# Partial immunization of trees 

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#### Abstract

For a graph $G$ and an integer-valued function $\tau$ on its vertex set, a dynamic monopoly is a set of vertices of $G$ such that iteratively adding to it vertices $u$ of $G$ that have at least $\tau(u)$ neighbors in it eventually yields the vertex set of $G$. We study the problem of maximizing the minimum order of a dynamic monopoly by increasing the threshold values of individual vertices subject to vertex-dependent lower and upper bounds, and fixing the total increase. We solve this problem efficiently for trees, which extends a result of Khoshkhah and Zaker (On the largest dynamic monopolies of graphs with a given average threshold, Canadian Mathematical Bulletin 58 (2015) 306-316).


Keywords: Dynamic monopoly; vaccination

## 1 Introduction

As a simple model for an infection process within a network [12, 13, 16] one can consider a graph $G$ in which each vertex $u$ is assigned a non-negative integral threshold value $\tau(u)$ quantifying how many infected neighbors of $u$ are required to spread the infection to $u$. In this setting, a dynamic monopoly of $(G, \tau)$ is a set $D$ of vertices such that an infection starting in $D$ spreads to all of $G$, and the smallest order $\operatorname{dyn}(G, \tau)$ of such a dynamic monopoly measures the vulnerability of $G$ for the given threshold values.

Khoshkhah and Zaker [17] consider the maximum of $\operatorname{dyn}(G, \tau)$ over all choices for the function $\tau$ such that the average threshold is at most some positive real $\bar{\tau}$. They show that this maximum equals

$$
\begin{equation*}
\max \left\{k: \sum_{i=1}^{k}\left(d_{G}\left(u_{i}\right)+1\right) \leq n(G) \bar{\tau}\right\} \tag{1}
\end{equation*}
$$

where $u_{1}, \ldots, u_{n(G)}$ is a linear ordering of the vertices of $G$ with non-decreasing vertex degrees $d_{G}\left(u_{1}\right) \leq \ldots \leq d_{G}\left(u_{n(G)}\right)$. To obtain this simple formula one has to allow $d_{G}(u)+1$ as a threshold value for vertices $u$, a value that makes these vertices completely immune to the infection, and forces every dynamic monopoly to contain them. Requiring $\tau(u) \leq d_{G}(u)$ for every vertex $u$ of $G$ leads to a harder problem; Khoshkhah and Zaker [17] show hardness for planar graphs
and describe an efficient algorithm for trees. In the present paper we consider their problem with additional vertex-dependent lower and upper bounds on the threshold values. As our main result, we describe an efficient algorithm for trees based on a completely different approach than the one in (17.

In order to phrase the problem and our results exactly, and to discuss further related work, we introduce some terminology. Let $G$ be a finite, simple, and undirected graph. A threshold function for $G$ is a function from the vertex set $V(G)$ of $G$ to the set of integers. For notational simplicity, we allow negative threshold values. Let $\tau \in \mathbb{Z}^{V(G)}$ be a threshold function for $G$. For a set $D$ of vertices of $G$, the hull $H_{(G, \tau)}(D)$ of $D$ in $(G, \tau)$ is the smallest set $H$ of vertices of $G$ such that $D \subseteq H$, and $u \in H$ for every vertex $u$ of $G$ with $\left|H \cap N_{G}(u)\right| \geq \tau(u)$. Clearly, the set $H_{(G, \tau)}(D)$ is obtained by starting with $D$, and iteratively adding vertices $u$ that have at least $\tau(u)$ neighbors in the current set as long as possible. With this notation, the set $D$ is a dynamic monopoly of $(G, \tau)$ if $H_{(G, \tau)}(D)$ equals the vertex set of $G$, and $\operatorname{dyn}(G, \tau)$ is the minimum order of such a set. A dynamic monopoly of $(G, \tau)$ of order $\operatorname{dyn}(G, \tau)$ is minimum. The parameter $\operatorname{dyn}(G, \tau)$ is computationally hard [5, 10]; next to general bounds [1, 9, 15] efficient algorithms are only known for essentially tree-structured instances [2,5, 7, 8, 10 .

We can now phrase the problem we consider: For a given graph $G$, two functions $\tau, \iota_{\max } \in$ $\mathbb{Z}^{V(G)}$, and a non-negative integer budget $b$, let $\operatorname{vacc}\left(G, \tau, \iota_{\max }, b\right)$ be defined as

$$
\begin{equation*}
\max \left\{\operatorname{dyn}(G, \tau+\iota): \iota \in \mathbb{Z}^{V(G)}, 0 \leq \iota \leq \iota_{\max }, \text { and } \iota(V(G))=b\right\}, \tag{2}
\end{equation*}
$$

where inequalities between functions are meant pointwise, and $\iota(V(G))=\sum_{u \in V(G)} \iota(u)$. The function $\iota$ is the increment of the original threshold function $\tau$. The final threshold function $\tau+\iota$ must lie between $\tau$ and $\tau+\iota_{\max }$, which allows to incorporate vertex-dependent lower and upper bounds. Note that no such increment $\iota$ exists if $\iota_{\max }(V(G))$ is strictly less than $b$, in which $\operatorname{case} \operatorname{vacc}\left(G, \tau, \iota_{\max }, b\right)$ equals $\max \emptyset=-\infty$. Note that we require $\iota(V(G))=b$ in (2), which determines the average final threshold as $(\tau(V(G))+b) / n(G)$. Since $\operatorname{dyn}(G, \rho) \leq \operatorname{dyn}\left(G, \rho^{\prime}\right)$ for every two threshold functions $\rho$ and $\rho^{\prime}$ for $G$ with $\rho \leq \rho^{\prime}$, for $\iota_{\max }(V(G)) \geq b$, the value in (2) remains the same when replacing ' $\iota(V(G))=b$ ' with ' $\iota(V(G)) \leq b$ ' provided that $b \leq$ $\iota_{\text {max }}(V(G))$.

The results of Khoshkhah and Zaker [17] mentioned above can be phrased by saying
(i) that $\operatorname{vacc}\left(G, 0, d_{G}+1, n(G) \bar{\tau}\right)$ equals (1) whenever $n(G) \bar{\tau}$ is a non-negative integer at most

$$
\sum_{u \in V(G)}\left(d_{G}(u)+1\right)=2 m(G)+n(G), \text { where } m(G) \text { is the size of } G, \text { and }
$$

(ii) that $\operatorname{vacc}\left(T, 0, d_{T}, b\right)$ can be determined efficiently whenever $T$ is a tree.

Our main result is the following.
Theorem 1.1. For a given tuple $\left(T, \tau, \iota_{\max }, b\right)$, where $T$ is a tree of order $n, \tau, \iota_{\max } \in \mathbb{Z}^{V(G)}$, and $b$ is an integer with $0 \leq b \leq \iota_{\max }(V(T))$, the value $\operatorname{vacc}\left(T, \tau, \iota_{\max }, b\right)$ as well as an increment $\iota \in \mathbb{Z}^{V(G)}$ with $0 \leq \iota \leq \iota_{\text {max }}$ and $\iota(V(G))=b$ such that $\operatorname{vacc}\left(T, \tau, \iota_{\max }, b\right)=\operatorname{dyn}(T, \tau+\iota)$ can be determined in time $O\left(n^{2}(b+1)^{2}\right)$.

While our approach relies on dynamic programming, Khoshkhah and Zaker show (ii) using the following result in combination with a minimum cost flow algorithm.

Theorem 1.2 (Khoshkhah and Zaker [17]). For a given tree $T$, and a given integer $b$ with $0 \leq b \leq 2 m(T)$, there is a matching $M$ of $T$ such that $\operatorname{vacc}\left(T, 0, d_{T}, b\right)=\operatorname{dyn}\left(G, \tau_{M}\right)$ and $\tau_{M}(V(T)) \leq b$, where

$$
\tau_{M}: V(T) \rightarrow \mathbb{Z}: u \mapsto \begin{cases}d_{T}(u) & , u \text { is incident with a vertex in } M, \text { and } \\ 0 & , \text { otherwise. }\end{cases}
$$

We believe that the threshold function $\tau_{M}$ considered in Theorem 1.2 is a good choice in general, and pose the following.

Conjecture 1.3. For a given graph $G$, and a given integer $b$ with $0 \leq b \leq 2 m(G)$, there is a matching $M$ of $G$ such that $\operatorname{vacc}\left(G, 0, d_{G}, b\right) \leq 2 \operatorname{dyn}\left(G, \tau_{M}\right)$ and $\tau_{M}(V(G)) \leq b$, where $\tau_{M}$ is as in Theorem 1.2 (with $T$ replaced by $G$ ).

As a second result we show Conjecture 1.3 for some regular graphs.
Theorem 1.4. Conjecture 1.3 holds if $G$ is $r$-regular and $b \geq(2 r-1)(r+1)$.
Before we proceed to the proofs of Theorems 1.1 and 1.4 , we mention some further related work. Centeno and Rautenbach [6] establish bounds for the problems considered in [17]. In [14, Ehard and Rautenbach consider the following two variants of (2) for a given triple ( $G, \tau, b$ ), where $G$ is a graph, $\tau$ is a threshold function for $G$, and $b$ is a non-negative integer:

$$
\max \left\{\operatorname{dyn}(G-X, \tau): X \in\binom{V(G)}{b}\right\} \quad \text { and } \quad \max \left\{\operatorname{dyn}\left(G, \tau_{X}\right): X \in\binom{V(G)}{b}\right\}
$$

where

$$
\tau_{X}(u)= \begin{cases}d_{G}(u)+1 & , \text { if } u \in X \\ \tau(u) & , \text { if } u \in V(G) \backslash X,\end{cases}
$$

and $\binom{V(G)}{b}$ denotes the set of all $b$-element subsets of $V(G)$. For both variants, they describe efficient algorithms for trees. In [3] Bhawalkar et al. study so-called anchored $k$-cores. For a given graph $G$, and a positive integer $k$, the $k$-core of $G$ is the largest induced subgraph of $G$ of minimum degree at least $k$. It is easy to see that the vertex set of the $k$-core of $G$ equals $V(G) \backslash H_{(G, \tau)}(\emptyset)$ for the special threshold function $\tau=d_{G}-k+1$. Now, the anchored $k$-core problem [3] is to determine

$$
\begin{equation*}
\max \left\{\left|V(G) \backslash H_{\left(G, \tau_{X}\right)}(\emptyset)\right|: X \in\binom{V(G)}{b}\right\} \tag{3}
\end{equation*}
$$

for a given graph $G$ and non-negative integer $b$. Bhawalkar et al. show that (3) is hard to approximate in general, but can be determined efficiently for $k=2$, and for graphs of bounded treewidth. Vaccination problems in random settings were studied in [4, 11, 16].

## 2 Proofs of Theorem 1.1 and Theorem 1.4

Throughout this section, let $T$ be a tree rooted in some vertex $r$, and let $\tau, \iota_{\max } \in \mathbb{Z}^{V(T)}$ be two functions. For a vertex $u$ of $T$, and a function $\rho \in \mathbb{Z}^{V(T)}$, let $V_{u}$ be the subset of $V(T)$ containing
$u$ and its descendants, let $T_{u}$ be the subtree of $T$ induced by $V_{u}$, and let $\rho^{\rightarrow u} \in \mathbb{Z}^{V(T)}$ be the function with

$$
\rho^{\rightarrow u}(v)= \begin{cases}\rho(v) & , \text { if } v \in V(T) \backslash\{u\}, \text { and } \\ \rho(v)-1 & , \text { if } v=u\end{cases}
$$

Below we consider threshold functions of the form $\left.\rho\right|_{V_{u}}+\left.\rho^{\prime}\right|_{V_{u}}$ for the subtrees $T_{u}$, where $\rho$ and $\rho^{\prime}$ are defined on sets containing $V_{u}$. For notational simplicity, we omit the restriction to $V_{u}$ and write ' $\rho+\rho^{\prime}$ ' instead of ' $\left.\rho\right|_{V_{u}}+\left.\rho^{\prime}\right|_{V_{u}}$ ' in these cases. For an integer $k$ and a non-negative integer $b$, let $[k]$ be the set of positive integers at most $k$, and let

$$
\mathcal{P}_{k}(b)=\left\{\left(b_{1}, \ldots, b_{k}\right) \in \mathbb{N}_{0}^{k}: b_{1}+\cdots+b_{k}=b\right\}
$$

be the set of ordered partitions of $b$ into $k$ non-negative integers.
Our approach to show Theorem 1.1 is similar as in [14] and relies on recursive expressions for the following two quantities: For a vertex $u$ of $T$ and a non-negative integer $b$, let

- $x_{0}(u, b)$ be the maximum of $\operatorname{dyn}\left(T_{u}, \tau+\iota\right)$ over all $\iota \in \mathbb{Z}^{V_{u}}$ with $0 \leq \iota(v) \leq \iota_{\max }(v)$ for every $v \in V_{u}$, and $\iota\left(V_{u}\right)=b$, and
- $x_{1}(u, b)$ be the maximum of $\operatorname{dyn}\left(T_{u},(\tau+\iota)^{\rightarrow u}\right)$ over all $\iota \in \mathbb{Z}^{V_{u}}$ with $0 \leq \iota(v) \leq \iota_{\max }(v)$ for every $v \in V_{u}$, and $\iota\left(V_{u}\right)=b$.

The increment $\iota$ captures the local increases of the thresholds within $V_{u}$. The value $x_{1}(u, b)$ corresponds to a situation, where the infection reaches the parent of $u$ before it reaches $u$, that is, the index 0 or 1 indicates the amount of help that $u$ receives from outside of $V_{u}$.

Note that $x_{j}(u, b)=-\infty$ if and only if $b>\iota_{\max }\left(V_{u}\right)$ for both $j$ in $\{0,1\}$. If $b \leq \iota_{\max }\left(V_{u}\right)$, then let $\iota_{0}(u, b), \iota_{1}(u, b) \in \mathbb{Z}^{V_{u}}$ with $0 \leq \iota_{j}(u, b) \leq \iota_{\max }$, and $\iota_{j}(u, b)\left(V_{u}\right)=b$ for both $j \in\{0,1\}$, be such that

$$
\begin{aligned}
& x_{0}(u, b)=\operatorname{dyn}\left(T_{u}, \tau+\iota_{0}(u, b)\right) \text { and } \\
& x_{1}(u, b)=\operatorname{dyn}\left(T_{u},\left(\tau+\iota_{1}(u, b)\right)^{\rightarrow u}\right),
\end{aligned}
$$

where, if possible, let $\iota_{0}(u, b)=\iota_{1}(u, b)$. As we show in Corollary 2.4 below, $\iota_{0}(u, b)$ always equals $\iota_{1}(u, b)$, which is a key fact for our approach.

Lemma 2.1. $x_{0}(u, b) \geq x_{1}(u, b)$, and if $x_{0}(u, b)=x_{1}(u, b)$, then $\iota_{0}(u, b)=\iota_{1}(u, b)$.
Proof. If $x_{1}(u, b)=-\infty$, then the statement is trivial. Hence, we may assume that $x_{1}(u, b)>$ $-\infty$, which implies that the function $\iota_{1}(u, b)$ is defined. Let $D$ be a minimum dynamic monopoly of $\left(T_{u}, \tau+\iota_{1}(u, b)\right)$. By the definition of $x_{0}(u, b)$, we have $x_{0}(u, b) \geq|D|$. Since $D$ is a dynamic monopoly of $\left(T_{u},\left(\tau+\iota_{1}(u, b)\right)^{\rightarrow u}\right)$, we obtain $x_{0}(u, b) \geq|D| \geq \operatorname{dyn}\left(T_{u},\left(\tau+\iota_{1}(u, b)\right)^{\rightarrow u}\right)=$ $x_{1}(u, b)$. Furthermore, if $x_{0}(u, b)=x_{1}(u, b)$, then $x_{0}(u, b)=|D|=\operatorname{dyn}\left(T_{u}, \tau+\iota_{1}(u, b)\right)$, which implies $\iota_{0}(u, b)=\iota_{1}(u, b)$.

Lemma 2.2. If $u$ is a leaf of $T$, and $b$ is an integer with $0 \leq b \leq \iota_{\max }(u)$, then, for $j \in\{0,1\}$,

$$
\begin{aligned}
x_{j}(u, b) & = \begin{cases}0 & , \text { if } \tau(u)+b-j \leq 0, \\
1 & , \text { otherwise, and }\end{cases} \\
\iota_{j}(u, b)(u) & =b .
\end{aligned}
$$

Proof. These equalities follow immediately from the definitions.
Lemma 2.3. Let $u$ be a vertex of $T$ that is not a leaf, and let $b$ be a non-negative integer. If $v_{1}, \ldots, v_{k}$ are the children of $u$, and $\iota_{0}\left(v_{i}, b_{i}\right)=\iota_{1}\left(v_{i}, b_{i}\right)$ for every $i \in[k]$ and every integer $b_{i}$ with $0 \leq b_{i} \leq \iota_{\max }\left(V_{v_{i}}\right)$, then, for $j \in\{0,1\}$,

$$
\begin{align*}
x_{j}(u, b) & =z_{j}(u, b), \text { and }  \tag{4}\\
\iota_{0}(u, b) & =\iota_{1}(u, b), \text { if } b \leq \iota_{\max }\left(V_{u}\right), \tag{5}
\end{align*}
$$

where $z_{j}(u, b)$ is defined as

$$
\max \left\{\delta_{j}\left(b_{u}, b_{1}, \ldots, b_{k}\right)+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right):\left(b_{u}, b_{1}, \ldots, b_{k}\right) \in \mathcal{P}_{k+1}(b) \text { with } b_{u} \leq \iota_{\max }(u)\right\}
$$

and, for $\left(b_{u}, b_{1}, \ldots, b_{k}\right) \in \mathcal{P}_{k+1}(b)$ with $b_{u} \leq \iota_{\max }(u)$,

$$
\delta_{j}\left(b_{u}, b_{1}, \ldots, b_{k}\right):= \begin{cases}0 & , \text { if }\left|\left\{i \in[k]: x_{0}\left(v_{i}, b_{i}\right)=x_{1}\left(v_{i}, b_{i}\right)\right\}\right| \geq \tau(u)+b_{u}-j, \text { and } \\ 1 & , \text { otherwise } .\end{cases}
$$

Proof. By symmetry, it suffices to consider the case $j=0$.
First, suppose that $b>\iota_{\max }\left(V_{u}\right)$. If $\left(b_{u}, b_{1}, \ldots, b_{k}\right) \in \mathcal{P}_{k+1}(b)$ with $b_{u} \leq \iota_{\max }(u)$, then $b_{i}>\iota_{\max }\left(V_{v_{i}}\right)$ for some $i \in[k]$, which implies $z_{0}(u, b)=-\infty=x_{0}(u, b)$.

Now, let $b \leq n\left(T_{u}\right)$, which implies $x_{0}(u, b)>-\infty$. The following two claims complete the proof of (4).

Claim 1. $x_{0}(u, b) \geq z_{0}(u, b)$.
Proof of Claim 1. It suffices to show that $x_{0}(u, b) \geq \delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right)$ for every choice of $\left(b_{u}, b_{1}, \ldots, b_{k}\right)$ in $\mathcal{P}_{k+1}(b)$ with $b_{u} \leq \iota_{\max }(u)$ and $b_{i} \leq \iota_{\max }\left(V_{v_{i}}\right)$ for every $i \in[k]$. Let $\left(b_{u}, b_{1}, \ldots, b_{k}\right)$ be one such an element. Let $\iota_{u} \in \mathbb{Z}^{V_{u}}$ be defined as

$$
\iota_{u}(v)= \begin{cases}b_{u} & , \text { if } v=u, \text { and }  \tag{6}\\ 0 & , \text { otherwise },\end{cases}
$$

and let $\iota=\iota_{u}+\sum_{i=1}^{k} \iota_{1}\left(v_{i}, b_{i}\right)$, where $\iota_{1}\left(v_{i}, b_{i}\right)(u)$ is set to 0 for every $i \in[k]$. Since $\iota\left(V_{u}\right)=b$ and $0 \leq \iota \leq \iota_{\text {max }}$, we have $x_{0}(u, b) \geq \operatorname{dyn}\left(T_{u}, \tau+\iota\right)$.

Let $D$ be a minimum dynamic monopoly of $\left(T_{u}, \tau+\iota\right)$, that is, $|D| \leq x_{0}(u, b)$. For each $i \in[k]$, it follows that the set $D_{i}=D \cap V_{v_{i}}$ is a dynamic monopoly of $\left(T_{v_{i}},(\tau+\iota)^{\rightarrow v_{i}}\right)$. Since,
restricted to $V_{v_{i}}$, the two functions $(\tau+\iota)^{\rightarrow v_{i}}$ and $\left(\tau+\iota_{1}\left(v_{i}, b_{i}\right)\right)^{\rightarrow v_{i}}$ coincide, we obtain

$$
\left|D_{i}\right| \geq \operatorname{dyn}\left(T_{v_{i}},\left(\tau+\iota_{1}\left(v_{i}, b_{i}\right)\right)^{\rightarrow v_{i}}\right) \geq x_{1}\left(v_{i}, b_{i}\right) .
$$

If $\delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)=0$, then $|D| \geq \sum_{i=1}^{k}\left|D_{i}\right| \geq \delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right)$. Similarly, if $u \in D$, then $|D|=1+\sum_{i=1}^{k}\left|D_{i}\right| \geq \delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right)$. Therefore, we may assume that $\delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)=1$ and that $u \notin D$. This implies that there is some $\ell \in[k]$ with $x_{0}\left(v_{\ell}, b_{\ell}\right)>x_{1}\left(v_{\ell}, b_{\ell}\right)$ such that $D_{\ell}=D \cap V_{v_{\ell}}$ is a dynamic monopoly of ( $\left.T_{v_{\ell}}, \tau+\iota\right)$. Since, by assumption, $\iota_{0}\left(v_{\ell}, b_{\ell}\right)=\iota_{1}\left(v_{\ell}, b_{\ell}\right)$, we obtain that, restricted to $V_{v_{\ell}}$, the two functions $\tau+\iota$ and $\tau+\iota_{0}\left(v_{\ell}, b_{\ell}\right)$ coincide, which implies $\left|D_{\ell}\right| \geq \operatorname{dyn}\left(T_{v_{\ell}}, \tau+\iota_{0}\left(v_{\ell}, b_{\ell}\right)\right)=x_{0}\left(v_{\ell}, b_{\ell}\right) \geq 1+x_{1}\left(v_{\ell}, b_{\ell}\right)$. Therefore, also in this case, $|D|=\left|D_{\ell}\right|+\sum_{i \in[k] \backslash\{\ell\}}\left|D_{i}\right| \geq \delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right)$.

Claim 2. $x_{0}(u, b) \leq z_{0}(u, b)$.
Proof of Claim圆 Let $\iota=\iota_{0}(u, b)$, that is, $x_{0}(u, b)=\operatorname{dyn}\left(T_{u}, \tau+\iota\right)$. Let $b_{i}=\iota\left(V_{v_{i}}\right)$ for every $i \in[k]$, and let $b_{u}=b-\sum_{i=1}^{k} b_{i}$. Clearly, $\left(b_{u}, b_{1}, \ldots, b_{k}\right) \in \mathcal{P}_{k+1}(b)$ and $b_{u} \leq \iota_{\max }(u)$. Let $D_{i}$ be a minimum dynamic monopoly of $\left(T_{v_{i}},(\tau+\iota)^{\rightarrow v_{i}}\right)$ for every $i \in[k]$. By the definition of $x_{1}\left(v_{i}, b_{i}\right)$, we obtain $\left|D_{i}\right| \leq x_{1}\left(v_{i}, b_{i}\right)$. Let $D=\{u\} \cup \bigcup_{i=1}^{k} D_{i}$. The set $D$ is a dynamic monopoly of $\left(T_{u}, \tau+\iota\right)$, which implies $x_{0}(u, b) \leq|D|$.

If $\delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)=1$, then

$$
x_{0}(u, b) \leq|D|=1+\sum_{i=1}^{k}\left|D_{i}\right| \leq \delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right) \leq z_{0}(u, b) .
$$

Therefore, we may assume that $\delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)=0$. By symmetry, we may assume that $x_{0}\left(v_{i}, b_{i}\right)=x_{1}\left(v_{i}, b_{i}\right)$ for every $i \in\left[\tau(u)+b_{u}\right]$. Let $D_{i}^{\prime}$ be a minimum dynamic monopoly of $\left(T_{v_{i}}, \tau+\iota\right)$ for every $i \in\left[\tau(u)+b_{u}\right]$. By the definition of $x_{0}\left(v_{i}, b_{i}\right)$, we obtain $\left|D_{i}^{\prime}\right| \leq x_{0}\left(v_{i}, b_{i}\right)=$ $x_{1}\left(v_{i}, b_{i}\right)$. Let $D^{\prime}=\underset{i \in\left[\tau(u)+b_{u}\right]}{\bigcup} D_{i}^{\prime} \cup \bigcup_{\left.i \in[k] \backslash \tau(u)+b_{u}\right]}^{\bigcup} D_{i}$. The set $D^{\prime}$ is a dynamic monopoly of ( $T_{u}, \tau+\iota$ ). This implies

$$
x_{0}(u, b) \leq\left|D^{\prime}\right|=\sum_{i \in\left[\tau(u)+b_{u}\right]}\left|D_{i}^{\prime}\right|+\sum_{i \in[k] \backslash\left[\tau(u)+b_{u}\right]}\left|D_{i}\right| \leq \sum_{i \in[k]} x_{1}\left(v_{i}, b_{i}\right) \leq z_{0}(u, b),
$$

which completes the proof of the claim.
It remains to show (5). If $x_{0}(u, b)=x_{1}(u, b)$, then (5) follows from Lemma 2.1. Hence, we may assume that $x_{0}(u, b)>x_{1}(u, b)$. Since, by definition,

$$
\delta_{1}\left(b_{u}, b_{1}, \ldots, b_{k}\right) \leq \delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right) \leq \delta_{1}\left(b_{u}, b_{1}, \ldots, b_{k}\right)+1
$$

for every $\left(b_{u}, b_{1}, \ldots, b_{k}\right) \in \mathcal{P}_{k+1}(b)$ with $b_{u} \leq \iota_{\max }(u)$, we obtain $z_{1}(u, b) \leq z_{0}(u, b) \leq z_{1}(u, b)+1$.

Together with (4), the inequality $x_{0}(u, b)>x_{1}(u, b)$ implies that

$$
\begin{aligned}
& x_{0}(u, b)=z_{0}(u, b)>z_{1}(u, b)=x_{1}(u, b) \text { and } \\
& z_{1}(u, b)=z_{0}(u, b)-1 .
\end{aligned}
$$

Let $\left(b_{u}, b_{1}, \ldots, b_{k}\right) \in \mathcal{P}_{k+1}(b)$ with $b_{u} \leq \iota_{\max }(u)$ be such that

$$
z_{0}(u, b)=\delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right) .
$$

We obtain

$$
\begin{aligned}
z_{1}(u, b) & \geq \delta_{1}\left(b_{u}, b_{1}, \ldots, b_{k}\right)+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right) \\
& \geq \delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)-1+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right) \\
& =z_{0}(u, b)-1 \\
& =z_{1}(u, b),
\end{aligned}
$$

which implies $z_{1}(u, b)=\delta_{1}\left(b_{u}, b_{1}, \ldots, b_{k}\right)+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right)$, that is, the same choice of $\left(b_{u}, b_{1}, \ldots, b_{k}\right)$ in $\mathcal{P}_{k+1}(b)$ with $b_{u} \leq \iota_{\max }(u)$ maximizes the terms defining $z_{0}(u, b)$ and $z_{1}(u, b)$.

Since $z_{0}(u, b)>z_{1}(u, b)$, we obtain $\delta_{1}\left(b_{u}, b_{1}, \ldots, b_{k}\right)=0$ and $\delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)=1$, which, by the definition of $\delta_{j}$, implies that there are exactly $\tau(u)+b_{u}-1$ indices $i$ in $[k]$ with $x_{0}\left(v_{i}, b_{i}\right)=$ $x_{1}\left(v_{i}, b_{i}\right)$. By symmetry, we may assume that $x_{0}\left(v_{i}, b_{i}\right)=x_{1}\left(v_{i}, b_{i}\right)$ for $i \in\left[\tau(u)+b_{u}-1\right]$ and $x_{0}\left(v_{i}, b_{i}\right)>x_{1}\left(v_{i}, b_{i}\right)$ for $i \in[k] \backslash\left[\tau(u)+b_{u}-1\right]$.

Let $\iota=\iota_{u}+\sum_{i=1}^{k} \iota_{0}\left(v_{i}, b_{i}\right)$, where $\iota_{0}\left(v_{i}, b_{i}\right)(u)$ is set to 0 for every $i \in[k]$ and $\iota_{u}$ is as in (6). Note that, by assumption, we have $\iota=\iota_{u}+\sum_{i=1}^{k} \iota_{1}\left(v_{i}, b_{i}\right)$. Let $D$ be a minimum dynamic monopoly of ( $T_{u}, \tau+\iota$ ). By the definition of $x_{0}(u, b)$, we have $|D| \leq x_{0}(u, b)$. Let $D_{i}=D \cap V_{v_{i}}$ for every $i \in[k]$. Since $D_{i}$ is a dynamic monopoly of $\left(T_{v_{i}},(\tau+\iota)^{\rightarrow v_{i}}\right)$ for every $i \in[k]$, we obtain $\left|D_{i}\right| \geq x_{1}\left(v_{i}, b_{i}\right)$. Note that

- either $u \in D$,
- or $u \notin D$ and there is some index $\ell \in[k] \backslash\left[\tau(u)+b_{u}-1\right]$ such that $D_{\ell}=D \cap V_{v_{\ell}}$ is a dynamic monopoly of ( $T_{v_{\ell}}, \tau+\iota$ ).

In the first case, we obtain

$$
z_{0}(u, b)=x_{0}(u, b) \geq|D|=1+\sum_{i=1}^{k}\left|D_{i}\right| \geq 1+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right)=z_{0}(u, b),
$$

and, in the second case, we obtain $\left|D_{\ell}\right| \geq x_{0}\left(v_{\ell}, b_{\ell}\right) \geq x_{1}\left(v_{\ell}, b_{\ell}\right)+1$, and, hence,

$$
z_{0}(u, b)=x_{0}(u, b) \geq|D|=\left|D_{\ell}\right|+\sum_{i \in[k] \backslash\{\ell\}}\left|D_{i}\right| \geq 1+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right)=z_{0}(u, b) .
$$

In both cases we obtain $|D|=x_{0}(u, b)$, which implies that $\iota_{0}(u, b)$ may be chosen equal to $\iota$.
Now, let $D^{-}$be a minimum dynamic monopoly of $\left(T_{u},(\tau+\iota)^{\rightarrow u}\right)$. By the definition of $x_{1}(u, b)$, we have $\left|D^{-}\right| \leq x_{1}(u, b)$. Let $D_{i}^{-}=D^{-} \cap V_{v_{i}}$ for every $i \in[k]$. Since $D_{i}^{-}$is a dynamic monopoly of $\left(T_{v_{i}},(\tau+\iota)^{\rightarrow v_{i}}\right)$ for every $i \in[k]$, we obtain $\left|D_{i}^{-}\right| \geq x_{1}\left(v_{i}, b_{i}\right)$. Now,

$$
z_{1}(u, b)=x_{1}(u, b) \geq\left|D^{-}\right| \geq \sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right)=z_{1}(u, b),
$$

which implies that $\left|D^{-}\right|=x_{1}(u, b)$, and that $\iota_{1}(u, b)$ may be chosen equal to $\iota$. Altogether, the two functions $\iota_{0}(u, b)$ and $\iota_{1}(u, b)$ may be chosen equal, which implies (5).

Applying induction using Lemma 2.2 and Lemma [2.3, we obtain the following.
Corollary 2.4. $\iota_{0}(u, b)=\iota_{1}(u, b)$ for every vertex $u$ of $T$, and every integer $b$ with $0 \leq b \leq$ $\iota_{\max }\left(V_{u}\right)$.

Apart from the specific values of $x_{0}(u, b)$ and $x_{1}(u, b)$, the arguments in the proof of Lemma 2.3 also yield feasible recursive choices for $\iota_{0}(u, b)$. In fact, if

$$
x_{0}(u, b)=\delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)+\sum_{i=1}^{k} x_{1}\left(v_{i}, b_{i}\right)>-\infty
$$

for $\left(b_{u}, b_{1}, \ldots, b_{k}\right) \in \mathcal{P}_{k+1}(b)$ with $b_{u} \leq \iota_{\max }(u)$, and $\iota_{u}$ is as in (6), then $\iota_{u}+\sum_{i=1}^{k} \iota_{0}\left(v_{i}, b_{i}\right)$ is a feasible choice for $\iota_{0}(u, b)$.

Our next lemma explains how to efficiently compute the expressions in Lemma 2.3.
Lemma 2.5. Let $u$ be a vertex of $T$ that is not a leaf, let $b$ be an integer with $0 \leq b \leq \iota_{\max }\left(V_{u}\right)$, and let $v_{1}, \ldots, v_{k}$ be the children of $u$. If the values $x_{1}\left(v_{i}, b_{i}\right)$ are given for every $i \in[k]$ and every integer $b_{i}$ with $0 \leq b_{i} \leq \iota_{\max }\left(V_{v_{i}}\right)$, then $x_{0}(u, b)$ and $x_{1}(u, b)$ can be computed in time $O\left(k^{2}(b+1)^{2}\right)$.

Proof. By symmetry, it suffices to explain how to compute $z_{0}(u, b)$.
For $p \in\{0\} \cup[k]$, an integer $p_{=}$, an integer $b^{\prime} \in\{0\} \cup[b]$, and $b_{u} \in\{0\} \cup\left[\min \left\{\iota_{\max }(u), b^{\prime}\right\}\right]$, let $M\left(p, p_{=}, b^{\prime}, b_{u}\right)$ be defined as the maximum of the expression $\sum_{i=1}^{p} x_{1}\left(v_{i}, b_{i}\right)$ over all $\left(b_{1}, \ldots, b_{p}\right) \in$ $\mathcal{P}_{p}\left(b^{\prime}-b_{u}\right)$ such that $p_{=}$equals $\left|\left\{i \in[p]: x_{0}\left(v_{i}, b_{i}\right)=x_{1}\left(v_{i}, b_{i}\right)\right\}\right|$. Clearly, $M\left(p, p_{=}, b^{\prime}, b_{u}\right)=-\infty$ if $p<p_{=}$or $p_{=}<0$ or $b^{\prime}-b_{u}>\sum_{i=1}^{p} \iota_{\max }\left(V_{v_{i}}\right)$, and

$$
M\left(0,0, b^{\prime}, b_{u}\right)= \begin{cases}0 & , \text { if } b^{\prime}=b_{u}, \text { and } \\ -\infty & , \text { otherwise }\end{cases}
$$

For $p \in[k]$, the value of $M\left(p, p_{=}, b^{\prime}, b_{u}\right)$ is the maximum of the following two values:

- The maximum of $M\left(p-1, p_{=}-1, b_{\leq p-1}, b_{u}\right)+x_{1}\left(v_{p}, b_{p}\right)$ over all $\left(b_{\leq p-1}, b_{p}\right) \in \mathcal{P}_{2}\left(b^{\prime}-b_{u}\right)$ with $x_{0}\left(v_{p}, b_{p}\right)=x_{1}\left(v_{p}, b_{p}\right)$, and
- the maximum of $M\left(p-1, p_{=}, b_{\leq p-1}, b_{u}\right)+x_{1}\left(v_{p}, b_{p}\right)$ over all $\left(b_{\leq p-1}, b_{p}\right) \in \mathcal{P}_{2}\left(b^{\prime}-b_{u}\right)$ with $x_{0}\left(v_{p}, b_{p}\right)>x_{1}\left(v_{p}, b_{p}\right)$,
which implies that $M\left(p, p_{=}, b^{\prime}, b_{u}\right)$ can be determined in $O\left(b^{\prime}+1\right)$ time given the values

$$
M\left(p-1, p_{=}, b_{\leq p-1}, b_{u}\right), M\left(p-1, p_{=}-1, b_{\leq p-1}, b_{u}\right), x_{0}\left(v_{p}, b_{p}\right), \text { and } x_{1}\left(v_{p}, b_{p}\right)
$$

Altogether, the values $M\left(k, p_{=}, b, b_{u}\right)$ for all $p_{=} \in\{0\} \cup[k]$ can be determined in time $O\left(k^{2}(b+1)\right)$.
For $b_{u} \in\{0\} \cup\left[\min \left\{\iota_{\max }(u), b\right\}\right]$, let $m\left(b_{u}\right)$ be the maximum of the two expressions

$$
1+\max \left\{M\left(k, p_{=}, b, b_{u}\right): p_{=} \in\{0\} \cup\left[\tau(u)-b_{u}-1\right]\right\}
$$

and

$$
\max \left\{M\left(k, p_{=}, b, b_{u}\right): p_{=} \in[k] \backslash\left[\tau(u)-b_{u}-1\right]\right\} .
$$

Now, by the definition of $\delta_{0}\left(b_{u}, b_{1}, \ldots, b_{k}\right)$, the value of $z_{0}(u, b)$ equals $\max \left\{m\left(b_{u}\right): b_{u} \in\{0\} \cup\right.$ $\left.\left[\min \left\{\iota_{\max }(u), b\right\}\right]\right\}$. Hence, $z_{0}(u, b)$ can be computed in time $O\left(k^{2}(b+1)^{2}\right)$.

We proceed to the proof of our first theorem.
Proof of Theorem [1.1. Given $\left(T, \tau, \iota_{\max }, b\right)$, Lemma 2.2 to Lemma 2.5 imply that the values of $x_{0}\left(u, b^{\prime}\right)$ and of $x_{1}\left(u, b^{\prime}\right)$ for all $u \in V(T)$ and all $b^{\prime} \in\{0\} \cup[b]$ can be determined in time

$$
O\left(\sum_{u \in V(T)} d_{T}(u)^{2}(b+1)^{2}\right) .
$$

It is a simple folklore exercise that $\sum_{u \in V(T)} d_{T}(u)^{2} \leq n^{2}-n$ for every tree $T$ of order $n$, which implies the statement about the running time. Since $\operatorname{vacc}\left(T, \tau, \iota_{\text {max }}, b\right)=x_{0}(r, b)$, the statement about the value of $\operatorname{vacc}\left(T, \tau, \iota_{\max }, b\right)$ follows. The statement about the increment $\iota$ follows easily from the remark after Corollary [2.4 concerning the function $\iota_{0}(u, b)$, and the proof of Lemma [2.5, where, next to the values $M\left(p, p_{=}, b^{\prime}, b_{u}\right)$, one may also memorize suitable increments.

We conclude with the proof of our second theorem.
Proof of Theorem [1.4. Let $G$ be an $r$-regular graph of order $n$, and let $b$ be an integer with $(2 r-1)(r+1) \leq b \leq r n=2 m(G)$.

Let $\iota \in \mathbb{Z}^{V(G)}$ with $0 \leq \iota \leq d_{G}$ and $\iota(V(G))=b$ be such that $\operatorname{vacc}\left(G, 0, d_{G}, b\right)=\operatorname{dyn}(G, \iota)$. By a result of Ackerman et al. [1],

$$
\operatorname{vacc}\left(G, 0, d_{G}, b\right)=\operatorname{dyn}(G, \iota) \leq \sum_{u \in V(G)} \frac{\iota(u)}{d_{G}(u)+1}=\frac{\iota(V(G))}{r+1}=\frac{b}{r+1} .
$$

First, suppose that the matching number $\nu$ of $G$ satisfies $2 r \nu>b$. In this case, $G$ has a matching $M$ with $\tau_{M}(V(G))=2 r|M| \leq b$ and $2 r(|M|+1) \geq b+1$, where $\tau_{M}$ is as in the statement. We obtain $2 \operatorname{dyn}\left(G, \tau_{M}\right) \geq 2|M| \geq 2\left(\frac{b+1}{2 r}-1\right) \geq \frac{b}{r+1} \geq \operatorname{vacc}\left(G, 0, d_{G}, b\right)$. Next, suppose that $2 r \nu \leq b$. If $M$ is a maximum matching and $D$ is a minimum vertex cover, then $|D| \leq 2|M|$. Since $D$ is a dynamic monopoly of $\left(G, d_{G}\right)$, we obtain $2 \operatorname{dyn}\left(G, \tau_{M}\right) \geq 2|M| \geq|D| \geq \operatorname{dyn}\left(G, d_{G}\right) \geq$ $\operatorname{vacc}\left(G, 0, d_{G}, b\right)$, that is, $2 \operatorname{dyn}\left(G, \tau_{M}\right) \geq \operatorname{vacc}\left(G, 0, d_{G}, b\right)$ holds in both cases.

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