# An upper bound of the number of distinct powers in binary words 

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#### Abstract

A power is a word of the form $\underbrace{u u \ldots u}$, where $u$ is a word and $\underbrace{}_{k \text { times }}$ $k$ is a positive integer and a square is a word of the form $u u$. Fraenkel and Simpson conjectured in 1998 that the number of distinct squares in a word is bounded by the length of the word. This conjecture was proven recently by Brlek and Li. Besides, there exists a stronger upper bound for binary words conjectured by Jonoska, Manea and Seki stating that for a word of length $n$ over the alphabet $\{a, b\}$, if we let $k$ be the least of the number of a's and the number of b's and $k \geq 2$, then the number of distinct squares is upper bounded by $\frac{2 k-1}{2 k+2} n$. In this article, we prove this conjecture by giving a stronger statement on the number of distinct powers in a binary word.


## 1 Introduction

A power is a word of the form $\underbrace{u u \ldots u}$, where $u$ is a word and $k$ is a positive $\underbrace{}_{k \text { times }}$
integer; the power is also called a $k$-power and $k$ is its exponent. The upper bound of the number of distinct $k$-powers in a finite word was studied in [7910] and the best known result is as follows:

Theorem 1 (Theorem 1 and Theorem 2 in [10] ) For every finite word $w$, let $m(w)$ denote the number of distinct nonempty powers of exponent at least 2 in $w$, let $m_{k}(w)$ denote the number of distinct nonempty $k$-powers in $w$, let $|w|$ denote the length of $w$ and let $|\operatorname{Alph}(w)|$ denote the number of distinct letters in $w$, then one has

$$
m(w) \leq|w|-|\operatorname{Alph}(w)| ;
$$

Moreover, for any integer $k \geq 2$,

$$
m_{k}(w) \leq \frac{|w|-|\operatorname{Alph}(w)|}{k-1}
$$

Particularly, a square is a 2-power and upper bound of the number of distinct squares in a finite word was studied in 4|5|8|3|11|2. A conjecture of Fraenkel
and Simpson [4] states that the number of distinct squares in a word is bounded by the length of the word. This conjecture is confirmed by the previous theorem. Besides, there exists a stronger upper bound for binary words conjectured by Jonoska, Manea and Seki [6]:

Theorem 2 Let $w$ be a finite word over the alphabet $\{a, b\}$, let $s(w)$ denote the number of distinct squares in $w$ and let $k$ denote the least of the number of a's and the number of $b$ 's in $w$. If $k \geq 2$, then one has

$$
s(w) \leq \frac{2 k-1}{2 k+2}|w| .
$$

In this article, we confirm this conjecture by proving the following result:
Theorem 3 Let $w$ be a finite word over the alphabet $\{a, b\}$ such that $w=$ $a^{r_{1}} b a^{r_{2}} b a^{r_{3}} b \ldots a^{r_{k}} b a^{r_{k+1}}$, where $r_{1}, r_{2}, \ldots, r_{k+1}$ are nonnegative integers and $a^{n}=$ $\underbrace{a a \ldots a}$. Let $\sigma$ be a permutation of $1,2, \ldots, k+1$ such that $r_{\sigma(1)} \leq r_{\sigma(2)} \leq \ldots \leq$ $n$ times
$r_{\sigma(k+1)}$, let $m(w)$ denote the number of distinct nonempty powers of exponent at least 2 in $w$ and let $|w|$ denote the length of $w$. Then one has

$$
m(w)+r_{\sigma(k)} \leq|w|-2
$$

## 2 Preliminaries

Let $\sum$ be an alphabet and $\sum^{*}$ be the set of all words over $\sum$. Let $w \in \sum^{*}$. By $|w|$, we denote its length. A word of length 0 is called the empty word and it is denoted by $\varepsilon$. A word $u$ is a factor of $w$ if $w=p u s$ for some words $p, s$. When $p=\varepsilon$ (resp. $s=\varepsilon$ ), $u$ is called a prefix (resp. suffix) of $w$. The set of all nonempty factors of $w$ is denoted by $\operatorname{Fac}(w)$. The number of occurrences of a factor $u \in w$ is denoted by $|w|_{u}$.

Let $w$ be a finite word. For any integer $i$ satisfying $1 \leq i \leq|w|$, let $L_{w}(i)$ be the set of all length- $i$ factors of $w$ and let $C_{w}(i)$ be the cardinality of $L_{w}(i)$. For any natural number $k$, we define the $k$-th power of a finite word $u$ to be $u^{k}=u u \cdots u$ and it consists of the concatenation of $k$ copies of $u$. A finite word $w$ is said to be primitive if it is not a power of another word, that is, $w=u^{k}$ implies $k=1$. A square is a 2 -power, that is a word $w$ satisfying $w=u u$ for a certain word $u$.

For a finite word $w$, let $\operatorname{Prim}(w)$ denote the set of primitive factors of $w$, let

$$
\begin{gathered}
M(w)=\left\{p^{i} \mid p^{i} \in \operatorname{Fac}(w), p \in \operatorname{Prim}(w), i \in \mathbf{N}, i \geq 2\right\}, \\
S(w)=\left\{p^{2 i} \mid p^{2 i} \in \operatorname{Fac}(w), p \in \operatorname{Prim}(w), i \in \mathbf{N}, i \geq 1\right\}, \\
N S(w)=\left\{p^{2 i+1} \mid p^{2 i+1} \in \operatorname{Fac}(w), p \in \operatorname{Prim}(w), i \in \mathbf{N}, i \geq 1\right\},
\end{gathered}
$$

and let $m(w), s(w)$ and $n s(w)$ be respectively the cardinality of $M(w), S(w)$ and $N S(w)$. Obviously, $M(w)=S(w) \cup N S(w), m(w)=s(w)+n s(w)$ and $s(w)$ is
the number of distinct nonempty squares in $w$.

Here we recall some elementary definitions and proprieties concerning graphs from Berge 1 .

A directed graph consists of a nonempty set of vertices $V$ and a set of edges $E$. A vertex $a$ represents an endpoint of an edge and an edge joins two vertices $a, b$ in order. A chain is a sequence of edges $e_{1}, e_{2}, \cdots, e_{k}$, such that there exists a sequence of vertices $v_{1}, v_{2}, \cdots, v_{k+1}$ and that for each $i$ satisfying $1 \leq i \leq k, e_{i}$ is either directed from $v_{i}$ to $v_{i+1}$ or from $v_{i+1}$ to $v_{i}$. A cycle is a finite chain such that $v_{k+1}=v_{1}$. A path is a sequence of edges $e_{1}, e_{2}, \cdots, e_{k}$, such that there exists a sequence of vertices $v_{1}, v_{2}, \cdots, v_{k+1}$ and that for each $i$ satisfying $1 \leq i \leq k$, $e_{i}$ is directed from $v_{i}$ to $v_{i+1}$. A circuit is a finite path such that $v_{k+1}=v_{1}$.

A cycle or a circuit is called elementary if, apart from $v_{1}$ and $v_{k+1}$, every vertex which it meets is distinct. A directed graph is called weakly connected if for any couple of vertices $a, b$ in this graph, there exists a chain connecting $a$ and $b$.

Let $G$ be a weakly connected graph and let $\left\{e_{1}, e_{2} \cdots e_{l}\right\},\left\{v_{1}, v_{2} \cdots v_{s}\right\}$ denote respectively the edge set and the vertex set of $G$. The number $\chi(G)=l-s+1$ is called the cyclomatic number of $G$.

Let $C$ be a cycle in $G$. A vector $\mu(C)=\left(c_{1}, c_{2} \cdots c_{l}\right)$ in the $l$-dimensional space $\mathbb{R}^{l}$ is called the vector-cycle corresponding to $C$ if $c_{i}$ is the number of visits of the edge $e_{i}$ in the cycle $C$ for all $i$ satisfying $1 \leq i \leq l$. The cycle $C_{1}, C_{2}, \cdots, C_{k}, \ldots$ are said to be independent if their corresponding vectors are linearly independent.

Theorem 4 (Theorem 2, Chapter 4 in [1]) the cyclomatic number of a graph is the maximum number of independent cycles in this graph.

## 3 Rauzy graphs

Let $w$ be a finite word. For any integer $i$ satisfying $1 \leq i \leq|w|$, the $i$-th Rauzy graph $\Gamma_{w}(i)$ of $w$ is defined to be an directed graph whose vertex set is $L_{w}(i)$ and the edge set is $L_{w}(i+1)$; an edge $e \in L_{w}(i+1)$ starts at the vertex $u$ and ends at the vertex $v$, if $u$ is a prefix and $v$ is a suffix of $e$. Let us define $\Gamma_{w}=\cup_{n=1}^{k-1} \Gamma_{w}(n)$.

Let $\Gamma_{w}(i)$ be a Rauzy graph of $w$ for some $i$, a sub-graph on $\Gamma_{w}(i)$ is called a small circuit if it is an elementary circuit and the number of its vertices is no larger than $i$.

Lemma 5 (Lemma 8 in [2]) Let $w$ be a finite word and let $\Gamma_{w}(i)$ be a Rauzy graph of $w$ for some $i$. Then all small circuits on $\Gamma_{w}(i)$ are independent.

Lemma 6 (Lemma 6 in [10]) Let $w$ be a finite word, then there exists an injection from $M(w)$ to the set of all small circuits on $\Gamma_{w}$.

Example 7 Let us consider the word $u=a b a a a b a a a a b a a b a$, the Rauzy graph $\Gamma_{u}(4)$ is as follows:


In this graph, there are three circuits: $C_{1}=\{\{a a a b, a a b a, a b a a, b a a a\},\{a a a b a, a a b a a, a b a a a, b a a a b\}\}$, $C_{2}=\{\{a a b a, a b a a, b a a b\},\{a a b a a, a b a a b, b a a b a\}\}$ and $C_{3}=\{\{a a a b, a a b a, a b a a, b a a a, a a a a\},\{a a a b a, a a b a a, a b a a a, b a a a a, a a a a b\}\}$. Two of them are small, they are $C_{1}$ and $C_{2}$, while $C_{3}$ is not small.

## 4 Proof of Theorem 3

Lemma 8 Let $w \in\{a, b\}^{*}$ and let $i$ be an integer satisfying $1 \leq i \leq|w|$. If there exists an elementary circuit on $\Gamma_{w}(i)$ containing the edge $a^{i} b$, then this circuit is not a small circuit and it is independent with all the small circuits on $\Gamma_{w}(i)$. Let $C_{s p}(i)$ denote one of these circuits (if any).

Proof. We first prove that, if there exists a small circuit passing through the vertex $a^{i}$, then it should be the sub-graph $\left\{\left\{a^{i}\right\},\left\{a^{i+1}\right\}\right\}$ of $\Gamma_{w}(i)$. In fact, if there exists a path $e_{1}, e_{2}, \ldots, e_{k}$ from $a^{i}$ to $a^{i}$ satisfying $k \leq i$, let $p=l_{1} l_{2} \ldots l_{k}$ be a word such that $l_{j}$ is the last letter of $e_{j}$ for all $j$ satisfying that $1 \leq j \leq k$. From the hypothesis that $e_{1}, e_{2}, \ldots, e_{k}$ form a circuit, we can deduce that $a^{i}$ is a suffix of $a^{i} p$. Moreover, as $|p|=k \leq i, l_{j}=a$ for all $j$. Consequently, $e_{j}=a^{i+1}$ for all $j$. Thus, from the unicity of each edge, we prove that there exists only one edge on the path and the graph is given by $\left\{\left\{a^{i}\right\},\left\{a^{i+1}\right\}\right\}$.

If their exists an elementary circuit $C$ on $\Gamma_{w}(i)$ containing the edge $a^{i} b$, from the fact that $a^{i} b$ cannot be contained in any small circuit on $\Gamma_{w}(i)$, we conclude that $C$ is not a small circuit and independent with all the small circuits on $\Gamma_{w}(i)$.

Lemma 9 Let $w \in\{a, b\}^{*}$ such that $w=a^{r_{1}} b a^{r_{2}} b a^{r_{3}} b \ldots a^{r_{k}} b a^{r_{k+1}}$ with $r_{j} \geq 0$ for all $j$ satisfying $1 \leq j \leq k+1$. Let $\sigma$ be a permutation of $1,2, \ldots, k+1$ such that $r_{\sigma(1)} \leq r_{\sigma(2)} \leq \ldots \leq r_{\sigma(k+1)}$. Then for any integer $i$ satisfying $1 \leq i \leq \sigma(k)$, there exists a $C_{s p}(i)$ on $\Gamma_{w}(i)$.

Proof. From the hypothesis that $i \leq r_{\sigma(k)} \leq r_{\sigma(k+1)}$, there exists a nonempty word $X$ such that $a^{i} X a^{i} \in \operatorname{Fac}(w)$ and that

$$
X= \begin{cases}b Y b & \text { if }|X| \geq 2 \\ b & \text { otherwise }\end{cases}
$$

Moreover, we can suppose that $a^{i} \notin \operatorname{Fac}(Y)$. Indeed, there exists a circuit on $\Gamma_{a^{i} X a^{i}}(i)$ containing the edge $a^{i} b$ and $\Gamma_{a^{i} X a^{i}}(i)$ is a sub-graph of $\Gamma_{w}(i)$. Thus, there exists a $C_{s p}(i)$ on $\Gamma_{w}(i)$.

Lemma 10 Let $w \in\{a, b\}^{*}$ and let $I_{w}$ be the cardinality of

$$
S_{w}=\left\{i\left|1 \leq i \leq|w|, \text { there exists a circuit } C_{s p}(i) \text { on } \Gamma_{w}(i)\right\}\right.
$$

then

$$
m(w)+I_{w} \leq|w|-|\operatorname{Alph}(w)|
$$

Proof. Let $\mathrm{sc}_{w}(i)$ denote the number of small circuits on $\Gamma_{w}(i)$. From Lemma 5 , Lemma 8 and Theorem 4 for any $i \in S_{w}$,

$$
\mathrm{sc}_{w}(i)+1 \leq C_{w}(i+1)-C_{w}(i)+1
$$

and for any $i \notin S_{w}$,

$$
\mathrm{sc}_{w}(i) \leq C_{w}(i+1)-C_{w}(i)+1
$$

Consequently,

$$
\sum_{i=1}^{|w|} \operatorname{sc}_{w}(i)+I_{w} \leq \sum_{i=1}^{|w|} C_{w}(i+1)-C_{w}(i)+1=|w|-|\operatorname{Alph}(w)|
$$

Moreover, from 6 $m(w) \leq \sum_{i=1}^{|w|} \mathrm{sc}(i)$, thus, $m(w)+I_{w} \leq|w|-|\operatorname{Alph}(w)|$.
Proof (of Theorem[3). It is a direct consequence of Lemma 10 and Lemma 9.

## 5 Proof of Theorem 2

In this section, Let $w \in\{a, b\}^{*}$ such that $|w|_{a} \geq|w|_{b}$, that $|w|_{b}=k$ and that $w=a^{r_{1}} b a^{r_{2}} b a^{r_{3}} b \ldots a^{r_{k}} b a^{r_{k+1}}$ with $r_{j} \geq 0$ for all $j$ satisfying $1 \leq j \leq k+1$.

Let $\delta$ be an integer such that $|w|_{a}=k+\delta$. From the hypothesis $|w|_{a} \geq|w|_{b}$, we can suppose $\delta \geq 0$ and $|w|=2 k+\delta$. Let $\delta=n(k+1)+i$ with $n \geq 0$ and $0 \leq i \leq k$.

Let $\sigma$ be a permutation of $1,2, \ldots, k+1$ such that $r_{\sigma(1)} \leq r_{\sigma(2)} \leq \ldots \leq r_{\sigma(k+1)}$. For any real number $x$, let $\lfloor x\rfloor$ to be the integer part of $x$.

Lemma 11 (Proposition 2 in [6]) Theorem 图holds if $k \leq 9$.

## Lemma 12

$$
s(w)+1+\left\lfloor\frac{3 n}{2}+\frac{3 i}{2(k+1)}\right\rfloor \leq|w|-2 \Longrightarrow s(w) \leq \frac{2 k-1}{2 k+2}|w| .
$$

Proof.

$$
\begin{aligned}
& s(w) \leq \frac{2 k-1}{2 k+2}|w| \\
& \Longleftarrow s(w) \leq\left\lfloor\frac{2 k-1}{2 k+2}|w|\right\rfloor \\
& \Longleftarrow s(w) \leq|w|-\left\lfloor\frac{3}{2 k+2}(2 k+\delta)\right\rfloor \\
& \Longleftarrow s(w) \leq|w|-\left\lfloor\frac{3 k}{k+1}+\frac{3 \delta}{2(k+1)}\right\rfloor \\
& \Longleftarrow s(w) \leq|w|-2-\left\lfloor\frac{k-2}{k+1}+\frac{3 \delta}{2(k+1)}\right\rfloor \\
& \Longleftarrow s(w)+\left\lfloor\frac{k-2}{k+1}+\frac{3 \delta}{2(k+1)}\right\rfloor \leq|w|-2 \\
& \Longleftarrow s(w)+1+\left\lfloor\frac{3 \delta}{2(k+1)}\right\rfloor \leq|w|-2 .
\end{aligned}
$$

From the definition, $\delta=n(k+1)+i$, thus,

$$
s(w)+1+\left\lfloor\frac{3 n}{2}+\frac{3 i}{2(k+1)}\right\rfloor \leq|w|-2 \Longrightarrow s(w) \leq \frac{2 k-1}{2 k+2}|w| .
$$

## Lemma 13

$$
\begin{equation*}
s(w)+1+\left\lfloor\frac{3 n}{2}+\frac{3 i}{2(k+1)}\right\rfloor \leq m(w)+r_{\sigma(k)} \Longrightarrow s(w) \leq \frac{2 k-1}{2 k+2}|w| \tag{1}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
1+\left\lfloor\frac{3(n+1)}{2}\right\rfloor \leq\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor+r_{\sigma(k)} \Longrightarrow s(w) \leq \frac{2 k-1}{2 k+2}|w| . \tag{2}
\end{equation*}
$$

Proof. The first part is a direct consequence of Theorem 3 and Lemma 12 For the second part, from the hypothesis that $i \leq k$, we have

$$
\left\lfloor\frac{3 n}{2}+\frac{3 i}{2(k+1)}\right\rfloor \leq\left\lfloor\frac{3(n+1)}{2}\right\rfloor
$$

Moreover, from the fact that

$$
\left\{a^{2 i+1} \left\lvert\, 1 \leq i \leq\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor\right.\right\} \subset N S(w),
$$

we have

$$
s(w)+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor \leq m(s) .
$$

Lemma 14 If $r_{\sigma(k)} \geq n+2$, then

$$
1+\left\lfloor\frac{3(n+1)}{2}\right\rfloor \leq r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor .
$$

Proof. If $r_{\sigma(k)} \geq n+2$, then $r_{\sigma(k+1)} \geq n+2$.
If $n$ is odd, let $n=2 t+1$, then

$$
\begin{aligned}
r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor & \geq(2 t+1)+2+\left\lfloor\frac{(2 t+1)+2-1}{2}\right\rfloor \\
& \geq 3 t+4 \\
& \geq\left\lfloor\frac{3((2 t+1)+1)}{2}\right\rfloor+1
\end{aligned}
$$

If $n$ is even, let $n=2 t$, then

$$
\begin{aligned}
r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor & \geq(2 t)+2+\left\lfloor\frac{(2 t)+2-1}{2}\right\rfloor \\
& \geq 3 t+2 \\
& \geq\left\lfloor\frac{3((2 t)+1)}{2}\right\rfloor+1
\end{aligned}
$$

Lemma 15 If $r_{\sigma(k)} \leq n$ and $k \geq 8$ then

$$
1+\left\lfloor\frac{3(n+1)}{2}\right\rfloor \leq r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor .
$$

Proof. If $r_{\sigma(k)} \leq n$, let us suppose $r_{\sigma(k)}=m$. Then

$$
\sum_{j=1}^{k} r_{\sigma(j)} \leq m k
$$

From the fact that $\sum_{j=1}^{k+1} r_{\sigma(j)}=k+\delta$,

$$
r_{\sigma(k+1)} \geq k+\delta-m k \geq n(k+1)-m k+k
$$

Thus,

$$
\begin{aligned}
& 1+\left\lfloor\frac{3(n+1)}{2}\right\rfloor \leq r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor \\
& \Longleftarrow 1+\frac{3(n+1)}{2} \leq r_{\sigma(k)}+\frac{r_{\sigma(k+1)}-1}{2}-1 \\
& \Longleftarrow \frac{5}{2}+\frac{3 n}{2} \leq m+\frac{n(k+1)-m k+k-1}{2}-1 \\
& \Longleftarrow 3 n+5 \leq 2 m+n(k+1)-m k+k-1-2 \\
& \Longleftarrow 3 n+5 \leq n(k+1)-m(k-2)+k-3 \\
& \Longleftarrow 3 n+5 \leq n(k+1)-n(k-2)+k-3(*) \\
& \Longleftarrow 3 n+5 \leq 3 n+k-3 \\
& \Longleftarrow 8 \leq k .
\end{aligned}
$$

The relation $\left(^{*}\right)$ holds because $k-2 \geq 0$ and $m \leq n$ from hypothesis.
Lemma 16 If $r_{\sigma(k)}=n+1$ and $r_{\sigma(k+1)} \geq n+4$, then

$$
1+\left\lfloor\frac{3(n+1)}{2}\right\rfloor \leq r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor .
$$

Proof. If $n$ is odd, let $n=2 t+1$, then

$$
\begin{aligned}
r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor & \geq(2 t+1)+1+\left\lfloor\frac{(2 t+1)+4-1}{2}\right\rfloor \\
& \geq 3 t+4 \\
& \geq\left\lfloor\frac{3((2 t+1)+1)}{2}\right\rfloor+1
\end{aligned}
$$

If $n$ is even, let $n=2 t$, then

$$
\begin{aligned}
r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor & \geq(2 t)+1+\left\lfloor\frac{(2 t)+4-1}{2}\right\rfloor \\
& \geq 3 t+2 \\
& \geq\left\lfloor\frac{3((2 t)+1)}{2}\right\rfloor+1
\end{aligned}
$$

Lemma 17 If $r_{\sigma(k)}=n+1, r_{\sigma(k+1)} \leq n+3$ and $k \geq 10$ then

$$
s(w)+1+\left\lfloor\frac{3 n}{2}+\frac{3 i}{2(k+1)}\right\rfloor \leq s(w)+r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor \leq m(w)+r_{\sigma(k)} .
$$

Proof. From the fact that

$$
\sum_{j=1}^{k+1} r_{\sigma(j)}=|w|_{a}=k+\delta=k+n(k+1)+i
$$

we have

$$
k+n(k+1)+i \leq\left(\sum_{j=1}^{k} n+1\right)+n+3 .
$$

Consequently, $i \leq 3$ and $\frac{3 i}{2(k+1)}<\frac{1}{2}$. Thus, $\left\lfloor\frac{3 n}{2}+\frac{3 i}{2(k+1)}\right\rfloor=\left\lfloor\frac{3 n}{2}\right\rfloor$.
We only need to prove

$$
\left\lfloor\frac{3 n}{2}\right\rfloor+1 \leq r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor .
$$

If $n$ is odd, let $n=2 t+1$, then

$$
\begin{aligned}
r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor & \geq(2 t+1)+1+\left\lfloor\frac{(2 t+1)+1-1}{2}\right\rfloor \\
& \geq 3 t+2 \\
& \geq\left\lfloor\frac{3(2 t+1)}{2}\right\rfloor+1 .
\end{aligned}
$$

If $n$ is even, let $n=2 t$, then

$$
\begin{aligned}
r_{\sigma(k)}+\left\lfloor\frac{r_{\sigma(k+1)}-1}{2}\right\rfloor & \geq(2 t)+1+\left\lfloor\frac{(2 t)+1-1}{2}\right\rfloor \\
& \geq 3 t+1 \\
& \geq\left\lfloor\frac{3((2 t))}{2}\right\rfloor+1
\end{aligned}
$$

Proof (of Theorem (2). From Lemma 13, Lemma 14, Lemma 15, Lemma 16] and Lemma 17, Theorem 2 holds if:

$$
\begin{aligned}
& r_{\sigma(k)} \geq n+2 \\
& r_{\sigma(k)} \leq n \text { and } k \geq 8 \\
& r_{\sigma(k)}=n+1 \text { and } r_{\sigma(k+1)} \geq n+4 ; \\
& r_{\sigma(k)}=n+1, r_{\sigma(k+1)} \leq n+3 \text { and } k \geq 10 .
\end{aligned}
$$

Combining with Lemma 11 we conclude.

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