An upper bound of the number of distinct powers in binary words

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Abstract. A power is a word of the form $\underbrace{uu...u}_{k \text{ times}}$, where u is a word and

k is a positive integer and a square is a word of the form uu. Fraenkel and Simpson conjectured in 1998 that the number of distinct squares in a word is bounded by the length of the word. This conjecture was proven recently by Brlek and Li. Besides, there exists a stronger upper bound for binary words conjectured by Jonoska, Manea and Seki stating that for a word of length n over the alphabet $\{a, b\}$, if we let k be the least of the number of a's and the number of b's and $k \ge 2$, then the number of distinct squares is upper bounded by $\frac{2k-1}{2k+2}n$. In this article, we prove this conjecture by giving a stronger statement on the number of distinct powers in a binary word.

1 Introduction

A power is a word of the form $\underbrace{uu...u}_{k \text{ times}}$, where u is a word and k is a positive

integer; the power is also called a k-power and k is its *exponent*. The upper bound of the number of distinct k-powers in a finite word was studied in [7,9,10] and the best known result is as follows:

Theorem 1 (Theorem 1 and Theorem 2 in [10]) For every finite word w, let m(w) denote the number of distinct nonempty powers of exponent at least 2 in w, let $m_k(w)$ denote the number of distinct nonempty k-powers in w, let |w| denote the length of w and let |Alph(w)| denote the number of distinct letters in w, then one has

$$m(w) \le |w| - |\operatorname{Alph}(w)|;$$

Moreover, for any integer $k \geq 2$,

$$m_k(w) \le \frac{|w| - |\operatorname{Alph}(w)|}{k - 1}.$$

Particularly, a square is a 2-power and upper bound of the number of distinct squares in a finite word was studied in [4,5,8,3,11,2]. A conjecture of Fraenkel

and Simpson [4] states that the number of distinct squares in a word is bounded by the length of the word. This conjecture is confirmed by the previous theorem. Besides, there exists a stronger upper bound for binary words conjectured by Jonoska, Manea and Seki [6]:

Theorem 2 Let w be a finite word over the alphabet $\{a, b\}$, let s(w) denote the number of distinct squares in w and let k denote the least of the number of a's and the number of b's in w. If $k \ge 2$, then one has

$$s(w) \leq \frac{2k-1}{2k+2}|w|.$$

In this article, we confirm this conjecture by proving the following result:

Theorem 3 Let w be a finite word over the alphabet $\{a, b\}$ such that $w = a^{r_1}ba^{r_2}ba^{r_3}b...a^{r_k}ba^{r_{k+1}}$, where $r_1, r_2, ..., r_{k+1}$ are nonnegative integers and $a^n = aa...a$. Let σ be a permutation of 1, 2, ..., k+1 such that $r_{\sigma(1)} \leq r_{\sigma(2)} \leq ... \leq n$ times

 $r_{\sigma(k+1)}$, let m(w) denote the number of distinct nonempty powers of exponent at least 2 in w and let |w| denote the length of w. Then one has

$$m(w) + r_{\sigma(k)} \le |w| - 2.$$

2 Preliminaries

Let \sum be an *alphabet* and \sum^* be the set of all words over \sum . Let $w \in \sum^*$. By |w|, we denote its *length*. A word of length 0 is called the *empty word* and it is denoted by ε . A word u is a *factor* of w if w = pus for some words p, s. When $p = \varepsilon$ (resp. $s = \varepsilon$), u is called a *prefix* (resp. *suffix*) of w. The set of all nonempty factors of w is denoted by Fac(w). The number of occurrences of a factor $u \in w$ is denoted by $|w|_u$.

Let w be a finite word. For any integer i satisfying $1 \le i \le |w|$, let $L_w(i)$ be the set of all length-i factors of w and let $C_w(i)$ be the cardinality of $L_w(i)$. For any natural number k, we define the k-th power of a finite word u to be $u^k = uu \cdots u$ and it consists of the concatenation of k copies of u. A finite word w is said to be primitive if it is not a power of another word, that is, $w = u^k$ implies k = 1. A square is a 2-power, that is a word w satisfying w = uu for a certain word u.

For a finite word w, let Prim(w) denote the set of primitive factors of w, let

$$M(w) = \left\{ p^{i} | p^{i} \in \operatorname{Fac}(w), p \in \operatorname{Prim}(w), i \in \mathbf{N}, i \geq 2 \right\},$$

$$S(w) = \left\{ p^{2i} | p^{2i} \in \operatorname{Fac}(w), p \in \operatorname{Prim}(w), i \in \mathbf{N}, i \geq 1 \right\},$$

$$NS(w) = \left\{ p^{2i+1} | p^{2i+1} \in \operatorname{Fac}(w), p \in \operatorname{Prim}(w), i \in \mathbf{N}, i \geq 1 \right\}$$

and let m(w), s(w) and ns(w) be respectively the cardinality of M(w), S(w) and NS(w). Obviously, $M(w) = S(w) \cup NS(w), m(w) = s(w) + ns(w)$ and s(w) is

the number of distinct nonempty squares in w.

Here we recall some elementary definitions and proprieties concerning graphs from Berge [1].

A directed graph consists of a nonempty set of vertices V and a set of edges E. A vertex a represents an endpoint of an edge and an edge joins two vertices a, b in order. A chain is a sequence of edges e_1, e_2, \dots, e_k , such that there exists a sequence of vertices v_1, v_2, \dots, v_{k+1} and that for each *i* satisfying $1 \le i \le k, e_i$ is either directed from v_i to v_{i+1} or from v_{i+1} to v_i . A cycle is a finite chain such that $v_{k+1} = v_1$. A path is a sequence of edges e_1, e_2, \dots, e_k , such that there exists a sequence of vertices v_1, v_2, \dots, v_{k+1} and that for each *i* satisfying $1 \le i \le k$, e_i is directed from v_i to v_{i+1} . A circuit is a finite path such that $v_{k+1} = v_1$.

A cycle or a circuit is called *elementary* if, apart from v_1 and v_{k+1} , every vertex which it meets is distinct. A directed graph is called *weakly connected* if for any couple of vertices a, b in this graph, there exists a chain connecting a and b.

Let G be a weakly connected graph and let $\{e_1, e_2 \cdots e_l\}, \{v_1, v_2 \cdots v_s\}$ denote respectively the edge set and the vertex set of G. The number $\chi(G) = l - s + 1$ is called the *cyclomatic number* of G.

Let C be a cycle in G. A vector $\mu(C) = (c_1, c_2 \cdots c_l)$ in the *l*-dimensional space \mathbb{R}^l is called the *vector-cycle corresponding to* C if c_i is the number of visits of the edge e_i in the cycle C for all *i* satisfying $1 \leq i \leq l$. The cycle $C_1, C_2, \cdots, C_k, \ldots$ are said to be *independent* if their corresponding vectors are linearly independent.

Theorem 4 (Theorem 2, Chapter 4 in [1]) the cyclomatic number of a graph is the maximum number of independent cycles in this graph.

3 Rauzy graphs

Let w be a finite word. For any integer i satisfying $1 \le i \le |w|$, the *i*-th Rauzy graph $\Gamma_w(i)$ of w is defined to be an directed graph whose vertex set is $L_w(i)$ and the edge set is $L_w(i+1)$; an edge $e \in L_w(i+1)$ starts at the vertex u and ends at the vertex v, if u is a prefix and v is a suffix of e. Let us define $\Gamma_w = \bigcup_{n=1}^{k-1} \Gamma_w(n)$.

Let $\Gamma_w(i)$ be a Rauzy graph of w for some i, a sub-graph on $\Gamma_w(i)$ is called a *small circuit* if it is an elementary circuit and the number of its vertices is no larger than i.

Lemma 5 (Lemma 8 in [2]) Let w be a finite word and let $\Gamma_w(i)$ be a Rauzy graph of w for some i. Then all small circuits on $\Gamma_w(i)$ are independent.

Lemma 6 (Lemma 6 in [10]) Let w be a finite word, then there exists an injection from M(w) to the set of all small circuits on Γ_w .

Example 7 Let us consider the word $u = abaaabaaabaabaa, the Rauzy graph <math>\Gamma_u(4)$ is as follows:



In this graph, there are three circuits: $C_1 = \{\{aaab, aaba, abaa, baaa\}, \{aaaba, aabaa, abaaa, baaab\}\}, C_2 = \{\{aaba, abaa, baaa\}, \{aabaa, abaab, baaba\}\}$ and $C_3 = \{\{aaab, aaba, abaa, baaa, aaaa\}, \{aaaba, aabaa, abaaa, abaaa, baaaa, aaaab\}\}$. Two of them are small, they are C_1 and C_2 , while C_3 is not small. \Box

4 Proof of Theorem 3

Lemma 8 Let $w \in \{a, b\}^*$ and let *i* be an integer satisfying $1 \le i \le |w|$. If there exists an elementary circuit on $\Gamma_w(i)$ containing the edge $a^i b$, then this circuit is not a small circuit and it is independent with all the small circuits on $\Gamma_w(i)$. Let $C_{sp}(i)$ denote one of these circuits (if any).

Proof. We first prove that, if there exists a small circuit passing through the vertex a^i , then it should be the sub-graph $\{\{a^i\}, \{a^{i+1}\}\}\$ of $\Gamma_w(i)$. In fact, if there exists a path $e_1, e_2, ..., e_k$ from a^i to a^i satisfying $k \leq i$, let $p = l_1 l_2 ... l_k$ be a word such that l_j is the last letter of e_j for all j satisfying that $1 \leq j \leq k$. From the hypothesis that $e_1, e_2, ..., e_k$ form a circuit, we can deduce that a^i is a suffix of $a^i p$. Moreover, as $|p| = k \leq i$, $l_j = a$ for all j. Consequently, $e_j = a^{i+1}$ for all j. Thus, from the unicity of each edge, we prove that there exists only one edge on the path and the graph is given by $\{\{a^i\}, \{a^{i+1}\}\}$.

If their exists an elementary circuit C on $\Gamma_w(i)$ containing the edge $a^i b$, from the fact that $a^i b$ cannot be contained in any small circuit on $\Gamma_w(i)$, we conclude that C is not a small circuit and independent with all the small circuits on $\Gamma_w(i)$.

Lemma 9 Let $w \in \{a, b\}^*$ such that $w = a^{r_1}ba^{r_2}ba^{r_3}b...a^{r_k}ba^{r_{k+1}}$ with $r_j \ge 0$ for all j satisfying $1 \le j \le k + 1$. Let σ be a permutation of 1, 2, ..., k + 1 such that $r_{\sigma(1)} \le r_{\sigma(2)} \le ... \le r_{\sigma(k+1)}$. Then for any integer i satisfying $1 \le i \le \sigma(k)$, there exists a $C_{sp}(i)$ on $\Gamma_w(i)$.

Proof. From the hypothesis that $i \leq r_{\sigma(k)} \leq r_{\sigma(k+1)}$, there exists a nonempty word X such that $a^i X a^i \in Fac(w)$ and that

$$X = \begin{cases} bYb \text{ if } |X| \ge 2; \\ b \quad \text{otherwise.} \end{cases}$$

Moreover, we can suppose that $a^i \notin \operatorname{Fac}(Y)$. Indeed, there exists a circuit on $\Gamma_{a^i X a^i}(i)$ containing the edge $a^i b$ and $\Gamma_{a^i X a^i}(i)$ is a sub-graph of $\Gamma_w(i)$. Thus, there exists a $C_{sp}(i)$ on $\Gamma_w(i)$.

Lemma 10 Let $w \in \{a, b\}^*$ and let I_w be the cardinality of

$$S_w = \{i | 1 \leq i \leq |w|, \text{ there exists a circuit } C_{sp}(i) \text{ on } \Gamma_w(i) \},\$$

then

$$m(w) + I_w \le |w| - |\mathrm{Alph}(w)|.$$

Proof. Let $sc_w(i)$ denote the number of small circuits on $\Gamma_w(i)$. From Lemma 5, Lemma 8 and Theorem 4, for any $i \in S_w$,

$$sc_w(i) + 1 \le C_w(i+1) - C_w(i) + 1;$$

and for any $i \notin S_w$,

$$sc_w(i) \le C_w(i+1) - C_w(i) + 1.$$

Consequently,

$$\sum_{i=1}^{|w|} \operatorname{sc}_{w}(i) + I_{w} \le \sum_{i=1}^{|w|} C_{w}(i+1) - C_{w}(i) + 1 = |w| - |\operatorname{Alph}(w)|.$$

Moreover, from 6, $m(w) \leq \sum_{i=1}^{|w|} \operatorname{sc}(i)$, thus, $m(w) + I_w \leq |w| - |\operatorname{Alph}(w)|$. \Box

Proof (of Theorem 3). It is a direct consequence of Lemma 10 and Lemma 9. \Box

5 Proof of Theorem 2

In this section, Let $w \in \{a, b\}^*$ such that $|w|_a \ge |w|_b$, that $|w|_b = k$ and that $w = a^{r_1}ba^{r_2}ba^{r_3}b...a^{r_k}ba^{r_{k+1}}$ with $r_j \ge 0$ for all j satisfying $1 \le j \le k+1$.

Let δ be an integer such that $|w|_a = k + \delta$. From the hypothesis $|w|_a \ge |w|_b$, we can suppose $\delta \ge 0$ and $|w| = 2k + \delta$. Let $\delta = n(k+1) + i$ with $n \ge 0$ and $0 \le i \le k$.

Let σ be a permutation of 1, 2, ..., k+1 such that $r_{\sigma(1)} \leq r_{\sigma(2)} \leq ... \leq r_{\sigma(k+1)}$. For any real number x, let $\lfloor x \rfloor$ to be the integer part of x.

Lemma 11 (Proposition 2 in [6]) Theorem 2 holds if $k \leq 9$.

${\rm Lemma}~12$

$$s(w) + 1 + \lfloor \frac{3n}{2} + \frac{3i}{2(k+1)} \rfloor \le |w| - 2 \implies s(w) \le \frac{2k-1}{2k+2}|w|.$$

Proof.

$$\begin{split} s(w) &\leq \frac{2k-1}{2k+2}|w| \\ &\Leftarrow s(w) \leq \lfloor \frac{2k-1}{2k+2}|w| \rfloor \\ &\Leftarrow s(w) \leq |w| - \lfloor \frac{3}{2k+2}(2k+\delta) \rfloor \\ &\Leftarrow s(w) \leq |w| - \lfloor \frac{3k}{k+1} + \frac{3\delta}{2(k+1)} \rfloor \\ &\Leftarrow s(w) \leq |w| - 2 - \lfloor \frac{k-2}{k+1} + \frac{3\delta}{2(k+1)} \rfloor \\ &\Leftarrow s(w) + \lfloor \frac{k-2}{k+1} + \frac{3\delta}{2(k+1)} \rfloor \leq |w| - 2 \\ &\Leftarrow s(w) + 1 + \lfloor \frac{3\delta}{2(k+1)} \rfloor \leq |w| - 2. \end{split}$$

From the definition, $\delta = n(k+1) + i$, thus,

$$s(w) + 1 + \lfloor \frac{3n}{2} + \frac{3i}{2(k+1)} \rfloor \le |w| - 2 \implies s(w) \le \frac{2k-1}{2k+2} |w|.$$

Lemma 13

$$s(w) + 1 + \lfloor \frac{3n}{2} + \frac{3i}{2(k+1)} \rfloor \le m(w) + r_{\sigma(k)} \implies s(w) \le \frac{2k-1}{2k+2} |w|.$$
(1)

Moreover,

$$1 + \lfloor \frac{3(n+1)}{2} \rfloor \le \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor + r_{\sigma(k)} \implies s(w) \le \frac{2k - 1}{2k + 2} |w|.$$
(2)

Proof. The first part is a direct consequence of Theorem 3 and Lemma 12. For the second part, from the hypothesis that $i \leq k$, we have

$$\lfloor \frac{3n}{2} + \frac{3i}{2(k+1)} \rfloor \le \lfloor \frac{3(n+1)}{2} \rfloor.$$

Moreover, from the fact that

$$\left\{a^{2i+1}|1\leq i\leq \lfloor\frac{r_{\sigma(k+1)}-1}{2}\rfloor\right\}\subset NS(w),$$

we have

$$s(w) + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor \le m(s).$$

Lemma 14 If $r_{\sigma(k)} \ge n+2$, then

$$1 + \lfloor \frac{3(n+1)}{2} \rfloor \le r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor.$$

 $\begin{array}{l} \textit{Proof. If } r_{\sigma(k)} \geq n+2, \, \text{then } r_{\sigma(k+1)} \geq n+2. \\ \text{If } n \text{ is odd, let } n=2t+1, \, \text{then} \end{array}$

$$r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor \ge (2t+1) + 2 + \lfloor \frac{(2t+1) + 2 - 1}{2} \rfloor$$
$$\ge 3t + 4$$
$$\ge \lfloor \frac{3((2t+1) + 1)}{2} \rfloor + 1.$$

If n is even, let n = 2t, then

$$r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor \ge (2t) + 2 + \lfloor \frac{(2t) + 2 - 1}{2} \rfloor$$
$$\ge 3t + 2$$
$$\ge \lfloor \frac{3((2t) + 1)}{2} \rfloor + 1.$$

Lemma 15 If $r_{\sigma(k)} \leq n$ and $k \geq 8$ then

$$1 + \lfloor \frac{3(n+1)}{2} \rfloor \le r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor.$$

Proof. If $r_{\sigma(k)} \leq n$, let us suppose $r_{\sigma(k)} = m$. Then

$$\sum_{j=1}^{k} r_{\sigma(j)} \le mk.$$

From the fact that $\sum_{j=1}^{k+1} r_{\sigma(j)} = k + \delta$, $r_{\sigma(k+1)} \ge k + \delta - mk$

$$\hat{\sigma}_{\sigma(k+1)} \ge k + \delta - mk \ge n(k+1) - mk + k.$$

Thus,

$$1 + \lfloor \frac{3(n+1)}{2} \rfloor \leq r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor$$

$$\iff 1 + \frac{3(n+1)}{2} \leq r_{\sigma(k)} + \frac{r_{\sigma(k+1)} - 1}{2} - 1$$

$$\iff \frac{5}{2} + \frac{3n}{2} \leq m + \frac{n(k+1) - mk + k - 1}{2} - 1$$

$$\iff 3n + 5 \leq 2m + n(k+1) - mk + k - 1 - 2$$

$$\iff 3n + 5 \leq n(k+1) - m(k-2) + k - 3$$

$$\iff 3n + 5 \leq n(k+1) - n(k-2) + k - 3 \quad (*)$$

$$\iff 3n + 5 \leq 3n + k - 3$$

$$\iff 8 \leq k.$$

The relation (*) holds because $k - 2 \ge 0$ and $m \le n$ from hypothesis. Lemma 16 If $r_{\sigma(k)} = n + 1$ and $r_{\sigma(k+1)} \ge n + 4$, then

$$1 + \lfloor \frac{3(n+1)}{2} \rfloor \le r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor.$$

Proof. If n is odd, let n = 2t + 1, then

$$\begin{aligned} r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor &\geq (2t+1) + 1 + \lfloor \frac{(2t+1) + 4 - 1}{2} \rfloor \\ &\geq 3t + 4 \\ &\geq \lfloor \frac{3((2t+1) + 1)}{2} \rfloor + 1. \end{aligned}$$

If n is even, let n = 2t, then

$$r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor \ge (2t) + 1 + \lfloor \frac{(2t) + 4 - 1}{2} \rfloor$$
$$\ge 3t + 2$$
$$\ge \lfloor \frac{3((2t) + 1)}{2} \rfloor + 1.$$

Lemma 17 If $r_{\sigma(k)} = n + 1$, $r_{\sigma(k+1)} \le n + 3$ and $k \ge 10$ then

$$s(w) + 1 + \lfloor \frac{3n}{2} + \frac{3i}{2(k+1)} \rfloor \le s(w) + r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor \le m(w) + r_{\sigma(k)}.$$

Proof. From the fact that

$$\sum_{j=1}^{k+1} r_{\sigma(j)} = |w|_a = k + \delta = k + n(k+1) + i,$$

we have

$$k + n(k + 1) + i \le (\sum_{j=1}^{k} n + 1) + n + 3.$$

Consequently, $i \leq 3$ and $\frac{3i}{2(k+1)} < \frac{1}{2}$. Thus, $\lfloor \frac{3n}{2} + \frac{3i}{2(k+1)} \rfloor = \lfloor \frac{3n}{2} \rfloor$. We only need to prove

$$\lfloor \frac{3n}{2} \rfloor + 1 \le r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor.$$

If n is odd, let n = 2t + 1, then

$$r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor \ge (2t+1) + 1 + \lfloor \frac{(2t+1) + 1 - 1}{2} \rfloor$$
$$\ge 3t + 2$$
$$\ge \lfloor \frac{3(2t+1)}{2} \rfloor + 1.$$

If n is even, let n = 2t, then

$$r_{\sigma(k)} + \lfloor \frac{r_{\sigma(k+1)} - 1}{2} \rfloor \ge (2t) + 1 + \lfloor \frac{(2t) + 1 - 1}{2} \rfloor$$
$$\ge 3t + 1$$
$$\ge \lfloor \frac{3((2t))}{2} \rfloor + 1.$$

Proof (of Theorem 2). From Lemma 13, Lemma 14, Lemma 15, Lemma 16 and Lemma 17, Theorem 2 holds if:

$$\begin{split} & r_{\sigma(k)} \geq n+2; \\ & r_{\sigma(k)} \leq n \text{ and } k \geq 8; \\ & r_{\sigma(k)} = n+1 \text{ and } r_{\sigma(k+1)} \geq n+4; \\ & r_{\sigma(k)} = n+1, \ r_{\sigma(k+1)} \leq n+3 \text{ and } k \geq 10 \end{split}$$

Combining with Lemma 11, we conclude.

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