# Partial immunization of trees

Mitre C. Dourado<sup>1</sup>

Stefan Ehard<sup>2</sup> Lucia D. Penso<sup>2</sup>

Dieter Rautenbach<sup>2</sup>

 $^{1}$ Instituto de Matemática

Universidade Federal do Rio de Janeiro, Rio de Janeiro, Brazil, mitre@dcc.ufrj.br

<sup>2</sup> Institut für Optimierung und Operations Research, Universität Ulm, Ulm, Germany, {stefan.ehard,lucia.penso,dieter.rautenbach}@uni-ulm.de

#### Abstract

For a graph G and an integer-valued function  $\tau$  on its vertex set, a dynamic monopoly is a set of vertices of G such that iteratively adding to it vertices u of G that have at least  $\tau(u)$ neighbors in it eventually yields the vertex set of G. We study the problem of maximizing the minimum order of a dynamic monopoly by increasing the threshold values of individual vertices subject to vertex-dependent lower and upper bounds, and fixing the total increase. We solve this problem efficiently for trees, which extends a result of Khoshkhah and Zaker (On the largest dynamic monopolies of graphs with a given average threshold, Canadian Mathematical Bulletin 58 (2015) 306-316).

Keywords: Dynamic monopoly; vaccination

### 1 Introduction

As a simple model for an infection process within a network [12, 13, 16] one can consider a graph G in which each vertex u is assigned a non-negative integral threshold value  $\tau(u)$  quantifying how many infected neighbors of u are required to spread the infection to u. In this setting, a dynamic monopoly of  $(G, \tau)$  is a set D of vertices such that an infection starting in D spreads to all of G, and the smallest order dyn $(G, \tau)$  of such a dynamic monopoly measures the vulnerability of G for the given threshold values.

Khoshkhah and Zaker [17] consider the maximum of  $dyn(G, \tau)$  over all choices for the function  $\tau$  such that the average threshold is at most some positive real  $\bar{\tau}$ . They show that this maximum equals

$$\max\left\{k: \sum_{i=1}^{k} (d_G(u_i) + 1) \le n(G)\bar{\tau}\right\},$$
(1)

where  $u_1, \ldots, u_{n(G)}$  is a linear ordering of the vertices of G with non-decreasing vertex degrees  $d_G(u_1) \leq \ldots \leq d_G(u_{n(G)})$ . To obtain this simple formula one has to allow  $d_G(u)+1$  as a threshold value for vertices u, a value that makes these vertices completely immune to the infection, and forces every dynamic monopoly to contain them. Requiring  $\tau(u) \leq d_G(u)$  for every vertex u of G leads to a harder problem; Khoshkhah and Zaker [17] show hardness for planar graphs

and describe an efficient algorithm for trees. In the present paper we consider their problem with additional vertex-dependent lower and upper bounds on the threshold values. As our main result, we describe an efficient algorithm for trees based on a completely different approach than the one in [17].

In order to phrase the problem and our results exactly, and to discuss further related work, we introduce some terminology. Let G be a finite, simple, and undirected graph. A threshold function for G is a function from the vertex set V(G) of G to the set of integers. For notational simplicity, we allow negative threshold values. Let  $\tau \in \mathbb{Z}^{V(G)}$  be a threshold function for G. For a set D of vertices of G, the hull  $H_{(G,\tau)}(D)$  of D in  $(G,\tau)$  is the smallest set H of vertices of G such that  $D \subseteq H$ , and  $u \in H$  for every vertex u of G with  $|H \cap N_G(u)| \ge \tau(u)$ . Clearly, the set  $H_{(G,\tau)}(D)$  is obtained by starting with D, and iteratively adding vertices u that have at least  $\tau(u)$  neighbors in the current set as long as possible. With this notation, the set D is a dynamic monopoly of  $(G,\tau)$  if  $H_{(G,\tau)}(D)$  equals the vertex set of G, and  $dyn(G,\tau)$  is the minimum order of such a set. A dynamic monopoly of  $(G,\tau)$  of order  $dyn(G,\tau)$  is minimum. The parameter  $dyn(G,\tau)$  is computationally hard [5,10]; next to general bounds [1,9,15] efficient algorithms are only known for essentially tree-structured instances [2,5,7,8,10].

We can now phrase the problem we consider: For a given graph G, two functions  $\tau, \iota_{\max} \in \mathbb{Z}^{V(G)}$ , and a non-negative integer *budget b*, let  $\operatorname{vacc}(G, \tau, \iota_{\max}, b)$  be defined as

$$\max\left\{\operatorname{dyn}(G,\tau+\iota):\iota\in\mathbb{Z}^{V(G)}, 0\leq\iota\leq\iota_{\max}, \text{ and } \iota(V(G))=b\right\},\tag{2}$$

where inequalities between functions are meant pointwise, and  $\iota(V(G)) = \sum_{u \in V(G)} \iota(u)$ . The function  $\iota$  is the *increment* of the original threshold function  $\tau$ . The final threshold function  $\tau + \iota$  must lie between  $\tau$  and  $\tau + \iota_{\max}$ , which allows to incorporate vertex-dependent lower and upper bounds. Note that no such increment  $\iota$  exists if  $\iota_{\max}(V(G))$  is strictly less than b, in which case  $\operatorname{vacc}(G, \tau, \iota_{\max}, b)$  equals  $\max \emptyset = -\infty$ . Note that we require  $\iota(V(G)) = b$  in (2), which determines the average final threshold as  $(\tau(V(G)) + b)/n(G)$ . Since  $\operatorname{dyn}(G, \rho) \leq \operatorname{dyn}(G, \rho')$ for every two threshold functions  $\rho$  and  $\rho'$  for G with  $\rho \leq \rho'$ , for  $\iota_{\max}(V(G)) \geq b$ , the value in (2) remains the same when replacing  $\iota(V(G)) = b'$  with  $\iota(V(G)) \leq b'$  provided that  $b \leq \iota_{\max}(V(G))$ .

The results of Khoshkhah and Zaker [17] mentioned above can be phrased by saying

- (i) that  $\operatorname{vacc}(G, 0, d_G + 1, n(G)\bar{\tau})$  equals (1) whenever  $n(G)\bar{\tau}$  is a non-negative integer at most  $\sum_{u \in V(G)} (d_G(u) + 1) = 2m(G) + n(G), \text{ where } m(G) \text{ is the size of } G, \text{ and}$
- (ii) that  $vacc(T, 0, d_T, b)$  can be determined efficiently whenever T is a tree.

Our main result is the following.

**Theorem 1.1.** For a given tuple  $(T, \tau, \iota_{\max}, b)$ , where T is a tree of order  $n, \tau, \iota_{\max} \in \mathbb{Z}^{V(G)}$ , and b is an integer with  $0 \le b \le \iota_{\max}(V(T))$ , the value  $\operatorname{vacc}(T, \tau, \iota_{\max}, b)$  as well as an increment  $\iota \in \mathbb{Z}^{V(G)}$  with  $0 \le \iota \le \iota_{\max}$  and  $\iota(V(G)) = b$  such that  $\operatorname{vacc}(T, \tau, \iota_{\max}, b) = \operatorname{dyn}(T, \tau + \iota)$  can be determined in time  $O(n^2(b+1)^2)$ .

While our approach relies on dynamic programming, Khoshkhah and Zaker show (ii) using the following result in combination with a minimum cost flow algorithm. **Theorem 1.2** (Khoshkhah and Zaker [17]). For a given tree T, and a given integer b with  $0 \le b \le 2m(T)$ , there is a matching M of T such that  $vacc(T, 0, d_T, b) = dyn(G, \tau_M)$  and  $\tau_M(V(T)) \le b$ , where

 $au_M: V(T) \to \mathbb{Z}: u \mapsto \begin{cases} d_T(u) &, u \text{ is incident with a vertex in } M, and \\ 0 &, otherwise. \end{cases}$ 

We believe that the threshold function  $\tau_M$  considered in Theorem 1.2 is a good choice in general, and pose the following.

**Conjecture 1.3.** For a given graph G, and a given integer b with  $0 \le b \le 2m(G)$ , there is a matching M of G such that  $\operatorname{vacc}(G, 0, d_G, b) \le 2\operatorname{dyn}(G, \tau_M)$  and  $\tau_M(V(G)) \le b$ , where  $\tau_M$  is as in Theorem 1.2 (with T replaced by G).

As a second result we show Conjecture 1.3 for some regular graphs.

**Theorem 1.4.** Conjecture 1.3 holds if G is r-regular and  $b \ge (2r-1)(r+1)$ .

Before we proceed to the proofs of Theorems 1.1 and 1.4, we mention some further related work. Centeno and Rautenbach [6] establish bounds for the problems considered in [17]. In [14], Ehard and Rautenbach consider the following two variants of (2) for a given triple  $(G, \tau, b)$ , where G is a graph,  $\tau$  is a threshold function for G, and b is a non-negative integer:

$$\max\left\{\operatorname{dyn}(G-X,\tau): X \in \binom{V(G)}{b}\right\} \quad \text{and} \quad \max\left\{\operatorname{dyn}(G,\tau_X): X \in \binom{V(G)}{b}\right\},$$

where

$$\tau_X(u) = \begin{cases} d_G(u) + 1 & \text{, if } u \in X, \\ \tau(u) & \text{, if } u \in V(G) \setminus X, \end{cases},$$

and  $\binom{V(G)}{b}$  denotes the set of all *b*-element subsets of V(G). For both variants, they describe efficient algorithms for trees. In [3] Bhawalkar et al. study so-called anchored *k*-cores. For a given graph *G*, and a positive integer *k*, the *k*-core of *G* is the largest induced subgraph of *G* of minimum degree at least *k*. It is easy to see that the vertex set of the *k*-core of *G* equals  $V(G) \setminus H_{(G,\tau)}(\emptyset)$  for the special threshold function  $\tau = d_G - k + 1$ . Now, the anchored *k*-core problem [3] is to determine

$$\max\left\{ \left| V(G) \setminus H_{(G,\tau_X)}(\emptyset) \right| : X \in \binom{V(G)}{b} \right\},\tag{3}$$

for a given graph G and non-negative integer b. Bhawalkar et al. show that (3) is hard to approximate in general, but can be determined efficiently for k = 2, and for graphs of bounded treewidth. Vaccination problems in random settings were studied in [4, 11, 16].

### 2 Proofs of Theorem 1.1 and Theorem 1.4

Throughout this section, let T be a tree rooted in some vertex r, and let  $\tau, \iota_{\max} \in \mathbb{Z}^{V(T)}$  be two functions. For a vertex u of T, and a function  $\rho \in \mathbb{Z}^{V(T)}$ , let  $V_u$  be the subset of V(T) containing

u and its descendants, let  $T_u$  be the subtree of T induced by  $V_u$ , and let  $\rho^{\to u} \in \mathbb{Z}^{V(T)}$  be the function with

$$\rho^{\to u}(v) = \begin{cases} \rho(v) &, \text{ if } v \in V(T) \setminus \{u\}, \text{ and} \\ \rho(v) - 1 &, \text{ if } v = u. \end{cases}$$

Below we consider threshold functions of the form  $\rho|_{V_u} + \rho'|_{V_u}$  for the subtrees  $T_u$ , where  $\rho$  and  $\rho'$  are defined on sets containing  $V_u$ . For notational simplicity, we omit the restriction to  $V_u$  and write ' $\rho + \rho'$ ' instead of ' $\rho|_{V_u} + \rho'|_{V_u}$ ' in these cases. For an integer k and a non-negative integer b, let [k] be the set of positive integers at most k, and let

$$\mathcal{P}_k(b) = \left\{ (b_1, \dots, b_k) \in \mathbb{N}_0^k : b_1 + \dots + b_k = b \right\}$$

be the set of ordered partitions of b into k non-negative integers.

Our approach to show Theorem 1.1 is similar as in [14] and relies on recursive expressions for the following two quantities: For a vertex u of T and a non-negative integer b, let

- $x_0(u,b)$  be the maximum of  $dyn(T_u, \tau + \iota)$  over all  $\iota \in \mathbb{Z}^{V_u}$  with  $0 \le \iota(v) \le \iota_{\max}(v)$  for every  $v \in V_u$ , and  $\iota(V_u) = b$ , and
- $x_1(u,b)$  be the maximum of dyn  $(T_u, (\tau + \iota)^{\to u})$  over all  $\iota \in \mathbb{Z}^{V_u}$  with  $0 \le \iota(v) \le \iota_{\max}(v)$  for every  $v \in V_u$ , and  $\iota(V_u) = b$ .

The increment  $\iota$  captures the local increases of the thresholds within  $V_u$ . The value  $x_1(u, b)$  corresponds to a situation, where the infection reaches the parent of u before it reaches u, that is, the index 0 or 1 indicates the amount of help that u receives from outside of  $V_u$ .

Note that  $x_j(u,b) = -\infty$  if and only if  $b > \iota_{\max}(V_u)$  for both j in  $\{0,1\}$ . If  $b \le \iota_{\max}(V_u)$ , then let  $\iota_0(u,b), \iota_1(u,b) \in \mathbb{Z}^{V_u}$  with  $0 \le \iota_j(u,b) \le \iota_{\max}$ , and  $\iota_j(u,b)(V_u) = b$  for both  $j \in \{0,1\}$ , be such that

$$x_0(u,b) = \operatorname{dyn}\left(T_u, \tau + \iota_0(u,b)\right) \text{ and}$$
  
$$x_1(u,b) = \operatorname{dyn}\left(T_u, \left(\tau + \iota_1(u,b)\right)^{\to u}\right),$$

where, if possible, let  $\iota_0(u, b) = \iota_1(u, b)$ . As we show in Corollary 2.4 below,  $\iota_0(u, b)$  always equals  $\iota_1(u, b)$ , which is a key fact for our approach.

**Lemma 2.1.**  $x_0(u,b) \ge x_1(u,b)$ , and if  $x_0(u,b) = x_1(u,b)$ , then  $\iota_0(u,b) = \iota_1(u,b)$ .

Proof. If  $x_1(u, b) = -\infty$ , then the statement is trivial. Hence, we may assume that  $x_1(u, b) > -\infty$ , which implies that the function  $\iota_1(u, b)$  is defined. Let D be a minimum dynamic monopoly of  $(T_u, \tau + \iota_1(u, b))$ . By the definition of  $x_0(u, b)$ , we have  $x_0(u, b) \ge |D|$ . Since D is a dynamic monopoly of  $(T_u, (\tau + \iota_1(u, b))^{\rightarrow u})$ , we obtain  $x_0(u, b) \ge |D| \ge \operatorname{dyn}(T_u, (\tau + \iota_1(u, b))^{\rightarrow u}) = x_1(u, b)$ . Furthermore, if  $x_0(u, b) = x_1(u, b)$ , then  $x_0(u, b) = |D| = \operatorname{dyn}(T_u, \tau + \iota_1(u, b))$ , which implies  $\iota_0(u, b) = \iota_1(u, b)$ .

**Lemma 2.2.** If u is a leaf of T, and b is an integer with  $0 \le b \le \iota_{\max}(u)$ , then, for  $j \in \{0, 1\}$ ,

$$x_j(u,b) = \begin{cases} 0 & , if \tau(u) + b - j \le 0, \\ 1 & , otherwise, and \\ \iota_j(u,b)(u) = b. \end{cases}$$

*Proof.* These equalities follow immediately from the definitions.

**Lemma 2.3.** Let u be a vertex of T that is not a leaf, and let b be a non-negative integer. If  $v_1, \ldots, v_k$  are the children of u, and  $\iota_0(v_i, b_i) = \iota_1(v_i, b_i)$  for every  $i \in [k]$  and every integer  $b_i$  with  $0 \le b_i \le \iota_{\max}(V_{v_i})$ , then, for  $j \in \{0, 1\}$ ,

$$x_j(u,b) = z_j(u,b), and \tag{4}$$

$$\iota_0(u,b) = \iota_1(u,b), \text{ if } b \le \iota_{\max}(V_u), \tag{5}$$

where  $z_i(u, b)$  is defined as

$$\max\left\{\delta_j(b_u, b_1, \dots, b_k) + \sum_{i=1}^k x_1(v_i, b_i) : (b_u, b_1, \dots, b_k) \in \mathcal{P}_{k+1}(b) \text{ with } b_u \le \iota_{\max}(u)\right\},\$$

and, for  $(b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b)$  with  $b_u \leq \iota_{\max}(u)$ ,

$$\delta_j(b_u, b_1, \dots, b_k) := \begin{cases} 0 & , if \left| \left\{ i \in [k] : x_0(v_i, b_i) = x_1(v_i, b_i) \right\} \right| \ge \tau(u) + b_u - j, and \\ 1 & , otherwise. \end{cases}$$

*Proof.* By symmetry, it suffices to consider the case j = 0.

First, suppose that  $b > \iota_{\max}(V_u)$ . If  $(b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b)$  with  $b_u \leq \iota_{\max}(u)$ , then  $b_i > \iota_{\max}(V_{v_i})$  for some  $i \in [k]$ , which implies  $z_0(u, b) = -\infty = x_0(u, b)$ .

Now, let  $b \leq n(T_u)$ , which implies  $x_0(u, b) > -\infty$ . The following two claims complete the proof of (4).

Claim 1.  $x_0(u, b) \ge z_0(u, b)$ .

Proof of Claim 1. It suffices to show that  $x_0(u,b) \geq \delta_0(b_u,b_1,\ldots,b_k) + \sum_{i=1}^k x_1(v_i,b_i)$  for every choice of  $(b_u,b_1,\ldots,b_k)$  in  $\mathcal{P}_{k+1}(b)$  with  $b_u \leq \iota_{\max}(u)$  and  $b_i \leq \iota_{\max}(V_{v_i})$  for every  $i \in [k]$ . Let  $(b_u,b_1,\ldots,b_k)$  be one such an element. Let  $\iota_u \in \mathbb{Z}^{V_u}$  be defined as

$$\iota_u(v) = \begin{cases} b_u & \text{, if } v = u \text{, and} \\ 0 & \text{, otherwise,} \end{cases}$$
(6)

and let  $\iota = \iota_u + \sum_{i=1}^k \iota_1(v_i, b_i)$ , where  $\iota_1(v_i, b_i)(u)$  is set to 0 for every  $i \in [k]$ . Since  $\iota(V_u) = b$  and  $0 \le \iota \le \iota_{\max}$ , we have  $x_0(u, b) \ge \operatorname{dyn}(T_u, \tau + \iota)$ .

Let D be a minimum dynamic monopoly of  $(T_u, \tau + \iota)$ , that is,  $|D| \leq x_0(u, b)$ . For each  $i \in [k]$ , it follows that the set  $D_i = D \cap V_{v_i}$  is a dynamic monopoly of  $(T_{v_i}, (\tau + \iota)^{\rightarrow v_i})$ . Since,

restricted to  $V_{v_i}$ , the two functions  $(\tau + \iota)^{\rightarrow v_i}$  and  $(\tau + \iota_1(v_i, b_i))^{\rightarrow v_i}$  coincide, we obtain

$$|D_i| \ge \operatorname{dyn}\left(T_{v_i}, \left(\tau + \iota_1(v_i, b_i)\right)^{\to v_i}\right) \ge x_1(v_i, b_i).$$

If  $\delta_0(b_u, b_1, \dots, b_k) = 0$ , then  $|D| \ge \sum_{i=1}^k |D_i| \ge \delta_0(b_u, b_1, \dots, b_k) + \sum_{i=1}^k x_1(v_i, b_i)$ . Similarly, if  $u \in D$ , then  $|D| = 1 + \sum_{i=1}^k |D_i| \ge \delta_0(b_u, b_1, \dots, b_k) + \sum_{i=1}^k x_1(v_i, b_i)$ . Therefore, we may assume that  $\delta_0(b_u, b_1, \dots, b_k) = 1$  and that  $u \notin D$ . This implies that there is some  $\ell \in [k]$  with  $x_0(v_\ell, b_\ell) > x_1(v_\ell, b_\ell)$  such that  $D_\ell = D \cap V_{v_\ell}$  is a dynamic monopoly of  $(T_{v_\ell}, \tau + \iota)$ . Since, by assumption,  $\iota_0(v_\ell, b_\ell) = \iota_1(v_\ell, b_\ell)$ , we obtain that, restricted to  $V_{v_\ell}$ , the two functions  $\tau + \iota$  and  $\tau + \iota_0(v_\ell, b_\ell)$  coincide, which implies  $|D_\ell| \ge dyn(T_{v_\ell}, \tau + \iota_0(v_\ell, b_\ell)) = x_0(v_\ell, b_\ell) \ge 1 + x_1(v_\ell, b_\ell)$ . Therefore, also in this case,  $|D| = |D_\ell| + \sum_{i \in [k] \setminus \{\ell\}} |D_i| \ge \delta_0(b_u, b_1, \dots, b_k) + \sum_{i=1}^k x_1(v_i, b_i)$ .

### Claim 2. $x_0(u, b) \le z_0(u, b)$ .

Proof of Claim 2. Let  $\iota = \iota_0(u, b)$ , that is,  $x_0(u, b) = \operatorname{dyn}(T_u, \tau + \iota)$ . Let  $b_i = \iota(V_{v_i})$  for every  $i \in [k]$ , and let  $b_u = b - \sum_{i=1}^k b_i$ . Clearly,  $(b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b)$  and  $b_u \leq \iota_{\max}(u)$ . Let  $D_i$  be a minimum dynamic monopoly of  $(T_{v_i}, (\tau + \iota)^{\rightarrow v_i})$  for every  $i \in [k]$ . By the definition of  $x_1(v_i, b_i)$ , we obtain  $|D_i| \leq x_1(v_i, b_i)$ . Let  $D = \{u\} \cup \bigcup_{i=1}^k D_i$ . The set D is a dynamic monopoly of  $(T_u, \tau + \iota)$ , which implies  $x_0(u, b) \leq |D|$ .

If  $\delta_0(b_u, b_1, ..., b_k) = 1$ , then

$$x_0(u,b) \le |D| = 1 + \sum_{i=1}^k |D_i| \le \delta_0(b_u, b_1, \dots, b_k) + \sum_{i=1}^k x_1(v_i, b_i) \le z_0(u, b)$$

Therefore, we may assume that  $\delta_0(b_u, b_1, \ldots, b_k) = 0$ . By symmetry, we may assume that  $x_0(v_i, b_i) = x_1(v_i, b_i)$  for every  $i \in [\tau(u) + b_u]$ . Let  $D'_i$  be a minimum dynamic monopoly of  $(T_{v_i}, \tau + \iota)$  for every  $i \in [\tau(u) + b_u]$ . By the definition of  $x_0(v_i, b_i)$ , we obtain  $|D'_i| \leq x_0(v_i, b_i) = x_1(v_i, b_i)$ . Let  $D' = \bigcup_{i \in [\tau(u) + b_u]} D'_i \cup \bigcup_{i \in [k] \setminus [\tau(u) + b_u]} D_i$ . The set D' is a dynamic monopoly of  $(T_u, \tau + \iota)$ . This implies

$$x_0(u,b) \le |D'| = \sum_{i \in [\tau(u)+b_u]} |D'_i| + \sum_{i \in [k] \setminus [\tau(u)+b_u]} |D_i| \le \sum_{i \in [k]} x_1(v_i,b_i) \le z_0(u,b),$$

which completes the proof of the claim.

It remains to show (5). If  $x_0(u,b) = x_1(u,b)$ , then (5) follows from Lemma 2.1. Hence, we may assume that  $x_0(u,b) > x_1(u,b)$ . Since, by definition,

$$\delta_1(b_u, b_1, \dots, b_k) \le \delta_0(b_u, b_1, \dots, b_k) \le \delta_1(b_u, b_1, \dots, b_k) + 1$$

for every  $(b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b)$  with  $b_u \leq \iota_{\max}(u)$ , we obtain  $z_1(u, b) \leq z_0(u, b) \leq z_1(u, b) + 1$ .

Together with (4), the inequality  $x_0(u,b) > x_1(u,b)$  implies that

$$x_0(u,b) = z_0(u,b) > z_1(u,b) = x_1(u,b)$$
 and  
 $z_1(u,b) = z_0(u,b) - 1.$ 

Let  $(b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b)$  with  $b_u \leq \iota_{\max}(u)$  be such that

$$z_0(u,b) = \delta_0(b_u, b_1, \dots, b_k) + \sum_{i=1}^k x_1(v_i, b_i).$$

We obtain

$$z_{1}(u,b) \geq \delta_{1}(b_{u},b_{1},\ldots,b_{k}) + \sum_{i=1}^{k} x_{1}(v_{i},b_{i})$$
  
$$\geq \delta_{0}(b_{u},b_{1},\ldots,b_{k}) - 1 + \sum_{i=1}^{k} x_{1}(v_{i},b_{i})$$
  
$$= z_{0}(u,b) - 1$$
  
$$= z_{1}(u,b),$$

which implies  $z_1(u, b) = \delta_1(b_u, b_1, \dots, b_k) + \sum_{i=1}^k x_1(v_i, b_i)$ , that is, the same choice of  $(b_u, b_1, \dots, b_k)$ in  $\mathcal{P}_{k+1}(b)$  with  $b_u \leq \iota_{\max}(u)$  maximizes the terms defining  $z_0(u, b)$  and  $z_1(u, b)$ .

Since  $z_0(u,b) > z_1(u,b)$ , we obtain  $\delta_1(b_u, b_1, \ldots, b_k) = 0$  and  $\delta_0(b_u, b_1, \ldots, b_k) = 1$ , which, by the definition of  $\delta_j$ , implies that there are exactly  $\tau(u) + b_u - 1$  indices i in [k] with  $x_0(v_i, b_i) = x_1(v_i, b_i)$ . By symmetry, we may assume that  $x_0(v_i, b_i) = x_1(v_i, b_i)$  for  $i \in [\tau(u) + b_u - 1]$  and  $x_0(v_i, b_i) > x_1(v_i, b_i)$  for  $i \in [k] \setminus [\tau(u) + b_u - 1]$ .

Let  $\iota = \iota_u + \sum_{i=1}^k \iota_0(v_i, b_i)$ , where  $\iota_0(v_i, b_i)(u)$  is set to 0 for every  $i \in [k]$  and  $\iota_u$  is as in (6). Note

that, by assumption, we have  $\iota = \iota_u + \sum_{i=1}^k \iota_1(v_i, b_i)$ . Let D be a minimum dynamic monopoly of  $(T_u, \tau + \iota)$ . By the definition of  $x_0(u, b)$ , we have  $|D| \leq x_0(u, b)$ . Let  $D_i = D \cap V_{v_i}$  for every  $i \in [k]$ . Since  $D_i$  is a dynamic monopoly of  $(T_{v_i}, (\tau + \iota)^{\rightarrow v_i})$  for every  $i \in [k]$ , we obtain  $|D_i| \geq x_1(v_i, b_i)$ . Note that

- either  $u \in D$ ,
- or  $u \notin D$  and there is some index  $\ell \in [k] \setminus [\tau(u) + b_u 1]$  such that  $D_\ell = D \cap V_{v_\ell}$  is a dynamic monopoly of  $(T_{v_\ell}, \tau + \iota)$ .

In the first case, we obtain

$$z_0(u,b) = x_0(u,b) \ge |D| = 1 + \sum_{i=1}^k |D_i| \ge 1 + \sum_{i=1}^k x_1(v_i,b_i) = z_0(u,b),$$

and, in the second case, we obtain  $|D_{\ell}| \ge x_0(v_{\ell}, b_{\ell}) \ge x_1(v_{\ell}, b_{\ell}) + 1$ , and, hence,

$$z_0(u,b) = x_0(u,b) \ge |D| = |D_\ell| + \sum_{i \in [k] \setminus \{\ell\}} |D_i| \ge 1 + \sum_{i=1}^k x_1(v_i,b_i) = z_0(u,b).$$

In both cases we obtain  $|D| = x_0(u, b)$ , which implies that  $\iota_0(u, b)$  may be chosen equal to  $\iota$ .

Now, let  $D^-$  be a minimum dynamic monopoly of  $(T_u, (\tau + \iota)^{\to u})$ . By the definition of  $x_1(u, b)$ , we have  $|D^-| \leq x_1(u, b)$ . Let  $D_i^- = D^- \cap V_{v_i}$  for every  $i \in [k]$ . Since  $D_i^-$  is a dynamic monopoly of  $(T_{v_i}, (\tau + \iota)^{\to v_i})$  for every  $i \in [k]$ , we obtain  $|D_i^-| \geq x_1(v_i, b_i)$ . Now,

$$z_1(u,b) = x_1(u,b) \ge |D^-| \ge \sum_{i=1}^k x_1(v_i,b_i) = z_1(u,b),$$

which implies that  $|D^-| = x_1(u, b)$ , and that  $\iota_1(u, b)$  may be chosen equal to  $\iota$ . Altogether, the two functions  $\iota_0(u, b)$  and  $\iota_1(u, b)$  may be chosen equal, which implies (5).

Applying induction using Lemma 2.2 and Lemma 2.3, we obtain the following.

**Corollary 2.4.**  $\iota_0(u,b) = \iota_1(u,b)$  for every vertex u of T, and every integer b with  $0 \le b \le \iota_{\max}(V_u)$ .

Apart from the specific values of  $x_0(u, b)$  and  $x_1(u, b)$ , the arguments in the proof of Lemma 2.3 also yield feasible recursive choices for  $\iota_0(u, b)$ . In fact, if

$$x_0(u,b) = \delta_0(b_u, b_1, \dots, b_k) + \sum_{i=1}^k x_1(v_i, b_i) > -\infty$$

for  $(b_u, b_1, \ldots, b_k) \in \mathcal{P}_{k+1}(b)$  with  $b_u \leq \iota_{\max}(u)$ , and  $\iota_u$  is as in (6), then  $\iota_u + \sum_{i=1}^k \iota_0(v_i, b_i)$  is a feasible choice for  $\iota_0(u, b)$ .

Our next lemma explains how to efficiently compute the expressions in Lemma 2.3.

**Lemma 2.5.** Let u be a vertex of T that is not a leaf, let b be an integer with  $0 \le b \le \iota_{\max}(V_u)$ , and let  $v_1, \ldots, v_k$  be the children of u. If the values  $x_1(v_i, b_i)$  are given for every  $i \in [k]$  and every integer  $b_i$  with  $0 \le b_i \le \iota_{\max}(V_{v_i})$ , then  $x_0(u, b)$  and  $x_1(u, b)$  can be computed in time  $O(k^2(b+1)^2)$ .

*Proof.* By symmetry, it suffices to explain how to compute  $z_0(u, b)$ .

For  $p \in \{0\} \cup [k]$ , an integer  $p_{=}$ , an integer  $b' \in \{0\} \cup [b]$ , and  $b_u \in \{0\} \cup [\min\{\iota_{\max}(u), b'\}]$ , let  $M(p, p_{=}, b', b_u)$  be defined as the maximum of the expression  $\sum_{i=1}^{p} x_1(v_i, b_i)$  over all  $(b_1, \ldots, b_p) \in \mathcal{P}_p(b'-b_u)$  such that  $p_{=}$  equals  $\left|\left\{i \in [p] : x_0(v_i, b_i) = x_1(v_i, b_i)\right\}\right|$ . Clearly,  $M(p, p_{=}, b', b_u) = -\infty$  if  $p < p_{=}$  or  $p_{=} < 0$  or  $b' - b_u > \sum_{i=1}^{p} \iota_{\max}(V_{v_i})$ , and

$$M(0,0,b',b_u) = \begin{cases} 0 & , \text{ if } b' = b_u, \text{ and} \\ -\infty & , \text{ otherwise.} \end{cases}$$

For  $p \in [k]$ , the value of  $M(p, p_{=}, b', b_u)$  is the maximum of the following two values:

- The maximum of  $M(p-1, p_{=}-1, b_{\leq p-1}, b_u) + x_1(v_p, b_p)$  over all  $(b_{\leq p-1}, b_p) \in \mathcal{P}_2(b'-b_u)$ with  $x_0(v_p, b_p) = x_1(v_p, b_p)$ , and
- the maximum of  $M(p-1, p_{=}, b_{\leq p-1}, b_u) + x_1(v_p, b_p)$  over all  $(b_{\leq p-1}, b_p) \in \mathcal{P}_2(b'-b_u)$  with  $x_0(v_p, b_p) > x_1(v_p, b_p)$ ,

which implies that  $M(p, p_{=}, b', b_{u})$  can be determined in O(b' + 1) time given the values

$$M(p-1, p_{=}, b_{\leq p-1}, b_u), M(p-1, p_{=}-1, b_{\leq p-1}, b_u), x_0(v_p, b_p), \text{ and } x_1(v_p, b_p).$$

Altogether, the values  $M(k, p_{=}, b, b_u)$  for all  $p_{=} \in \{0\} \cup [k]$  can be determined in time  $O(k^2(b+1))$ .

For  $b_u \in \{0\} \cup [\min\{\iota_{\max}(u), b\}]$ , let  $m(b_u)$  be the maximum of the two expressions

$$1 + \max\left\{M(k, p_{=}, b, b_{u}) : p_{=} \in \{0\} \cup [\tau(u) - b_{u} - 1]\right\}$$

and

$$\max\Big\{M(k, p_{=}, b, b_{u}) : p_{=} \in [k] \setminus [\tau(u) - b_{u} - 1]\Big\}.$$

Now, by the definition of  $\delta_0(b_u, b_1, \dots, b_k)$ , the value of  $z_0(u, b)$  equals  $\max \{m(b_u) : b_u \in \{0\} \cup [\min\{\iota_{\max}(u), b\}]\}$ . Hence,  $z_0(u, b)$  can be computed in time  $O(k^2(b+1)^2)$ .

We proceed to the proof of our first theorem.

Proof of Theorem 1.1. Given  $(T, \tau, \iota_{\max}, b)$ , Lemma 2.2 to Lemma 2.5 imply that the values of  $x_0(u, b')$  and of  $x_1(u, b')$  for all  $u \in V(T)$  and all  $b' \in \{0\} \cup [b]$  can be determined in time

$$O\left(\sum_{u\in V(T)} d_T(u)^2(b+1)^2\right).$$

It is a simple folklore exercise that  $\sum_{u \in V(T)} d_T(u)^2 \leq n^2 - n$  for every tree T of order n, which implies the statement about the running time. Since  $\operatorname{vacc}(T, \tau, \iota_{\max}, b) = x_0(r, b)$ , the statement about the value of  $\operatorname{vacc}(T, \tau, \iota_{\max}, b)$  follows. The statement about the increment  $\iota$  follows easily from the remark after Corollary 2.4 concerning the function  $\iota_0(u, b)$ , and the proof of Lemma 2.5, where, next to the values  $M(p, p_{=}, b', b_u)$ , one may also memorize suitable increments.

We conclude with the proof of our second theorem.

Proof of Theorem 1.4. Let G be an r-regular graph of order n, and let b be an integer with  $(2r-1)(r+1) \le b \le rn = 2m(G)$ .

Let  $\iota \in \mathbb{Z}^{V(G)}$  with  $0 \leq \iota \leq d_G$  and  $\iota(V(G)) = b$  be such that  $\operatorname{vacc}(G, 0, d_G, b) = \operatorname{dyn}(G, \iota)$ . By a result of Ackerman et al. [1],

$$\operatorname{vacc}(G, 0, d_G, b) = \operatorname{dyn}(G, \iota) \le \sum_{u \in V(G)} \frac{\iota(u)}{d_G(u) + 1} = \frac{\iota(V(G))}{r + 1} = \frac{b}{r + 1}.$$

First, suppose that the matching number  $\nu$  of G satisfies  $2r\nu > b$ . In this case, G has a matching M with  $\tau_M(V(G)) = 2r|M| \leq b$  and  $2r(|M|+1) \geq b+1$ , where  $\tau_M$  is as in the statement. We obtain  $2dyn(G, \tau_M) \geq 2|M| \geq 2\left(\frac{b+1}{2r}-1\right) \geq \frac{b}{r+1} \geq vacc(G, 0, d_G, b)$ . Next, suppose that  $2r\nu \leq b$ . If M is a maximum matching and D is a minimum vertex cover, then  $|D| \leq 2|M|$ . Since D is a dynamic monopoly of  $(G, d_G)$ , we obtain  $2dyn(G, \tau_M) \geq 2|M| \geq |D| \geq dyn(G, d_G) \geq vacc(G, 0, d_G, b)$ , that is,  $2dyn(G, \tau_M) \geq vacc(G, 0, d_G, b)$  holds in both cases.  $\Box$ 

## References

- [1] E. Ackerman, O. Ben-Zwi, G. Wolfovitz, Combinatorial model and bounds for target set selection, Theoretical Computer Science 411 (2010) 4017-4022.
- [2] O. Ben-Zwi, D. Hermelin, D. Lokshtanov, I. Newman, Treewidth governs the complexity of target set selection, Discrete Optimization 8 (2011) 87-96
- [3] K. Bhawalkar, J. Kleinberg, K. Lewi, T. Roughgarden, A. Sharma, Preventing unraveling in social networks: the anchored k-core problem, SIAM Journal on Discrete Mathematics 29 (2015) 1452-1475.
- [4] T. Britton, S. Janson, A. Martin-Löf, Graphs with specified degree distributions, simple epidemics, and local vaccination strategies, Advances in Applied Probability 39 (2007) 922-948.
- [5] C.C. Centeno, M.C. Dourado, L.D. Penso, D. Rautenbach, J.L. Szwarcfiter, Irreversible conversion of graphs, Theoretical Computer Science 412 (2011) 3693-3700.
- [6] C.C. Centeno, D. Rautenbach, Remarks on dynamic monopolies with given average thresholds, Discussiones Mathematicae Graph Theory 35 (2015) 133-140.
- [7] C.-Y. Chiang, L.-H. Huang, B.-J. Li. J. Wu, H.-G. Yeh, Some results on the target set selection problem, Journal of Combinatorial Optimization 25 (2013) 702-715.
- [8] F. Cicalese, G. Cordasco, L. Gargano, M. Milanič, J. Peters, U. Vaccaro, Spread of influence in weighted networks under time and budget constraints, Theoretical Computer Science 586 (2015) 40-58.
- [9] C.-L. Chang, Y.-D. Lyuu, Triggering cascades on strongly connected directed graphs, Theoretical Computer Science 593 (2015) 62-69.
- [10] N. Chen, On the approximability of influence in social networks, SIAM Journal on Discrete Mathematics 23 (2009) 1400-1415.
- [11] M. Deijfen, Epidemics and vaccination on weighted graphs, Mathematical Biosciences 232 (2011) 57-65.
- [12] P. Domingos, M. Richardson, Mining the network value of customers, Proceedings of the 7th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining, (2001) 57-66.

- [13] P.A. Dreyer Jr., F.S. Roberts, Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion, Discrete Applied Mathematics 157 (2009) 1615-1627.
- [14] S. Ehard, D. Rautenbach, Vaccinate your trees!, arXiv:1801.08705.
- [15] M. Gentner, D. Rautenbach, Dynamic monopolies for degree proportional thresholds in connected graphs of girth at least five and trees, Theoretical Computer Science 667 (2017) 93-100.
- [16] D. Kempe, J. Kleinberg, E. Tardos, Maximizing the spread of influence through a social network, Theory of Computing 11 (2015) 105-147.
- [17] K. Khoshkhah, M. Zaker, On the largest dynamic monopolies of graphs with a given average threshold, Canadian Mathematical Bulletin 58 (2015) 306-316.