# Exact generalized Turán number for $K_3$ versus suspension of $P_4$

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#### Abstract

Let  $P_4$  denote the path graph on 4 vertices. The suspension of  $P_4$ , denoted by  $\hat{P}_4$ , is the graph obtained via adding an extra vertex and joining it to all four vertices of  $P_4$ . In this note, we demonstrate that for  $n \geq 8$ , the maximum number of triangles in any *n*-vertex graph not containing  $\hat{P}_4$  is  $\lfloor n^2/8 \rfloor$ . Our method uses simple induction along with computer programming to prove a base case of the induction hypothesis.

Keywords: generalized Turán problem, suspension of a graph, computer programming.

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### 1 Introduction

The generalized Turán number ex(n, T, H) is defined as the maximum number of copies of T in an *n*-vertex graph not containing H as a (not necessarily induced) subgraph. When  $T = K_2$ , this is the Turán number ex(n, H) of the graph. The first systematic study of ex(n, T, H) for  $T \neq K_2$  was carried out by Alon and Shikhelman [1].

In more recent years, several researchers have studied the asymptotic behavior of  $ex(n, K_3, H)$  for the case  $T = K_3$  (see, for example [2, 3, 5]). It is known that when  $\chi(H) > 3$ ,  $ex(n, K_3, H) \sim {\binom{\chi(H)-1}{3}}/{(\chi(H)-1)^2 \cdot n^2}$ , where  $\chi(H)$  denotes the chromatic number of H [1, 6]. Alon and Shikhelman [1] extensively study the case when  $\chi(H) = 2$ .

Mubayi and the author [7] initiated the study of  $ex(n, K_3, H)$  for a simple family of graphs H with  $\chi(H) = 3$ . For any graph G, they denoted the suspension  $\hat{G}$  as the graph obtained from G by adding a new vertex v and joining it with all vertices of G. They proceeded to analyze the asymptotic behavior of  $ex(n, K_3, \hat{G})$  for different bipartite graphs G.

One of the several bipartite graphs they consider is the path  $P_4$  on four vertices. It was shown that for any  $n \ge 4$ ,

$$\frac{n^2}{8} - O(1) \le \exp(n, K_3, \hat{P}_4) < \frac{n^2}{8} + 3n.$$
(1.1)

An exact result for sufficiently large n was given by Gerbner [4] using the technique of progressive induction. In particular, they prove that for a number  $K \leq 1575$  and  $n \geq 525 + 4K$ ,

$$\exp(n, K_3, \widehat{P}_4) = |n^2/8|.$$
 (1.2)

They mention that a proof of the upper bound of (1.2) for n = 8, 9, 10, 11 together with induction would suffice to prove (1.2) for every  $n \ge 8$ . In this note, we leverage this idea to determine the exact value of  $ex(n, K_3, \hat{P}_4)$  for every  $n \ge 4$ , thus closing the gap in the literature for this extremal problem.

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**Theorem 1.1.** For  $n \ge 8$ ,  $ex(n, K_3, \hat{P}_4) = \lfloor n^2/8 \rfloor$ . For n = 4, 5, 6, 7 the values of  $ex(n, K_3, \hat{P}_4)$  are 4, 4, 5, 8 respectively.

The lower bound constructions for Theorem 1.1 are different for the cases  $n \in \{4, 5, 6, 7\}$  and  $n \ge 8$ .

Figure 1.1 illustrates graphs on n vertices for  $n \in \{4, 5, 6, 7\}$  that achieve the maximum number of triangles. In fact, we shall see later in Section 3.1 that these constructions are unique up to isomorphism.



Figure 1.1: Graphs on 4, 5, 6, 7 vertices and 4, 4, 5, 8 triangles, respectively.

The general lower bound construction considered in [4, 7] (for  $n \ge 8$ ) was the complete bipartite graph  $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$  with a matching in any of the even parts. A short case analysis shows that the total number of triangles in these graphs is given by  $\lfloor n^2/8 \rfloor$ , hence proving the lower bound in Theorem 1.1 for general n.

Thus, the main goal of this manuscript is to prove that these lower bounds on  $ex(n, K_3, \hat{P}_4)$  are tight.

This work is organized as follows. We present some preliminaries in Section 2. Then, we show the upper bound of Theorem 1.1 for  $n \ge 5$  in Section 3. Finally, we make some concluding remarks regarding uniqueness of the lower bound constructions in Section 4.

### 2 Preliminaries

Throughout the rest of this paper, we assume without loss of generality that all graphs are *edge-minimal*. This implies that every edge of the graphs considered must lie in a triangle, as we can simply delete edges that do not help forming a triangle. We also assume that the vertex set of any *n*-vertex graph in the rest of this section is  $\{0, \ldots, n-1\}$ , and abuse notation to represent a  $K_3$  on vertex subset  $\{a, b, c\}$  as simply *abc*.

Let n(G), e(G) and t(G) denote the number of vertices, edges and triangles in G, respectively.

Now we recall some definitions and state a two important lemmas from [4] and [7] which are instrumental in our proof.

**Definition 2.1** (Triangle-connectivity). For a graph G, two edges e and e' are said to be triangleconnected if there is a sequence of triangles  $\{T_1, \ldots, T_k\}$  of G such that  $e \in T_1$ ,  $e' \in T_k$ , and  $T_i$  and  $T_{i+1}$ share a common edge for every  $1 \le i \le k-1$ . A subgraph  $H \subseteq G$  is triangle-connected if e and e' are triangle-connected for every edges e and e' of H.

**Definition 2.2** (Triangle block). A subgraph  $H \subseteq G$  is a *triangle block* (or simply a *block*) if it is edge-maximally triangle-connected.

By definition, the triangle blocks of any graph G are edge-disjoint.

Let  $B_s$  denote the book graph on (s+2) vertices, consisting of s triangles all sharing a common edge. Let this common edge be called the base of the  $B_s$ . The following lemma characterizes the triangle blocks of any  $\hat{P}_4$ -free graph G.

**Lemma 2.3** ([7], Claim 5.3). Every triangle block of a  $\widehat{P}_4$ -free graph G is isomorphic to a  $K_4$  or a  $B_s$  for some  $s \ge 1$ .

*Proof.* Let  $H \subseteq G$  be an arbitrary triangle block. If H contains only one or two triangles, it is isomorphic to  $B_1$  or  $B_2$ . Suppose H contains at least three triangles. Let two of them be  $abx_1$  and  $abx_2$  (see Figure 2.1).



Figure 2.1: (left): third triangle on  $ax_1$ , (right): third triangle on ab

If another triangle is of the form  $ax_1y$  for some  $y \in V(H)$ , then there are two possible cases. If  $y \neq x_2$ , then  $N_H(a)$  contains the 4-path  $x_2bx_1y$ , a contradiction. Otherwise if  $y = x_2$ , then the vertices  $a, b, x_1, x_2$  create a  $K_4$ , and this  $K_4$  is a triangle block by itself.

Similarly, if a triangle contained any of the edges  $bx_1, ax_2, bx_2$ , we would end up with a  $K_4$ -block, and this block cannot be extended any further.

Therefore all triangles in H would intersect the edge ab, implying  $H \cong B_s$  for some  $s \ge 1$ .

**Lemma 2.4** ([4], Section 2). Suppose G is an n-vertex  $\widehat{P}_4$ -free graph containing no  $K_4$ . Then, we have  $t(G) \leq \lfloor n^2/8 \rfloor$ .

Proof. By Lemma 2.3, all triangle blocks of G are isomorphic to  $B_s$  for some  $s \ge 1$ . Let G' be obtained from G by deleting the base edges of each of the books (if s = 1, delete any arbitrary edge). As each triangle of G contains two distinct edges from G', we have t(G) = e(G')/2. By Mantel's theorem,  $e(G') \le \lfloor n^2/4 \rfloor$ , implying  $t(G) \le \frac{1}{2} \lfloor n^2/4 \rfloor$ , i.e.  $t(G) \le \lfloor n^2/8 \rfloor$ .

### 3 Upper bounds

In order to prove that  $ex(n, K_3, \widehat{P}_4) \leq K$  for some fixed n and K, we need to show that any n-vertex graph containing at least K + 1 triangles contains a copy of  $\widehat{P}_4$ .

#### **3.1** The cases $5 \le n \le 8$ : brute force

While a case-by-case analysis is tractable by hand for n = 5 for example, we quickly run into several possible configurations while trying to prove  $ex(8, K_3, \hat{P}_4) = 8$ . This is where we turn to a computergenerated check. For example, to prove that all 8-vertex graphs with more than 9 triangles is  $\hat{P}_4$ -free, we can assume that 012 and 013 are two triangles in some 8-vertex graph G containing 9 triangles. Then triangles that have an edge from the set  $\{02, 03, 12, 13\}$  and have a node from  $\{4, 5, 6, 7\}$  are excluded from G since any of these patterns form a  $\hat{P}_4$ . This excludes 16 triangles. Hence the plausible triangles that G may contain other than 012 and 013 are  $\binom{8}{3} - 18 = 38$  in number. We generate  $\binom{38}{7} \approx 1.26 \times 10^7$  possible graphs, filter out the ones that have exactly 9 triangles, and check for  $\hat{P}_4$ 's in each of them.

Our program is available at the Github repository in [8]. We run triangle\_count\_parallel.py for different pairs of (n,t) to figure out both the extremal number and all extremal configurations for *n*-vertex graphs with *t*-triangles. The results are compiled in the notebook triangle\_count.ipynb. Our computation shows that  $ex(n, K_3, \hat{P}_4) = 4, 5, 8, 8$  for n = 5, 6, 7, 8, respectively. The total computation time required for (n,t) = (8,9) on 7 threads of an Intel(R) Core(TM) i7-8550U laptop processor running at 1.80GHz was around 18 minutes.

### **3.2** The cases $9 \le n \le 11$ : identifying $K_4$

The main idea behind these cases is to follow the steps of the proof in [7], Section 5.2.

**Theorem 3.1.** Suppose  $(n,t) \in \{(9,11), (10,13), (11,16)\}$ , and G is an (edge-minimal) n-vertex graph with t triangles. Then G must contain a  $\widehat{P}_4$ .

*Proof.* For the sake of contradiction, assume that G was  $\hat{P}_4$ -free. If G was also  $K_4$ -free, then by Lemma 2.4,  $t(G) \leq \lfloor n^2/8 \rfloor = 10, 12, 15$  for n = 9, 10, 11, contradicting our initial assumption on t(G).

Therefore G must contain a  $K_4$ . Let this  $K_4$  be induced by vertex subset  $S = \{u_0, u_1, u_2, u_3\} \subset V(G)$ . Define  $X_i := N(u_i) - S$  for  $0 \le i \le 3$ . As G[S] is a triangle block,  $X_i \cap X_j = \emptyset$  for every  $i \ne j$ . Further,  $\sum_{i=0}^3 |X_i| \le n-4$ . Without loss of generality assume  $|X_0| \le \cdots \le |X_3|$ . Now we consider each case separately.

- Case 1. (n,t) = (9,11): In this case,  $\sum_{i=0}^{3} |X_i| \leq 5$ . If  $|X_1| > 0$ , by edge-minimality we would have  $|X_1| \geq 2$ , implying  $|X_1| + |X_2| + |X_3| \geq 6$ , a contradiction. Thus,  $|X_0| = |X_1| = 0$ , and by a similar argument,  $|X_2| \leq 2$ . This means the vertex  $u_2$  lies in at most one triangle outside of G[S]. Let G' be obtained by deleting  $\{u_0, u_1, u_2\}$  from G. Clearly n(G') = 6 and  $t(G') \geq t(G) 5 = 6$ . As  $ex(6, K_3, \hat{P}_4) = 5$  by the discussion in Section 3.1, G' has a  $\hat{P}_4$ , a contradiction.
- Case 2. (n,t) = (10,13): Here,  $\sum_{i=0}^{3} |X_i| \le 6$ . By a similar analysis as before, we can infer that  $|X_0| = 0$  and  $|X_1| \le 2$ . If  $|X_1| = 0$ , we could consider  $G' = G \{u_0, u_1\}$ , which would have n(G') = 8 and t(G') = 13 4 = 9, which would lead us to a  $\hat{P}_4$  since  $\exp(8, K_3, \hat{P}_4) = 8$  by the calculation in Section 3.1. Thus, we have  $|X_0| = 0$ ,  $|X_1| = 2$ , and hence  $|X_2| = |X_3| = 2$ . Now, if we consider G'' = G S, we have n(G'') = 6 and t(G'') = 13 4 3 = 6, again implying that G'' has a  $\hat{P}_4$ .
- Case 3. (n,t) = (11,16): For this pair of (n,t), we have  $\sum_{i=0}^{3} |X_i| \leq 7$ , implying  $|X_0| = 0$  again. Since  $u_0$  lies in exactly three triangles of G[S],  $G' = G - \{u_0\}$  has n(G') = 10 and t(G') = 13, leading us to the previous case.

In either of the three cases, we obtain a contradiction, finishing the proof for these cases.

### **3.3** The case $n \ge 12$ : identifying $K_4$

Now that we have proved  $ex(n, K_3, \hat{P}_4) = \lfloor n^2/8 \rfloor$  for  $8 \leq n \leq 11$ , we are now ready to handle the general case using induction on n. Our proof follows the idea of [4] with a more careful analysis to obtain the desired bound.

Proof of Theorem 1.1 for  $n \ge 12$ . Let us assume that  $ex(k, K_3, \widehat{P}_4) = \lfloor k^2/8 \rfloor$  for all  $8 \le k \le n-1$ . We note that a simple case analysis leads to

$$\lfloor n^2/8 \rfloor - \lfloor (n-1)^2/8 \rfloor \ge \lfloor n/4 \rfloor \lfloor n^2/8 \rfloor - \lfloor (n-4)^2/8 \rfloor = n-2.$$
(3.1)

For the sake of contradiction, suppose G is an n-vertex  $\widehat{P}_4$ -free graph with  $t(G) \ge \lfloor n^2/8 \rfloor + 1$ . For a subset  $U \subset V(G)$ , let us denote by t(U) the number of triangles containing at least one vertex from U. By (3.1), we may assume that

$$|U| = 1 \implies t(U) \ge \lfloor n/4 \rfloor + 1,$$
  

$$|U| = 4 \implies t(U) \ge n - 1.$$
(3.2)

Now, notice that by Lemma 2.4, G must contain a  $K_4$ . As in the previous section, let  $S = \{u_0, u_1, u_2, u_3\}$  induce this  $K_4$ , and denote  $X_i = N(u_i) - S$  for  $0 \le i \le 3$ . Again,  $|X_i \cap X_j| = \emptyset$  for every  $i \ne j$ . Observe

that  $t(S) = \sum_{i=0}^{3} e(X_i) + 4$ , and so by (3.2),

$$\sum_{i=0}^{3} e(X_i) \ge n - 5.$$

On the other hand, since each  $X_i$  is  $P_4$ -free, we have  $\sum_{i=0}^3 e(X_i) \leq \sum_{i=0}^3 |X_i| \leq n-4$ . Hence,

$$\sum_{i=0}^{3} e(X_i) \in \{n-5, n-4\}$$
(3.3)

This implies that  $e(X_i) = |X_i|$  for at least three  $u_i \in S$ . Assume that  $e(X_i) = |X_i|$  for  $0 \le i \le 2$  and  $e(X_3) \in \{|X_3| - 1, |X_3|\}$ . This also means that  $G[X_i]$  are vertex-disjoint unions of triangles for  $0 \le i \le 2$ , and  $X_3$  is a union of triangles and a star on r vertices for some  $r \ge 0$ . Further, (3.2) gives us the bound

$$|X_i| \ge \lfloor n/4 \rfloor - 2. \tag{3.4}$$

We now continue with a more detailed analysis of the neighborhoods of vertices in G. In what follows, let  $x_i$  denote the size of  $X_i$ . For a subset  $A \subset V(G)$ , let  $\mathcal{T}(A)$  denote the set of triangles in G[A]. We now consider two cases.

**Case 1:**  $\sum_{i=0}^{4} x_i = n - 5$ . In this case, note that since  $\sum_{i=0}^{3} e(X_i) = n - 5$ , we have  $e(X_3) = x_3$ . Thus, the subgraphs  $G[X_i]$  are all disjoint unions of triangles, and there is exactly one vertex y in  $V(G) - \bigcup_i X_i \cup S$ , and thus  $3 \mid n - 5$ , implying  $n \equiv 2 \mod 3$ . Moreover, (3.4) implies  $x_i \geq 3$ , and hence  $n \geq 17$ .

Now, observe that for  $G' = G - \{y\}$ ,

$$\sum_{v \in V(G)} \deg v = \sum_{i=0}^{3} \sum_{v \le \mathcal{T}(X_i)} (\deg_{G'} v + \deg_{G'} w + \deg_{G'} z) + \sum_{v \in S} \deg v + 2 \deg y.$$
(3.5)

We proceed by upper bounding each term of (3.5) separately.

• Let  $vwz \in \mathcal{T}(X_0)$ . For any  $j \neq 0$ , as  $N(v) - X_0 - S - \{y\}$  cannot contain two adjacent vertices from the same  $X_j$ , v can only be adjacent to at most one vertex from each triangle of  $X_j$ . Finally, v is adjacent to exactly three nodes from  $X_0 \cup S$ , leading to

$$\deg_{G'} v + \deg_{G'} w + \deg_{G'} z \le 3\left(\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3}{3}\right) + 9 = (x_1 + x_2 + x_3) + 9.$$

By repeating the same argument over all  $x_i/3$  triangles from  $\mathcal{T}(X_i)$ , we have

$$\sum_{wwz\in\mathcal{T}(X_i)} (\deg_{G'} v + \deg_{G'} w + \deg_{G'} z) \le \frac{x_i}{3} \sum_{j\neq i} x_j + 3x_i.$$

• As y is not adjacent to any vertex of S, we have

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$$\sum_{v \in S} \deg v = (x_0 + x_1 + x_2 + x_3) + 12 = n + 7.$$

• For each  $i, N(y) \cap X_i$  has at most  $x_i/3$  vertices, as otherwise by the pigeonhole principle we would have  $v, w \in N(y) \cap X_i$  that are adjacent, leading to a triangle yvw sharing an edge with the  $K_4$ containing  $u_i, v$  and w. Further, y does not have a neighbor in S. Thus,

$$\deg y \le \frac{x_0 + x_1 + x_2 + x_3}{3} = \frac{n - 5}{3}.$$

Putting these inequalities together and noting that  $3t(G) \leq \sum_{v \in V(G)} \deg v$ , (3.5) gives us

$$3\lfloor n^2/8 \rfloor + 3 \le 3t(G) \le \frac{2}{3} \sum_{i < j} x_i x_j + 3(x_0 + x_1 + x_2 + x_3) + (n+7) + \frac{2}{3}(n-5)$$
$$= \frac{1}{3}(n-5)^2 - \frac{1}{3} \sum_{i=0}^3 x_i^2 + \frac{14n-34}{3}.$$

On the other hand, we note that by the Cauchy-Schwarz inequality,  $\sum_{i=0}^{3} x_i^2 \geq \frac{1}{4}(n-5)^2$ . Therefore,

$$3\lfloor n^2/8 \rfloor + 3 \le \frac{1}{4}(n-5)^2 + \frac{14n-34}{3} = \frac{1}{12}(3n^2 + 26n - 61),$$

A contradiction to  $n \ge 17$ . This completes the proof in this case.

**Case 2:**  $\sum_{i=0}^{4} x_i = n - 4$ . In this case, recall that  $G[X_i]$  are disjoint unions of triangles for  $0 \le i \le 2$ , and  $X_3$  is a union of triangles and a star on  $r \ge 0$  vertices. Let us denote this star as  $S^* = \{c, \ell_1, \ldots, \ell_{r-1}\}$  where c is the center and  $\ell_j$  the leaves.

We now continue with the exact same analysis of the neighborhoods of vertices in G as in the previous case. For a subset  $A \subset V(G)$ , let  $\mathcal{T}(A)$  denote the set of triangles in G[A]. First, we note that

$$\sum_{v \in V(G)} \deg v = \sum_{i=0}^{2} \sum_{v \le x \in \mathcal{T}(X_i)} (\deg v + \deg w + \deg z) + \sum_{v \in X_3} \deg v + \sum_{v \in S} \deg v.$$
(3.6)

Let us now upper bound each term in (3.6) separately.

• Let  $vwz \in \mathcal{T}(X_0)$ . Clearly  $N(v) - X_0 - S$  cannot contain two adjacent vertices from the same  $X_j$ ,  $j \neq 0$ . Therefore, v can only be adjacent with at most one vertex from each triangle of  $X_j$  for  $j \neq 0$ . Moreover,  $N(v) \cap S^*$ ,  $N(w) \cap S^*$  and  $N(z) \cap S^*$  are disjoint, implying

$$\deg v + \deg w + \deg z \le 3\left(\frac{x_1}{3} + \frac{x_2}{3} + \frac{x_3 - r}{3}\right) + r + 9 = (x_1 + x_2 + x_3) + 9.$$

Similar inequalities hold for each of the  $x_i/3$  triangles in  $\mathcal{T}(X_i), 0 \leq i \leq 2$ . In particular, we have

$$\sum_{vwz \in \mathcal{T}(X_i)} (\deg v + \deg w + \deg z) \le \frac{x_i}{3} \sum_{j \ne i} x_j + 3x_i$$

• Let  $v \in X_3$ . Then,  $N(v) - X_3 - S$  can have at most one vertex from each triangle of  $X_i$ . Thus,

$$\deg v \le \begin{cases} \frac{1}{3}(x_0 + x_1 + x_2) + 3, & v \notin S^*, \\ \frac{1}{3}(x_0 + x_1 + x_2) + r, & v = c, \\ \frac{1}{3}(x_0 + x_1 + x_2) + 2, & v \in S^* - \{c\} \end{cases}$$

Thus, if  $r \geq 1$ ,

$$\sum_{v \in X_3} \deg v \le \frac{x_3(x_0 + x_1 + x_2)}{3} + 3(x_3 - r) + r + 2(r - 1) = \frac{x_3(x_0 + x_1 + x_2)}{3} + 3x_3 - 2,$$

and if r = 0,

$$\sum_{v \in X_3} \deg v \le \frac{x_3(x_0 + x_1 + x_2)}{3} + 3x_3.$$

We use the latter inequality as it holds for any value of r.

• Finally, we have

$$\sum_{v \in S} \deg v = (x_0 + x_1 + x_2 + x_3) + 12 = n + 8$$

Therefore, (3.6) along with  $3t(G) \leq \sum_{v \in V(G)} \deg v$ , gives us

$$3t(G) \le \frac{2}{3} \sum_{i < j} x_i x_j + 3(x_0 + x_1 + x_2 + x_3) + n + 8.$$
(3.7)

$$=\frac{1}{3}(n-4)^2 - \frac{1}{3}\sum_{i=0}^3 x_i^2 + 4n - 4$$
(3.8)

Observe that by Cauchy-Schwarz,  $\sum_{i=0}^{3} x_i^2 \ge \frac{1}{4}(n-4)^2$ . Hence, (3.8) implies,

$$3t(G) \le \frac{1}{4}(n-4)^2 + 4n - 4 \implies t(G) \le \frac{1}{12}n(n+8).$$

By  $t(G) \ge \lfloor n^2/8 \rfloor + 1$ , this implies  $n \le 14$ . Note that as  $n-4 = \sum_{i=0}^{3} x_i \ge 9 + x_3$ , we would have  $x_3 \le 1$ . By (3.4), this would mean  $x_3 = 1$ . However, this contradicts edge-minimality of G, as the edge between  $u_3$  and the only vertex of  $X_3$  would not be incident to any triangle in G, again leading to a contradiction in this case.

This completes the proof of the induction step, implying  $ex(n, K_3, \hat{P}_4) \leq \lfloor n^2/8 \rfloor$  for all  $n \geq 12$ .

### 4 Concluding Remarks: Uniqueness

For  $n \ge 8$ , one may ask whether the lower bound construction of  $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$  with a matching in any of the even parts is unique or not. In particular, our proof of Theorem 1.1 implies that if the extremal construction contained a  $K_4$ , then  $\lfloor n^2/8 \rfloor \le \frac{1}{12}n(n+8)$ . This implies  $n \le 16$ , and indeed, setting  $x_i = 3$  for every *i* leads us to an equality case in Case 2.

Our proof therefore gives us the following construction from Figure 4.1 for n = 16 consisting entirely of  $K_4$ -blocks: consider a  $K_4$  given by  $S = \{u_0, u_1, u_2, u_3\}$ . For  $0 \le i \le 3$ , let  $N(u_i) - S$  consist of the triangles  $b_i o_i r_i$ , where the  $b_i$ 's are colored blue,  $o_i$ 's olive and  $r_i$ 's red. Suppose the blue, red and olive vertices each form a  $K_4$  (the diagonal edges are omitted in Figure 4.1 for clarity). Clearly each vertex neighborhood has 6 edges, leading to a total of  $16 \cdot 6/3 = 32$  triangles, and hence this graph is a valid extremal configuration for n = 16.



Figure 4.1: A 16-vertex graph with 32 triangles consisting of only  $K_4$ -blocks.

It seems many extremal constructions are possible for smaller values of n whenever divisibility and structural constraints are satisfied. For example, when n = 8, we enumerate in our repository [8] all extremal constructions with 8 triangles programmatically, and these constructions are comprised of either two edge-disjoint  $K_4$ 's, or only books. However, our proof of Theorem 1.1 provides uniqueness of the extremal configuration for  $n \ge 17$ .

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### References

- Noga Alon and Clara Shikhelman. Many T copies in H-free graphs. Journal of Combinatorial Theory, Series B, 121:146–172, 2016.
- [2] Beka Ergemlidze, Ervin Győri, Abhishek Methuku, and Nika Salia. A note on the maximum number of triangles in a C<sub>5</sub>-free graph. Journal of Graph Theory, 90(3):227–230, 2019.
- [3] Beka Ergemlidze and Abhishek Methuku. Triangles in  $C_5$ -free graphs and hypergraphs of girth six. arXiv preprint arXiv:1811.11873, 2018.
- [4] Dániel Gerbner. A note on the number of triangles in graphs without the suspension of a path on four vertices. *Discrete Mathematics Letters*, 10:32–34, 2022.
- [5] Ervin Győri and Hao Li. The maximum number of triangles in  $C_{2k+1}$ -free graphs. Combinatorics, Probability and Computing, 21(1-2):187–191, 2012.
- [6] Jie Ma and Yu Qiu. Some sharp results on the generalized Turán numbers. European Journal of Combinatorics, 84:103026, 2020.
- [7] Dhruv Mubayi and Sayan Mukherjee. Triangles in graphs without bipartite suspensions. Discrete Mathematics, 346(6):113355, 2023.
- [8] Sayan Mukherjee. Exact generalized Turán number for  $K_3$  versus suspension of  $P_4$ , June 2023. Code available at https://github.com/Potla1995/hatP4Free.