let $\delta(G) = \delta$. We define the closed neighborhood of a vertex $v \in V(H)$ in the given subgraph H as, $N_H[v] = \{u \mid u = v \text{ or } uv \in E(H)\}$. The open neighborhood of v in a given subgraph H, denoted $N_H(v)$, is defined as $N_H(v) = \{u \mid u \neq v \text{ and } uv \in E(H)\}$. We may also use N[v] and N(v) if the subgraph H is unambiguous. Let g(G) = g be the girth of G or the length of the smallest cycle in the graph G.

We define the *distance* from u and v in a graph G or digraph \overrightarrow{G} as the minimum number of edges or arcs on a path from u to v. We denote this as $\rho_G(u, v)$ or $\rho_{\overrightarrow{G}}(u, v)$. If there does not exist a path from u to v, we say that $\rho_G(u, v) = \infty$ or $\rho_{\overrightarrow{G}}(u, v) = \infty$. We define the *diameter* of G or \overrightarrow{G} to be diam $(G) = \max\{\rho_G(u, v) \mid u, v \in V(G)\}$ and diam $(\overrightarrow{G}) = \max\{\rho_{\overrightarrow{G}}(u, v) \mid u, v \in V(\overrightarrow{G})\}$ respectively. If diam $(G) < \infty$, we call G connected. An edge $e \in E(G)$ (or an arc $a \in A(\overrightarrow{G})$) is called a *bridge* if diam $(G) < \infty$ and diam $(G \setminus e) = \infty$ (similar for \overrightarrow{G}). If a graph contains no bridges, we call it *bridgeless*. If diam $(\overrightarrow{G}) < \infty$, then we call \overrightarrow{G} strongly connected.

A classical result, due to Robbins [25], states that every bridgeless graph has a strongly connected orientation. There may be many such orientations of a graph. A natural next question is what it may mean to find a "good" such orientation. Many notions of an objective for optimality of such orientations may be considered. For the purposes of this paper, given a graph G, let $\overrightarrow{\mathcal{G}}$ represent the set of all strongly connected orientations of G. We wish to minimize the oriented diameter of a graph G, defined as the following:

$$\overrightarrow{\operatorname{diam}}(G) = \min_{\overrightarrow{G} \in \overrightarrow{\mathcal{G}}} \operatorname{diam}\left(\overrightarrow{G}\right).$$

It was shown by Chvátal and Thomassen [5] that finding the oriented diameter of a given graph is NP-complete. In the same paper, Chvátal and Thomassen found that for the class of bridgeless graphs with diameter d, $\overrightarrow{\text{diam}}(G) \leq 2d^2 + 2d$ and constructed bridgeless graphs of diameter d for which every strong orientation admits a diameter of at least $\frac{1}{2}d^2 + d$. The upper bound was improved by Babu, Benson, Rajendraprasad and Vaka [1] to $1.373d^2 + 6.971d - 1$.

The paper by Chvátal and Thomassen [5] has led to further investigation of such bounds on the oriented diameter given certain graph parameters, including the diameter [10, 15, 19], the radius [4], the domination number [11, 20], the maximum degree [8], the minimum degree [2, 7, 26], the number of edges of the graph [6], and other graph classes [3, 12, 13, 14, 15, 17, 18, 21, 22, 23, 24, 27, 28]. See the survey by Koh and Tay [16] for more information on some of these results.

Erdős, Pach, Pollack and Tuza [9] proved that the diameter of connected graphs of order n and minimum degree δ is at most $\frac{3n}{\delta+1} + O(1)$. Bau and Dankelmann [2] sought to investigate a similar bound for the oriented diameter and proved that given a bridgeless graph G of order n and minimum degree δ , $\frac{3n}{\delta+1} \leq \overrightarrow{\operatorname{diam}}(G) \leq \frac{11n}{\delta+1}$. The upper bound was improved to $\frac{7n}{\delta+1}$ by Surmacs [26].

In this paper, we will consider upper bounds on the oriented diameter of a graph considering both the minimum degree δ and the girth g of a graph. In particular we will prove the following theorem.

Theorem 1.1. Given G = (V, E), a bridgeless graph of order n and minimum degree δ , there is a polynomial in δ and g, $h(\delta, g)$ of degree $\lfloor \frac{g-1}{2} \rfloor$, for which, given any choice of $\varepsilon > 0$,

$$\overrightarrow{diam}(G) \le (2g + \varepsilon)\frac{n}{h(\delta, g)} + c.$$

We will also show that in the case of general bridgeless graphs, that $\overline{\operatorname{diam}}(G) \leq (2g + \varepsilon)\frac{n}{\delta+1} + O(1)$. Since bridgeless graphs have a girth $g \geq 3$, we find that if we choose $0 < \varepsilon < 1$, this gives an improvement on the bound found in the paper by Surmacs [26].

2. Preliminaries

Given a vertex $v \in V(G)$, a natural number g, and a path P, let $\mathcal{N}(g, v) = \{u \mid \rho_G(u, v) \leq \lfloor \frac{g}{2} \rfloor - 1\}$ and $\mathcal{N}(g, v, P) = \{u \mid \rho_{G \setminus E(P)}(u, v) \leq \lfloor \frac{g}{2} \rfloor - 1\}.$

Lemma 2.1. Given a graph G with minimum degree $\delta > 3$, girth g, a path $P = p_0 p_1 \dots p_\ell$, for which $\rho_G(p_i, p_j) = |j - i|$, and a vertex $x \notin V(P)$,

$$|\mathcal{N}(g, x, P)| \ge 1 + \delta + \sum_{i=1}^{\lfloor \frac{g-1}{2} \rfloor - 1} \delta(\delta - 3)^i.$$

Proof. Given a vertex $x \in V(G)$ for which $x \notin V(P)$, $G[\mathcal{N}(g, x)]$ is a tree. If not, there would be a cycle of length less than g in G a contradiction to g being the girth. Since $G[\mathcal{N}(g, x, P)] \subseteq G[\mathcal{N}(g, x)]$, $G[\mathcal{N}(g, x, P)]$ is also a tree.

We will construct the set $\mathcal{N}(g, x, P)$. Note that $x \in \mathcal{N}(g, x, P)$. Since $x \notin V(P), N(x) \subseteq \mathcal{N}(g, x, P)$ and $|N(x)| \ge \delta$, so $|\{u \mid \rho_{G \setminus E(P)}(v, u) = 1\}| \ge \delta$. For each vertex $v_1 \in N(x)$, if $v_1 \notin V(P)$, then $|N_{G \setminus E(P)}(v_1)| \ge \delta$. If $v_1 \in V(P)$, either one or two of the edges incident to v_1 are in E(P), so $|N_{G \setminus E(P)}(v_1)| \ge (\delta - 2)$. Since $x \in N(v_1)$ we have that $|\{u \mid \rho_{G \setminus E(P)}(v, u) = 2\}| \ge \delta(\delta - 3)$. Since $\mathcal{N}(g, x, P)$ is a tree, as long as $1 \le i \le \lfloor \frac{g-1}{2} \rfloor - 1$, we can perform a similar analysis to show that $|\{u \mid \rho_{G \setminus E(P)}(v, u) = i + 1\}| \ge \delta(\delta - 3)^i$. Hence, $|\mathcal{N}(g, x, P)| \ge 1 + \delta + \sum_{i=1}^{\lfloor \frac{g-1}{2} \rfloor - 1} \delta(\delta - 3)^i$.

3. Introduction of Main Lemma

Let
$$h(\delta, g) = 1 + \delta + \sum_{i=1}^{\lfloor \frac{g-1}{2} \rfloor - 1} \delta(\delta - 3)^i$$
. For any $\varepsilon > 0$, let $L = \lceil \frac{g-1}{\varepsilon} \rceil$.

Lemma 3.1. Given a bridgeless graph G with |G| = n, girth g and minimum degree $\delta = \delta(G)$, there exists a set of increasing bridgeless subgraphs $H_0 \subset H_1 \subset H_2 \subset \ldots H_k \subseteq G$, vertex sets $B_0 \subset B_1 \subset \ldots$ for which $B_i \subseteq V(H_i)$, and a set of forests F_i for which the following hold:

- 1. For all $v \in V(G)$, $\rho_G(v, H_k) < L \cdot g$,
- 2. for all $i, |F_i| \ge h(\delta, g)|B_i|$, and
- 3. $|H_i| \leq (2g + \varepsilon)|B_i|$.

Proof. We will prove by induction on B_i, F_i , and H_i . For some $v_0 \in V(G)$, let $B_0 = \{v_0\}, F_0 = G[\mathcal{N}(g, v_0)]$, and $H_0 = (\{v_0\}, \emptyset)$. Certainly property 3 holds. Note that F_0 is a tree of order $\sum_{\alpha=0}^{\lfloor \frac{g-1}{2} \rfloor} \delta^{\alpha} \geq h(\delta, g)$, so property 2 holds. If property 1 holds, we are done.

Consider B_i , F_i , H_i for which properties 2 and 3 hold and property 1 does not yet hold. Since property 1 does not yet hold, there exists a vertex, v, for which $\rho_G(v, H_i) = L \cdot g$. Let p_0 be a vertex in H_i for which $\rho_G(v, p_0) = L \cdot g$. Consider a path of shortest length between p_0 and v, call this path $P = p_0 p_1 \dots p_{Lg-1} v$ with $v = p_{Lg}$. Let $e_i = p_{i-1}p_i$. Let $H'_i = H_i$. Call $e_j \in E(P)$ covered if e_j is not a bridge in $H'_i \cup P$. Let $P_j = p_0 \dots p_j$ and $P'_j = p_j p_{j+1} \dots p_{Lg}$. We consider a set of edges $E(P_j)$ to be covered if no edge $e \in E(P_j)$ is a bridge in $H'_i \cup P_j$. We will build a set of vertices $cov(P) \subseteq V(G) \setminus (V(P) \cup V(H_i)$ which is incident to all the edges used to cover E(P).

To expand H'_i , note that e_1 is not covered in $H'_i \cup P$. Since G is bridgeless, there must be a path from H'_i to P'_1 . Consider a path of length $\rho_{G \setminus E(P)}(H'_i, P'_1)$, call it Q. Note that the two end vertices of Q are the only vertices in V(Q)which can intersect with V(P). Let p_β be the end vertex of Q on $P \setminus p_0$. Add Q and P_β to H'_i . Add the set of interior vertices of Q, $V(\overline{Q})$, to cov(P), a set of vertices which will eventually be incident to all the edges used to cover P. Label the vertices in cov(P) as q_r such that $r = \rho_{G \setminus E(P)}(H_i, q_r)$. Let $B'_i = B_i$. We will now consider an algorithm that will add to cov(P), B'_i , and H'_i .

- 1. If there is no longer an edge left uncovered, terminate the algorithm.
- 2. If there is an uncovered edge in P, consider the edge e_j with the smallest index j that is not yet covered. Since G is bridgeless, there exists a path from H'_i to P'_j of length $\rho_{G \setminus E(P)}(H'_i, P'_j)$, call it R. Add $V(\overline{R})$ to cov(P). Label the vertices $v \in V(\overline{R})$ as q_r where $r = |cov(P)| + \rho_{G \setminus E(P)}(H'_i, v)$. Add R and P_j to H'_i .
- 3. If for all pairs of vertices $q_{m_1}, q_{m_2} \in cov(P)$ we have $\rho_{G \setminus E(P)}(q_{m_1}, H_i) \ge m_1$ and $\rho_{G \setminus E(P)}(q_{m_1}, q_{m_2}) \ge |m_2 m_1|$, then return to step 1. If this was not the case, consider one of the following augmentations.
 - (a) If $\rho_{G \setminus E(P)}(q_{m_1}, H_i) = s < m_1$, remove $\{q_1, \dots, q_{m_1-1}\}$ and any edges incident to that vertex set from H'_i and cov(P). Consider a path S, which is edge disjoint from P between q_{m-1} and H_i of length $\rho_{G \setminus E(P)}(q_{m_1}, H_i) = s$. Add this path to H'_i , add the vertices in $V(\overline{S})$ to cov(P), and label them q_ℓ such that $\ell = \rho_{G \setminus E(P)}(H_i, q_\ell)$. For values from m_1 to t, where t is the highest current label r for q_r in cov(P), relabel $q_{m_1} \dots q_t = q_s \dots q_{t-(m_1-s)}$. After relabeling, return to step 3.
 - (b) If $\rho_{G \setminus E(P)}(q_{m_1}, q_{m_2}) = s < |m_2 m_1|$, without loss of generality, let $m_1 < m_2$. Remove the vertices $q_{m_1+1}, \ldots, q_{m_2-1}$ from H'_i and cov(P). Consider a path S, which is edge disjoint from P between q_{m_1} and q_{m_2} of length $\rho_{G \setminus E(P)}(q_{m_1}, q_{m_2}) = s$. Add this path to H'_i ,



Figure 3.1: The left graph is an example of subgraph H' where step 3a will be executed. The right is H_i' after execution of 3a.



Figure 3.2: The left graph is an example of subgraph H' where step 3b will be executed. The right is H'_i after execution of 3b.

add the vertices in $V(\overline{S})$ to cov(P). Label the newly added vertices $q_{m_1+1}, \ldots, q_{m_1+s-1}$ and relabel $q_{m_2+1} \ldots q_t = q_{m_1+s} \ldots q_{t-((m_2-m_1)-s)}$. After relabeling, return to step 3.

Any step for which step 3a or step 3b executes, there was a strict reduction in |cov(P)|. On the path P, since $\rho(H_i, p_j) = j$, there must be at least 1 vertex in cov(P), so at some point we must leave step 3 of the algorithm. Any time step 2 executes, there is a strict increase in the number of edges in P that are covered. Since P is finite, at some point the algorithm must return to step 1 and terminate.

Let $H_{i+1} = H'_i$, $B_{i+1} = \{B_i \cup q_r | r \equiv 0 \mod g\}$, and $F_{i+1} = F_i \bigcup \cup_{b \in B_{i+1} \setminus B_i} \mathcal{N}(g, b, P)$. Now we will show that Properties 2 and 3 of Lemma 3.1 hold.

To prove Property 3 holds, first remember that $L = \lceil \frac{g-1}{\varepsilon} \rceil \geq \frac{g-1}{\varepsilon}$, hence $g-1 \leq L\varepsilon$. We will have two cases: $L \leq |B_{i+1} \setminus B_i|$ and $L > |B_{i+1} \setminus B_i|$. If

 $L \leq |B_{i+1} \setminus B_i|$, the following holds:

$$|H_{i+1}| \le |H_{i+1}| + |P| + |cov(P)| \tag{3.1}$$

$$\leq |H_i| + gL + g|B_{i+1} \setminus B_i| + (g-1)$$
(3.2)

$$\leq |H_i| + g|B_{i+1} \setminus B_i| + g|B_{i+1} \setminus B_i| + L\varepsilon \tag{3.3}$$

$$\leq |H_i| + g|B_{i+1} \setminus B_i| + g|B_{i+1} \setminus B_i| + |B_{i+1} \setminus B_i|\varepsilon$$
(3.4)

$$\leq |H_i| + (2g + \varepsilon)|B_{i+1} \setminus B_i| \tag{3.5}$$

$$\leq (2g+\varepsilon)|B_i| + (2g+\varepsilon)|B_{i+1} \setminus B_i|$$
(3.6)

$$\leq (2g+\varepsilon)|B_{i+1}|. \tag{3.7}$$

To prove property 2, note that for each $b \in B_{i+1} \setminus B_i$, $\rho_{G \setminus E(P)}(b, H_i) \geq b_{i+1}$ g, otherwise we would have augmented cov(P) in step 3a the algorithm, so $\rho_{G\setminus E(P)}(b, B_i) \geq g$. For any pair of vertices $b_1, b_2 \in B_{i+1} \setminus B_i, \rho_{G\setminus E(P)}(b_1, b_2) \geq g$. g, otherwise we would have augmented cov(P) in step 3b of the algorithm. Hence, $\mathcal{N}(g, b_1, P) \cap \mathcal{N}(g, b_2, P) = \emptyset$. So,

$$|F_{i+1}| \ge |F_i| + \left| \bigcup_{b \in B_{i+1} \setminus B_i} \mathcal{N}(g, b, P) \right|$$
(3.8)

$$\geq |B_i| h(\delta, g) + |B_{i+1} \setminus B_i| h(\delta, g)$$
(3.9)

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$$\geq |B_{i+1}| h(\delta, g). \tag{3.10}$$

In the case that $L > |B_{i+1} \setminus B_i|$, redefine B_{i+1} to be $B_i \bigcup \bigcup_{c=1}^{L} p_{cg}$. Since $Lg = g|B_{i+1} \setminus B_i|$, the computation above from 3.1 to 3.7 holds. See that by definition of P, for any $b \in B_{i+1} \setminus B_i$, $\rho_G(b, H_i) \ge g$, so $\mathcal{N}(g, b) \cap V(H_i) = \emptyset$. For any $b_1, b_2 \in B_{i+1} \setminus B_i$, $\rho_G(b_1, b_2) \ge g$, hence $\mathcal{N}(g, b_1) \cap \mathcal{N}(g, b_2) = \emptyset$. It follows that

$$|F_{i+1}| \ge |F_i| + \left| \bigcup_{b \in B_{i+1} \setminus B_i} \mathcal{N}(g, b) \right|$$
(3.11)

$$\geq |B_i| h(\delta, g) + |B_{i+1} \setminus B_i| h(\delta, g)$$
(3.12)

$$\geq |B_{i+1}| h(\delta, g). \tag{3.13}$$

Hence, Property 2 of Lemma 3.1 holds in this case.

Since H_i and B_i are increasing subgraphs and vertex sets, and our graph G is a finite graph, eventually 1 will hold. When this happens, let i = k.

Now we wish to use Lemma 3.1 to create an orientation on a subgraph of G with a small diameter. First, we need to consider the following theorem by Robbins.

Theorem 3.2 (Robbins[25]). A graph is bridgeless if and only if it admits a strong orientation.

Lemma 3.3. Let $H_k \subseteq G$, $B_k \subseteq V(G)$, and $F_k \subseteq G$, and Properties 1, 2, and 3 of Lemma 3.1 hold. There exists an orientation of H_k , $\overrightarrow{H_k}$ for which $diam\left(\overrightarrow{H_k}\right) \leq (2g + \varepsilon) \frac{n}{h(\delta,g)}$.

Proof. By Property 2 of Lemma 3.1 we have that $h(\delta, g)|B_k| \leq |F_k| \leq n$, so we find that $|B_k| \leq \frac{n}{h(\delta,g)}$. In conjunction with Property 3 of Lemma 3.1, we find that $|H_k| \leq (2g + \varepsilon)|B_k| \leq (2g + \varepsilon)\frac{n}{h(\delta,g)}$.

Hence, there exists a bridgeless subraph H_k for which $|H_k| \leq (2g + \varepsilon) \frac{n}{h(\delta,g)}$. By Theorem 3.2, there is strong orientation of H_k , $\overrightarrow{H_k}$. Note that diam $\left(\overrightarrow{H_k}\right) \leq |H_k| \leq (2g + \varepsilon) \frac{n}{h(\delta,g)}$.

We now wish to extend our result in Lemma 3.3 for $H_k \subseteq G$ to G. To do so, we will need to consider an extension to the following two lemmas, one by Fomin et al. [11] and one by Bau et al. [2]

Lemma 3.4 (Fomin, Matamala, Prisner and Rapaport [11]). Let G be a bridgeless graph and H a bridgeless subgraph of G with $\rho_G(v, H) \leq 1$ for all $v \in V(G)$. Given an orientation \overrightarrow{H} such that diam $(\overrightarrow{H}) = d$, then G has an orientation of d+4.

Lemma 3.5 (Bau and Dankelmann[2]). Let G be a bridgeless graph and H a bridgeless subgraph of G such that $\rho_G(v, H) \leq 2$ for all $v \in V(G)$. Let \overrightarrow{H} be a strongly connected orientation of H of diameter d. Then there exists a strongly connected orientation of G of diameter at most d + 12 that extends the orientation of \overrightarrow{H} .

We have that for any $v \in V(G)$, $\rho_G(v, H_k) \leq Lg$. Since Lg > 2, we will need to extend this lemma as seen below.

Lemma 3.6. Let G be a bridgeless graph, H a bridgeless subgraph of G, and let s be an integer such that $s \ge 2$ and for all $v \in V(G)$, $\rho_G(v, H) \le s$. Let \overrightarrow{H} be a strongly connected orientation of H of diameter d. Then there exists a strongly connected orientation of G of diameter at most $d + 4\binom{s+1}{2}$ that extends the orientation of \overrightarrow{H} .

Proof. Let $H \subseteq G$ be a bridgeless subgraph with an orientation \overrightarrow{H} such that diam $(\overrightarrow{H}) = d$ and $\rho_G(v, H) \leq k$ for all $v \in V(G)$. Let $V_1 := \{v \mid \rho_G(v, H) = 1\}$. Given a vertex $v \in V_1$, label one of its neighbors in H as x. Let $\overrightarrow{H'} = \overrightarrow{H}$, we will continue to augment $\overrightarrow{H'}$ throughout the proof. We will call $\overrightarrow{H'}$ extendable at step i if for any $v \in V(\overrightarrow{H'})$, $\rho_{\overrightarrow{H'}}(v, H) + \rho_{\overrightarrow{H'}}(H, v) \leq 2i$ and $\rho_G(H, v) \leq i$.

Assume there is a vertex z for which $\rho_G(\vec{H}, z) = s$. First, we will show that there exists a graph $\vec{H'}$ that is extendable at step 1. If there is a vertex

 $v \in V_1 \setminus V\left(\overrightarrow{H'}\right)$ for which $\rho_{G\setminus vx}(v, H) = 1$, there exists some vertex $y, y \neq x$ for which $vy \in E(G)$. Let $\overrightarrow{H'} = \overrightarrow{H} \cup \overrightarrow{xvy}$. Repeat this until there are no longer vertices $v \in V_1 \setminus V\left(\overrightarrow{H'}\right)$ for which $\rho_{G\setminus vx}(v, H) = 1$. Note that for any $v \in V\left(\overrightarrow{H'}\right), \rho_{\overrightarrow{H'}}(v, H) + \rho_{\overrightarrow{H'}}(H, v) \leq 2$ and $\rho_G(v, H) \leq 1$, so $\overrightarrow{H'}$ is extendable at step 1.

We will show that for any $1 \leq i < 2s$, if $\overrightarrow{H'}$ is extendable at step i, then it is also extendable at step i + 1. If there is a vertex $v \in V_1 \setminus V\left(\overrightarrow{H'}\right)$ for which $\rho_{G\setminus vx}(v,H) = i$, let Q be a path of length i from v to H which does not include vx. Consider a vertex $v' \in V(Q)$ for which $v' \in V\left(\overrightarrow{H'}\right)$ and $\rho_{G\setminus vx}(v',v)$ is minimized. If $v' \in V(H)$, add $\overrightarrow{Q} \cup \overrightarrow{xv}$ to $\overrightarrow{H'}$. See that for all $v \in V\left(\overrightarrow{H'}\right)$, $\rho_{\overrightarrow{H'}}(v,H) + \rho_{\overrightarrow{H'}}(H,v) \leq 2i$ and $\rho_G(v,H) \leq i$, so $\overrightarrow{H'}$ is extendable at step i.

If $v' \notin V(H)$, let Q' be the subpath of Q from v to v'. Since $\overrightarrow{H'}$ is extendable at step i, there exists an integer j for which |j| < i, $\rho_{\overrightarrow{H'}}(v', H) \leq i - j$, and $\rho_{\overrightarrow{H'}}(H, v') \leq i + j$. If $j \geq 0$, add $\overleftarrow{Q'} \cup \overrightarrow{vx}$ to $\overrightarrow{H'}$. If j < 0, add $\overrightarrow{Q'} \cup \overrightarrow{xv}$ to $\overrightarrow{H'}$. See in each case that for all $v \in V\left(\overrightarrow{H'}\right)$, $\rho_{\overrightarrow{H'}}(v, H) + \rho_{\overrightarrow{H'}}(H, v) \leq 2i$ and $\rho_G(v, H) \leq i$.

Once we have an extendable subgraph $\overrightarrow{H'}$ at step 2s, and have considered all vertices $v \in V_1 \setminus V\left(\overrightarrow{H'}\right)$ for which $\rho_{G \setminus e}(v, H) \leq 2s$, there are no more vertices $v \in V_1 \setminus V\left(\overrightarrow{H'}\right)$. If there were a vertex $v \in V_1 \setminus V\left(\overrightarrow{H'}\right)$ for which $\rho_{G \setminus e}(v, H) > 2s$, notice that this would mean there exists a vertex $v' \in V(G)$ for which $\rho_G(v', H) > s$, a contradiction to the assumption of the lemma.

Since $\overrightarrow{H'}$ was extendable at step 2s, for any $v \in V\left(\overrightarrow{H'}\right)$, $\rho_{\overrightarrow{H'}}(v,H) \leq 2s$ and $\rho_{\overrightarrow{H'}}(H,v) \leq 2s$, so $\overrightarrow{\operatorname{diam}}(H') \leq \overrightarrow{\operatorname{diam}}(H) + 4s$.

We will now prove Theorem 1.1.

Proof. In Lemma 3.1 we showed that there is a bridgeless subgraph $H_k \subseteq G$ such that for any $v \in V(G)$, $\rho_G(v, H_k) \leq Lg$ and

$$\overrightarrow{\operatorname{diam}}(H_k) \le (2g + \varepsilon) \frac{n}{h(\delta, g)}$$

By a combination of this and Lemma 3.6 with $s = L \cdot g$, we find

$$\overrightarrow{\operatorname{diam}}(G) \le \overrightarrow{\operatorname{diam}}(H_k) + \sum_{i=1}^{L_g} 4i \le (2g + \varepsilon) \frac{n}{h(\delta, g)} + 4 \binom{Lg + 1}{2}.$$

Corollary 3.7. In Theorem 1.1, if g = 3 and $0 < \varepsilon < 1$,

$$\overrightarrow{diam}(G) \leq (2g+\varepsilon)\frac{n}{h(\delta,g)} + 4\binom{Lg+1}{2} < 7\frac{n}{\delta+1} + O(1).$$

This is an improvement on the current bound by Surmacs [26]. It is still left as an open question whether this is the smallest possible upper bound in the case without girth. The same question could be asked when including girth as well.

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