# ON COMPLEXITIES OF MINUS DOMINATION

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**Abstract.** A function  $f: V \to \{-1,0,1\}$  is a minus-domination function of a graph G = (V,E) if the values over the vertices in each closed neighborhood sum to a positive number. The weight of f is the sum of f(x) over all vertices  $x \in V$ . The minus-domination number  $\gamma^-(G)$  is the minimum weight over all minus-domination functions. The size of a minus domination is the number of vertices that are assigned 1. In this paper we show that the minus-domination problem is fixed-parameter tractable for d-degenerate graphs when parameterized by the size of the minus-dominating set and by d. The minus-domination problem is polynomial for graphs of bounded rankwidth and for strongly chordal graphs. It is NP-complete for splitgraphs. Unless P = NP there is no fixed-parameter algorithm for minus-domination.

## 1 Introduction

A fresh breeze seems to be blowing through the area of domination problems. This research area is aroused anew by the recent fixed-parameter investigations (see, eg, [2,6,21,22]).

Let G=(V,E) be a graph and let  $f:V\to S$  be a function that assigns some integer from  $S\subseteq \mathbb{Z}$  to every vertex of G. For a subset  $W\subseteq V$  we write

$$f(W) = \sum_{x \in W} f(x).$$

The function f is a domination function if for every vertex x, f(N[x]) > 0, where  $N[x] = \{x\} \cup N(x)$  is the closed neighborhood of x. The weight of f is defined as the value f(V).

In this manner, the ordinary domination problem is described by a domination function that assigns a value of  $\{0,1\}$  to each element of V. A signed domination function assigns a value of  $\{-1,1\}$  to each vertex x. The minimal weight of a dominating and signed dominating function are denoted by  $\gamma(G)$  and  $\gamma_s(G)$ . In this paper we look at the minus-domination problem.

**Definition 1.** Let G = (V, E) be a graph. A function  $f : V \to \{-1, 0, 1\}$  is a minus-domination function if f(N[x]) > 0 for every vertex x.

In the minus-domination problem one tries to minimize the weight of a minus-domination function. The minimal weight of a minus-domination function is denoted as  $\gamma^-(G)$ . Notice that the weight may be negative. For example, consider a  $K_4$  and add one new vertex for every edge, adjacent to the endpoints of that edge. Assign a value 1 to every vertex of the  $K_4$  and assign a value -1 to each of the six other vertices. This is a valid signed-domination function and its weight is -2.

The problem to determine the value of  $\gamma^-(G)$  is NP-complete, even when restricted to bipartite graphs, chordal graphs and planar graphs with maximal degree four [3,4]. Sharp bounds for the minimum weight are obtained in, eg, [16].

Damaschke shows that, unless P = NP, the value of  $\gamma^-$  cannot be approximated in polynomial time within a factor  $1 + \epsilon$ , for some  $\epsilon > 0$ , not even for graphs with all degrees at most four [3, Theorem 3].

Famous open problems are the complexity of the minus-domination problem for splitgraphs and for strongly chordal graphs. In this paper we settle these questions.

### 2 Planar graphs

Determining the smallest weight of a minus-dominating function is NP-complete, even when restricted to planar graphs [4].

Let G = (V, E) be a graph and let  $f : V \to S$  be a domination function. Following Zheng et al. we define the <u>size of f</u> as the number of vertices  $x \in V$  with f(x) > 0. We denote the size of a minus-dominating function f as size(f).

Consider signed-domination functions of size at most k. It is easy to see that  $|V(G)| = O(k^2)$  (see [21]). It follows that the signed domination problem parameterized by the size is fixed-parameter tractable. This is not so clear for the minus domination problem. For example, consider a star and assign to the center a value of 1 and to every leaf a value of zero. This is a valid minus-domination function with size 1 but the number of vertices is unbounded.

**Theorem 1.** For planar graphs the minus-domination problem, parameterized by the size, is fixed-parameter tractable.

*Proof.* Let  $f: V \rightarrow \{-1, 0, 1\}$  be a minus-domination function. Let

$$D = \{ x \mid x \in V \text{ and } f(x) = 1 \}.$$

Then D is a dominating set in G. It follows that, for all graphs G,

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\gamma^{-}(G) \leq \gamma(G) \leq \min \{ \text{ size}(f) \mid f \text{ is a minus-dominating function } \}.
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The first subexponential fixed-parameter algorithm for domination in planar graphs appeared in [1]. In this paper the authors prove that, if G is a planar graph with  $\gamma(G) \leqslant k$ , then the treewidth of G is  $O(\sqrt{k})$ . Using a treedecomposition of bounded treewidth one can solve the domination problem in  $O(2^{15.13\sqrt{k}} \cdot k + n^3 + k^4)$  time (or conclude that  $\gamma(G) > k$ ). The results were generalized to some nonplanar classes of graphs by Demaine, et al.

The minus-domination problem with size bounded by k can be formulated in monadic second-order logic. By Courcelle's theorem, any such problem can be solved in linear time on graphs of bounded treewidth (see, eg, [10,12]). This proves the theorem.

### 2.1 d-Degenerate graphs

**Definition 2.** A graph is d-degenerate if each of its induced subgraphs has a vertex of degree at most d.

Graphs with bounded degeneracy contain, eg, graphs that are embeddable on some fixed surface, families of graphs that exclude some minor, graphs of bounded treewidth, etc. [20].

In this section we show that, for each fixed d, the minus domination problem, parameterized by the number k of vertices that receive a 1, is fixed-parameter tractable for d-degenerate graphs.

In this section, when considering a partition of the vertices, we allow that some parts of the partition are empty.

In the minus domination problem one searches for a partition of the vertices into three parts, say red, white and blue. The red vertices are assigned -1, white are 1 and blue are 0. Zheng et al. proved a lemma similar to the one below for the signed domination problem in [22, Theorem 2] and [21, Lemma 6].

**Lemma 1.** Assume that G = (V, E) has a minus-dominating function with size at most k. Let R, W and B be the coloring of the vertices into red, white and blue, defined by this minus-domination function. Then

$$|W \cup R| = O(k^2)$$
.

*Proof.* By assumption, the minus-domination function colors at most k vertices white. Consider the subgraph G' induced by the red and white vertices. Consider a vertex x of G'. Then at least half of its neighbors is colored white, otherwise its closed neighborhood has weight at most zero. Since there are at most k white vertices, each vertex of G' has degree less than 2k.

Notice also that each red vertex has at least two white neighbors. Since there are only k white vertices, and each white vertex has degree less than 2k, the number of red vertices is less than  $2k^2$ . This proves the lemma.

For algorithmic purposes one usually considers the following generalization of the domination problem. Consider graphs of which each vertex is either colored black or white. In the parameterized black-and-white domination problem the objective is to find a set D of at most k vertices such that

for each black vertex 
$$x$$
,  $N[x] \cap D \neq \emptyset$ .

Obviously, the domination problem is a special case, in which each vertex is black.

For the minus-domination problem we describe an algorithm for a black-and-white version, where the vertices with a 0 or -1 are black and such that each closed neighborhood of a black vertex has a positive weight. To see that this solves the minus-domination problem, just consider the case where all vertices are black.

Alon and Gutner prove, in their seminal paper, that the domination problem is fixed-parameter tractable for d-degenerate graphs [2]. The main ingredient of their paper is the following lemma.

**Lemma 2.** Let G = (V, E) be a d-degenerate black-and-white colored graph. Let B and W be the set of black and white vertices. If |B| > (4d + 2)k then the set

$$\Omega = \left\{ \begin{array}{ll} x \mid x \in V \quad \text{and} \quad |N[x] \cap B| \geqslant \frac{|B|}{k} \end{array} \right\} \quad \text{satisfies} \quad |\Omega| \leqslant (4d+2)k.$$

To prove that the minus-domination problem, parameterized by the size, is fixed-parameter tractable for d-degenerate graphs, we adapt the proof of [2, Theorem 1].

**Theorem 2.** For each d and k, there exists a linear algorithm for finding a minus-domination of size at most k in a d-degenerate black-and-white graph, if such a set exists.

*Proof.* Let B and W be the set of black and white vertices. First assume that  $|B| \le (4d + 2)k$ . If there is a minus-domination function of size at most k then there are k vertices (assigned 1) that dominate all vertices in B.

The algorithm tries all possible subsets  $R \subseteq B$  for the set of red vertices (those are assigned -1). Number the closed neighborhoods of the vertices in  $R \cup B$ , say

$$N_1, \ldots, N_t$$

where  $t=|B\cup R|\leqslant (4d+2)k$ . Define an equivalence relation on the vertices of  $V\setminus R$  by making two vertices equivalent if they are contained in exactly the same subsets  $N_i$ . For each equivalence class that contains more than k vertices which are not red, remove all of them except at most k vertices. This kernelization reduces the graph to an instance H with at most g(k,d) vertices, for some function g.

Consider all subsets of V(H) with at most k vertices of which none is red. Give these vertices the value 1 and the remaining vertices that are not red the value 0. Check if this is a valid minus-domination.

Now assume that |B| > (4d + 2)k. Then, by Lemma 2,  $|\Omega| \le (4d + 2)k$ . Notice that at least one vertex of  $\Omega$  is assigned 1 in any minus-domination function of size k. In that case the algorithm grows a search tree of size at most  $(4d + 2)^k \cdot k!$  before it arrives at the previous case (see [2]).

# 3 Cographs

A minus domination with bounded size can be formulated in monadic second-order logic without quantification over subsets of edges. It follows that there is a linear-time algorithm to solve the problem for graphs of bounded treewidth or rankwidth (or cliquewidth) [14]. It is less obvious that  $\gamma^-$  is computable for bounded rankwidth when there is no restriction on the size. In this section we adapt a method of Yeh and Chang to show this.

It is well-known that the graphs of rankwidth one are the distance-hereditary graphs. We first analyze the complexity of the minus-domination problem for the class of cographs. Cographs form a proper subclass of the class of distance-hereditary graphs.

We denote a path with four vertices by  $P_4$ .

**Definition 3.** A cograph is a graph without induced  $P_4$ .

Cographs are characterized by the property that each induced subgraph with at least two vertices is either a join or a union of two smaller cographs. It follows that cographs admit a decomposition tree (T, f) where T is a rooted binary tree and where f is a bijection from the vertices of G to the leaves of T. Each internal

node is labeled as  $\otimes$  or  $\oplus$ . When the label is  $\otimes$  then all vertices of the left subtree are adjacent to all vertices of the right subtree. A node that is labeled as  $\otimes$  is called a join-node. When the label is  $\oplus$  there is no edge between vertices of the right and left subtree. A node that is labeled as  $\oplus$  is called a union-node. One refers to a decomposition tree of this type as a cotree. A cotree for a cograph can be obtained in linear time.

**Theorem 3.** There exists an efficient algorithm that computes  $\gamma^-$  for cographs.

*Proof.* Let G = (V, E) be a cograph. We assume that a cotree for G is a part of the input. Consider a subtree T' and let  $W \subseteq V$  be the set of vertices that are mapped to the leaves in T'.

For three numbers (a, b, c), an (a, b, c)-function is a function  $f: W \to \{-1, 0, 1\}$  such that f assigns a vertices the value -1, b vertices the value 0 and c vertices the value 1. Obviously, we have that a + b + c = |W|.

For an integer t, let

$$\zeta(t, a, b, c) = \max |\{x \mid x \in W \text{ and } f(N[x] \cap W) + t > 0 \text{ and } where f is an } (a, b, c)\text{-function } \}|. (1)$$

When the set is empty we let  $\zeta(t, a, b, c) = -\infty$ .

Notice that a minus-domination function with minimum weight can be computed when  $\zeta$  is known for the root node, that is, when W = V. Namely,

$$\gamma^{-}(G) = \min \{ -a + c \mid a + b + c = n \text{ and } \zeta(0, a, b, c) = n \}.$$
 (2)

We show how the values  $\zeta(t,a,b,c)$  can be computed. Assume that G is the union of two cographs  $G_1=(V_1,E_1)$  and  $G_2=(V_2,E_2)$ . We denote the  $\zeta$ -values for  $G_1$  and  $G_2$  by  $\zeta_1$  and  $\zeta_2$ . Then

$$\zeta(t, a, b, c) = \max \{ \zeta_1(t, a_1, b_1, c_1) + \zeta_2(t, a_2, b_2, c_2)$$
 where  $a_1 + a_2 = a$   $b_1 + b_2 = b$   $c_1 + c_2 = c \}$ . (3)

Now assume that G is the join of  $G_1$  and  $G_2$ . Then

$$\zeta(t,a,b,c) = \max \left\{ \begin{array}{l} \zeta_1(t-c_2+a_2,a_1,b_1,c_1) + \zeta_2(t-c_1+a_1,a_2,b_2,c_2) \\ \text{where} \quad a_1+a_2=a \quad b_1+b_2=b \quad c_1+c_2=c \end{array} \right\}.$$

This proves the theorem.

*Remark 1.* Notice that complete multipartite graphs are cographs. Formulas for the signed and minus domination number of complete multipartite graphs appear in a recent paper by H. Liang.

By similar methods we obtain a polynomial algorithm for minus domination on distance-hereditary graphs. For brevity we put the proof of the next theorem in an appendix.

**Theorem 4.** There exists a polynomial algorithm that computes  $\gamma^-$  for distance-hereditary graphs.

Remark 2. It is not hard to see that similar results can be derived for graphs of bounded rankwidth, that is,  $\gamma^-$  is computable in polynomial time for graphs of bounded rankwidth (see, eg, [12]). The rankwidth appears as a function in the exponent of n. Graphs of bounded treewidth are contained in the class of bounded rankwidth and so a similar statement holds for graphs of bounded treewidth. At the moment we do not believe that there is a fixed-parameter algorithm, parameterized by treewidth or rankwidth, to compute  $\gamma^-$ . The results of [22, Section 4.2] seem wrong.<sup>5</sup>

# 4 Strongly chordal graphs

The minus domination problem is NP-complete for chordal graphs. In this section we show that the problem can be solved in polynomial time for strongly chordal graphs.

A graph is chordal if it has no induced cycle of length more than three. A chord in a cycle is an edge that runs between two vertices that are not consecutive in the cycle. Let  $C = [x_1, \ldots, x_{2k}]$  be an even cycle of length 2k. A chord  $\{x_i, x_j\}$  in C is odd if the distance in C between  $x_i$  and  $x_j$  is odd.

**Definition 4.** A chordal graph G is strongly chordal if each cycle in G of even length at least 6 has an odd chord.

There are many characterizations of strongly chordal graphs [5,11]. Perhaps the best known examples of strongly chordal graphs are the interval graphs.

In strongly chordal graphs the domination number is equal to the 2-packing number (see, eg, [19, Theorem 7.4.4]). It follows that the domination number for strongly chordal graphs is polynomial [5].

**Theorem 5.** The minus domination problem for strongly chordal graphs can be solved in  $O(\min \{n^2, m \log n\})$  time. Here n is the number of vertices and m is the number of edges of the graph.

<sup>&</sup>lt;sup>5</sup> We communicated with the authors of [22] and our ideas about it are now in agreement.

*Proof.* Farber describes a linear programming formulation for the domination problem. In this linear programming formulation we can change the variables from  $x_i$  to  $z_i = x_i + 1$ . This changes the constraints  $-1 \leqslant x_i \leqslant 1$  into  $0 \leqslant z_i \leqslant 2$ . The linear program becomes

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i=1}^n z_i \\ \\ \text{subject to} & \displaystyle\sum_{i\in N[k]} z_i\geqslant b_k \\ \\ \text{and} & 0\leqslant z_i\leqslant 2 \end{array} \qquad \text{for each i.}$$

In our case, the variable  $b_k$  is equal to |N[k]| + 1.

The closed neighborhood matrix of a strongly chordal graph is totally balanced. By [8,9,13] (see also, eg, [19, Theorem A.3.4]) the integer program and its linear relaxation have the same value.

To deal with the constraints  $z_i \leq 2$  we write the LP as

$$\label{eq:minimize} \begin{array}{ll} \text{Minimize} & \mathbf{j}^T \cdot \mathbf{z} \\ \text{subject to} & \begin{pmatrix} A \\ -I \end{pmatrix} \mathbf{z} \geqslant \begin{pmatrix} \mathbf{b} \\ -2 \cdot \mathbf{j} \end{pmatrix} \qquad \text{and} \quad \mathbf{z} \geqslant \mathbf{0}. \end{array}$$

Here, the matrix A is the closed neigborhood matrix, and the vector  $\mathbf{b}$  is equal to

$$\mathbf{b} = \mathbf{j} + A\mathbf{j}.$$

The dual of this LP is

$$\label{eq:maximize} \begin{split} & \textbf{Maximize} \quad \textbf{b}^{\mathsf{T}} \cdot \textbf{y}_1 - 2\textbf{j}^{\mathsf{T}} \cdot \textbf{y}_2 \\ & \text{subject to} \quad A\textbf{y}_1 \leqslant \textbf{j} + \textbf{y}_2 \quad \text{and} \qquad \qquad \textbf{y}_1 \geqslant \textbf{0} \quad \text{and} \quad \textbf{y}_2 \geqslant \textbf{0}. \end{split}$$

Notice that

$$y_{2,k} = max \{ 0, -1 + \sum_{i \in N[k]} y_{1,i} \}$$
 for all k.

The complementary slackness conditions are as follows.

$$\begin{array}{ccc} y_{1,k}>0 & \Rightarrow & \displaystyle\sum_{i\in N[k]} z_i = 1 + |N[k]| \\ \\ \displaystyle\sum_{i\in N[k]} y_{1,i}>1 & \Rightarrow & z_k = 2, \quad \text{and} \\ \\ z_k>0 & \Rightarrow & \displaystyle\sum_{i\in N[k]} y_{1,i}\geqslant 1. \end{array}$$

Solving the linear problem can be done in  $O(n^{3.5} \log n)$ . Farber's method can be used to bring it down to  $O(n^2)$  or  $m \log n$ ), which is the time needed to compute a strong elimination ordering. We omit the details; see Remark 4.

Remark 3. When G is strongly chordal then  $G^2$  is that also [15]. A simple vertex of G is simplicial in  $G^2$ . The weighted 2-packing problem in G asks for the maximal weight independent set in  $G^2$ . This can be solved in linear time [7]. It uses the fact that in any chordal graph, with integer weights on the vertices, the maximal weight of an independent set equals the minimal number of cliques that have the property that every vertex is covered at least as many times by cliques as its weight.

**Corollary 1.** The exists a linear-time algorithm that solves minus domination on interval graphs.

*Remark 4.* After the publication of our draft on arXiv, one of the authors of their paper, quoted in the footnote, drew our attention to their result. The authors claim a linear algorithm for minus domination on strongly chordal graphs. (Here, they assume that a strong elimination ordering is a part of the input). <sup>6</sup>

# 5 Splitgraphs

In this section we show that the minus-domination problem is NP-complete for splitgraphs. We reduce the (3,2)-hitting set problem to the minus-domination problem. The (3,2)-hitting set problem is defined as follows (see, eg, [17]).

Instance: Let  $\ensuremath{\mathfrak{C}}$  be a collection of sets, each containing exactly three elements

from a universe U.

**Ouestion:** Find a smallest set  $U' \subseteq U$  such that for each  $C \in \mathcal{C}$ .

$$|C \cap U'| \ge 2$$
.

**Lemma 3.** The (3, 2)-hitting set is NP-complete.

*Proof.* The reduction is from vertex cover, ie, (2,1)-hitting set. The (2,1)-hitting set is defined similar as above, except that in this case every subset has two elements and the problem is to find a subset U' which hits every subset at least once.

Consider an instance of (2,1)-hitting set. Let  $\mathfrak C$  be the collection of 2-element subsets of a universe U. Add four vertices to the universe, say  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$ . Add  $\alpha$  to every subset of  $\mathfrak C$  and add two subsets,  $\{\alpha,\beta,\gamma\}$  and  $\{\alpha,\beta,\delta\}$ . We claim that any solution of this (3,2)-hitting set problem has  $\alpha$  in the hitting set. If not, then  $\{\beta,\gamma,\delta\}$  is a subset of the (3,2)-hitting set. In that case we may replace the elements  $\beta$ ,  $\gamma$  and  $\delta$  with  $\alpha$  and  $\beta$ . Then we obtain a (3,2)-hitting set with fewer elements.

Thus, we may assume that  $\alpha$  is in the (3,2)-hitting set. But now the problem is equivalent to the (2,1)-hitting set, since every adapted subset contains  $\alpha$ .

<sup>&</sup>lt;sup>6</sup> C. Lee and M. Chang, Variations of Y-dominating functions on graphs, *Discrete Mathematics* **308** (2008), pp. 4185–4204.

**Theorem 6.** The minus-domination problem is NP-complete for splitgraphs.

*Proof.* Consider an instance of the (3,2)-hitting set. We first construct a split-graph where U is the clique and where each element  $C \in \mathcal{C}$  is a vertex of the independent set, and adjacent exactly to the three vertices of C in the clique. Next, we extend the splitgraph by adding auxiliary vertices in the clique and the independent set, respectively. Precisely, we add a set X of  $|U| + |\mathcal{C}| + 1$  vertices in the clique, and for each vertex x in X, we add a distinct vertex x' in the independent set that connects with x. This completes the description of the splitgraph. Let V be the set of vertices of this graph, that is

$$V = U \cup X \cup \{ x' \mid x \in X \} \cup \mathcal{C}.$$

Consider a minus-domination function f of minimal weight. Notice that, we may assume that for each vertex x in X, f(x) = 1. Otherwise, by considering the closed neighborhood N[x'], we require f(x') + f(x) > 0, so that f(x') = 1 and f(x) = 0; in such a case, we can reset f(x') as 0 and f(x) as 1, while maintaining validity (i.e., positive total weight for each close neighborhood) and optimality (ie, minimum total weight) of the assignment.

Notice that for any function  $f: V \rightarrow \{-1, 0, 1\}$  we have that

$$\forall_{x \in X} f(x) = 1 \Rightarrow \forall_{u \in U} f(N[u]) > 0$$

no matter what values the vertices  $u \in U$  or  $C \in \mathcal{C}$  are assigned.

We may now, further assume that for each vertex C in the independent set

$$f(C) \leqslant \min \{ f(u) \mid u \in N(C) \}. \tag{5}$$

If this were not the case, then we could swap the value f(C) with the value of a vertex in N(C) and obtain a minus-domination function of at most the same weight, satisfying (5). Note that after the change, we cannot have f(N(C)) = 0.

We claim that there is a domination function of minimal weight with f(C)=-1 for every  $C\in \mathcal{C}$ . To see that, consider the following cases. If f(N(C))=3, then we have f(C)=-1. If f(N(C))=2, then N(C) has two ones and one zero. Also in that case we have f(C)=-1. The only case that is left is where N(C) contains one 1 and two zeroes and f(C)=0. In that case we may change the value of a zero in N(C) to one, and f(C) to -1. Repeated application of this type of exchange produces a minus domination function of the same weight and satisfying the claim.

So, we may assume that for  $C \in \mathcal{C}$ , f(C) = -1 and that for each vertex  $u \in U$ ,  $f(u) \in \{0, 1\}$ . Since f(C) = -1, the minus-domination function has at least two plus ones in N(C).

This proves the theorem.

#### 5.1 Minus domination is not FPT

Consider the following problem.

**Instance:** A graph G.

Question: Does G have a minus domination of weight at most 0?

Following Hattingh et al., we call this 'the zero minus-domination problem.' Consider the graph L in Figure 1.

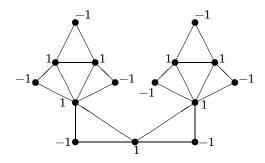


Fig. 1: The graph L. It has  $\gamma^{-}(L) = -1$ .

**Lemma 4.** The graph L has minus-domination weight  $\gamma^-(L) = -1$ . The minus-domination function that achieves this weight is unique; it is the one depicted in Figure 1.

**Theorem 7.** The zero minus-domination problem is NP-complete.

*Proof.* Let H be a graph and let G be the union of H and k disjoint copies of L. Obviously

$$\gamma^{-}(G) = \gamma^{-}(H) + k \cdot \gamma^{-}(L) = \gamma^{-}(H) - k$$
.

It follows that  $\gamma^-(G) \leqslant 0$  if and only if  $\gamma^-(H) \leqslant k$ . By Theorem 6, given a graph H and a positive k it is NP-complete to decide whether  $\gamma^-(H) \leqslant k$ .

**Theorem 8.** The minus-domination problem is not fixed-parameter tractable, unless P = NP.

*Proof.* Assume there exists an algorithm which runs in time  $O(f(k) \cdot n^c)$  and that determines whether a graph G has a minus domination of weight at most k. Then the zero minus-domination problem would be solvable in polynomial time.

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## A Distance-hereditary graphs

Distance-hereditary graphs are the graphs of rankwidth one (see, eg, [12]). They were introduced in 1977 by Howorka as those graphs in which, for every pair of nonadjacent vertices, all the cordless paths between them have the same length. They have a decomposition tree  $(\mathsf{T},\mathsf{f})$  where  $\mathsf{T}$  is a rooted binary tree and  $\mathsf{f}$  is a bijection from the vertices to the leaves of  $\mathsf{T}$ . For each branch, the 'twinset' of that branch is defined as those vertices in the leaves that have neighbors in leaves outside that branch. Each twinset induces a cograph. Each internal node of  $\mathsf{T}$  is labeled as  $\oplus$  or  $\otimes$ . When the label is  $\otimes$  then all the vertices in the twinset of the left branch are adjacent to all the vertices in the twinset of the right branch. When the label is  $\oplus$  there are no edges between vertices mapped to different branches. The twinset of a parent is either empty, or the twinset of one of the two children or the union of the twinsets at the two children.

**Theorem 9.** There exists a polynomial algorithm that computes  $\gamma^-$  for distance-hereditary graphs.

*Proof.* Consider a branch B and let W be the set of vertices that are mapped to leaves of B. Let Q be the twinset of B, that is, the set of vertices in W that have neighbors in  $V \setminus W$ .

An (a, b, c)-function is a function  $f: W \to \{-1, 0, 1\}$  such that f assigns to a vertices of Q the value -1, to b vertices of Q the value 0 and to c vertices of Q the value 1. Furthermore,

for all 
$$x \in W \setminus Q$$
  $f(N[x]) > 0$ . (6)

For an integer t let  $\zeta(t, a, b, c)$  be defined as

$$\zeta(t, a, b, c) = \max |\{x \mid x \in Q \text{ and } f(N[x] \cap W) + t > 0 \text{ and } where f \text{ is an } (a, b, c)\text{-function }\}|.$$
 (7)

It is a nice, easy exercise to see that the arguments given in the proof of Theorem 3 extend to show that these definitions lead to an efficient computation of  $\gamma^-$  for distance-hereditary graphs. For brevity we omit the details.