# Exact generalized Turán number for $K_{3}$ versus suspension of $P_{4}$ 

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#### Abstract

Let $P_{4}$ denote the path graph on 4 vertices. The suspension of $P_{4}$, denoted by $\widehat{P}_{4}$, is the graph obtained via adding an extra vertex and joining it to all four vertices of $P_{4}$. In this note, we demonstrate that for $n \geq 8$, the maximum number of triangles in any $n$-vertex graph not containing $\widehat{P}_{4}$ is $\left\lfloor n^{2} / 8\right\rfloor$. Our method uses simple induction along with computer programming to prove a base case of the induction hypothesis.


Keywords: generalized Turán problem, suspension of a graph, computer programming.
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## 1 Introduction

The generalized Turán number ex $(n, T, H)$ is defined as the maximum number of copies of $T$ in an $n$ vertex graph not containing $H$ as a (not necessarily induced) subgraph. When $T=K_{2}$, this is the Turán number ex $(n, H)$ of the graph. The first systematic study of $\operatorname{ex}(n, T, H)$ for $T \neq K_{2}$ was carried out by Alon and Shikhelman [1].

In more recent years, several researchers have studied the asymptotic behavior of ex $\left(n, K_{3}, H\right)$ for the case $T=K_{3}$ (see, for example [2, [3, 5). It is known that when $\chi(H)>3$, ex $\left(n, K_{3}, H\right) \sim$ $\binom{\chi(H)-1}{3} /(\chi(H)-1)^{2} \cdot n^{2}$, where $\chi(H)$ denotes the chromatic number of $H$ [1, 6. Alon and Shikhelman [1] extensively study the case when $\chi(H)=2$.

Mubayi and the author [7 initiated the study of $\operatorname{ex}\left(n, K_{3}, H\right)$ for a simple family of graphs $H$ with $\chi(H)=3$. For any graph $G$, they denoted the suspension $\widehat{G}$ as the graph obtained from $G$ by adding a new vertex $v$ and joining it with all vertices of $G$. They proceeded to analyze the asymptotic behavior of $\operatorname{ex}\left(n, K_{3}, \widehat{G}\right)$ for different bipartite graphs $G$.

One of the several bipartite graphs they consider is the path $P_{4}$ on four vertices. It was shown that for any $n \geq 4$,

$$
\begin{equation*}
\frac{n^{2}}{8}-O(1) \leq \operatorname{ex}\left(n, K_{3}, \widehat{P}_{4}\right)<\frac{n^{2}}{8}+3 n \tag{1.1}
\end{equation*}
$$

An exact result for sufficiently large $n$ was given by Gerbner [4] using the technique of progressive induction. In particular, they prove that for a number $K \leq 1575$ and $n \geq 525+4 K$,

$$
\begin{equation*}
\operatorname{ex}\left(n, K_{3}, \widehat{P}_{4}\right)=\left\lfloor n^{2} / 8\right\rfloor . \tag{1.2}
\end{equation*}
$$

They mention that a proof of the upper bound of (1.2) for $n=8,9,10,11$ together with induction would suffice to prove (1.2) for every $n \geq 8$. In this note, we leverage this idea to determine the exact value of $\operatorname{ex}\left(n, K_{3}, \widehat{P}_{4}\right)$ for every $n \geq 4$, thus closing the gap in the literature for this extremal problem.

[^0]Theorem 1.1. For $n \geq 8$, ex $\left(n, K_{3}, \widehat{P}_{4}\right)=\left\lfloor n^{2} / 8\right\rfloor$. For $n=4,5,6,7$ the values of ex $\left(n, K_{3}, \widehat{P}_{4}\right)$ are 4, 4, 5, 8 respectively.

The lower bound constructions for Theorem 1.1] are different for the cases $n \in\{4,5,6,7\}$ and $n \geq 8$.
Figure 1.1 illustrates graphs on $n$ vertices for $n \in\{4,5,6,7\}$ that achieve the maximum number of triangles. In fact, we shall see later in Section 3.1 that these constructions are unique up to isomorphism.


Figure 1.1: Graphs on $4,5,6,7$ vertices and $4,4,5,8$ triangles, respectively.

The general lower bound construction considered in [4, 7] (for $n \geq 8$ ) was the complete bipartite graph $K_{\lfloor n / 2\rfloor,\lceil n / 2\rceil}$ with a matching in any of the even parts. A short case analysis shows that the total number of triangles in these graphs is given by $\left\lfloor n^{2} / 8\right\rfloor$, hence proving the lower bound in Theorem 1.1 for general $n$.

Thus, the main goal of this manuscript is to prove that these lower bounds on ex $\left(n, K_{3}, \widehat{P}_{4}\right)$ are tight.
This work is organized as follows. We present some preliminaries in Section 2 Then, we show the upper bound of Theorem 1.1 for $n \geq 5$ in Section 3 Finally, we make some concluding remarks regarding uniqueness of the lower bound constructions in Section 4.

## 2 Preliminaries

Throughout the rest of this paper, we assume without loss of generality that all graphs are edge-minimal. This implies that every edge of the graphs considered must lie in a triangle, as we can simply delete edges that do not help forming a triangle. We also assume that the vertex set of any $n$-vertex graph in the rest of this section is $\{0, \ldots, n-1\}$, and abuse notation to represent a $K_{3}$ on vertex subset $\{a, b, c\}$ as simply $a b c$.
Let $n(G), e(G)$ and $t(G)$ denote the number of vertices, edges and triangles in $G$, respectively.
Now we recall some definitions and state a two important lemmas from 4] and 7] which are instrumental in our proof.

Definition 2.1 (Triangle-connectivity). For a graph $G$, two edges $e$ and $e^{\prime}$ are said to be triangleconnected if there is a sequence of triangles $\left\{T_{1}, \ldots, T_{k}\right\}$ of $G$ such that $e \in T_{1}, e^{\prime} \in T_{k}$, and $T_{i}$ and $T_{i+1}$ share a common edge for every $1 \leq i \leq k-1$. A subgraph $H \subseteq G$ is triangle-connected if $e$ and $e^{\prime}$ are triangle-connected for every edges $e$ and $e^{\prime}$ of $H$.

Definition 2.2 (Triangle block). A subgraph $H \subseteq G$ is a triangle block (or simply a block) if it is edge-maximally triangle-connected.

By definition, the triangle blocks of any graph $G$ are edge-disjoint.
Let $B_{s}$ denote the book graph on $(s+2)$ vertices, consisting of $s$ triangles all sharing a common edge. Let this common edge be called the base of the $B_{s}$. The following lemma characterizes the triangle blocks of any $\widehat{P}_{4}$-free graph $G$.

Lemma 2.3 (7], Claim 5.3). Every triangle block of a $\widehat{P}_{4}$-free graph $G$ is isomorphic to a $K_{4}$ or a $B_{s}$ for some $s \geq 1$.

Proof. Let $H \subseteq G$ be an arbitrary triangle block. If $H$ contains only one or two triangles, it is isomorphic to $B_{1}$ or $B_{2}$. Suppose $H$ contains at least three triangles. Let two of them be $a b x_{1}$ and $a b x_{2}$ (see Figure 2.1).


Figure 2.1: (left): third triangle on $a x_{1}$, (right): third triangle on $a b$
If another triangle is of the form $a x_{1} y$ for some $y \in V(H)$, then there are two possible cases. If $y \neq x_{2}$, then $N_{H}(a)$ contains the 4-path $x_{2} b x_{1} y$, a contradiction. Otherwise if $y=x_{2}$, then the vertices $a, b, x_{1}, x_{2}$ create a $K_{4}$, and this $K_{4}$ is a triangle block by itself.

Similarly, if a triangle contained any of the edges $b x_{1}, a x_{2}, b x_{2}$, we would end up with a $K_{4}$-block, and this block cannot be extended any further.

Therefore all triangles in $H$ would intersect the edge $a b$, implying $H \cong B_{s}$ for some $s \geq 1$.
Lemma $2.4\left([4)\right.$, Section 2). Suppose $G$ is an n-vertex $\widehat{P}_{4}$-free graph containing no $K_{4}$. Then, we have $t(G) \leq\left\lfloor n^{2} / 8\right\rfloor$.

Proof. By Lemma [2.3, all triangle blocks of $G$ are isomorphic to $B_{s}$ for some $s \geq 1$. Let $G^{\prime}$ be obtained from $G$ by deleting the base edges of each of the books (if $s=1$, delete any arbitrary edge). As each triangle of $G$ contains two distinct edges from $G^{\prime}$, we have $t(G)=e\left(G^{\prime}\right) / 2$. By Mantel's theorem, $e\left(G^{\prime}\right) \leq\left\lfloor n^{2} / 4\right\rfloor$, implying $t(G) \leq \frac{1}{2}\left\lfloor n^{2} / 4\right\rfloor$, i.e. $t(G) \leq\left\lfloor n^{2} / 8\right\rfloor$.

## 3 Upper bounds

In order to prove that $\operatorname{ex}\left(n, K_{3}, \widehat{P}_{4}\right) \leq K$ for some fixed $n$ and $K$, we need to show that any $n$-vertex graph containing at least $K+1$ triangles contains a copy of $\widehat{P}_{4}$.

### 3.1 The cases $5 \leq n \leq 8$ : brute force

While a case-by-case analysis is tractable by hand for $n=5$ for example, we quickly run into several possible configurations while trying to prove $\operatorname{ex}\left(8, K_{3}, \widehat{P}_{4}\right)=8$. This is where we turn to a computergenerated check. For example, to prove that all 8 -vertex graphs with more than 9 triangles is $\widehat{P}_{4}$-free, we can assume that 012 and 013 are two triangles in some 8 -vertex graph $G$ containing 9 triangles. Then triangles that have an edge from the set $\{02,03,12,13\}$ and have a node from $\{4,5,6,7\}$ are excluded from $G$ since any of these patterns form a $\widehat{P}_{4}$. This excludes 16 triangles. Hence the plausible triangles that $G$ may contain other than 012 and 013 are $\binom{8}{3}-18=38$ in number. We generate $\binom{38}{7} \approx 1.26 \times 10^{7}$ possible graphs, filter out the ones that have exactly 9 triangles, and check for $\widehat{P}_{4}$ 's in each of them.
Our program is available at the Github repository in [8]. We run triangle_count_parallel.py for different pairs of $(n, t)$ to figure out both the extremal number and all extremal configurations for $n$ vertex graphs with $t$-triangles. The results are compiled in the notebook triangle_count.ipynb. Our computation shows that $\operatorname{ex}\left(n, K_{3}, \widehat{P}_{4}\right)=4,5,8,8$ for $n=5,6,7,8$, respectively. The total computation time required for $(n, t)=(8,9)$ on 7 threads of an Intel(R) Core(TM) i7-8550U laptop processor running at 1.80 GHz was around 18 minutes.

### 3.2 The cases $9 \leq n \leq 11$ : identifying $K_{4}$

The main idea behind these cases is to follow the steps of the proof in [7], Section 5.2.
Theorem 3.1. Suppose $(n, t) \in\{(9,11),(10,13),(11,16)\}$, and $G$ is an (edge-minimal) $n$-vertex graph with $t$ triangles. Then $G$ must contain a $\widehat{P}_{4}$.

Proof. For the sake of contradiction, assume that $G$ was $\widehat{P}_{4}$-free. If $G$ was also $K_{4}$-free, then by Lemma 2.4, $t(G) \leq\left\lfloor n^{2} / 8\right\rfloor=10,12,15$ for $n=9,10,11$, contradicting our initial assumption on $t(G)$.

Therefore $G$ must contain a $K_{4}$. Let this $K_{4}$ be induced by vertex subset $S=\left\{u_{0}, u_{1}, u_{2}, u_{3}\right\} \subset V(G)$. Define $X_{i}:=N\left(u_{i}\right)-S$ for $0 \leq i \leq 3$. As $G[S]$ is a triangle block, $X_{i} \cap X_{j}=\varnothing$ for every $i \neq j$. Further, $\sum_{i=0}^{3}\left|X_{i}\right| \leq n-4$. Without loss of generality assume $\left|X_{0}\right| \leq \cdots \leq\left|X_{3}\right|$. Now we consider each case separately.

- Case 1. $(n, t)=(9,11)$ : In this case, $\sum_{i=0}^{3}\left|X_{i}\right| \leq 5$. If $\left|X_{1}\right|>0$, by edge-minimality we would have $\left|X_{1}\right| \geq 2$, implying $\left|X_{1}\right|+\left|X_{2}\right|+\left|X_{3}\right| \geq 6$, a contradiction. Thus, $\left|X_{0}\right|=\left|X_{1}\right|=0$, and by a similar argument, $\left|X_{2}\right| \leq 2$. This means the vertex $u_{2}$ lies in at most one triangle outside of $G[S]$. Let $G^{\prime}$ be obtained by deleting $\left\{u_{0}, u_{1}, u_{2}\right\}$ from $G$. Clearly $n\left(G^{\prime}\right)=6$ and $t\left(G^{\prime}\right) \geq t(G)-5=6$. As ex $\left(6, K_{3}, \widehat{P}_{4}\right)=5$ by the discussion in Section 3.1, $G^{\prime}$ has a $\widehat{P}_{4}$, a contradiction.
- Case 2. $(n, t)=(10,13)$ : Here, $\sum_{i=0}^{3}\left|X_{i}\right| \leq 6$. By a similar analysis as before, we can infer that $\left|X_{0}\right|=0$ and $\left|X_{1}\right| \leq 2$. If $\left|X_{1}\right|=0$, we could consider $G^{\prime}=G-\left\{u_{0}, u_{1}\right\}$, which would have $n\left(G^{\prime}\right)=8$ and $t\left(G^{\prime}\right)=13-4=9$, which would lead us to a $\widehat{P}_{4}$ since $\operatorname{ex}\left(8, K_{3}, \widehat{P}_{4}\right)=8$ by the calculation in Section 3.1 Thus, we have $\left|X_{0}\right|=0,\left|X_{1}\right|=2$, and hence $\left|X_{2}\right|=\left|X_{3}\right|=2$. Now, if we consider $G^{\prime \prime}=G-S$, we have $n\left(G^{\prime \prime}\right)=6$ and $t\left(G^{\prime \prime}\right)=13-4-3=6$, again implying that $G^{\prime \prime}$ has a $\widehat{P}_{4}$.
- Case 3. $(n, t)=(11,16)$ : For this pair of $(n, t)$, we have $\sum_{i=0}^{3}\left|X_{i}\right| \leq 7$, implying $\left|X_{0}\right|=0$ again. Since $u_{0}$ lies in exactly three triangles of $G[S], G^{\prime}=G-\left\{u_{0}\right\}$ has $n\left(G^{\prime}\right)=10$ and $t\left(G^{\prime}\right)=13$, leading us to the previous case.

In either of the three cases, we obtain a contradiction, finishing the proof for these cases.

### 3.3 The case $n \geq 12$ : identifying $K_{4}$

Now that we have proved ex $\left(n, K_{3}, \widehat{P}_{4}\right)=\left\lfloor n^{2} / 8\right\rfloor$ for $8 \leq n \leq 11$, we are now ready to handle the general case using induction on $n$. Our proof follows the idea of [4] with a more careful analysis to obtain the desired bound.

Proof of Theorem 1.1 for $n \geq 12$. Let us assume that $\operatorname{ex}\left(k, K_{3}, \widehat{P}_{4}\right)=\left\lfloor k^{2} / 8\right\rfloor$ for all $8 \leq k \leq n-1$. We note that a simple case analysis leads to

$$
\begin{align*}
& \left\lfloor n^{2} / 8\right\rfloor-\left\lfloor(n-1)^{2} / 8\right\rfloor \geq\lfloor n / 4\rfloor \\
& \left\lfloor n^{2} / 8\right\rfloor-\left\lfloor(n-4)^{2} / 8\right\rfloor=n-2 \tag{3.1}
\end{align*}
$$

For the sake of contradiction, suppose $G$ is an $n$-vertex $\widehat{P}_{4}$-free graph with $t(G) \geq\left\lfloor n^{2} / 8\right\rfloor+1$. For a subset $U \subset V(G)$, let us denote by $t(U)$ the number of triangles containing at least one vertex from $U$. By (3.1), we may assume that

$$
\begin{align*}
& |U|=1 \Longrightarrow t(U) \geq\lfloor n / 4\rfloor+1  \tag{3.2}\\
& |U|=4 \Longrightarrow t(U) \geq n-1
\end{align*}
$$

Now, notice that by Lemma 2.4. $G$ must contain a $K_{4}$. As in the previous section, let $S=\left\{u_{0}, u_{1}, u_{2}, u_{3}\right\}$ induce this $K_{4}$, and denote $X_{i}=N\left(u_{i}\right)-S$ for $0 \leq i \leq 3$. Again, $\left|X_{i} \cap X_{j}\right|=\varnothing$ for every $i \neq j$. Observe
that $t(S)=\sum_{i=0}^{3} e\left(X_{i}\right)+4$, and so by (3.2),

$$
\sum_{i=0}^{3} e\left(X_{i}\right) \geq n-5
$$

On the other hand, since each $X_{i}$ is $P_{4}$-free, we have $\sum_{i=0}^{3} e\left(X_{i}\right) \leq \sum_{i=0}^{3}\left|X_{i}\right| \leq n-4$. Hence,

$$
\begin{equation*}
\sum_{i=0}^{3} e\left(X_{i}\right) \in\{n-5, n-4\} \tag{3.3}
\end{equation*}
$$

This implies that $e\left(X_{i}\right)=\left|X_{i}\right|$ for at least three $u_{i} \in S$. Assume that $e\left(X_{i}\right)=\left|X_{i}\right|$ for $0 \leq i \leq 2$ and $e\left(X_{3}\right) \in\left\{\left|X_{3}\right|-1,\left|X_{3}\right|\right\}$. This also means that $G\left[X_{i}\right]$ are vertex-disjoint unions of triangles for $0 \leq i \leq 2$, and $X_{3}$ is a union of triangles and a star on $r$ vertices for some $r \geq 0$. Further, (3.2) gives us the bound

$$
\begin{equation*}
\left|X_{i}\right| \geq\lfloor n / 4\rfloor-2 \tag{3.4}
\end{equation*}
$$

We now continue with a more detailed analysis of the neighborhoods of vertices in $G$. In what follows, let $x_{i}$ denote the size of $X_{i}$. For a subset $A \subset V(G)$, let $\mathcal{T}(A)$ denote the set of triangles in $G[A]$. We now consider two cases.

Case 1: $\sum_{i=0}^{4} x_{i}=n-5$. In this case, note that since $\sum_{i=0}^{3} e\left(X_{i}\right)=n-5$, we have $e\left(X_{3}\right)=x_{3}$. Thus, the subgraphs $G\left[X_{i}\right]$ are all disjoint unions of triangles, and there is exactly one vertex $y$ in $V(G)-\bigcup_{i} X_{i} \cup S$, and thus $3 \mid n-5$, implying $n \equiv 2 \bmod 3$. Moreover, (3.4) implies $x_{i} \geq 3$, and hence $n \geq 17$.

Now, observe that for $G^{\prime}=G-\{y\}$,

$$
\begin{equation*}
\sum_{v \in V(G)} \operatorname{deg} v=\sum_{i=0}^{3} \sum_{v w z \in \mathcal{T}\left(X_{i}\right)}\left(\operatorname{deg}_{G^{\prime}} v+\operatorname{deg}_{G^{\prime}} w+\operatorname{deg}_{G^{\prime}} z\right)+\sum_{v \in S} \operatorname{deg} v+2 \operatorname{deg} y \tag{3.5}
\end{equation*}
$$

We proceed by upper bounding each term of (3.5) separately.

- Let $v w z \in \mathcal{T}\left(X_{0}\right)$. For any $j \neq 0$, as $N(v)-X_{0}-S-\{y\}$ cannot contain two adjacent vertices from the same $X_{j}, v$ can only be adjacent to at most one vertex from each triangle of $X_{j}$. Finally, $v$ is adjacent to exactly three nodes from $X_{0} \cup S$, leading to

$$
\operatorname{deg}_{G^{\prime}} v+\operatorname{deg}_{G^{\prime}} w+\operatorname{deg}_{G^{\prime}} z \leq 3\left(\frac{x_{1}}{3}+\frac{x_{2}}{3}+\frac{x_{3}}{3}\right)+9=\left(x_{1}+x_{2}+x_{3}\right)+9
$$

By repeating the same argument over all $x_{i} / 3$ triangles from $\mathcal{T}\left(X_{i}\right)$, we have

$$
\sum_{v w z \in \mathcal{T}\left(X_{i}\right)}\left(\operatorname{deg}_{G^{\prime}} v+\operatorname{deg}_{G^{\prime}} w+\operatorname{deg}_{G^{\prime}} z\right) \leq \frac{x_{i}}{3} \sum_{j \neq i} x_{j}+3 x_{i} .
$$

- As $y$ is not adjacent to any vertex of $S$, we have

$$
\sum_{v \in S} \operatorname{deg} v=\left(x_{0}+x_{1}+x_{2}+x_{3}\right)+12=n+7
$$

- For each $i, N(y) \cap X_{i}$ has at most $x_{i} / 3$ vertices, as otherwise by the pigeonhole principle we would have $v, w \in N(y) \cap X_{i}$ that are adjacent, leading to a triangle yvw sharing an edge with the $K_{4}$ containing $u_{i}, v$ and $w$. Further, $y$ does not have a neighbor in $S$. Thus,

$$
\operatorname{deg} y \leq \frac{x_{0}+x_{1}+x_{2}+x_{3}}{3}=\frac{n-5}{3}
$$

Putting these inequalities together and noting that $3 t(G) \leq \sum_{v \in V(G)} \operatorname{deg} v$, (3.5) gives us

$$
\begin{aligned}
3\left\lfloor n^{2} / 8\right\rfloor+3 \leq 3 t(G) & \leq \frac{2}{3} \sum_{i<j} x_{i} x_{j}+3\left(x_{0}+x_{1}+x_{2}+x_{3}\right)+(n+7)+\frac{2}{3}(n-5) \\
& =\frac{1}{3}(n-5)^{2}-\frac{1}{3} \sum_{i=0}^{3} x_{i}^{2}+\frac{14 n-34}{3}
\end{aligned}
$$

On the other hand, we note that by the Cauchy-Schwarz inequality, $\sum_{i=0}^{3} x_{i}^{2} \geq \frac{1}{4}(n-5)^{2}$. Therefore,

$$
3\left\lfloor n^{2} / 8\right\rfloor+3 \leq \frac{1}{4}(n-5)^{2}+\frac{14 n-34}{3}=\frac{1}{12}\left(3 n^{2}+26 n-61\right)
$$

A contradiction to $n \geq 17$. This completes the proof in this case.

Case 2: $\sum_{i=0}^{4} x_{i}=n-4$. In this case, recall that $G\left[X_{i}\right]$ are disjoint unions of triangles for $0 \leq i \leq 2$, and $X_{3}$ is a union of triangles and a star on $r \geq 0$ vertices. Let us denote this star as $S^{*}=\left\{c, \ell_{1}, \ldots, \ell_{r-1}\right\}$ where $c$ is the center and $\ell_{j}$ the leaves.
We now continue with the exact same analysis of the neighborhoods of vertices in $G$ as in the previous case. For a subset $A \subset V(G)$, let $\mathcal{T}(A)$ denote the set of triangles in $G[A]$. First, we note that

$$
\begin{equation*}
\sum_{v \in V(G)} \operatorname{deg} v=\sum_{i=0}^{2} \sum_{v w z \in \mathcal{T}\left(X_{i}\right)}(\operatorname{deg} v+\operatorname{deg} w+\operatorname{deg} z)+\sum_{v \in X_{3}} \operatorname{deg} v+\sum_{v \in S} \operatorname{deg} v \tag{3.6}
\end{equation*}
$$

Let us now upper bound each term in (3.6) separately.

- Let $v w z \in \mathcal{T}\left(X_{0}\right)$. Clearly $N(v)-X_{0}-S$ cannot contain two adjacent vertices from the same $X_{j}$, $j \neq 0$. Therefore, $v$ can only be adjacent with at most one vertex from each triangle of $X_{j}$ for $j \neq 0$. Moreover, $N(v) \cap S^{*}, N(w) \cap S^{*}$ and $N(z) \cap S^{*}$ are disjoint, implying

$$
\operatorname{deg} v+\operatorname{deg} w+\operatorname{deg} z \leq 3\left(\frac{x_{1}}{3}+\frac{x_{2}}{3}+\frac{x_{3}-r}{3}\right)+r+9=\left(x_{1}+x_{2}+x_{3}\right)+9
$$

Similar inequalities hold for each of the $x_{i} / 3$ triangles in $\mathcal{T}\left(X_{i}\right), 0 \leq i \leq 2$. In particular, we have

$$
\sum_{v w z \in \mathcal{T}\left(X_{i}\right)}(\operatorname{deg} v+\operatorname{deg} w+\operatorname{deg} z) \leq \frac{x_{i}}{3} \sum_{j \neq i} x_{j}+3 x_{i}
$$

- Let $v \in X_{3}$. Then, $N(v)-X_{3}-S$ can have at most one vertex from each triangle of $X_{i}$. Thus,

$$
\operatorname{deg} v \leq \begin{cases}\frac{1}{3}\left(x_{0}+x_{1}+x_{2}\right)+3, & v \notin S^{*} \\ \frac{1}{3}\left(x_{0}+x_{1}+x_{2}\right)+r, & v=c \\ \frac{1}{3}\left(x_{0}+x_{1}+x_{2}\right)+2, & v \in S^{*}-\{c\}\end{cases}
$$

Thus, if $r \geq 1$,

$$
\sum_{v \in X_{3}} \operatorname{deg} v \leq \frac{x_{3}\left(x_{0}+x_{1}+x_{2}\right)}{3}+3\left(x_{3}-r\right)+r+2(r-1)=\frac{x_{3}\left(x_{0}+x_{1}+x_{2}\right)}{3}+3 x_{3}-2
$$

and if $r=0$,

$$
\sum_{v \in X_{3}} \operatorname{deg} v \leq \frac{x_{3}\left(x_{0}+x_{1}+x_{2}\right)}{3}+3 x_{3}
$$

We use the latter inequality as it holds for any value of $r$.

- Finally, we have

$$
\sum_{v \in S} \operatorname{deg} v=\left(x_{0}+x_{1}+x_{2}+x_{3}\right)+12=n+8
$$

Therefore, (3.6) along with $3 t(G) \leq \sum_{v \in V(G)} \operatorname{deg} v$, gives us

$$
\begin{align*}
3 t(G) & \leq \frac{2}{3} \sum_{i<j} x_{i} x_{j}+3\left(x_{0}+x_{1}+x_{2}+x_{3}\right)+n+8 .  \tag{3.7}\\
& =\frac{1}{3}(n-4)^{2}-\frac{1}{3} \sum_{i=0}^{3} x_{i}^{2}+4 n-4 \tag{3.8}
\end{align*}
$$

Observe that by Cauchy-Schwarz, $\sum_{i=0}^{3} x_{i}^{2} \geq \frac{1}{4}(n-4)^{2}$. Hence, (3.8) implies,

$$
3 t(G) \leq \frac{1}{4}(n-4)^{2}+4 n-4 \Longrightarrow t(G) \leq \frac{1}{12} n(n+8) .
$$

By $t(G) \geq\left\lfloor n^{2} / 8\right\rfloor+1$, this implies $n \leq 14$. Note that as $n-4=\sum_{i=0}^{3} x_{i} \geq 9+x_{3}$, we would have $x_{3} \leq 1$. By (3.4), this would mean $x_{3}=1$. However, this contradicts edge-minimality of $G$, as the edge between $u_{3}$ and the only vertex of $X_{3}$ would not be incident to any triangle in $G$, again leading to a contradiction in this case.
This completes the proof of the induction step, implying ex $\left(n, K_{3}, \widehat{P}_{4}\right) \leq\left\lfloor n^{2} / 8\right\rfloor$ for all $n \geq 12$.

## 4 Concluding Remarks: Uniqueness

For $n \geq 8$, one may ask whether the lower bound construction of $K_{\lfloor n / 2\rfloor,\lceil n / 2\rceil}$ with a matching in any of the even parts is unique or not. In particular, our proof of Theorem 1.1 implies that if the extremal construction contained a $K_{4}$, then $\left\lfloor n^{2} / 8\right\rfloor \leq \frac{1}{12} n(n+8)$. This implies $n \leq 16$, and indeed, setting $x_{i}=3$ for every $i$ leads us to an equality case in Case 2 .

Our proof therefore gives us the following construction from Figure 4.1 for $n=16$ consisting entirely of $K_{4}$-blocks: consider a $K_{4}$ given by $S=\left\{u_{0}, u_{1}, u_{2}, u_{3}\right\}$. For $0 \leq i \leq 3$, let $N\left(u_{i}\right)-S$ consist of the triangles $b_{i} o_{i} r_{i}$, where the $b_{i}$ 's are colored blue, $o_{i}$ 's olive and $r_{i}$ 's red. Suppose the blue, red and olive vertices each form a $K_{4}$ (the diagonal edges are omitted in Figure 4.1 for clarity). Clearly each vertex neighborhood has 6 edges, leading to a total of $16 \cdot 6 / 3=32$ triangles, and hence this graph is a valid extremal configuration for $n=16$.


Figure 4.1: A 16 -vertex graph with 32 triangles consisting of only $K_{4}$-blocks.
It seems many extremal constructions are possible for smaller values of $n$ whenever divisibility and structural constraints are satisfied. For example, when $n=8$, we enumerate in our repository [8] all extremal constructions with 8 triangles programmatically, and these constructions are comprised of either two edge-disjoint $K_{4}$ 's, or only books. However, our proof of Theorem 1.1 provides uniqueness of the extremal configuration for $n \geq 17$.

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