# Large Girth and Small Oriented Diameter Graphs 

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#### Abstract

In 2015, Dankelmann and Bau proved that for every bridgeless graph $G$ of order $n$ and minimum degree $\delta$ there is an orientation of diameter at most $11 \frac{n}{\delta+1}+9$. In 2016, Surmacs reduced this bound to $7 \frac{n}{\delta+1}$. In this paper, we consider the girth of a graph $g$ and show that for any $\varepsilon>0$ there is a bound of the form $(2 g+\varepsilon) \frac{n}{h(\delta, g)}+O(1)$, where $h(\delta, g)$ is a polynomial. Letting $g=3$ and $\varepsilon<1$ gives an inprovement on the result by Surmacs.

Keywords: diameter, oriented diameter, orientation, oriented graph, distance, size, girth


## 1. Definitions

Let $G=(V, E)$ denote a finite simple graph with vertex set $V$ and edge set $E \subseteq\binom{V}{2}$. Given $G=(V, E)$, a subgraph $H$ of $G$, denoted $H \subseteq G$, is a graph $H=\left(V^{\prime}, E^{\prime}\right)$ for which $V^{\prime} \subseteq V$ and $E^{\prime} \subseteq E \cap\binom{V^{\prime}}{2}$. By $|G|$ we mean the order of $G,|V(G)|$. A digraph $\vec{G}=(V, A)$ is a graph with a vertex set $V$ and an arc set $A$ where each arc is oriented and the orientation of the arc $a$ with ends $u$ and $v$ is in the direction from $u$ to $v$ will be denoted as $\overrightarrow{u v}$. If a set of $\operatorname{arcs} A$ when considered to be unordered is the set $E$, we call $\vec{G}$ an orientation of the graph $G$. A path is defined as $P=(V, E)$, where $V=\left\{v_{0}, v_{1}, \ldots, v_{n}\right\}$ and $E=\left\{x_{0} x_{1}, x_{1} x_{2}, \ldots, x_{n-1} x_{n}\right\}$. We will denote this path $P=v_{0} v_{1} \ldots v_{n}$.

[^0]Given such a path $P$, a cycle is defined as a graph $G=\left(V(P), E(P) \cup\left\{v_{0} v_{n}\right\}\right)$. Given an unoriented path $P=v_{0} v_{1} \ldots v_{n}$, we denote using $\vec{P}$ the corresponding oriented path from $v_{0}$ to $v_{n}$, we will denote using $\overleftarrow{P}$ the oriented path from $v_{n}$ to $v_{0}$. Denote the interior of a path $\bar{P}=v_{1} \ldots v_{n-1}$. Given a graph $G$ and an edge set $E^{\prime} \subseteq E$, define $G \backslash E^{\prime}=\left(V, E \backslash E^{\prime}\right)$. Given an edge set containing a single edge, $E^{\prime}=\{e\}$, we may leave off the brackets, i.e. $G \backslash\{e\}=G \backslash e=(V, E \backslash\{e\})$. We define a forest as a graph containing no cycles. A connected forest is called a tree.

For a set $B \subseteq V(G)$, the induced subgraph of $G$ on the vertex set $B$ is denoted by $G[B]$. That is, $G[B]=\left(B,\binom{B}{2} \cap V\right)$. Given $G$ a simple graph and $v \in V(G)$, the degree of $v$ in $G$ is the number of vertices adjacent to $v$, denoted $\operatorname{deg}(v)=|\{u v \mid u \in V(G), u \neq v, u v \in E(G)\}|$. The minimum degree of a graph $G$ is $\delta(G)=\min \{\operatorname{deg}(v) \mid v \in V(G)\}$. If the graph $G$ is unambiguous, we let $\delta(G)=\delta$. We define the closed neighborhood of a vertex $v \in V(H)$ in the given subgraph $H$ as, $N_{H}[v]=\{u \mid u=v$ or $u v \in E(H)\}$. The open neighborhood of $v$ in a given subgraph $H$, denoted $N_{H}(v)$, is defined as $N_{H}(v)=\{u \mid u \neq v$ and $u v \in E(H)\}$. We may also use $N[v]$ and $N(v)$ if the subgraph $H$ is unambiguous. Let $g(G)=g$ be the girth of $G$ or the length of the smallest cycle in the graph $G$.

We define the distance from $u$ and $v$ in a graph $G$ or digraph $\vec{G}$ as the minimum number of edges or arcs on a path from $u$ to $v$. We denote this as $\rho_{G}(u, v)$ or $\rho_{\vec{G}}(u, v)$. If there does not exist a path from $u$ to $v$, we say that $\rho_{G}(u, v)=\infty$ or $\rho_{\vec{G}}(u, v)=\infty$. We define the diameter of $G$ or $\vec{G}$ to be $\operatorname{diam}(G)=$ $\max \left\{\rho_{G}(u, v) \mid u, v \in V(G)\right\}$ and $\operatorname{diam}(\vec{G})=\max \left\{\rho_{\vec{G}}(u, v) \mid u, v \in V(\vec{G})\right\}$ respectively. If $\operatorname{diam}(G)<\infty$, we call $G$ connected. An edge $e \in E(G)$ (or an arc $a \in A(\vec{G})$ ) is called a bridge if $\operatorname{diam}(G)<\infty$ and $\operatorname{diam}(G \backslash e)=$ $\infty$ (similar for $\vec{G}$ ). If a graph contains no bridges, we call it bridgeless. If $\operatorname{diam}(\vec{G})<\infty$, then we call $\vec{G}$ strongly connected.

A classical result, due to Robbins [25], states that every bridgeless graph has a strongly connected orientation. There may be many such orientations of a graph. A natural next question is what it may mean to find a "good" such
orientation. Many notions of an objective for optimality of such orientations may be considered. For the purposes of this paper, given a graph $G$, let $\overrightarrow{\mathcal{G}}$ represent the set of all strongly connected orientations of $G$. We wish to minimize the oriented diameter of a graph $G$, defined as the following:

$$
\overrightarrow{\operatorname{diam}}(G)=\min _{\overrightarrow{\vec{G}} \in \overrightarrow{\mathcal{G}}} \operatorname{diam}(\vec{G}) .
$$

It was shown by Chvátal and Thomassen [5] that finding the oriented diameter of a given graph is NP-complete. In the same paper, Chvátal and Thomassen found that for the class of bridgeless graphs with diameter $d, \overrightarrow{\operatorname{diam}}(G) \leq 2 d^{2}+2 d$ and constructed bridgeless graphs of diameter $d$ for which every strong orientation admits a diameter of at least $\frac{1}{2} d^{2}+d$. The upper bound was improved by Babu, Benson, Rajendraprasad and Vaka 1] to $1.373 d^{2}+6.971 d-1$.

The paper by Chvátal and Thomassen [5] has led to further investigation of such bounds on the oriented diameter given certain graph parameters, including the diameter [10, 15, 19], the radius [4], the domination number [11, 20], the maximum degree [8], the minimum degree [2, 7, 26|, the number of edges of the graph 6], and other graph classes [3, 12, 13, 14, 15, 17, 18, 21, 22, 23, 24, 27, 28]. See the survey by Koh and Tay [16] for more information on some of these results.

Erdős, Pach, Pollack and Tuza [9] proved that the diameter of connected graphs of order $n$ and minimum degree $\delta$ is at most $\frac{3 n}{\delta+1}+O(1)$. Bau and Dankelmann 2] sought to investigate a similar bound for the oriented diameter and proved that given a bridgeless graph $G$ of order $n$ and minimum degree $\delta$, $\frac{3 n}{\delta+1} \leq \overrightarrow{\operatorname{diam}}(G) \leq \frac{11 n}{\delta+1}$. The upper bound was improved to $\frac{7 n}{\delta+1}$ by Surmacs [26].

In this paper, we will consider upper bounds on the oriented diameter of a graph considering both the minimum degree $\delta$ and the girth $g$ of a graph. In particular we will prove the following theorem.

Theorem 1.1. Given $G=(V, E)$, a bridgeless graph of order $n$ and minimum degree $\delta$, there is a polynomial in $\delta$ and $g, h(\delta, g)$ of degree $\left\lfloor\frac{g-1}{2}\right\rfloor$, for which, given any choice of $\varepsilon>0$,

$$
\overrightarrow{\operatorname{diam}}(G) \leq(2 g+\varepsilon) \frac{n}{h(\delta, g)}+c
$$

We will also show that in the case of general bridgeless graphs, that $\overrightarrow{\operatorname{diam}}(G) \leq$ $(2 g+\varepsilon) \frac{n}{\delta+1}+O(1)$. Since bridgeless graphs have a girth $g \geq 3$, we find that if we choose $0<\varepsilon<1$, this gives an improvement on the bound found in the paper by Surmacs [26].

## 2. Preliminaries

Given a vertex $v \in V(G)$, a natural number $g$, and a path $P$, let $\mathcal{N}(g, v)=$ $\left\{u \left\lvert\, \rho_{G}(u, v) \leq\left\lfloor\frac{g}{2}\right\rfloor-1\right.\right\}$ and $\mathcal{N}(g, v, P)=\left\{u \left\lvert\, \rho_{G \backslash E(P)}(u, v) \leq\left\lfloor\frac{g}{2}\right\rfloor-1\right.\right\}$.

Lemma 2.1. Given a graph $G$ with minimum degree $\delta>3$, girth $g$, a path $P=p_{0} p_{1} \ldots p_{\ell}$, for which $\rho_{G}\left(p_{i}, p_{j}\right)=|j-i|$, and a vertex $x \notin V(P)$,

$$
|\mathcal{N}(g, x, P)| \geq 1+\delta+\sum_{i=1}^{\left\lfloor\frac{g-1}{2}\right\rfloor-1} \delta(\delta-3)^{i}
$$

Proof. Given a vertex $x \in V(G)$ for which $x \notin V(P), G[\mathcal{N}(g, x)]$ is a tree. If not, there would be a cycle of length less than $g$ in $G$ a contradiction to $g$ being the girth. Since $G[\mathcal{N}(g, x, P)] \subseteq G[\mathcal{N}(g, x)], G[\mathcal{N}(g, x, P)]$ is also a tree.

We will construct the set $\mathcal{N}(g, x, P)$. Note that $x \in \mathcal{N}(g, x, P)$. Since $x \notin V(P), N(x) \subseteq \mathcal{N}(g, x, P)$ and $|N(x)| \geq \delta$, so $\left|\left\{u \mid \rho_{G \backslash E(P)}(v, u)=1\right\}\right| \geq \delta$. For each vertex $v_{1} \in N(x)$, if $v_{1} \notin V(P)$, then $\left|N_{G \backslash E(P)}\left(v_{1}\right)\right| \geq \delta$. If $v_{1} \in V(P)$, either one or two of the edges incident to $v_{1}$ are in $E(P)$, so $\left|N_{G \backslash E(P)}\left(v_{1}\right)\right| \geq$ $(\delta-2)$. Since $x \in N\left(v_{1}\right)$ we have that $\left|\left\{u \mid \rho_{G \backslash E(P)}(v, u)=2\right\}\right| \geq \delta(\delta-3)$. Since $\mathcal{N}(g, x, P)$ is a tree, as long as $1 \leq i \leq\left\lfloor\frac{g-1}{2}\right\rfloor-1$, we can perform a similar analysis to show that $\left|\left\{u \mid \rho_{G \backslash E(P)}(v, u)=i+1\right\}\right| \geq \delta(\delta-3)^{i}$. Hence, $|\mathcal{N}(g, x, P)| \geq 1+\delta+\sum_{i=1}^{\left\lfloor\frac{g-1}{2}\right\rfloor-1} \delta(\delta-3)^{i}$.

## 3. Introduction of Main Lemma

Let $h(\delta, g)=1+\delta+\sum_{i=1}^{\left\lfloor\frac{g-1}{2}\right\rfloor-1} \delta(\delta-3)^{i}$. For any $\varepsilon>0$, let $L=\left\lceil\frac{g-1}{\varepsilon}\right\rceil$.
Lemma 3.1. Given a bridgeless graph $G$ with $|G|=n$, girth $g$ and minimum degree $\delta=\delta(G)$, there exists a set of increasing bridgeless subgraphs $H_{0} \subset H_{1} \subset$ $H_{2} \subset \ldots H_{k} \subseteq G$, vertex sets $B_{0} \subset B_{1} \subset \ldots$ for which $B_{i} \subseteq V\left(H_{i}\right)$, and a set of forests $F_{i}$ for which the following hold:

1. For all $v \in V(G), \rho_{G}\left(v, H_{k}\right)<L \cdot g$,
2. for all $i,\left|F_{i}\right| \geq h(\delta, g)\left|B_{i}\right|$, and
3. $\left|H_{i}\right| \leq(2 g+\varepsilon)\left|B_{i}\right|$.

Proof. We will prove by induction on $B_{i}, F_{i}$, and $H_{i}$. For some $v_{0} \in V(G)$, let $B_{0}=\left\{v_{0}\right\}, F_{0}=G\left[\mathcal{N}\left(g, v_{0}\right)\right]$, and $H_{0}=\left(\left\{v_{0}\right\}, \emptyset\right)$. Certainly property 3 holds. Note that $F_{0}$ is a tree of order $\sum_{\alpha=0}^{\left\lfloor\frac{g-1}{2}\right\rfloor} \delta^{\alpha} \geq h(\delta, g)$, so property 2 holds. If property 1 holds, we are done.

Consider $B_{i}, F_{i}, H_{i}$ for which properties 2 and 3 hold and property 1 does not yet hold. Since property 1 does not yet hold, there exists a vertex, $v$, for which $\rho_{G}\left(v, H_{i}\right)=L \cdot g$. Let $p_{0}$ be a vertex in $H_{i}$ for which $\rho_{G}\left(v, p_{0}\right)=L \cdot g$. Consider a path of shortest length between $p_{0}$ and $v$, call this path $P=p_{0} p_{1} \ldots p_{L g-1} v$ with $v=p_{L g}$. Let $e_{i}=p_{i-1} p_{i}$. Let $H_{i}^{\prime}=H_{i}$. Call $e_{j} \in E(P)$ covered if $e_{j}$ is not a bridge in $H_{i}^{\prime} \cup P$. Let $P_{j}=p_{0} \ldots p_{j}$ and $P_{j}^{\prime}=p_{j} p_{j+1} \ldots p_{L g}$. We consider a set of edges $E\left(P_{j}\right)$ to be covered if no edge $e \in E\left(P_{j}\right)$ is a bridge in $H_{i}^{\prime} \cup P_{j}$. We will build a set of vertices $\operatorname{cov}(P) \subseteq V(G) \backslash\left(V(P) \cup V\left(H_{i}\right)\right.$ which is incident to all the edges used to cover $E(P)$.

To expand $H_{i}^{\prime}$, note that $e_{1}$ is not covered in $H_{i}^{\prime} \cup P$. Since $G$ is bridgeless, there must be a path from $H_{i}^{\prime}$ to $P_{1}^{\prime}$. Consider a path of length $\rho_{G \backslash E(P)}\left(H_{i}^{\prime}, P_{1}^{\prime}\right)$, call it $Q$. Note that the two end vertices of $Q$ are the only vertices in $V(Q)$ which can intersect with $V(P)$. Let $p_{\beta}$ be the end vertex of $Q$ on $P \backslash p_{0}$. Add $Q$ and $P_{\beta}$ to $H_{i}^{\prime}$. Add the set of interior vertices of $Q, V(\bar{Q})$, to $\operatorname{cov}(P)$, a set of vertices which will eventually be incident to all the edges used to cover $P$. Label the vertices in $\operatorname{cov}(P)$ as $q_{r}$ such that $r=\rho_{G \backslash E(P)}\left(H_{i}, q_{r}\right)$. Let $B_{i}^{\prime}=B_{i}$. We will now consider an algorithm that will add to $\operatorname{cov}(P), B_{i}^{\prime}$, and $H_{i}^{\prime}$.

1. If there is no longer an edge left uncovered, terminate the algorithm.
2. If there is an uncovered edge in $P$, consider the edge $e_{j}$ with the smallest index $j$ that is not yet covered. Since $G$ is bridgeless, there exists a path from $H_{i}^{\prime}$ to $P_{j}^{\prime}$ of length $\rho_{G \backslash E(P)}\left(H_{i}^{\prime}, P_{j}^{\prime}\right)$, call it $R$. Add $V(\bar{R})$ to $\operatorname{cov}(P)$. Label the vertices $v \in V(\bar{R})$ as $q_{r}$ where $r=|\operatorname{cov}(P)|+\rho_{G \backslash E(P)}\left(H_{i}^{\prime}, v\right)$. Add $R$ and $P_{j}$ to $H_{i}^{\prime}$.
3. If for all pairs of vertices $q_{m_{1}}, q_{m_{2}} \in \operatorname{cov}(P)$ we have $\rho_{G \backslash E(P)}\left(q_{m_{1}}, H_{i}\right) \geq$ $m_{1}$ and $\rho_{G \backslash E(P)}\left(q_{m_{1}}, q_{m_{2}}\right) \geq\left|m_{2}-m_{1}\right|$, then return to step 1. If this was not the case, consider one of the following augmentations.
(a) If $\rho_{G \backslash E(P)}\left(q_{m_{1}}, H_{i}\right)=s<m_{1}$, remove $\left\{q_{1}, \ldots q_{m_{1}-1}\right\}$ and any edges incident to that vertex set from $H_{i}^{\prime}$ and $\operatorname{cov}(P)$. Consider a path $S$, which is edge disjoint from $P$ between $q_{m-1}$ and $H_{i}$ of length $\rho_{G \backslash E(P)}\left(q_{m_{1}}, H_{i}\right)=s$. Add this path to $H_{i}^{\prime}$, add the vertices in $V(\bar{S})$ to $\operatorname{cov}(P)$, and label them $q_{\ell}$ such that $\ell=\rho_{G \backslash E(P)}\left(H_{i}, q_{\ell}\right)$. For values from $m_{1}$ to $t$, where $t$ is the highest current label $r$ for $q_{r}$ in $\operatorname{cov}(P)$, relabel $q_{m_{1}} \ldots q_{t}=q_{s} \ldots q_{t-\left(m_{1}-s\right)}$. After relabeling, return to step 3 .
(b) If $\rho_{G \backslash E(P)}\left(q_{m_{1}}, q_{m_{2}}\right)=s<\left|m_{2}-m_{1}\right|$, without loss of generality, let $m_{1}<m_{2}$. Remove the vertices $q_{m_{1}+1}, \ldots, q_{m_{2}-1}$ from $H_{i}^{\prime}$ and $\operatorname{cov}(P)$. Consider a path $S$, which is edge disjoint from $P$ between $q_{m_{1}}$ and $q_{m_{2}}$ of length $\rho_{G \backslash E(P)}\left(q_{m_{1}}, q_{m_{2}}\right)=s$. Add this path to $H_{i}^{\prime}$,


Figure 3.1: The left graph is an example of subgraph $H^{\prime}$ where step 3 a will be executed. The right is $H_{i}^{\prime}$ after execution of 3 a


Figure 3.2: The left graph is an example of subgraph $H^{\prime}$ where step 3 b will be executed. The right is $H_{i}^{\prime}$ after execution of 3 b
add the vertices in $V(\bar{S})$ to $\operatorname{cov}(P)$. Label the newly added vertices $q_{m_{1}+1}, \ldots, q_{m_{1}+s-1}$ and relabel $q_{m_{2}+1} \ldots q_{t}=q_{m_{1}+s} \ldots q_{t-\left(\left(m_{2}-m_{1}\right)-s\right)}$. After relabeling, return to step 3.

Any step for which step 3a or step 3b executes, there was a strict reduction in $|\operatorname{cov}(P)|$. On the path $P$, since $\rho\left(H_{i}, p_{j}\right)=j$, there must be at least 1 vertex in $\operatorname{cov}(P)$, so at some point we must leave step 3 of the algorithm. Any time step 2 executes, there is a strict increase in the number of edges in $P$ that are covered. Since $P$ is finite, at some point the algorithm must return to step 1 and terminate.

Let $H_{i+1}=H_{i}^{\prime}, B_{i+1}=\left\{B_{i} \cup q_{r} \mid r \equiv 0 \bmod g\right\}$, and $F_{i+1}=F_{i} \bigcup \cup_{b \in B_{i+1} \backslash B_{i}} \mathcal{N}(g, b, P)$. Now we will show that Properties 2 and 3 of Lemma 3.1 hold.

To prove Property 3 holds, first remember that $L=\left\lceil\frac{g-1}{\varepsilon}\right\rceil \geq \frac{g-1}{\varepsilon}$, hence $g-1 \leq L \varepsilon$. We will have two cases: $L \leq\left|B_{i+1} \backslash B_{i}\right|$ and $L>\left|B_{i+1} \backslash B_{i}\right|$. If
$L \leq\left|B_{i+1} \backslash B_{i}\right|$, the following holds:

$$
\begin{align*}
\left|H_{i+1}\right| & \leq\left|H_{i+1}\right|+|P|+|\operatorname{cov}(P)|  \tag{3.1}\\
& \leq\left|H_{i}\right|+g L+g\left|B_{i+1} \backslash B_{i}\right|+(g-1)  \tag{3.2}\\
& \leq\left|H_{i}\right|+g\left|B_{i+1} \backslash B_{i}\right|+g\left|B_{i+1} \backslash B_{i}\right|+L \varepsilon  \tag{3.3}\\
& \leq\left|H_{i}\right|+g\left|B_{i+1} \backslash B_{i}\right|+g\left|B_{i+1} \backslash B_{i}\right|+\left|B_{i+1} \backslash B_{i}\right| \varepsilon  \tag{3.4}\\
& \leq\left|H_{i}\right|+(2 g+\varepsilon)\left|B_{i+1} \backslash B_{i}\right|  \tag{3.5}\\
& \leq(2 g+\varepsilon)\left|B_{i}\right|+(2 g+\varepsilon)\left|B_{i+1} \backslash B_{i}\right|  \tag{3.6}\\
& \leq(2 g+\varepsilon)\left|B_{i+1}\right| . \tag{3.7}
\end{align*}
$$

To prove property 2 note that for each $b \in B_{i+1} \backslash B_{i}, \rho_{G \backslash E(P)}\left(b, H_{i}\right) \geq$ $g$, otherwise we would have augmented $\operatorname{cov}(P)$ in step 3a the algorithm, so $\rho_{G \backslash E(P)}\left(b, B_{i}\right) \geq g$. For any pair of vertices $b_{1}, b_{2} \in B_{i+1} \backslash B_{i}, \rho_{G \backslash E(P)}\left(b_{1}, b_{2}\right) \geq$ $g$, otherwise we would have augmented $\operatorname{cov}(P)$ in step 3b of the algorithm. Hence, $\mathcal{N}\left(g, b_{1}, P\right) \cap \mathcal{N}\left(g, b_{2}, P\right)=\emptyset$. So,

$$
\begin{align*}
\left|F_{i+1}\right| & \geq\left|F_{i}\right|+\left|\bigcup_{b \in B_{i+1} \backslash B_{i}} \mathcal{N}(g, b, P)\right|  \tag{3.8}\\
& \geq\left|B_{i}\right| h(\delta, g)+\left|B_{i+1} \backslash B_{i}\right| h(\delta, g)  \tag{3.9}\\
& \geq\left|B_{i+1}\right| h(\delta, g) \tag{3.10}
\end{align*}
$$

In the case that $L>\left|B_{i+1} \backslash B_{i}\right|$, redefine $B_{i+1}$ to be $B_{i} \bigcup \cup_{c=1}^{L} p_{c g}$. Since $L g=g\left|B_{i+1} \backslash B_{i}\right|$, the computation above from 3.1 to 3.7 holds. See that by definition of $P$, for any $b \in B_{i+1} \backslash B_{i}, \rho_{G}\left(b, H_{i}\right) \geq g$, so $\mathcal{N}(g, b) \cap V\left(H_{i}\right)=\emptyset$. For any $b_{1}, b_{2} \in B_{i+1} \backslash B_{i}, \rho_{G}\left(b_{1}, b_{2}\right) \geq g$, hence $\mathcal{N}\left(g, b_{1}\right) \cap \mathcal{N}\left(g, b_{2}\right)=\emptyset$. It follows that

$$
\begin{align*}
\left|F_{i+1}\right| & \geq\left|F_{i}\right|+\left|\bigcup_{b \in B_{i+1} \backslash B_{i}} \mathcal{N}(g, b)\right|  \tag{3.11}\\
& \geq\left|B_{i}\right| h(\delta, g)+\left|B_{i+1} \backslash B_{i}\right| h(\delta, g)  \tag{3.12}\\
& \geq\left|B_{i+1}\right| h(\delta, g) \tag{3.13}
\end{align*}
$$

Hence, Property 2 of Lemma 3.1 holds in this case.
Since $H_{i}$ and $B_{i}$ are increasing subgraphs and vertex sets, and our graph $G$ is a finite graph, eventually 1 will hold. When this happens, let $i=k$.

Now we wish to use Lemma 3.1 to create an orientation on a subgraph of $G$ with a small diameter. First, we need to consider the following theorem by Robbins.

Theorem 3.2 (Robbins 25]). A graph is bridgeless if and only if it admits a strong orientation.

Lemma 3.3. Let $H_{k} \subseteq G, B_{k} \subseteq V(G)$, and $F_{k} \subseteq G$, and Properties 1, 2, and 3 of Lemma 3.1 hold. There exists an orientation of $H_{k}, \overrightarrow{H_{k}}$ for which $\operatorname{diam}\left(\overrightarrow{H_{k}}\right) \leq(2 g+\varepsilon) \frac{n}{h(\delta, g)}$.

Proof. By Property 2 of Lemma 3.1 we have that $h(\delta, g)\left|B_{k}\right| \leq\left|F_{k}\right| \leq n$, so we find that $\left|B_{k}\right| \leq \frac{n}{h(\delta, g)}$. In conjunction with Property 3 of Lemma 3.1 we find that $\left|H_{k}\right| \leq(2 g+\varepsilon)\left|B_{k}\right| \leq(2 g+\varepsilon) \frac{n}{h(\delta, g)}$.

Hence, there exists a bridgeless subraph $H_{k}$ for which $\left|H_{k}\right| \leq(2 g+\varepsilon) \frac{n}{h(\delta, g)}$. By Theorem 3.2, there is strong orientation of $H_{k}, \overrightarrow{H_{k}}$. Note that $\operatorname{diam}\left(\overrightarrow{H_{k}}\right) \leq$ $\left|H_{k}\right| \leq(2 g+\varepsilon) \frac{n}{h(\delta, g)}$.

We now wish to extend our result in Lemma 3.3 for $H_{k} \subseteq G$ to $G$. To do so, we will need to consider an extension to the following two lemmas, one by Fomin et al. 11] and one by Bau et al. [2]
Lemma 3.4 (Fomin, Matamala, Prisner and Rapaport 11]). Let $G$ be a bridgeless graph and $H$ a bridgeless subgraph of $G$ with $\rho_{G}(v, H) \leq 1$ for all $v \in V(G)$. Given an orientation $\vec{H}$ such that $\operatorname{diam}(\vec{H})=d$, then $G$ has an orientation of $d+4$.

Lemma 3.5 (Bau and Dankelmann[2]). Let $G$ be a bridgeless graph and $H$ $a$ bridgeless subgraph of $G$ such that $\rho_{G}(v, H) \leq 2$ for all $v \in V(G)$. Let $\vec{H}$ be a strongly connected orientation of $H$ of diameter $d$. Then there exists a strongly connected orientation of $G$ of diameter at most $d+12$ that extends the orientation of $\vec{H}$.

We have that for any $v \in V(G), \rho_{G}\left(v, H_{k}\right) \leq L g$. Since $L g>2$, we will need to extend this lemma as seen below.

Lemma 3.6. Let $G$ be a bridgeless graph, $H$ a bridgeless subgraph of $G$, and let $s$ be an integer such that $s \geq 2$ and for all $v \in V(G), \rho_{G}(v, H) \leq s$. Let $\vec{H}$ be a strongly connected orientation of $H$ of diameter $d$. Then there exists a strongly connected orientation of $G$ of diameter at most $d+4\binom{s+1}{2}$ that extends the orientation of $\vec{H}$.
Proof. Let $H \subseteq G$ be a bridgeless subgraph with an orientation $\vec{H}$ such that $\operatorname{diam}(\vec{H})=d$ and $\rho_{G}(v, H) \leq k$ for all $v \in V(G)$. Let $V_{1}:=\left\{v \mid \rho_{G}(v, H)=\right.$ 1\}. Given a vertex $v \in V_{1}$, label one of its neighbors in $H$ as $x$. Let $\overrightarrow{H^{\prime}}=\vec{H}$, we will continue to augment $\overrightarrow{H^{\prime}}$ throughout the proof. We will call $\overrightarrow{H^{\prime}}$ extendable at step $i$ if for any $v \in V\left(\overrightarrow{H^{\prime}}\right), \rho_{\overrightarrow{H^{\prime}}}(v, H)+\rho_{\overrightarrow{H^{\prime}}}(H, v) \leq 2 i$ and $\rho_{G}(H, v) \leq i$.

Assume there is a vertex $z$ for which $\rho_{G}(\vec{H}, z)=s$. First, we will show that there exists a graph $\overrightarrow{H^{\prime}}$ that is extendable at step 1. If there is a vertex
$v \in V_{1} \backslash V\left(\overrightarrow{H^{\prime}}\right)$ for which $\rho_{G \backslash v x}(v, H)=1$, there exists some vertex $y, y \neq x$ for which $v y \in E(G)$. Let $\overrightarrow{H^{\prime}}=\vec{H} \cup \overrightarrow{x v y}$. Repeat this until there are no longer vertices $v \in V_{1} \backslash V\left(\overrightarrow{H^{\prime}}\right)$ for which $\rho_{G \backslash v x}(v, H)=1$. Note that for any $v \in V\left(\overrightarrow{H^{\prime}}\right), \rho_{\overrightarrow{H^{\prime}}}(v, H)+\rho_{\overrightarrow{H^{\prime}}}(H, v) \leq 2$ and $\rho_{G}(v, H) \leq 1$, so $\overrightarrow{H^{\prime}}$ is extendable at step 1.

We will show that for any $1 \leq i<2 s$, if $\overrightarrow{H^{\prime}}$ is extendable at step $i$, then it is also extendable at step $i+1$. If there is a vertex $v \in V_{1} \backslash V\left(\overrightarrow{H^{\prime}}\right)$ for which $\rho_{G \backslash v x}(v, H)=i$, let $Q$ be a path of length $i$ from $v$ to $H$ which does not include $v x$. Consider a vertex $v^{\prime} \in V(Q)$ for which $v^{\prime} \in V\left(\overrightarrow{H^{\prime}}\right)$ and $\rho_{G \backslash v x}\left(v^{\prime}, v\right)$ is minimized. If $v^{\prime} \in V(H)$, add $\vec{Q} \cup \overrightarrow{x v}$ to $\overrightarrow{H^{\prime}}$. See that for all $v \in V\left(\overrightarrow{H^{\prime}}\right)$, $\rho_{\overrightarrow{H^{\prime}}}(v, H)+\rho_{\overrightarrow{H^{\prime}}}(H, v) \leq 2 i$ and $\rho_{G}(v, H) \leq i$, so $\overrightarrow{H^{\prime}}$ is extendable at step $i$.

If $v^{\prime} \notin V(H)$, let $Q^{\prime}$ be the subpath of $Q$ from $v$ to $v^{\prime}$. Since $\overrightarrow{H^{\prime}}$ is extendable at step $i$, there exists an integer $j$ for which $|j|<i, \rho \overrightarrow{H^{\prime}}{ }^{\prime}\left(v^{\prime}, H\right) \leq i-j$, and $\rho_{\overrightarrow{H^{\prime}}}\left(H, v^{\prime}\right) \leq i+j$. If $j \geq 0$, add $\overleftarrow{Q^{\prime}} \cup \overrightarrow{v x}$ to $\overrightarrow{H^{\prime}}$. If $j<0$, add $\overrightarrow{Q^{\prime}} \cup \overrightarrow{x v}$ to $\overrightarrow{H^{\prime}}$. See in each case that for all $v \in V\left(\overrightarrow{H^{\prime}}\right), \rho_{\overrightarrow{H^{\prime}}}(v, H)+\rho_{\overrightarrow{H^{\prime}}}(H, v) \leq 2 i$ and $\rho_{G}(v, H) \leq i$.

Once we have an extendable subgraph $\overrightarrow{H^{\prime}}$ at step $2 s$, and have considered all vertices $v \in V_{1} \backslash V\left(\overrightarrow{H^{\prime}}\right)$ for which $\rho_{G \backslash e}(v, H) \leq 2 s$, there are no more vertices $v \in V_{1} \backslash V\left(\overrightarrow{H^{\prime}}\right)$. If there were a vertex $v \in V_{1} \backslash V\left(\overrightarrow{H^{\prime}}\right)$ for which $\rho_{G \backslash e}(v, H)>2 s$, notice that this would mean there exists a vertex $v^{\prime} \in V(G)$ for which $\rho_{G}\left(v^{\prime}, H\right)>s$, a contradiction to the assumption of the lemma.

Since $\overrightarrow{H^{\prime}}$ was extendable at step $2 s$, for any $v \in V\left(\overrightarrow{H^{\prime}}\right), \rho_{\overrightarrow{H^{\prime}}}(v, H) \leq 2 s$ and $\rho_{\overrightarrow{H^{\prime}}}(H, v) \leq 2 s$, so $\overrightarrow{\operatorname{diam}}\left(H^{\prime}\right) \leq \overrightarrow{\operatorname{diam}}(H)+4 s$.

We will now prove Theorem 1.1 .
Proof. In Lemma 3.1 we showed that there is a bridgeless subgraph $H_{k} \subseteq G$ such that for any $v \in V(G), \rho_{G}\left(v, H_{k}\right) \leq L g$ and

$$
\overrightarrow{\operatorname{diam}}\left(H_{k}\right) \leq(2 g+\varepsilon) \frac{n}{h(\delta, g)} .
$$

By a combination of this and Lemma 3.6 with $s=L \cdot g$, we find

$$
\overrightarrow{\operatorname{diam}}(G) \leq \overrightarrow{\operatorname{diam}}\left(H_{k}\right)+\sum_{i=1}^{L g} 4 i \leq(2 g+\varepsilon) \frac{n}{h(\delta, g)}+4\binom{L g+1}{2}
$$

Corollary 3.7. In Theorem 1.1, if $g=3$ and $0<\varepsilon<1$,

$$
\overrightarrow{\operatorname{diam}}(G) \leq(2 g+\varepsilon) \frac{n}{h(\delta, g)}+4\binom{L g+1}{2}<7 \frac{n}{\delta+1}+O(1)
$$

This is an improvement on the current bound by Surmacs [26]. It is still left as an open question whether this is the smallest possible upper bound in the case without girth. The same question could be asked when including girth as well.

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