# Convexifying Multilinear Sets with Cardinality Constraints: Structural Properties, Nested Case and Extensions 

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August 31, 2021


#### Abstract

The problem of minimizing a multilinear function of binary variables is a well-studied NP-hard problem. The set of solutions of the standard linearization of this problem is called the multilinear set. We study a cardinality constrained version of it with upper and lower bounds on the number of nonzero variables. We call the set of solutions of the standard linearization of this problem a multilinear set with cardinality constraints. We characterize a set of conditions on these multilinear terms (called properness) and observe that under these conditions the convex hull description of the set is tractable via an extended formulation. We then give an explicit polyhedral description of the convex hull when the multilinear terms have a nested structure. Our description has an exponential number of inequalities which can be separated in polynomial time. Finally, we generalize these inequalities to obtain valid inequalities for the general case.


## 1 Introduction

In this paper, we study the convex hull of the set

$$
X=\left\{(x, \delta) \in\{0,1\}^{n} \times\{0,1\}^{m}: \delta_{i}=\prod_{j \in S_{i}} x_{j}, i=1, \ldots, m, L \leq \sum_{j=1}^{n} x_{j} \leq U\right\},
$$

where $m, n$ are positive integers, $S_{i} \subseteq J=\{1, \ldots, n\}$ for $i=1, \ldots, m$ and $L, U$ are integers such that $0 \leq L \leq U \leq n$. We call $X$ the multilinear set with cardinality constraints. We investigate the structural properties of $\operatorname{conv}(X)$, give a polyhedral characterization in the special case that the sets $S_{i}$ are nested, i.e., $S_{1} \subset S_{2} \subset \cdots \subset S_{m}$, and give a family of valid inequalities for the non-nested case.

The problem of minimizing a polynomial objective function of binary variables subject to polynomial constraints is called the binary polynomial optimization problem, and is often solved by formulating it as an
integer linear programming problem. The first step in creating such a formulation is to replace each polynomial function by an equivalent (for all $x \in\{0,1\}^{n}$ ) multilinear expression of the form

$$
f(x)=\beta+\sum_{i=1}^{m} \gamma_{i} \prod_{j \in S_{i}} x_{j}
$$

where $S_{i} \subseteq\{1, \ldots, n\}$ for $i=1, \ldots, m, \beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}^{m}$. Minimizing $f(x)$ over the set of binary $x$-vectors is equivalent to minimizing the linear expression $\beta+\sum_{i=1}^{m} \gamma_{i} \delta_{i}$ over the set $Y$, obtained from $X$ by dropping the cardinality constraints:

$$
Y=\left\{(x, \delta) \in\{0,1\}^{n} \times\{0,1\}^{m}: \delta_{i}=\prod_{j \in S_{i}} x_{j}, i=1, \ldots, m\right\} .
$$

Set $Y$ is called the multilinear set and is well-studied in mixed-integer nonlinear optimization because of the connection to binary polynomial optimization.

The convex hull of the multilinear set is called the multilinear polytope 9, and several classes of valid inequalities for the multilinear polytope have been proposed recently [6, 1, 11, 10, 12]. The boolean quadric polytope [22] is equal to $\operatorname{conv}(Y)$ when $\left|S_{i}\right|=2$ for $i=1, \ldots, m$. A complete polyhedral characterization of $\operatorname{conv}(Y)$ has been given in some cases, for example, when the multilinear set is defined by a single nonlinear monomial $(m=1)$, see [5], or when the multilinear sets are associated with certain acyclic hypergraphs [10, 12, 8. When the nonlinear monomials have a nested structure, i.e., the sets $S_{i}$ have a nested structure, then the convex hull equals the 2-link polytope, which is obtained by augmenting the standard linearization constraints with the 2-link inequalities [6. This result follows from the work of Fischer, Fischer and McCormick [18]. The problem of minimizing a linear function over $Y$ contains as a special case the maximum monomial agreement problem which has been analyzed in the context of machine learning [14, 13, 15, 16] and solved via branch-and-bound methods and heuristics.

Mehrotra 21 studied the boolean quadric polytope with an upper bound constraint on the number of nonzero $x_{j}$ variables, i.e., the set $X$ with $\left|S_{i}\right|=2$ and $L=0$. When the nonlinear monomials have a nested structure, Fischer, Fischer and McCormick [18] gave a polyhedral description of the convex hull of $V=\{(x, \delta) \in Y: x \in \mathcal{M}\}$, where $\mathcal{M}$ is the independent set polytope of a matroid over $n$ elements, generalizing earlier results by Buchheim and Klein [2] and Fischer and Fischer [17]. When the matroid is a uniform matroid, $V$ is the same as $X$ with $L=0$. In this paper, we study the case when the nonlinear monomials have a nested structure and $L>0$. Our results do not follow from the work of Fischer, Fischer and McCormick mentioned above. Minimizing a linear function over $X$ generalizes the cardinality constrained maximum monomial agreement problem, which was studied in the context of binary classification in machine learning by Dash, Günlük and Wei 7.

When $m \geq 2$ and the sets $S_{i}$ are not nested, $X$ is quite a complicated object. We have given a complete characterization of the convex hull of $X$ when $m=2$ in 3]. This generalizes the work of Crama and

Rodríguez-Heck [6] who showed that the 2 -link polytope is equal to $\operatorname{conv}(Y)$ when $m=2$. In this paper, we give a general family of valid inequalities for the case $m \geq 2$.

The paper is organized as follows. In Section 2, we study general multilinear sets with cardinality constraints and give some facetial conditions of the convex hull under a set of properness assumptions. In Section 3, we propose new valid inequalities and give a complete polyhedral description of the convex hull for cases when $\left\{S_{i}\right\}_{i=1}^{m}$ are nested. In Section 4, we give necessary conditions and sufficient conditions for a set of valid inequalities to define facets of the convex hull for the nested case. In Section 5 , we generalize these valid inequalities to the non-nested case.

## 2 Preliminaries

Let $I=\{1, \ldots, m\}, J=\{1, \ldots, n\}, 0 \leq l \leq u$ and $u \geq 2$. Let $S_{1}, \ldots, S_{m}$ be distinct subsets of $J$ with $1 \leq\left|S_{i}\right| \leq n-l$ for $i=1, \ldots, m$. Note that the assumptions imply that $n-l \geq 1$. Define $\mathcal{S}:=\left\{S_{i}\right\}_{i \in I}$. We will study the set

$$
X^{l, u}:=\left\{(z, \delta) \in\{0,1\}^{n} \times\{0,1\}^{m}: \delta_{i}=\prod_{j \in S_{i}}\left(1-z_{j}\right), i \in I, l \leq \sum_{j \in J} z_{j} \leq u\right\}
$$

which is equivalent to the set $X$ in the previous section (let $z_{j}=1-x_{j}, l=n-U$ and $u=n-L$ ).
The standard linearization of the set $X^{l, u}$ is given by the following system of inequalities:

$$
\begin{array}{rlrl}
l \leq \sum_{j \in J} z_{j} & \leq u, & \\
z_{j}+\delta_{i} & \leq 1, & & j \in S_{i}, i \in I, \\
\delta_{i}+\sum_{j \in S_{i}} z_{j} & \geq 1, & & i \in I, \\
\delta_{i} & \geq 0, & & i \in I, \\
1 \geq z_{j} & \geq 0, & & j \in J, \tag{5}
\end{array}
$$

We say that $\mathcal{S}$ is closed under nonempty intersection if for each pair $S_{i}, S_{j} \in \mathcal{S}$ such that $S_{i} \cap S_{j} \neq \emptyset$, their intersection $S_{i} \cap S_{j}$ is also contained in $\mathcal{S}$. Let $\Delta^{l, u}=\operatorname{proj}_{\delta}\left(X^{l, u}\right)$ denote the orthogonal projection of $X^{l, u}$ onto the space of $\delta$ variables. The next result gives a simple characterization of the convex hull of the set

$$
X^{l, u}(\bar{\delta})=\left\{z \in\{0,1\}^{n}:(z, \bar{\delta}) \in X^{l, u}\right\}
$$

for each $\bar{\delta} \in \Delta^{l, u}$ under the assumption that $\mathcal{S}$ is closed under nonempty intersection.
Lemma 1. If $\mathcal{S}$ is closed under nonempty intersection, then for each $\bar{\delta} \in \Delta^{l, u}$, there exists a subset $I^{*}$ of $I$ and disjoint subsets $J_{0}$ and $\left\{J_{i}: i \in I^{*}\right\}$ of $J$ such that $\operatorname{conv}\left(X^{l, u}(\bar{\delta})\right)$ is defined by the inequalities

$$
\begin{equation*}
z_{j}=0, \quad j \in J_{0} \tag{6}
\end{equation*}
$$

$$
\begin{array}{rr}
\sum_{j \in J_{i}} z_{j} \geq 1, & i \in I^{*}, \\
l \leq \sum_{j \in J} z_{j} \leq u & \\
0 \leq z_{j} \leq 1, & j \in J \tag{9}
\end{array}
$$

Proof. Let $\bar{\delta} \in \Delta^{l, u}$. Then a binary vector $z \in X^{l, u}(\bar{\delta})$ if and only if (8) is satisfied and

$$
\begin{equation*}
\prod_{j \in S_{i}}\left(1-z_{j}\right)=\bar{\delta}_{i}, \quad i \in I \tag{10}
\end{equation*}
$$

Let $I_{0}:=\left\{i \in I: \bar{\delta}_{i}=0\right\}$ and let $J_{0}:=\bigcup_{i \in I: \bar{\delta}_{i}=1} S_{i}$. Note that (10) is equivalent to (16) and the inequalities

$$
\begin{equation*}
\sum_{j \in S_{i}} z_{j} \geq 1, \quad i \in I_{0} \tag{11}
\end{equation*}
$$

Let $I^{*} \subseteq I_{0}$ denote the index set of minimal elements (with respect to inclusion) of $\left\{S_{i}: i \in I_{0}\right\}$. Then replacing $I_{0}$ by $I^{*}$ in (11) yields an equivalent set of constraints.

For each $i \in I^{*}$, let

$$
J_{i}:=S_{i} \backslash J_{0}
$$

Then $\emptyset \neq J_{i} \subseteq S_{i}$. The nonemptyness of $J_{i}$ for $i \in I^{*} \subseteq I_{0}$ follows from the fact that $\bar{\delta}_{i}=0$ and for some $j \in S_{i}$ we must have $z_{j}=1$. But for all $j \in J_{0}$, we must have $z_{j}=0$. Therefore, any nonzero binary vector $z$ that satisfies equations (6) will also satisfy (11) if and only if (7) is satisfied.

We next argue that $J_{0}$ and $\left\{J_{i}\right\}_{i \in I^{*}}$ are disjoint sets. By definition, $J_{0} \cap J_{i}=\emptyset$, for all $i \in I^{*}$. For any $i_{1}<i_{2} \in I^{*}$, if $J_{i_{1}} \cap J_{i_{2}} \neq \emptyset$, then we must have $\emptyset \neq S_{i_{1}} \cap S_{i_{2}} \in \mathcal{S}$, as $\mathcal{S}$ is closed under nonempty intersection. Therefore $S_{i_{1}} \cap S_{i_{2}}=S_{i_{3}}$ for some $i_{3} \in I$, and $S_{i_{3}} \supseteq J_{i_{1}} \cap J_{i_{2}}$. Then $S_{i_{3}} \backslash J_{0} \neq \emptyset$, and therefore $\bar{\delta}_{i_{3}}=0$. Consequently, $i_{3} \in I_{0}$, which contradicts the fact that $S_{i_{1}}$ (or $S_{i_{2}}$ ) is a minimal element in $\left\{S_{i}: i \in I_{0}\right\}$. We have shown that if $z \in\{0,1\}$, then $z \in X^{l, u}(\bar{\delta})$ if and only if $z$ satisfies the constraints in (6)-(9). Note that the constraint matrix associated with this system of inequalities is totally unimodular. This is because each $z_{j}$ occurs once in (8) and possibly once more in (6) or (7) and therefore the rows of the associated constraint matrix admits an equitable row bi-coloring [4]. Therefore the polyhedron defined by (6)-(9) is an integral polyhedron, and has only 0-1 vertices. The result follows.

The previous result implies that if $\mathcal{S}$ is closed under nonempty intersection and $\Delta^{l, u}$ has polynomially many elements, optimizing a linear function over $X^{l, u}$ can be formulated as a linear program of polynomial size using Balas' disjunctive model [1]. In particular, optimizing a linear function over $X^{l, u}$ is equivalent to optimizing linear functions over $\operatorname{conv}\left(X^{l, u}(\bar{\delta})\right)$ for all $\bar{\delta} \in \Delta^{l, u}$. However, we are interested in characterizing $\operatorname{conv}\left(X^{l, u}\right)$ in the original space in order to deal with problems where $X^{l, u}$ appears as a substructure.

### 2.1 Proper families

We next present a definition where we call $\mathcal{S}$ that defines $X^{l, u}$ a proper family if it satisfies some simple conditions. We will then show that inequalities that define $\operatorname{conv}\left(X^{l, u}\right)$ satisfy certain properties if $\mathcal{S}$ is a proper family.

Definition 1. A family $\mathcal{S}=\left\{S_{i}\right\}_{i \in I}$ of subsets of $J$ is called a proper family if it satisfies the following properties:

1. $\Delta^{l, u}$ is a set of exactly $m+1$ affinely independent vectors in $\mathbb{R}^{m}$;
2. $\mathcal{S}$ is closed under nonempty intersection.

Note that if $\mathcal{S}$ is a proper family, then it is closed under nonempty intersection and the size of $\Delta^{l, u}$ is polynomial in $m$ and consequently a polynomial-sized extended formulation of $\operatorname{conv}\left(X^{l, u}\right)$ can be obtained using Balas' disjunctive model. In particular, we will show that if $\mathcal{S}$ is proper, then we can characterize $\operatorname{conv}\left(X^{l, u}\right)$ by enumerating a set of valid inequalities. We next present three examples of proper families $\mathcal{S}$ together with the corresponding sets $I^{*}$ and $J_{i}$ for $i \in I^{*} \cup\{0\}$, for each $\delta \in \Delta^{l, u}$.

Example 1. If $S_{1}, S_{2}, \ldots, S_{m}$ are nested subsets of $J, l \leq n-\left|S_{m}\right|$ and $u \geq 2$, then $\mathcal{S}=\left\{S_{1}, S_{2}, \ldots, S_{m}\right\}$ is proper. In this case, $S_{1} \subset S_{2} \subset \ldots \subset S_{m}$ and $\Delta^{l, u}=\left\{\delta \in\{0,1\}^{m}: \delta_{1} \geq \delta_{2} \geq \ldots \geq \delta_{m}\right\}$ is a set of $m+1$ affinely independent points in $\mathbb{R}^{m}$. For each $\delta \in \Delta^{l, u}, J_{0}$ and $\left\{J_{i}\right\}_{i \in I^{*}}$ are defined as follows.

| $\delta$ | $J_{0}$ | $\left\{J_{i}\right\}_{i \in I^{*}}$ |
| :---: | :---: | :---: |
| $(0,0, \ldots, 0)$ | $\emptyset$ | $\left\{S_{1}\right\}$ |
| $(\underbrace{1, \ldots, 1}_{\text {first } p \text { entries }}, 0, \ldots, 0)$ for some $1 \leq p \leq m-1$ | $S_{p}$ | $\left\{S_{p+1} \backslash S_{p}\right\}$ |
| $(1,1, \ldots, 1)$ | $S_{m}$ | $\emptyset$ |

Example 2. If $S_{1}, S_{2}$ are two disjoint subsets of $J, l \leq n-\left|S_{1} \cup S_{2}\right|$ and $u \geq 2$, then $\mathcal{S}=\left\{S_{1}, S_{2}, S_{1} \cup S_{2}\right\}$ is proper. In this case, $\Delta^{l, u}=\{(0,0,0),(1,0,0),(0,1,0),(1,1,1)\}$ is a set of 4 affinely independent points in $\mathbb{R}^{3}$. For each $\delta \in \Delta^{l, u}, J_{0}$ and $\left\{J_{i}\right\}_{i \in I^{*}}$ are defined as follows.

| $\delta$ | $J_{0}$ | $\left\{J_{i}\right\}_{i \in I^{*}}$ |
| :---: | :---: | :---: |
| $(0,0,0)$ | $\emptyset$ | $\left\{S_{1}, S_{2}\right\}$ |
| $(1,0,0)$ | $S_{1}$ | $\left\{S_{2}\right\}$ |
| $(0,1,0)$ | $S_{2}$ | $\left\{S_{1}\right\}$ |
| $(1,1,1)$ | $S_{1} \cup S_{2}$ | $\emptyset$ |

Example 3. If $S_{1}$ and $S_{2}$ are two subsets of $J$ satisfying $S_{1} \cap S_{2} \neq \emptyset, S_{1} \nsubseteq S_{2}, S_{2} \nsubseteq S_{1}, l \leq n-\left|S_{1} \cup S_{2}\right|$ and $u \geq 2$, then $\mathcal{S}=\left\{S_{1} \cap S_{2}, S_{1}, S_{2}, S_{1} \cup S_{2}\right\}$ is proper. In this case, $\Delta^{l, u}=\{(0,0,0,0),(1,0,0,0),(1,1,0,0)$,
$(1,0,1,0),(1,1,1,1)\}$ is a set of 5 affinely independent points in $\mathbb{R}^{4}$. For each $\delta \in \Delta^{l, u}, J_{0}$ and $\left\{J_{i}\right\}_{i \in I^{*}}$ are defined as follows.

| $\delta$ | $J_{0}$ | $\left\{J_{i}\right\}_{i \in I^{*}}$ |
| :---: | :---: | :---: |
| $(0,0,0,0)$ | $\emptyset$ | $\left\{S_{1} \cap S_{2}\right\}$ |
| $(1,0,0,0)$ | $S_{1} \cap S_{2}$ | $\left\{S_{1} \backslash S_{2}, S_{2} \backslash S_{1}\right\}$ |
| $(1,1,0,0)$ | $S_{1}$ | $\left\{S_{2} \backslash S_{1}\right\}$ |
| $(1,0,1,0)$ | $S_{2}$ | $\left\{S_{1} \backslash S_{2}\right\}$ |
| $(1,1,1,1)$ | $S_{1} \cup S_{2}$ | $\emptyset$ |

We next present an alternate way to certify that $\mathcal{S}$ is a proper family. We say that $\mathcal{S}$ is closed under union if for each pair $S_{i}, S_{j} \in \mathcal{S}$, their union $S_{i} \cup S_{j}$ is also contained in $\mathcal{S}$.

Proposition 2. Let $\mathcal{S}$ be a family of nonempty subsets of $J$ that is closed under union and nonempty intersection. Then, $\mathcal{S}$ is a proper family provided that $\Delta^{l, u}=\Delta^{0, n}$.

Proof. It suffices to show that $\Delta^{0, n}$ satisfies property 1 in Definition 1 if $\mathcal{S}$ is closed under union and nonempty intersection. We show this by induction on $m$. When $m=1$, then we have $\Delta^{0, n}=\{0,1\}$ and the statement holds.

For the inductive step, assume the statement holds for all $\mathcal{S}$ with $m \leq k$ for a given $k \geq 1$. We will next show that the statement then also holds for $k+1$. Let $\mathcal{S}^{\prime}=\left\{S_{i}\right\}_{i \in I^{\prime}}$ be a family of distinct nonempty subsets of $J$ that are closed under union and nonempty intersection with $m^{\prime}:=\left|I^{\prime}\right|=k+1$. Without loss of generality, assume $S_{1}$ is a minimal set (with respect to inclusion) in $\mathcal{S}^{\prime}$. Let $I_{1}:=\left\{i \in I^{\prime}: S_{i} \nsupseteq S_{1}\right\}=$ $\left\{i \in I^{\prime}: S_{i} \cap S_{1}=\emptyset\right\}$ and $I_{2}:=\left\{i \in I^{\prime}: S_{i} \supsetneq S_{1}\right\}, \mathcal{S}_{1}:=\left\{S_{i}\right\}_{i \in I_{1}}$ and $\mathcal{S}_{2}:=\left\{S_{i} \backslash S_{1}\right\}_{i \in I_{2}}$. Note that both $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ are closed under union and nonempty intersection.

Define

$$
\begin{aligned}
& \Delta^{\prime}:=\left\{\delta \in\{0,1\}^{I^{\prime}}: \exists z \in\{0,1\}^{n} \text { s.t. } \delta_{i}=\prod_{j \in S_{i}}\left(1-z_{j}\right), i \in I^{\prime}\right\}, \\
& \Delta_{1}:=\left\{\delta \in\{0,1\}^{I_{1}}: \exists z \in\{0,1\}^{n} \text { s.t. } \delta_{i}=\prod_{j \in S_{i}}\left(1-z_{j}\right), i \in I_{1}\right\}, \\
& \Delta_{2}:=\left\{\delta \in\{0,1\}^{I_{2}}: \exists z \in\{0,1\}^{n} \text { s.t. } \delta_{i}=\prod_{j \in S_{i} \backslash S_{1}}\left(1-z_{j}\right), i \in I_{2}\right\} .
\end{aligned}
$$

Family $\mathcal{S}_{2}$ cannot be empty as $S_{1}$ is minimal and $\mathcal{S}^{\prime}$ is closed under union with $m^{\prime}=k+1 \geq 2$. Now we consider two cases.

First assume that $\mathcal{S}_{1}=\emptyset$, then $\left|\mathcal{S}_{2}\right|=\left|\mathcal{S}^{\prime} \backslash\left\{S_{1}\right\}\right|=k$. Therefore, by inductive hypothesis, $\Delta_{2}$ contains exactly $k+1\left(=m^{\prime}\right)$ affinely independent points. Then

$$
\Delta^{\prime}=\{0\} \cup\left\{(1, \delta): \delta \in \Delta_{2}\right\}
$$

is a set of $m^{\prime}+1$ affinely independent points.
Next, consider the case when $\mathcal{S}_{1} \neq \emptyset$. In this case, $\mathcal{S}_{1}$ and $\mathcal{S}_{2}$ are closed under union and nonempty intersection with $m_{1}:=\left|I_{1}\right| \leq k$ and $m_{2}:=\left|I_{2}\right| \leq k$. Without loss of generality, assume $I_{1}=\left\{2,3, \ldots, m_{1}+\right.$ $1\}$ and $I_{2}=\left\{m_{1}+2, \ldots, m^{\prime}\right\}$. By inductive hypothesis, we have that $\Delta_{1}$ and $\Delta_{2}$ contain exactly $m_{1}+1$ and $m_{2}+1$ affinely independent points, respectively. Observe that $\mathcal{S}^{\prime}=\left\{S_{1}\right\} \cup \mathcal{S}_{1} \cup\left\{S \cup S_{1}\right\}_{S \in \mathcal{S}_{2}}$. Since $\mathcal{S}$ is closed under union, for each $S \in \mathcal{S}_{1}$, there exists $i \in I_{2}$ such that $S \cup S_{1}=S_{i}$. It follows that for all $\delta \in \Delta^{\prime}$ with $\delta_{1}=1$, for each $i_{1} \in I_{1}$, there exists $i_{2} \in I_{2}$ such that $\delta_{i_{1}}=\delta_{i_{2}}$. Therefore, there exists a mapping $F: \Delta_{2} \rightarrow \Delta_{1}$ such that

$$
\Delta^{\prime}=\left\{\left(0, \delta^{1}, 0\right): \delta^{1} \in \Delta_{1}\right\} \cup\left\{\left(1, F\left(\delta^{2}\right), \delta^{2}\right): \delta^{2} \in \Delta_{2}\right\}
$$

Then it is easy to verify that $\Delta^{\prime}$ is a set of $m^{\prime}+1\left(=\left|\Delta_{1}\right|+\left|\Delta_{2}\right|\right)$ affinely independent points.
It is easy to see that given a family $\mathcal{S}=\left\{S_{i}\right\}_{i \in I}$, the condition $\Delta^{l, u}=\Delta^{0, n}$ holds provided that $l \leq$ $n-\left|\bigcup_{i \in I} S_{i}\right|$ and

$$
u \geq \max _{\tilde{I} \subseteq I}\left\{|\tilde{I}|: S_{i} \backslash S_{j} \neq \emptyset, S_{j} \backslash S_{i} \neq \emptyset \text { for any } i \neq j \in \tilde{I}\right\}
$$

When the sets are nested, as in Example 1, these conditions simply reduce to $l \leq n-\left|S_{m}\right|$ and $u \geq 1$.

### 2.2 Properties of valid inequalities for $X^{l, u}$

Notice that an inequality $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ is valid for $X^{l, u}$ if and only if

$$
\gamma \geq \max _{(z, \delta) \in X^{l, u}}\left\{\alpha^{T} z+\beta^{T} \delta\right\}=\max _{\bar{\delta} \in \Delta^{l, u}}\left\{\beta^{T} \bar{\delta}+\max _{z \in X^{l, u}(\bar{\delta})} \alpha^{T} z\right\}
$$

In other words, it is valid if and only if

$$
\begin{equation*}
\gamma-\beta^{T} \bar{\delta} \geq \max _{z \in X^{l, u}(\bar{\delta})} \alpha^{T} z \tag{12}
\end{equation*}
$$

holds for all $\bar{\delta} \in \Delta^{l, u}$. We next characterize some properties of facet-defining inequalities for $\operatorname{conv}\left(X^{l, u}\right)$.
Lemma 3. Assume $\mathcal{S}$ is a proper family. Let $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ be a facet-defining inequality for conv $\left(X^{l, u}\right)$, and let $F$ be the associated facet. Then, either $F$ is defined by a facet-defining inequality of the form $\left(\beta^{\prime}\right)^{T} \delta \leq \gamma^{\prime}$ which also defines a facet of $\operatorname{conv}\left(\Delta^{l, u}\right)$, or (12) holds as equality for all $\bar{\delta} \in \Delta^{l, u}$.

Proof. Let $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ be a facet-defining inequality and assume that inequality (12) is strict for some $\bar{\delta} \in \Delta^{l, u}$. In this case, the facet $F$ does not contain any integral points of the form $(z, \bar{\delta})$ and consequently, for all integral points $(z, \delta) \in F$ we have $\delta \in \Delta^{l, u} \backslash\{\bar{\delta}\}$.
As $\mathcal{S}$ is a proper family, $\operatorname{conv}\left(\Delta^{l, u}\right)$ is a full-dimensional simplex in $\mathbb{R}^{m}$ with $m+1$ facets. Let $\left(\beta^{\prime}\right)^{T} \delta \leq \gamma^{\prime}$ be the (unique) facet-defining inequality for $\operatorname{conv}\left(\Delta^{l, u}\right)$ such that $\bar{\delta}$ is not contained in the corresponding facet
$F^{\prime}$. Note that all points in $\Delta^{l, u} \backslash\{\bar{\delta}\}$ satisfy $\left(\beta^{\prime}\right)^{T} \delta=\gamma^{\prime}$. As all integral points in $F$ have their $\delta$ components in $\Delta^{l, u} \backslash\{\bar{\delta}\}$, we conclude that all integral points in $F$ satisfy $\left(\beta^{\prime}\right)^{T} \delta=\gamma^{\prime}$. Therefore, $F$ is defined by the inequality $\left(\beta^{\prime}\right)^{T} \delta \leq \gamma^{\prime}$.

Given a proper family $\mathcal{S}$ with $\Delta^{l, u}=\left\{\boldsymbol{\delta}^{1}, \ldots, \boldsymbol{\delta}^{m+1}\right\}$ and a vector $\alpha \in \mathbb{R}^{n}$, let $A_{\mathcal{S}} \in \mathbb{R}^{(m+1) \times(m+1)}$ be the matrix with rows

$$
\left[A_{\mathcal{S}}\right]_{i}=\left[1,-\left(\delta^{i}\right)^{T}\right], \quad i=1, \ldots, m+1
$$

and $\nu_{\alpha} \in \mathbb{R}^{(m+1)}$ be the vector with entries

$$
\begin{equation*}
\left[\nu_{\alpha}\right]_{i}=\max _{z \in X^{l, u}\left(\boldsymbol{\delta}^{i}\right)} \alpha^{T} z, \quad i=1, \ldots, m+1 \tag{13}
\end{equation*}
$$

When (12) holds as equality for all $\bar{\delta} \in \Delta^{l, u}$, we can now write (12) in matrix form as $A_{\mathcal{S}}\binom{\gamma}{\beta}=\nu_{\alpha}$. Note that as $\mathcal{S}$ is proper, the vectors in $\Delta^{l, u}$ are affinely independent and therefore $A_{\mathcal{S}}$ is nonsingular. Then, for any given $\alpha \in \mathbb{R}^{n}$, we can construct a valid inequality $\alpha^{T} z+\beta_{\alpha}^{T} \delta \leq \gamma_{\alpha}$ for $X^{l, u}$ where

$$
\begin{equation*}
\binom{\gamma_{\alpha}}{\beta_{\alpha}}:=\left(A_{\mathcal{S}}\right)^{-1} \nu_{\alpha} \tag{14}
\end{equation*}
$$

Moreover, by Lemma 3 if $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ defines a facet $F$ of $\operatorname{conv}\left(X^{l, u}\right)$, then either $\beta=\beta_{\alpha}$ and $\gamma=\gamma_{\alpha}$, or, $F$ is defined by an inequality of the form $\left(\beta^{\prime}\right)^{T} \delta \leq \gamma^{\prime}$.

We will need the following definition in the next lemma.

Definition 2. For any two vectors $\alpha, \alpha^{\prime} \in \mathbb{R}^{n}$, we say $\alpha^{\prime}$ follows the pattern of $\alpha$ if

1. For each $j \in J,(i)$ if $\alpha_{j} \geq 0$, then $\alpha_{j}^{\prime} \geq 0$, and (ii) if $\alpha_{j} \leq 0$, then $\alpha_{j}^{\prime} \leq 0$;
2. For each pair $j_{1}, j_{2} \in J$, if $\alpha_{j_{1}} \geq \alpha_{j_{2}}$, then $\alpha_{j_{1}}^{\prime} \geq \alpha_{j_{2}}^{\prime}$.

By definition, it can be shown that if $\alpha^{\prime}$ follows the pattern of $\alpha$, then there exists an optimal solution of (13) such that it remains optimal if we replace $\alpha$ by $\alpha^{\prime}$ in (13). The next lemma has a similar flavor of this observation, and will be used to show that we can put a restriction on $\alpha$ when we consider any facet-defining inequality with coefficients defined by (14).

Lemma 4. Assume $\mathcal{S}$ is a proper family and $(\alpha, \beta, \gamma)$ satisfies (12) as equality for all $\bar{\delta} \in \Delta^{l, u}$. If two vectors $\alpha^{+}, \alpha^{-} \in \mathbb{R}^{n}$ both follow the pattern of $\alpha$ and $\alpha=\lambda \alpha^{+}+\mu \alpha^{-}$for some $\lambda \geq 0$ and $\mu \geq 0$, then

$$
\nu_{\alpha}=\lambda \nu_{\alpha^{+}}+\mu \nu_{\alpha^{-}}
$$

where $\nu_{\alpha}$ is defined as in (13).

Proof. As in the proof of Lemma 1 for a given $\bar{\delta} \in \Delta^{l, u}$ let $J_{0}:=\bigcup_{i \in I: \bar{\delta}_{i}=1} S_{i}, I_{0}:=\left\{i \in I: \bar{\delta}_{i}=0\right\}$, let $I^{*} \subseteq I_{0}$ denote the index set of minimal elements (with respect to inclusion) of $\left\{S_{i}: i \in I_{0}\right\}$, and let $J_{i}:=S_{i} \backslash J_{0}$ for $i \in I^{*}$. Note that the optimal value of $\max _{z \in X^{l, u}(\bar{\delta})} \alpha^{T} z$ is equal to $\sum_{i \in I^{*}} \max _{j \in J_{i}} \alpha_{j}$ (the sum of one largest $\alpha_{j}$ in $\left\{\alpha_{j}\right\}_{j \in J_{i}}$ for $i \in I^{*}$ ) plus the largest sum of at least ( $\max \left\{l-\left|I^{*}\right|, 0\right\}$ ) and up to $\left(u-\left|I^{*}\right|\right)$ largest remaining $\alpha_{j}$ values for $j \in J \backslash J_{0}$.

For $=1, \ldots, m+1$, let

$$
\begin{equation*}
\bar{z}^{i} \in \arg \max _{z \in X^{l}, u\left(\delta^{i}\right)} \alpha^{T} z . \tag{15}
\end{equation*}
$$

As $\alpha^{+}$and $\alpha^{-}$both follow the pattern of $\alpha, \bar{z}^{i}$ remains optimal for (15) after replacing $\alpha$ by $\alpha^{+}$or $\alpha^{-}$, i.e.,

$$
\bar{z}^{i} \in\left(\arg \max _{z \in X^{l}, u\left(\delta^{i}\right)}\left(\alpha^{+}\right)^{T} z\right) \cap\left(\arg \max _{z \in X^{l}, u\left(\boldsymbol{\delta}^{i}\right)}\left(\alpha^{-}\right)^{T} z\right),
$$

for $i=1, \ldots, m+1$. We next construct a matrix $\bar{Z}$ with columns $\bar{z}^{i}$ and observe that

$$
\bar{Z}^{T} \alpha=\nu_{\alpha}, \quad \bar{Z}^{T} \alpha^{+}=\nu_{\alpha^{+}}, \quad \text { and, } \bar{Z}^{T} \alpha^{-}=\nu_{\alpha^{-}} .
$$

Therefore, we have

$$
\nu_{\alpha}=\bar{Z}^{T} \alpha=\lambda \bar{Z}^{T} \alpha^{+}+\mu \bar{Z}^{T} \alpha^{-}=\lambda \nu_{\alpha^{+}}+\mu \nu_{\alpha^{-}} .
$$

Using this technical result, we next make an observation on the coefficients of facet-defining inequalities.
Lemma 5. Assume $\mathcal{S}$ is a proper family. Then each facet $F$ of $\operatorname{conv}\left(X^{l, u}\right)$ is defined by an inequality $\bar{\alpha}^{T} z+\bar{\beta}^{T} \delta \leq \bar{\gamma}$ where $\bar{\alpha} \in\{0, \kappa\}^{|J|}$ for some $\kappa \in \mathbb{R}$.

Proof. Assume that the claim does not hold. Then there is a facet $F$ such that any inequality $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ defining $F$ has the property that $\alpha$ has at least two distinct nonzero components. If $\operatorname{conv}\left(X^{l, u}\right)$ is fulldimensional, then there is a unique inequality (up to positive scaling) $\bar{\alpha}^{T} z+\bar{\beta}^{T} \delta \leq \bar{\gamma}$ defining $F$. If $\operatorname{conv}\left(X^{l, u}\right)$ is not full-dimensional, we chose $\bar{\alpha}^{T} z+\bar{\beta}^{T} \delta \leq \bar{\gamma}$ be an inequality defining $F$ such that $\bar{\alpha}$ has the smallest number $(\geq 2)$ of distinct nonzero components.
Let $\alpha_{\text {min }}$ denote the smallest nonzero component of $\bar{\alpha}$ and let $J_{\min }=\left\{j \in J: \bar{\alpha}_{j}=\alpha_{\text {min }}\right\}$. Let $\alpha^{+}$and $\alpha^{-}$ be obtained from $\bar{\alpha}$ as follows

$$
\alpha_{j}^{+}=\left\{\begin{array}{ll}
\bar{\alpha}_{j}+\epsilon, & \text { if } j \in J_{\min },  \tag{16}\\
\bar{\alpha}_{j}, & \text { otherwise },
\end{array} \quad \alpha_{j}^{-}= \begin{cases}\bar{\alpha}_{j}-\epsilon, & \text { if } j \in J_{\min }, \\
\bar{\alpha}_{j}, & \text { otherwise },\end{cases}\right.
$$

where $\epsilon>0$ is sufficiently small so that $\alpha^{+}$and $\alpha^{-}$follow the pattern of $\bar{\alpha}$. Then by Lemma 4 we have $\nu_{\bar{\alpha}}=\frac{1}{2} \nu_{\alpha^{+}}+\frac{1}{2} \nu_{\alpha^{-}}$and using (14), we can define two valid inequalities

$$
\begin{equation*}
\left(\alpha^{+}\right)^{T} z+\left(\beta_{\alpha^{+}}\right)^{T} \delta \leq \gamma_{\alpha^{+}}, \quad\left(\alpha^{-}\right)^{T} z+\left(\beta_{\alpha^{-}}\right)^{T} \delta \leq \gamma_{\alpha^{-}} . \tag{17}
\end{equation*}
$$

In this way, $(\bar{\beta}, \bar{\gamma})=\frac{1}{2}\left(\beta_{\alpha^{+}}, \gamma_{\alpha^{+}}\right)+\frac{1}{2}\left(\beta_{\alpha^{-}}, \gamma_{\alpha^{-}}\right)$. Consequently, $\bar{\alpha}^{T} z+\bar{\beta}^{T} \delta \leq \bar{\gamma}$ can be expressed as a strict convex combination of two valid inequalities. Moreover, these two inequalities are distinct (not a multiple of the original inequality) as $\left|\left\{\bar{\alpha}_{j}: \bar{\alpha}_{j} \neq 0, j \in J\right\}\right| \geq 2$. When $\operatorname{conv}\left(X^{l, u}\right)$ is full dimensional, this leads to a contradiction.

On the other hand, if $\operatorname{conv}\left(X^{l, u}\right)$ is contained in an affine subspace, then it is possible that both inequalities define the same facet as the original one. In this case, we can increase $\epsilon$ in (16) as much as possible while $\alpha^{+}$ and $\alpha^{-}$follow the pattern of $\alpha$. The largest such $\epsilon$ would give an $\alpha^{+}$or $\alpha^{-}$with one fewer distinct nonzero entries than $\alpha$. This again leads to a contradiction as $\alpha$ was assumed to have the smallest number of distinct nonzero components.

We conclude this section by showing that the convex hull of $X^{l, u}$ can simply be obtained from convex hulls of $X^{0, u}$ and $X^{l, n}$ provided that $\mathcal{S}$ satisfies some simple conditions.

Theorem 6. Assume $\mathcal{S}$ is a proper family and $\Delta^{l, u}=\Delta^{0, n}$. Then

$$
\operatorname{conv}\left(X^{l, u}\right)=\operatorname{conv}\left(X^{0, u}\right) \cap \operatorname{conv}\left(X^{l, n}\right)
$$

Proof. As $X^{l, u}=X^{0, u} \cap X^{l, n}$, we have $\operatorname{conv}\left(X^{l, u}\right) \subseteq \operatorname{conv}\left(X^{0, u}\right) \cap \operatorname{conv}\left(X^{l, n}\right)$. We next show that the reverse inclusion also holds. We first consider the case when $\operatorname{conv}\left(X^{l, u}\right)$ is not full-dimensional and argue that the affine hull of $\operatorname{conv}\left(X^{l, u}\right)$ is the same as that of $\operatorname{conv}\left(X^{0, u}\right) \cap \operatorname{conv}\left(X^{l, n}\right)$. Let $\alpha^{T} z+\beta^{T} \delta=\gamma$ be an equation satisfied by all points in $\operatorname{conv}\left(X^{l, u}\right)$. Consider now only one direction of the equation $\alpha^{T} z+\beta^{T} \delta \leq \gamma$. Using the notation defined in the proof of Lemma 3, we have $(\beta, \gamma)=\left(A_{\mathcal{S}}\right)^{-1} \nu_{\alpha}$ as $\alpha^{T} z+\beta^{T} \boldsymbol{\delta}^{i}=\gamma$ for any $z \in X^{l, u}\left(\boldsymbol{\delta}^{i}\right)$. Let $\alpha^{+}$and $\alpha^{-}$denote the nonnegative part and the nonpositive part of $\alpha$, respectively, i.e.,

$$
\alpha_{j}^{+}=\max \left\{\alpha_{j}, 0\right\}, \alpha_{j}^{-}=\min \left\{\alpha_{j}, 0\right\}, \quad j=1, \ldots, n
$$

Letting $\gamma^{+}=\gamma_{\alpha^{+}}, \beta^{+}=\beta_{\alpha^{+}}, \gamma^{-}=\gamma_{\alpha^{-}}, \beta^{-}=\beta_{\alpha^{-}}$as defined in (14), we see that the following inequalities are valid for $X^{l, u}$ :

$$
\left(\alpha^{+}\right)^{T} z+\left(\beta^{+}\right)^{T} \delta \leq \gamma^{+}, \quad\left(\alpha^{-}\right)^{T} z+\left(\beta^{-}\right)^{T} \delta \leq \gamma^{-}
$$

Moreover, as $\alpha=\alpha^{+}+\alpha^{-}$and both $\alpha^{+}$and $\alpha^{-}$follow the pattern of $\alpha$, by Lemma 4 we have $\nu_{\alpha}=\nu_{\alpha^{+}}+\nu_{\alpha^{-}}$, and therefore $\beta^{+}+\beta^{-}=\beta$ and $\gamma^{+}+\gamma^{-}=\gamma$. Note that when $\bar{\alpha} \geq 0, \nu_{\bar{\alpha}}$ does not depend on $l$ as its $i$-th entry is equal to $\sum_{i \in I^{*}}\left(\max _{j \in J_{i}} \bar{\alpha}_{j}\right)$ plus the sum of the $\left(u-\left|I^{*}\right|\right)$ largest remaining $\alpha_{j}$ values for $j \in J \backslash J_{0}$, where $I^{*},\left\{J_{i}\right\}_{i \in I^{*} \cup\{0\}}$ are associated with $\boldsymbol{\delta}^{i}$ (as defined in Lemma 1). It follows that

$$
\left[\nu_{\alpha}\right]_{i}=\max _{z \in X^{l, u}\left(\boldsymbol{\delta}^{i}\right)}\left(\alpha^{+}\right)^{T} z=\max _{z \in X^{0, u}\left(\boldsymbol{\delta}^{i}\right)}\left(\alpha^{+}\right)^{T} z, \quad i=1, \ldots, m+1
$$

This implies that $\left(\alpha^{+}\right)^{T} z+\left(\beta^{+}\right)^{T} \delta \leq \gamma^{+}$is valid for $\operatorname{conv}\left(X^{0, u}\right)$. Using a similar argument it is easy to see that $\left(\alpha^{-}\right)^{T} z+\left(\beta^{-}\right)^{T} \delta \leq \gamma^{-}$is valid for $\operatorname{conv}\left(X^{l, n}\right)$. Note that $(\alpha, \beta, \gamma)=\left(\alpha^{+}, \beta^{+}, \gamma^{+}\right)+\left(\alpha^{-}, \beta^{-}, \gamma^{-}\right)$. Combining both inequalities, we have $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ is valid for $\operatorname{conv}\left(X^{0, u}\right) \cap \operatorname{conv}\left(X^{l, n}\right)$.

When we consider the other direction $-\alpha^{T} z-\beta^{T} \delta \leq-\gamma$, by repeating the argument above for $(-\alpha,-\beta,-\gamma)$, we see that $-\alpha^{T} z-\beta^{T} \delta \leq-\gamma$ is valid for $\operatorname{conv}\left(X^{0, u}\right) \cap \operatorname{conv}\left(X^{l, n}\right)$. This implies that $\alpha^{T} z+\beta^{T} \delta=\gamma$ is valid for $\operatorname{conv}\left(X^{0, u}\right) \cap \operatorname{conv}\left(X^{l, n}\right)$.
We now consider an arbitrary facet $F$ of $\operatorname{conv}\left(X^{l, u}\right)$. Let $\Delta^{l, u}=\left\{\boldsymbol{\delta}^{i}\right\}_{i=1}^{m+1}$. As $\Delta^{l, u} \subseteq \Delta^{0, u}, \Delta^{l, n} \subseteq \Delta^{0, n}$, the assumption of the theorem implies that

$$
\begin{equation*}
\Delta^{l, u}=\Delta^{0, n}=\Delta^{0, u}=\Delta^{l, n} . \tag{18}
\end{equation*}
$$

By Lemmas 3 and 5 , we only need to discuss the following two cases:

1. $F$ can be defined by an inequality $\left(\beta^{\prime}\right)^{T} \delta \leq \gamma^{\prime}$ which also defines a facet of $\Delta^{l, u}$. In this case, by (18), we have $\Delta^{l, u}=\Delta^{l, n}=\Delta^{0, u}$, and $\left(\beta^{\prime}\right)^{T} \delta \leq \gamma^{\prime}$ is also valid for $\operatorname{conv}\left(X^{0, u}\right) \cap \operatorname{conv}\left(X^{l, n}\right)$.
2. $F$ can be defined by an inequality $\bar{\alpha}^{T} z+\bar{\beta}^{T} \delta \leq \bar{\gamma}$ where $\bar{\alpha}_{j} \in\{0, \kappa\}$ for some $\kappa \in \mathbb{R}$ and (12) holds as equality for all $\bar{\delta} \in \Delta^{l, u}$. If $\kappa \geq 0$, then $\bar{\alpha}^{T} z+\bar{\beta}^{T} \delta \leq \bar{\gamma}$ is valid for $\operatorname{conv}\left(X^{0, u}\right)$. On the other hand, if $\kappa \leq 0$, then $\bar{\alpha}^{T} z+\bar{\beta}^{T} \delta \leq \bar{\gamma}$ is valid for $\operatorname{conv}\left(X^{l, n}\right)$. In both cases, $\bar{\alpha}^{T} z+\bar{\beta}^{T} \delta \leq \bar{\gamma}$ is also valid for $\operatorname{conv}\left(X^{0, u}\right) \cap \operatorname{conv}\left(X^{l, n}\right)$.

We therefore conclude that any inequality valid for $\operatorname{conv}\left(X^{l, u}\right)$ is also valid for $\operatorname{conv}\left(X^{0, u}\right) \cap \operatorname{conv}\left(X^{l, n}\right)$, and consequently $\operatorname{conv}\left(X^{0, u}\right) \cap \operatorname{conv}\left(X^{l, n}\right) \subseteq \operatorname{conv}\left(X^{l, u}\right)$.

## 3 Convex hull description when $\mathcal{S}$ is a family of nested sets

In this section, we consider the special case when $\mathcal{S}=\left\{S_{i}\right\}_{i \in I}$ is a family of nested sets. In other words, we assume that $S_{1} \subset S_{2} \subset \ldots \subset S_{m} \subset J=\{1, \ldots, n\}$, and without loss of generality, we use $S_{i}=\left\{1, \ldots, k_{i}\right\}$ where $2 \leq k_{1}<k_{2}<\ldots<k_{m}$. Remember that $I=\{1, \ldots, m\}$. To avoid trivial cases (see Remark 13 below), we further assume that $u \geq 2$ and $l \leq n-\left|S_{m}\right|$ (i.e., $k_{m} \leq n-l$ ). For convenience, we define $S_{0}=\emptyset$, $S_{m+1}=J, \delta_{0}=1$ and $\delta_{m+1}=0$.
Without loss of generality, we also assume that $l<u$. Note that if $l=u$, then $z_{n}=u-\sum_{j \in J \backslash\{n\}} z_{j}$ and any problem of the form $\min \left\{c^{T} z+d^{T} \delta:(z, \delta) \in X^{l, u}\right\}$ is equivalent to

$$
\begin{aligned}
\min \left\{\sum_{j \in J \backslash\{n\}} c_{j} z_{j}+c_{n}\left(u-\sum_{j \in J \backslash\{n\}} z_{j}\right)+d^{T} \delta:\right. & \delta_{i}
\end{aligned}=\prod_{j \in S_{i}}\left(1-z_{j}\right), i \in I ;, 7\left(\sum_{j \in J \backslash\{n\}} z_{j} \leq u ; z_{j} \in\{0,1\}, j \in J \backslash\{n\}\right\},
$$

and we can then work in the projected space without variable $z_{n}$.

### 3.1 Basic properties of $\operatorname{conv}\left(X^{l, u}\right)$ and its continuous relaxation

Recall from Example 1 that $\mathcal{S}$ is a proper family. As $S_{i} \subset S_{i+1}$, all $(z, \delta) \in X^{l, u}$ satisfy $\delta_{i+1} \leq \delta_{i}$ for all $i<m$. Moreover, if $z_{j}=0$ for all $j \in S_{i+1} \backslash S_{i}$, then $\delta_{i+1}=\delta_{i}$. Consequently, the following inequalities are valid for $\operatorname{conv}\left(X^{l, u}\right)$ for all $i=1, \ldots, m-1$ :

$$
\begin{equation*}
\delta_{i}-\delta_{i+1}-\sum_{j \in S_{i+1} \backslash S_{i}}^{\delta_{i+1}-\delta_{i} \leq 0}, \tag{19}
\end{equation*}
$$

These inequalities are called 2-link inequalities by Crama and Rodríguez-Heck 6. When $\mathcal{S}$ is nested, Fischer, Fischer and McCormick [18] show that (19)-(20) along with the standard linearization (11)-(15) define the convex hull of $X^{0, n}$ (i.e. when $l=0, u=n$ ). Crama and Rodríguez-Heck [6] show the same result holds when $|\mathcal{S}|=2$ without assuming $\mathcal{S}$ is nested.

After adding (19)-(20) to the standard linearization of $X^{l, u}$, some of the initial inequalities (1)-(5) become redundant. We next give the subset of the inequalities (11)-(5) that give a correct formulation when combined with (19)-(20):

$$
\begin{array}{rlrl}
l \leq \sum_{j \in J} z_{j} & \leq u, & & \\
z_{j}+\delta_{i} & \leq 1, & & j \in S_{i}, i \in I, \\
1-\delta_{1}-\sum_{j \in S_{1}} z_{j} \leq 0, & & \\
-\delta_{m} & \leq 0, & & \\
-z_{j} & \leq 0, & & j \in J, \\
z_{j} & \leq 1, & &  \tag{26}\\
& & \\
\hline
\end{array}
$$

Note that unlike inequality (3), inequality (23) is only written for $S_{1}$ as (20) and (23) together imply the remaining inequalities in (3). Similarly, (19) and (24) imply that each $\delta_{i}$ is nonnegative.

Also note that given any $z \in\{0,1\}^{n}$ satisfying $l \leq \sum_{j \in J} z_{j} \leq u$, there exists a unique $\delta$ such that $(z, \delta) \in$ $X^{l, u}$. We next define this formally.

Definition 3. Given $U \subseteq J$ with $l \leq|U| \leq u$, we define the point $v^{U} \in X^{l, u}$ as follows:

$$
v^{U}=\left(z^{U}, \delta^{U}\right) \quad \text { where } \quad z_{j}^{U}=\left\{\begin{array}{ll}
1, & \text { if } j \in U, \\
0, & \text { otherwise },
\end{array} \quad \text { and } \quad \delta_{i}^{U}=\prod_{j \in S_{i}}\left(1-z_{j}^{U}\right)\right.
$$

Lemma 7. The polytope $\operatorname{conv}\left(X^{l, u}\right)$ is full-dimensional.

Proof. (sketch) We consider the two following cases separately:

If $l=0$, we argue that the following $m+n+1$ points in $\operatorname{conv}\left(X^{0, u}\right)$ are affinely independent:

$$
v^{\{2\}}=\left[\begin{array}{c}
e^{2} \\
0_{m}
\end{array}\right], \quad v^{\left\{k_{i}+1\right\}}=\left[\begin{array}{c}
e^{k_{i}+1} \\
d^{i}
\end{array}\right] \text { for } i \in I, \quad v^{\{1\}}=\left[\begin{array}{c}
e^{1} \\
0_{m}
\end{array}\right], \quad v^{\{1, j\}}=\left[\begin{array}{c}
e^{1}+e^{j} \\
0_{m}
\end{array}\right] \text { for } j \in J \backslash\{1\} .
$$

If, on the other hand, $l \geq 1$, then we let $Q=\{n-l+1, \ldots, n\} \subseteq J \backslash S_{m}$ where $|Q|=l$, and consider the following $m+n+1$ points in $\operatorname{conv}\left(X^{l, u}\right)$ :

$$
v^{\{2\} \cup Q}, \quad v^{\left\{k_{i}+1\right\} \cup Q} \text { for } i \in I, \quad v^{\{1\} \cup Q}, \quad v^{\{1, j\} \cup Q \backslash\{n\}} \text { for } j \in J \backslash\{1\} \backslash Q, \quad v^{\{1\} \cup Q \backslash\{j\}} \text { for } j \in Q,
$$

and argue that they are affinely independent. The detailed proof is presented in Appendix.

Lemma 7 also implies that $\operatorname{conv}\left(\Delta^{l, u}\right)$ is full-dimensional. In addition, it is easy to see that

$$
\begin{equation*}
\Delta^{l, u}=\left\{\delta \in\{0,1\}^{m}: \delta_{1} \geq \delta_{2} \geq \ldots \geq \delta_{m}\right\} \tag{27}
\end{equation*}
$$

Moreover, as the constraint matrix defining $\Delta^{l, u}$ above is totally unimodular, we also have

$$
\begin{equation*}
\operatorname{conv}\left(\Delta^{l, u}\right)=\left\{\delta \in \mathbb{R}^{m}: 1 \geq \delta_{1} \geq \delta_{2} \geq \ldots \geq \delta_{m} \geq 0\right\} \tag{28}
\end{equation*}
$$

From now on we will denote the $m+1$ vectors in $\Delta^{l, u}$ as

$$
\begin{equation*}
\Delta^{l, u}=\left\{\boldsymbol{\delta}^{[0]}, \boldsymbol{\delta}^{[1]}, \ldots, \boldsymbol{\delta}^{[m]}\right\} \tag{29}
\end{equation*}
$$

where $\boldsymbol{\delta}^{[0]}=\mathbf{0}$ and, for $i \in I$, the vector $\boldsymbol{\delta}^{[i]}$ has the first $i$ components equal to 1 and the rest equal to zero. Note that these vectors are affinely independent.

We start with characterizing facet-defining inequalities for $\operatorname{conv}\left(X^{l, u}\right)$ that have zero coefficients for all of the $z_{j}$ variables.

Lemma 8. If $\beta^{T} \delta \leq \gamma$ defines a facet of $\operatorname{conv}\left(X^{l, u}\right)$, then it is a multiple of an inequality from (19) or (24).

Proof. As $\operatorname{conv}\left(X^{l, u}\right)$ and $\operatorname{conv}\left(\Delta^{l, u}\right)$ are full-dimensional polytopes, if $\beta^{T} \delta \leq \gamma$ defines a facet of $\operatorname{conv}\left(X^{l, u}\right)$, then it also defines a facet of $\operatorname{conv}\left(\Delta^{l, u}\right)$. The only facet-defining inequality for $\operatorname{conv}\left(\Delta^{l, u}\right)$, see (28), that is not of the form (19) or (24) is $1 \geq \delta_{1}$. However, $1 \geq \delta_{1}$ cannot define a facet of $\operatorname{conv}\left(X^{l, u}\right)$ as it is implied by (22) and (25) for $i=1$ and any $j \in S_{1}$. Therefore, the only facet-defining inequalities of $\operatorname{conv}\left(\Delta^{l, u}\right)$ that can also define facets of $\operatorname{conv}\left(X^{l, u}\right)$ are of the form (19) and (24).

Under the assumptions (i) $u \geq 2$ and (ii) $l \leq n-\left|S_{m}\right|$, we have $\Delta^{l, u}=\Delta^{0, n}$ and consequently

$$
\operatorname{conv}\left(X^{l, u}\right)=\operatorname{conv}\left(X^{0, u}\right) \cap \operatorname{conv}\left(X^{l, n}\right)
$$

by Theorem 6. We next study $\operatorname{conv}\left(X^{0, u}\right)$ and $\operatorname{conv}\left(X^{l, n}\right)$ separately.

### 3.2 Trivial facets of $\operatorname{conv}\left(X^{0, u}\right)$

As $\operatorname{conv}\left(X^{0, u}\right)$ is full-dimensional, all facet-defining inequalities for $\operatorname{conv}\left(X^{0, u}\right)$ are uniquely defined up to multiplication by a positive scalar. We have already characterized all facets of the form $\beta^{T} \delta \leq \gamma$ in Lemma 8. We now characterize facet-defining inequalities of the form $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ for $\operatorname{conv}\left(X^{0, u}\right)$ with $\alpha \leq 0$ and $\alpha \neq 0$.

Lemma 9. Let $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ be a facet-defining inequality for $\operatorname{conv}\left(X^{0, u}\right)$. If $\alpha \leq 0$ and $\alpha \neq 0$, then the inequality is a multiple of one of the inequalities (20), (23) or (25).

Proof. Let $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ be a facet-defining inequality for $\operatorname{conv}\left(X^{0, u}\right)$ satisfying the conditions of the Lemma, and assume it defines the facet $F=\left\{(z, \delta) \in \operatorname{conv}\left(X^{0, u}\right): \alpha^{T} z+\beta^{T} \delta=\gamma\right\}$. By Lemma 5, we can assume without loss of generality that all nonzero components of $\alpha$ are equal to -1 . If $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ is the same as $z_{j} \geq 0$ for some $j \in J$, then it is one of the inequalities in (25) and the result follows. We will henceforth assume this is not the case.

If $\left|S_{i} \backslash S_{i-1}\right| \geq 2$ for any $i \in I$, then we will next argue that

$$
\begin{equation*}
\alpha_{j}=\alpha_{k} \text { for all distinct } j, k \in S_{i} \backslash S_{i-1} \tag{30}
\end{equation*}
$$

If this is not true, then we can assume, without loss of generality, that $\alpha_{j}=-1$ and $\alpha_{k}=0$ for some $j, k \in S_{i} \backslash S_{i-1}$. As $F$ is not defined by $z_{j} \geq 0$, we can assume that there is a point $(\bar{z}, \bar{\delta}) \in F$ with the property that $\bar{z}_{j}=1$. Consider the point $\left(z^{\prime}, \bar{\delta}\right)$ where the components of $z^{\prime}$ are the same as the components of $\bar{z}$, except that $z_{j}^{\prime}=0$ and $z_{k}^{\prime}=1$. It is easy to see that $\left(z^{\prime}, \bar{\delta}\right) \in X^{0, u}$, and $\alpha^{T} z^{\prime}=\alpha^{T} \bar{z}+1$ which implies that $\alpha^{T} z^{\prime}+\beta^{T} \bar{\delta}>\gamma$. This contradicts the fact that $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ is a valid inequality for $X^{0, u}$. Similarly, for any $j \in J \backslash S_{m}$, there exists a point $(\bar{z}, \bar{\delta}) \in F$ such that $\bar{z}_{j}=1$. If $\alpha_{j}=-1$, then constructing a new point by changing $\bar{z}_{j}$ to 0 shows that the inequality cannot be valid. Consequently, $\alpha_{j}=0$ for all $j \in J \backslash S_{m}$. As conv $\left(X^{0, u}\right)$ is full-dimensional, Lemma 3 and $\alpha \neq 0$ together imply that

$$
\begin{equation*}
\gamma-\beta^{T} \boldsymbol{\delta}^{[i]}=\max _{z \in X^{0, u}\left(\boldsymbol{\delta}^{[i]}\right)} \alpha^{T} z \tag{31}
\end{equation*}
$$

for $i=0, \ldots, m$. First note that as $\alpha \leq 0$ and $z \geq 0$, we have $\max _{z \in X^{0, u}\left(\boldsymbol{\delta}^{[m]}\right)} \alpha^{T} z=0$. Moreover, for $i=0, \ldots, m-1$, if $\bar{z} \in X^{0, u}\left(\boldsymbol{\delta}^{[i]}\right)$, then $\bar{z}_{j}=0$ for $j \in S_{i}$ and $\sum_{j \in S_{i+1} \backslash S_{i}} \bar{z}_{j} \geq 1$. Therefore,

$$
\max _{z \in X^{0, u}\left(\boldsymbol{\delta}^{[i]}\right)} \alpha^{T} z=\max _{j \in S_{i+1} \backslash S_{i}}\left\{\alpha_{j}\right\} .
$$

Consider $\theta \in \mathbb{R}^{m+1}$ where $\theta_{i}$ equals to the right-hand side of (31). Then $\theta_{m}=0$, and for $i=0, \ldots, m-1$ we have $\theta_{i} \in\{0,-1\}$, with $\theta_{i}=\alpha_{j}$ for all $j \in S_{i+1} \backslash S_{i}$. Then (31) implies that

$$
\gamma=\theta_{0}, \quad \gamma-\sum_{i=1}^{k} \beta_{i}=\theta_{k} \text { for } k \in\{1, \ldots, m-1\}, \quad \gamma-\sum_{i=1}^{m} \beta_{i}=0
$$

These equations have the unique solution:

$$
\begin{equation*}
\gamma=\theta_{0}, \quad \beta_{i}=\theta_{i-1}-\theta_{i} \text { for } i \in\{1, \ldots, m-1\}, \quad \beta_{m}=\theta_{m-1} \tag{32}
\end{equation*}
$$

We next observe that $\alpha_{j}=0$ for all $j \in J \backslash S_{m}$ and $\theta_{i} \leq 0$ for all $i \in\{0\} \cup I$, and therefore

$$
\begin{aligned}
\alpha^{T} z+\beta^{T} \delta & =\sum_{i=1}^{m} \theta_{i-1}\left(\sum_{j \in S_{i} \backslash S_{i-1}} z_{j}\right)+\sum_{i=1}^{m-1}\left(\theta_{i-1}-\theta_{i}\right) \delta_{i}+\theta_{m-1} \delta_{m} \\
& =\theta_{0}(\underbrace{\delta_{1}+\sum_{j \in S_{1}} z_{j}}_{\geq 1})+\sum_{i=1}^{m-1} \theta_{i}(\underbrace{\delta_{i+1}-\delta_{i}+\sum_{j \in S_{i+1} \backslash S_{i}} z_{j}}_{\geq 0}) \leq \theta_{0}+0=\gamma
\end{aligned}
$$

Therefore, inequality $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ is implied by inequalities (23) and (20). As it is facet-defining, it must indeed be one of them.

### 3.3 Convex hull description of $X^{0, u}$

We next derive a family of valid inequalities for $\operatorname{conv}\left(X^{0, u}\right)$ using the mixing procedure [20]. The inequalities we derive here apply when $\mathcal{S}=\left\{S_{i}\right\}_{i \in I}$ is a family of nested sets and as we show later, together with inequalities (21)-(26), they give a complete description of $\operatorname{conv}\left(X^{0, u}\right)$. Later in Section (5) we will generalize these inequalities for the case when $\mathcal{S}$ is not necessarily nested.

For some positive integer $k$, let $1>b_{k}>b_{k-1}>\ldots>b_{1}>0$, be given and let

$$
\begin{equation*}
Q=\left\{s \in \mathbb{R}, z \in \mathbb{Z}^{k}: s+z_{i} \geq b_{i} \quad \text { for } i=1, \ldots, k, s \geq 0\right\} \tag{33}
\end{equation*}
$$

Then, the following type I mixing inequality is known to be valid for $Q$ (see [20]):

$$
\begin{equation*}
s+b_{1} z_{1}+\sum_{i=2}^{k}\left(b_{i}-b_{i-1}\right) z_{i} \geq b_{k} \tag{34}
\end{equation*}
$$

The inequalities $s+z_{i} \geq b_{i}$ are called base inequalities and note that inequality (34) combines the mixedinteger rounding inequalities $s+b_{i} z_{i} \geq b_{i}$ associated with the base inequalities using a "telescopic" sum. We next derive some valid inequalities for $X^{0, u}$ to use as base inequalities for applying the mixing procedure.

Let $S^{\prime} \subseteq J$ and $i \in I$ be given and let $M>n$ be a fixed constant. Using the fact that $z_{j} \leq 1,1-\delta_{i}-z_{j} \geq 0$, and $\delta_{i} \leq 1$, for all $j \in S_{i}$, we can derive the following valid (base) inequality for $\operatorname{conv}\left(X^{0, u}\right)$ :

$$
\begin{aligned}
\frac{1}{M}\left(u-\sum_{j \in S^{\prime}} z_{j}\right)+\left(1-\delta_{i}\right) & =\frac{1}{M}\left(u-\sum_{j \in S^{\prime} \backslash S_{i}} z_{j}\right)+\frac{1}{M} \sum_{j \in S^{\prime} \cap S_{i}}\left(1-\delta_{i}-z_{j}\right)+\frac{1}{M}\left(M-\left|S^{\prime} \cap S_{i}\right|\right)\left(1-\delta_{i}\right) \\
& \geq \frac{1}{M}\left(u-\sum_{j \in S^{\prime} \backslash S_{i}} z_{j}\right) \geq \frac{1}{M}\left(u-\left|S^{\prime} \backslash S_{i}\right|\right)
\end{aligned}
$$

Moreover, if $\left|S^{\prime} \backslash S_{p}\right| \leq u-1$ for some $p \in I$, then the right-hand side of this valid inequality

$$
\frac{1}{M}\left(u-\sum_{j \in S^{\prime}} z_{j}\right)+\left(1-\delta_{i}\right) \geq \frac{1}{M}\left(u-\left|S^{\prime} \backslash S_{i}\right|\right)
$$

is strictly between 0 and 1 for all $i=p, p+1, \ldots, m$. Therefore, we can write a set of the form (33) using these inequalities as the base inequalities where we treat the term $\frac{1}{M}\left(u-\sum_{j \in S^{\prime}} z_{j}\right)$ as a nonnegative continuous variable and the term $\left(1-\delta_{i}\right)$ as an integer variable for all $i=p, p+1, \ldots, m$. Consequently, the resulting type I mixing inequality,
$\frac{1}{M}\left(u-\sum_{j \in S^{\prime}} z_{j}\right)+\frac{1}{M}\left(u-\left|S^{\prime} \backslash S_{p}\right|\right)\left(1-\delta_{p}\right)+\frac{1}{M} \sum_{i=p+1}^{m}\left(\left|S^{\prime} \backslash S_{i-1}\right|-\left|S^{\prime} \backslash S_{i}\right|\right)\left(1-\delta_{i}\right) \geq \frac{1}{M}\left(u-\left|S^{\prime} \backslash S_{m}\right|\right)$, which can be simplified to

$$
\frac{1}{M}\left(u-\sum_{j \in S^{\prime}} z_{j}\right) \geq \frac{1}{M}\left(u-\left|S^{\prime} \backslash S_{p}\right|\right) \delta_{p}+\frac{1}{M} \sum_{i=p+1}^{m}\left(\left|S^{\prime} \backslash S_{i-1}\right|-\left|S^{\prime} \backslash S_{i}\right|\right) \delta_{i}
$$

is valid for $\operatorname{conv}\left(X^{0, u}\right)$. After multiplying the inequality by $M$ and rearranging the terms, we obtain the following valid inequality for $\operatorname{conv}\left(X^{0, u}\right)$

$$
\begin{equation*}
\sum_{j \in S^{\prime}} z_{j}+\left(u-\left|S^{\prime} \backslash S_{p}\right|\right) \delta_{p}+\sum_{i=p+1}^{m}\left(\left|S^{\prime} \backslash S_{i-1}\right|-\left|S^{\prime} \backslash S_{i}\right|\right) \delta_{i} \leq u \tag{35}
\end{equation*}
$$

We next give an inequality description of $\operatorname{conv}\left(X^{0, u}\right)$ using the mixing inequalities.
Theorem 10. A complete inequality description of $\operatorname{conv}\left(X^{0, u}\right)$ is given by inequalities (21)-(26) together with inequalities (35) for all $p \in I$ and $S^{\prime} \subseteq J$ such that $\left|S^{\prime} \backslash S_{p}\right| \leq u-1$.

Proof. Let $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ be a facet-defining inequality for $\operatorname{conv}\left(X^{0, u}\right)$ and note that by Lemma 7 it has a unique representation up to multiplication. By Lemma 5 we can assume that either $\alpha \in\{0,1\}^{|J|}$ or $\alpha \in\{0,-1\}^{|J|}$. Furthermore, by Lemmas 8 and 9 we have established that if $\alpha \leq 0$ (including the case when $\alpha=0)$ the inequality $\alpha^{T} z+\beta^{T} \delta \leq \gamma$ has to be one of (21) $-(26)$. Therefore, the only remaining case to consider is when $\alpha \in\{0,1\}^{|J|}$ and $\alpha \neq 0$.
Let $\bar{S}:=\left\{j \in J: \alpha_{j}=1\right\}$ and therefore $\alpha^{T} z=\sum_{j \in \bar{S}} z_{j}$. Also remember that $\Delta^{0, u}=\left\{\boldsymbol{\delta}^{[0]}, \ldots, \boldsymbol{\delta}^{[m]}\right\}$ where the first $p \in I$ components of $\boldsymbol{\delta}^{[p]} \in\{0,1\}^{m}$ are 1 , and the rest components are 0 . Then by Lemma 3 the following equations must hold for all $\boldsymbol{\delta}^{[p]}$ with $p \in\{0, \ldots, m-1\}$,

$$
\begin{align*}
\gamma-\sum_{i=1}^{p} \beta_{i} & =\max \left\{\bar{\alpha}^{T} z:\left(z, \boldsymbol{\delta}^{[p]}\right) \in X^{0, u}\right\} \\
& =\max \left\{\sum_{j \in \bar{S}} z_{j}: \sum_{j \in J} z_{j} \leq u ; z_{j}=0, \forall j \in S_{p} ; \sum_{j \in S_{p+1} \backslash S_{p}} z_{j} \geq 1, z \in\{0,1\}^{|J|}\right\} \\
& =\min \left\{u-\mathbb{1}_{\left\{\bar{S} \cap S_{p+1} \backslash S_{p}=\emptyset\right\}},\left|\bar{S} \backslash S_{p}\right|\right\}, \tag{36}
\end{align*}
$$

where we define $\mathbb{1}_{A}$ to be 1 if condition $A$ is true, and 0 , otherwise. Similarly, for $\boldsymbol{\delta}^{[m]}$, we have

$$
\begin{equation*}
\gamma-\sum_{i=1}^{m} \beta_{i}=\min \left\{u,\left|\bar{S} \backslash S_{m}\right|\right\} \tag{37}
\end{equation*}
$$

Let $\bar{S}_{i}=\bar{S} \cap S_{i}$ for $i \in I$ and let $D_{1}=\bar{S}_{1}$ and $D_{i}=\bar{S}_{i} \backslash \bar{S}_{i-1}$ for $i \in\{2, \ldots, m\}$. Note that $\bar{S}=$ $\left(\bar{S} \backslash S_{m}\right) \cup\left(\bigcup_{i=1}^{m} D_{i}\right)$. The unique solution to equations (36) and (37) is therefore

$$
\begin{aligned}
\gamma & =\min \left\{u-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}},|\bar{S}|\right\} \\
\beta_{i} & = \begin{cases}\min \left\{u-\mathbb{1}_{\left\{D_{i}=\emptyset\right\}},\left|\bar{S} \backslash S_{i-1}\right|\right\}-\min \left\{u-\mathbb{1}_{\left\{D_{i+1}=\emptyset\right\}},\left|\bar{S} \backslash S_{i}\right|\right\} & \text { for } 1 \leq i \leq m-1, \\
\min \left\{u-\mathbb{1}_{\left\{D_{m}=\emptyset\right\}},\left|\bar{S} \backslash S_{m-1}\right|\right\}-\min \left\{u,\left|\bar{S} \backslash S_{m}\right|\right\} & \text { for } i=m .\end{cases}
\end{aligned}
$$

We now consider 3 cases:
Case 1: $\left|\bar{S} \backslash S_{m}\right| \geq u$. In this case, $\left|\bar{S} \backslash S_{i}\right| \geq u$ also holds for all $i \in I$ and

$$
\begin{aligned}
\gamma & =u-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}}, \\
\beta_{i} & =\left\{\begin{aligned}
\mathbb{1}_{\left\{D_{i+1}=\emptyset\right\}}-\mathbb{1}_{\left\{D_{i}=\emptyset\right\}}, & i \in\{1, \ldots, m-1\}, \\
-\mathbb{1}_{\left\{D_{m}=\emptyset\right\}}, & i=m .
\end{aligned}\right.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\alpha^{T} z+\beta^{T} \delta & =\sum_{j \in D_{1}} z_{j}-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}}(\underbrace{\delta_{1}}_{\geq 1-\sum_{j \in S_{1}} z_{j}})+\sum_{i=1}^{m-1}[\sum_{j \in D_{i+1}} z_{j}+\mathbb{1}_{\left\{D_{i+1}=\emptyset\right\}}(\underbrace{\delta_{i}-\delta_{i+1}}_{\leq \sum_{j \in S_{i+1} \backslash S_{i}} z_{j}})]+\sum_{j \in \bar{S} \backslash S_{m}} z_{j} \\
& \leq \underbrace{}_{\left.\leq \sum_{j \in S_{1} z_{j}-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}}}^{\sum_{j \in D_{1}} z_{j}-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}}\left(1-\sum_{j \in S_{1}} z_{j}\right.}\right)}+\sum_{i=1}^{m-1}[\underbrace{\left.\sum_{j \in D_{i+1}} z_{j}+\mathbb{1}_{\left\{D_{i+1}=\emptyset\right\}}\left(\sum_{j \in S_{i+1} \backslash S_{i}} z_{j}\right)\right]+\sum_{j \in \bar{S} \backslash S_{m}} z_{j}}_{\leq \sum_{j \in S_{i+1} \backslash S_{i}} z_{j}} \\
& \leq \sum_{j \in J} z_{j}-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}} \leq u-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}}=\gamma .
\end{aligned}
$$

In the first inequality above, we use inequalities (23) and (20) and in the second inequality we use the fact that if $\mathbb{1}_{\left\{D_{i}=\emptyset\right\}}=1$, then $\sum_{j \in D_{i}} z_{j}=0$ for all $i \in I$.
Therefore, inequalities (21)-(26) imply $\alpha^{T} z+\beta^{T} \delta \leq \gamma$.
Case 2a: $\left|\bar{S} \backslash S_{m}\right| \leq u-1$ and $|\bar{S}| \leq u-1$. In this case, equations (36) and (37) imply

$$
\gamma=|\bar{S}|, \quad \text { and } \quad \beta_{i}=\left|\bar{S} \backslash S_{i-1}\right|-\left|\bar{S} \backslash S_{i}\right|=\left|D_{i}\right|, \quad i \in I
$$

In this case, using inequalities (22) and (26), we can write

$$
\alpha^{T} z+\beta^{T} \delta=\sum_{i=1}^{m}\left[\sum_{j \in D_{i}}\left(z_{j}+\delta_{i}\right)\right]+\sum_{j \in \bar{S} \backslash S_{m}} z_{j} \leq \sum_{i=1}^{m}\left|D_{i}\right|+\left|\bar{S} \backslash S_{m}\right|=|\bar{S}|=\gamma
$$

Therefore, all points that satisfy equations (21)-(26) also satisfy $\alpha^{T} z+\beta^{T} \delta \leq \gamma$.

Case 2b: $\left|\bar{S} \backslash S_{m}\right| \leq u-1$ and $|\bar{S}| \geq u$. Let $h:=\min \left\{i \in I:\left|\bar{S} \backslash S_{i}\right| \leq u-1\right\}$. In this case,

$$
\begin{aligned}
\gamma & =u-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}}, \\
\beta_{i} & = \begin{cases}\mathbb{1}_{\left\{D_{i+1}=\emptyset\right\}}-\mathbb{1}_{\left\{D_{i}=\emptyset\right\}}, & i \in\{1, \ldots, h-1\}, \\
u-\mathbb{1}_{\left\{D_{h}=\emptyset\right\}}-\left|\bar{S} \backslash S_{h}\right|, & i=h, \\
\left|\bar{S} \backslash S_{i-1}\right|-\left|\bar{S} \backslash S_{i}\right|=\left|D_{i}\right|, & i \in\{h+1, \ldots, m\}\end{cases}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\alpha^{T} z+\beta^{T} \delta= & \sum_{j \in D_{1}} z_{j}-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}}(\underbrace{\delta_{1}}_{\geq 1-\sum_{j \in S_{1}} z_{j}})+\sum_{i=1}^{h-1}[\sum_{j \in D_{i+1}} z_{j}+\mathbb{1}_{\left\{D_{i+1}=\emptyset\right\}}(\underbrace{\delta_{i}-\delta_{i+1}}_{\leq \sum_{j \in S_{i+1} \backslash S_{i}} z_{j}})] \\
& +\left[\sum_{j \in D_{h+1}} z_{j}+\left(u-\left|\bar{S} \backslash S_{h}\right|\right) \delta_{h}\right]+\sum_{j=h+1}^{m}\left[\sum_{j \in D_{i+1}} z_{j}+\left|D_{i}\right| \delta_{i}\right]+\sum_{j \in \bar{S} \backslash S_{m}} z_{j} \\
\leq & \underbrace{\sum_{j \in S_{1}} z_{j}-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}}}_{\sum_{j \in D_{1}} z_{j}-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}}\left(1-\sum_{j \in S_{1}} z_{j}\right)}+\sum_{i=1}^{h-1}[\underbrace{\leq \sum_{j \in S_{i+1} \backslash S_{i}} z_{j}}_{\sum_{j \in D_{i+1}} z_{j}+\mathbb{1}_{\left\{D_{i+1}=\emptyset\right\}} \sum_{j \in S_{i+1} \backslash S_{i}} z_{j}} \\
& +\left[\sum_{j \in \bar{S}_{h+1} \backslash S_{h}} z_{j}+\left(u-\left|\bar{S} \backslash S_{h}\right|\right) \delta_{h}\right] \\
\leq & \sum_{j=h+1}^{m}\left[\sum_{j \in D_{i+1}} z_{j}+\left|D_{i}\right| \delta_{i}\right]+\sum_{j \in \bar{S} \backslash S_{m}} z_{j} \\
\leq & u-z_{j \in \bar{S} \cup S_{h}}+\left(u-\left|\bar{S} \backslash S_{h}\right|\right) \delta_{h}+\sum_{i=h+1}^{m}\left|D_{i}\right| \delta_{i}-\mathbb{1}_{\left\{D_{1}=\emptyset\right\}} \\
\leq &
\end{aligned}
$$

where the last inequality folows from the mixing inequality (35) with $S^{\prime}=\bar{S} \cup S_{h}$ and $p=h$.

### 3.4 Convex hull description of $X^{l, n}$

In [18], the authors study the convex hull description of the following set:

$$
\begin{equation*}
\left\{(x, \delta) \in\{0,1\}^{|J|+|I|}: \delta_{i}=\prod_{j \in S_{i}} x_{j} \quad \text { for } i \in I, \quad x \in P_{\mathcal{M}}\right\} \tag{38}
\end{equation*}
$$

where $\left\{S_{i}\right\}_{i \in I}$ is a family of nested subsets of a given set $J$ and $P_{\mathcal{M}}$ is the convex hull of incidence vectors associated with independent sets $\mathcal{U}$ of the matroid $\mathcal{M}=(J, \mathcal{U})$ defined on the ground set $J$. Note that if we let $\mathcal{U}$ be the set of all subsets of $J$ with cardinality at most $k$ for some $k \in \mathbb{Z}_{+}$, the constraint $x \in P_{\mathcal{M}}$ simply becomes $\sum_{j \in J} x_{j} \leq k$. Consequently, using this matroid in (38) leads to a set very similar to the one we have been studying. More precisely, taking $k=n-l$ to define the independent sets and replacing $x_{j}$ with $\left(1-z_{j}\right)$ for $j \in J$, gives the set $X^{l, n}$. Note that due to the complementation of the $x$ variables in (38), the upper bound on the sum of the $x$ variables becomes a lower bound on the sum of the $z$ variables.

Using the particular matroid described above, we next translate the results from 18 to our context. Remember that we use $S_{0}=\emptyset$ and $S_{m+1}=J$ for convenience.

Theorem 11 (Fischer, Fischer and McCormick [18]). Inequalities (21)-(26) together with

$$
\begin{equation*}
\sum_{j \in S^{\prime}} z_{j} \geq\left(\left|S^{\prime} \cup S_{p}\right|-n+l\right) \delta_{p}+\sum_{i=p+1}^{m}\left(\left|S^{\prime} \cup S_{i}\right|-\left|S^{\prime} \cup S_{i-1}\right|\right) \delta_{i} \tag{39}
\end{equation*}
$$

for all $p \in I$ and $S^{\prime} \subset J$ that satisfy $\left|S^{\prime} \cup S_{p-1}\right| \leq n-l<\left|S^{\prime} \cup S_{p}\right|$ give a complete description of conv $\left(X^{l, n}\right)$.

Notice that similar to inequalities (35), inequalities (39) above are also defined for subsets of $J$ and both (35) and (39) have the term $\sum_{j \in S^{\prime}} z_{j}$ as well as a telescopic sum involving the $\delta$ variables. We next show that (39) can also be derived using the mixing procedure. Let $S^{\prime} \subset J$ be fixed and let $M>n$ be a given constant. For any $i \in I$, the following (base) inequality is valid for $X^{l, n}$ :

$$
\begin{aligned}
\frac{1}{M} \sum_{j \in S^{\prime}} z_{j}+\left(1-\delta_{i}\right) & \geq \frac{1}{M}\left(\sum_{j \in S^{\prime} \cup S_{i}} z_{j}-\sum_{j \in S_{i} \backslash S^{\prime}} z_{j}\right)+\frac{\left|S_{i} \backslash S^{\prime}\right|}{M}\left(1-\delta_{i}\right) \\
& =\frac{1}{M}(\underbrace{\sum_{j \in J} z_{j}}_{\geq l}-\underbrace{\sum_{j \in J \backslash\left(S^{\prime} \cup S_{i}\right)} z_{j}}_{\leq\left|J \backslash\left(S^{\prime} \cup S_{i}\right)\right|})+\frac{1}{M} \sum_{j \in S_{i} \backslash S^{\prime}}(\underbrace{1-z_{j}-\delta_{i}}_{\geq 0}) \\
& \geq \frac{1}{M}\left(l-\left(n-\left|S^{\prime} \cup S_{i}\right|\right)\right)+0=\frac{\left|S^{\prime} \cup S_{i}\right|-n+l}{M} .
\end{aligned}
$$

When $\left|S^{\prime} \cup S_{p}\right| \geq n-l+1$, the right-hand side of the inequality is strictly between 0 and 1 , and treating the term $\frac{1}{M}\left(\sum_{j \in S^{\prime}} z_{j}\right)$ as a nonnegative continuous variable and $\left(1-\delta_{i}\right)$ as an integer variable, we can apply the type I mixing procedure to the base inequalities for $i=p, p+1, \ldots, m$ to obtain

$$
\frac{1}{M} \sum_{j \in S^{\prime}} z_{j}+\frac{\left|S^{\prime} \cup S_{p}\right|-n+l}{M}\left(1-\delta_{p}\right)+\sum_{i=p+1}^{m}\left(\frac{\left|S^{\prime} \cup S_{i}\right|-\left|S^{\prime} \cup S_{i-1}\right|}{M}\right)\left(1-\delta_{i}\right) \geq \frac{\left|S^{\prime} \cup S_{m}\right|-n+l}{M},
$$

which can be rewritten as

$$
\frac{1}{M} \sum_{j \in S^{\prime}} z_{j}-\frac{\left|S^{\prime} \cup S_{p}\right|-n+l}{M} \delta_{p}-\sum_{i=p+1}^{m}\left(\frac{\left|S^{\prime} \cup S_{i}\right|-\left|S^{\prime} \cup S_{i-1}\right|}{M}\right) \delta_{i} \geq 0
$$

Multiplying both sides by $M$ and rearranging the terms gives inequality (39). In Section 4 we will discuss the conditions under which these inequalities are facet-defining for $X^{l, n}$ and in Section 5 we will generalize these inequalities to the case when the sets in $\mathcal{S}$ are not necessarily nested.

We next present our main result:

Theorem 12. Let $\mathcal{S}=\left\{S_{i}\right\}_{i \in I}$ be a family of nested sets and assume that $u \geq 2$ and $l \leq n-\left|S_{|I|}\right|$. Then, $\operatorname{conv}\left(X^{l, u}\right)$ is defined by
(i) inequalities (21)-(26),
(ii) inequalities (35) for all $p \in I$ and $S^{\prime} \subseteq J$ such that $\left|S^{\prime} \backslash S_{p}\right| \leq u-1$, and,
(iii) inequalities (39) for all $p \in I$ and $S^{\prime} \subset J$ that satisfy $\left|S^{\prime} \cup S_{p-1}\right| \leq n-l<\left|S^{\prime} \cup S_{p}\right|$.

Moreover, given a point $(z, \delta) \notin \operatorname{conv}\left(X^{l, u}\right)$, a (most) violated inequality can be found in polynomial time.

Proof. Combining Theorems 6, 10 and 11it follows that $\operatorname{conv}\left(X^{l, u}\right)$ is given by inequalities (21)-(26) together with inequalities (35) and (39).

For the second part of the proof, note that there are a polynomial number of inequalities of the form (21)-(26) and there are an exponential number of mixing inequalities as one can write one for each $S^{\prime} \subset J$ and $p \in I$. However, for each $S^{\prime} \subseteq J$ and fixed $p \in I$ we can rewrite inequality (35) as

$$
\begin{equation*}
\sum_{j \in S^{\prime} \cap S_{p}} z_{j}+u \delta_{p}+\sum_{i=p+1}^{m} \sum_{j \in S^{\prime} \cap\left(S_{i} \backslash S_{i-1}\right)}\left(z_{j}+\delta_{i}-\delta_{p}\right)+\sum_{j \in S^{\prime} \backslash S_{m}}\left(z_{j}-\delta_{p}\right) \leq u . \tag{40}
\end{equation*}
$$

Given a fractional solution $(\hat{z}, \hat{\delta})$, let

$$
\pi_{j}= \begin{cases}\hat{z}_{j}, & \text { for } j \in S_{p} \\ \hat{z}_{j}+\hat{\delta}_{i}-\hat{\delta}_{p}, & \text { for } j \in S_{i} \backslash S_{i-1}, i=p+1, \ldots, m \\ \hat{z}_{j}-\hat{\delta}_{p}, & \text { for } j \in J \backslash S_{m}\end{cases}
$$

Then the left-hand side of (35) is maximized by

$$
S_{p}^{*}=\arg \max _{Q \subseteq J}\left[\sum_{j \in Q} \pi_{j}:\left|Q \backslash S_{p}\right| \leq u-1\right]
$$

which can be computed greedily by selecting $j \in J$ with the largest positive $\pi_{j}$ values while satisfying the cardinality constraint. Therefore, to separate from inequalities (35), one only needs to check $S^{\prime}=S_{p}^{*}$ for all $p \in I$. Similarly, inequalities (39) can be rewritten as

$$
\begin{equation*}
-\sum_{S^{\prime} \cap S_{p}} z_{j}+\left(\left|S_{p}\right|-n+l\right) \delta_{p}+\sum_{i=p+1}^{m}\left(\left|S_{i} \backslash S_{i-1}\right|\right) \delta_{i}+\sum_{i=p+1}^{m} \sum_{j \in S^{\prime} \cap\left(S_{i} \backslash S_{i-1}\right)}\left(\delta_{p}-\delta_{i}-z_{j}\right)+\sum_{j \in S^{\prime} \backslash S_{m}}\left(\delta_{p}-z_{j}\right) \leq 0 \tag{41}
\end{equation*}
$$

Given a fraction solution $(\hat{z}, \hat{\delta})$, we now define,

$$
\sigma_{j}= \begin{cases}-\hat{z}_{j}, & \text { for } j \in S_{p}  \tag{42}\\ \hat{\delta}_{p}-\hat{\delta}_{i}-\hat{z}_{j}, & \text { for } j \in S_{i} \backslash S_{i-1}, i=p+1, \ldots, m, \\ \hat{\delta}_{p}-\hat{z}_{j}, & \text { for } j \in J \backslash S_{m}\end{cases}
$$

Then the left-hand side of (41) is maximized by

$$
S_{p}^{* *}=\arg \max _{Q \subseteq J}\left\{\sum_{j \in Q} \sigma_{j}:\left|Q \cup S_{p-1}\right| \leq n-l<\left|Q \cup S_{p}\right|\right\}
$$

which can again be computed greedily by ordering the indices $j \in J$ according to the $\sigma_{j}$ values. Alternatively, one can solve the LP

$$
\max \left\{\sum_{j \in J} \sigma_{j} x_{j}: \sum_{j \in J \backslash S_{p-1}} x_{j} \leq n-l-\left|S_{p-1}\right|, \quad \sum_{j \in J \backslash S_{p}} x_{j} \geq n-l+1-\left|S_{p}\right|, \quad \mathbf{1} \geq x \geq \mathbf{0}\right\}
$$

which has a totally unimodular constraint matrix. Consequently, one only needs to check $S^{\prime}=S_{p}^{* *}$ for all $p \in I$ to separate from inequalities (39).

Remark 13. For the sake of completeness, we now consider the case when $\mathcal{S}$ is nested but $u \geq 2$ or $l \leq n-\left|S_{|I|}\right|$ does not hold. If $u=0$, then $X^{l, u}$ and its convex hull contains a single point. If $u=1$, then $\delta_{i}=1-\sum_{j \in S_{i}} z_{j}$ for all $i \in I$. These equations, together with $\sum_{j \in J} z_{j} \leq 1$, and $1 \geq z_{j} \geq 0$ for $j \in J$ give the convex hull description of $X^{l, u}$. For the case when $l>n-\left|S_{|I|}\right|$, consider $\mathcal{S}^{\prime}=\left\{S_{i}\right\}_{i \in I:\left|S_{i}\right| \leq n-l}$. In this case we have $\delta_{i}=0$ for all $i$ with $\left|S_{i}\right|>n-l$, and the multilinear set associated with $\mathcal{S}^{\prime}$ falls into the discussion of Theorem 12.

In [18], the authors show the separation of inequalities (39) can be solved in polynomial time by solving a submodular minimization problem.

## 4 Properties of facet-defining inequalities for the nested case

So far we have presented an inequality description of $X^{l, u}$ for the nested case using the description of $\operatorname{conv}\left(X^{0, u}\right)$ developed in Section 3.3 and the description of $\operatorname{conv}\left(X^{l, n}\right)$ presented earlier in [18. Not all inequalities in these exponential-size descriptions are facet-defining and in this section we present necessary and sufficient conditions for inequality of the form (35) or (39) to be facet-defining.

Theorem 14. Let $\mathcal{S}$ be nested and let $p \in I$ and $S^{\prime} \subseteq J$ be such that $\left|S^{\prime} \backslash S_{p}\right| \leq u-1$. Then, without loss of generality, the following conditions are necessary for the associated inequality (35) to define a facet of $\operatorname{conv}\left(X^{l, u}\right)$ :

$$
\begin{aligned}
& \text { U1. } S^{\prime} \supseteq S_{p} \\
& \text { U2. }\left|S^{\prime} \backslash S_{p-1}\right| \geq u \text { if } p \geq 2 \\
& \text { U3. }\left|S^{\prime}\right| \geq u+1
\end{aligned}
$$

Proof. If condition U1 is not satisfied, then replacing $S^{\prime}$ with $S^{\prime} \cup S_{p}$ in inequality (35) leads to a stronger inequality as $z_{j} \geq 0$ for all $j \in J$. Similarly, if condition 2 is not satisfied, then replacing $p$ with $p-1$ in inequality (35) leads to a stronger inequality as $\delta_{p} \leq \delta_{p-1}$.
If condition U3 is not satisfied, then $\left|S^{\prime}\right| \leq u$ and

$$
\begin{aligned}
\sum_{j \in S^{\prime}} z_{j}+(u & \left.-\left|S^{\prime} \backslash S_{p}\right|\right) \delta_{p}+\sum_{i=p+1}^{m}\left(\left|S^{\prime} \backslash S_{i-1}\right|-\left|S^{\prime} \backslash S_{i}\right|\right) \delta_{i} \\
& =\sum_{j \in S^{\prime} \cap S_{p}}\left(z_{j}+\delta_{p}\right)+\sum_{i=p+1}^{m} \sum_{j \in S^{\prime} \cap\left(S_{i} \backslash S_{i-1}\right)}\left(z_{j}+\delta_{i}\right)+\sum_{j \in S^{\prime} \backslash S_{m}} z_{j}+(\underbrace{u-\left|S^{\prime}\right|}_{\geq 0}) \delta_{p}
\end{aligned}
$$

$$
\leq\left|S^{\prime} \cap S_{p}\right|+\sum_{i=p+1}^{m}\left|S^{\prime} \cap\left(S_{i} \backslash S_{i-1}\right)\right|+\left|S^{\prime} \backslash S_{m}\right|+\left(u-\left|S^{\prime}\right|\right)=u
$$

where the last inequality is implied by the fact that $z_{j} \leq 1$ for all $j \in J$ and $z_{j}+\delta_{i} \leq 1$ for all $j \in S_{i}$, $i \in I$. Therefore, if condition U3 is not satisfied, then inequality (35) is implied by other valid inequalities. As $\operatorname{conv}\left(X^{l, u}\right)$ is full-dimensional, we conclude that conditions U1H3 are necessary for inequality (35) to define a facet.

Theorem 15. Let $\mathcal{S}$ be nested and let $p \in I$ and $S^{\prime} \subseteq J$ be such that $\left|S^{\prime} \backslash S_{p}\right| \leq u-1$. If $p<m$ or $\left|S_{m}\right|<n-l$, then conditions [1] [ 3 together with

$$
\text { U4. } S^{\prime} \cap\left(S_{p+1} \backslash S_{p}\right) \neq \emptyset \text { if } p \leq m-1
$$

are sufficient for inequality (35) to define a facet of $\operatorname{conv}\left(X^{l, u}\right)$.
Proof. (sketch) Assume that $S^{\prime} \subseteq J$ and $p \in I$ satisfy the conditions above. As $S^{\prime} \supseteq S_{p} \supseteq S_{1}$, we can assume $S^{\prime}=\left\{s_{1}, s_{2}, s_{3}, s_{4}, \ldots, s_{\left|S^{\prime}\right|}\right\}$ where $s_{1}=1, s_{2}=2$ and $2<s_{3}<s_{4}<\ldots<s_{\left|S^{\prime}\right|}$. We first show that the inequality

$$
\begin{equation*}
\sum_{j \in S^{\prime}} z_{j}+\left(u-\left|S^{\prime} \backslash S_{p}\right|\right) \delta_{p} \leq u \tag{43}
\end{equation*}
$$

defines an $(n+p-1)$-dimensional face of $\operatorname{conv}\left(X^{l, u}\right) \cap\left\{(z, \delta): \delta_{i}=0, i \in\{p+1, \ldots, m\}\right\}$. Let $Q=$ $\left\{s_{\left|S^{\prime}\right|-u+1}, \ldots, s_{\left|S^{\prime}\right|}\right\}$. Then $Q$ satisfies $|Q|=u \geq 2$ and $S^{\prime} \backslash S_{p} \subset Q \subseteq S^{\prime} \backslash S_{p-1}$. Note that $p<m$ or $\left|S_{m}\right|<n-l$ implies that $\left|J \backslash S_{p}\right| \geq l+1$. Let $R$ be a set satisfying $S^{\prime} \backslash S_{p} \subseteq R \subseteq J \backslash S_{p}$ and $|R|=\max \left\{l+1,\left|S^{\prime} \backslash S_{p}\right|\right\} \in[l+1, u]$. Define

$$
R^{\prime}= \begin{cases}R, & \text { if } l+1 \leq\left|S^{\prime} \backslash S_{p}\right|(\leq u-1), \text { i.e., } R=S^{\prime} \backslash S_{p} \\ R \backslash\left\{j_{0}\right\} \text { for some fixed } j_{0} \in R \backslash S^{\prime}, & \text { if } l+1>\left|S^{\prime} \backslash S_{p}\right|, \text { i.e., } R \backslash S^{\prime} \neq \emptyset\end{cases}
$$

Then $R^{\prime}$ satisfies $R^{\prime} \supseteq S^{\prime} \backslash S_{p}$ and $l \leq\left|R^{\prime}\right| \leq u-1$. Let $T$ be a set satisfying $|T|=u-1$ and $T \subseteq S^{\prime} \backslash S_{p-1}$. Consider points (using Definition 3) associated with the sets
$Q, \quad(Q \cup\{1\}) \backslash\{j\}$ for $j \in Q, \quad\left(Q \backslash\left\{s_{\left|S^{\prime}\right|-1}, s_{\left|S^{\prime}\right|}\right\}\right) \cup\{1, j\}$ for $j \in S^{\prime} \backslash Q \backslash\{1\}$,

$$
\begin{gathered}
R, \quad R \backslash\{j\} \text { for } j \in R \backslash S^{\prime}, \quad R^{\prime} \cup\{j\} \text { for } j \in J \backslash R \backslash S_{p}, \\
\left(Q \backslash\left\{s_{\left|S^{\prime}\right|}\right\}\right) \cup\{2\} \text { if } p \geq 2, \quad T \cup\left\{k_{i}\right\} \text { for } i \in\{2, \ldots, p-1\} .
\end{gathered}
$$

Note that some of the index sets used for defining the sets above can be empty, in which case the associated points are not considered. These $(n+p)$ points are feasible and satisfy $\delta_{i}=0$ for $i \in\{p+1, \ldots, m\}$ (as $\left(S^{\prime} \backslash S_{p}\right) \cap S_{p+1} \neq \emptyset$ by assumption U4), and lie on the hyperplane

$$
\sum_{j \in S^{\prime}} z_{j}+\left(u-\left|S^{\prime} \backslash S_{p}\right|\right) \delta_{p}=u
$$

In the rest of the proof (presented in Appendix), we first show that these points are affinely independent and therefore inequality (43) defines an $(n+p-1)$-dimensional face of $\operatorname{conv}\left(X^{l, u}\right) \cap\left\{(z, \delta): \delta_{i}=0, i \in\right.$ $\{p+1, \ldots, m\}\}$. We then lift the coefficients of $\delta_{p+1}, \ldots, \delta_{m}$ to conclude that inequality (35) is facet-defining.

Remark 16. For the case when $p=m$ and $\left|S_{m}\right|=n-l$, inequality (35) is facet-defining for conv $\left(X^{l, u}\right)$ if and only if $S^{\prime}=J$.

Results similar to Theorem 14 and 15 hold for valid inequalities (39) of the polytope $\operatorname{conv}\left(X^{l, n}\right)$. In [18], the conditions that $\left|S^{\prime} \cup S_{p-1}\right| \leq n-l$ and $n-l+1 \leq\left|S^{\prime} \cup S_{p}\right|$ are implicitly imposed on the choice of $p$ based on the rank function associated with the matroid. We next present a stronger characterization of the necessary conditions for these inequalities to be facet-defining.

Theorem 17. Let $\mathcal{S}$ be nested and let $p \in I$ and $S^{\prime} \subset J$ be such that $\left|S^{\prime} \cup S_{p-1}\right| \leq n-l<\left|S^{\prime} \cup S_{p}\right|$. Then the following conditions are necessary for inequality (39) to define a facet of $\operatorname{conv}\left(X^{l, u}\right)$ :

L1. $S^{\prime} \cap S_{p}=\emptyset$;
L2. $\left|S^{\prime}\right| \leq n-l-1$.

Proof. If condition L1 1 is not satisfied, then replacing $S^{\prime}$ with $S^{\prime} \backslash S_{p}$ in inequality (39) leads to a stronger inequality as $z_{j} \geq 0$ for all $j \in J$. If condition L 2 is not satisfied, then $\left|S^{\prime}\right| \geq n-l$. By valid inequalities (22), (26), $\delta_{p} \leq 1$ and $\sum_{j \in J} z_{j} \geq l$,

$$
\begin{aligned}
\left(\left|S^{\prime} \cup S_{p}\right|-n\right. & +l) \delta_{p}+\sum_{i=p+1}^{m}\left(\left|S^{\prime} \cup S_{i}\right|-\left|S^{\prime} \cup S_{i-1}\right|\right) \delta_{i}-\sum_{j \in S^{\prime}} z_{j} \\
& =(\underbrace{\left|S^{\prime}\right|-(n-l)}_{\geq 0}) \delta_{p}+\left(\left|S_{p} \backslash S^{\prime}\right|\right) \delta_{p}+\sum_{i=p+1}^{m}\left(\left|S_{i} \backslash S_{i-1} \backslash S^{\prime}\right|\right) \delta_{i}-\sum_{j \in S^{\prime}} z_{j} \\
& \leq\left|S^{\prime}\right|-(n-l)+\sum_{j \in S_{p} \backslash S^{\prime}}\left(1-z_{j}\right)+\sum_{i=p+1}^{m} \sum_{j \in S_{i} \backslash S_{i-1} \backslash S^{\prime}}\left(1-z_{j}\right)-\sum_{j \in S^{\prime}} z_{j} \\
& =\left|S^{\prime} \cup S_{m}\right|-(n-l)-\sum_{j \in S^{\prime} \cup S_{m}} z_{j} \\
& =\sum_{j \in J \backslash\left(S^{\prime} \cup S_{m}\right)} z_{j}+\left|S^{\prime} \cup S_{m}\right|-(n-l)-\sum_{j \in J} z_{j} \\
& \leq\left|J \backslash\left(S^{\prime} \cup S_{m}\right)\right|+\left|S^{\prime} \cup S_{m}\right|-(n-l)-l \\
& =0,
\end{aligned}
$$

where the first inequality is implied by the fact that $\delta_{p} \leq 1$ and $z_{j}+\delta_{i} \leq 1$, for all $j \in S_{i}, i \in I$ and the second inequality is implied by the fact that $\sum_{j \in J} z_{j} \geq l, z_{j} \leq 1$ for all $j \in J$. Therefore, if condition L 2
is not satisfied, then inequality (39) is implied by other valid inequalities. As $\operatorname{conv}\left(X^{l, u}\right)$ is full-dimensional, we conclude that conditions $\mathrm{L}[1$ and L 2 are necessary for inequality (39) to define a facet.

In [18, Proposition 23], the authors describe three conditions for inequality (39) to be facet-defining for $\operatorname{conv}\left(X^{l, n}\right)$. These conditions involve the rank function of the underlying matroid which, when translated to our context, has rank function

$$
r(S)=\min \{|S|, n-l\}
$$

for each subset $S$ of the ground set $J$. More precisely, these conditions are

C1. Inequality $\sum_{j \in S^{\prime}} x_{j} \leq r\left(S^{\prime}\right)$ is facet-defining for the set $\operatorname{conv}\left\{x \in\{0,1\}^{|J|}: \sum_{j \in J} x_{j} \leq n-l\right\}$;
C2. Set $S^{\prime}$ is closed [18, Definition 4] and non-separable [18, Definition 22], meaning
C2a. $r\left(S^{\prime}\right)<r\left(S^{\prime} \cup\{j\}\right)$ for all $j \in J \backslash S^{\prime}$,
C2b. $r\left(S^{\prime}\right)<r\left(S^{a}\right)+r\left(S^{b}\right)$ for all nonempty $S^{a} \subset S^{\prime}$ and $S^{b}=S^{\prime} \backslash S^{a}$;

C 3 . For all $i \in I, \delta_{i}$ has a strictly positive coefficient in (39), i.e.,

C3k. $p=1$, C3b. $\left|S^{\prime} \cup S_{1}\right|>n-l$ and C3c. $\left|S^{\prime} \cup S_{i-1}\right|<\left|S^{\prime} \cup S_{i}\right|$ for all $i \in\{2, \ldots, m\}$.

Notice that conditions C2a and C2b cannot hold simultaneously unless $S^{\prime}$ is equal to the set $J$, or it contains a single element, i.e., $S^{\prime}=\{j\}$ for some $j \in J$. Also note that condition C1 is satisfied in both cases, i.e. when $S^{\prime}=J$ or $\left|S^{\prime}\right|=1$. However, remember that Theorem 17 requires $S^{\prime} \subset J$ and therefore $S^{\prime} \neq J$. Therefore, the only remaining possible choices for $S^{\prime}$ are $S^{\prime}=\{j\}$ for some $j \in J$. Finally, condition C3b together with our starting assumption that $\left|S_{m}\right| \leq n-l$ implies that $m=p=1,\left|S_{1}\right|=n-l$ and $j \in J \backslash S_{1}$. In conclusion, we observe that conditions C1-C3 are satisfied only in the narrow case when the family $\mathcal{S}$ defining $\operatorname{conv}\left(X^{l, n}\right)$ contains a single set $S$ of cardinality $n-l$. In addition, the set $S^{\prime}$ must have cardinality one, containing a single element $j \in J \backslash S$.

In the next theorem, we give significantly less restrictive conditions for inequality (39) to be facet-defining for $\operatorname{conv}\left(X^{l, u}\right)$.

Theorem 18. Let $\mathcal{S}$ be nested and let $p \in I$ and $S^{\prime} \subset J$ be such that $\left|S^{\prime} \cup S_{p-1}\right| \leq n-l<\left|S^{\prime} \cup S_{p}\right|$. If $p<m$ or $\left|S_{m}\right|<n-l$, then conditions L 1 LR together with

L3. $S_{p+1} \backslash S_{p} \nsubseteq S^{\prime}$ if $p \leq m-1$
are sufficient for inequality (39) to define a facet of $\operatorname{conv}\left(X^{l, u}\right)$.

Proof. (sketch) Assume that $S^{\prime} \subseteq J$ and $p \in I$ satisfy the conditions above. Then the assumption $p<m$ or $\left|S_{m}\right|<n-l$ implies that $\left|S_{p}\right|<n-l$, and $S^{\prime} \neq \emptyset$ as $\left|S^{\prime} \cup S_{p}\right|>n-l$. Assume $S^{\prime}=\left\{s_{1}, \ldots, s_{\left|S^{\prime}\right|}\right\}$ with $s_{1}<\ldots<s_{\left|S^{\prime}\right|}$. We first show that the inequality

$$
\begin{equation*}
-\sum_{j \in S^{\prime}} z_{j}+\left(\left|S^{\prime} \cup S_{p}\right|-n+l\right) \delta_{p} \leq 0 \tag{44}
\end{equation*}
$$

defines an $(n+p-1)$-dimensional face of $\operatorname{conv}\left(X^{l, u}\right) \cap\left\{(z, \delta): \delta_{i}=0, i \in\{p+1, \ldots, m\}\right\}$.
Let $Q=S_{p} \cup\left\{s_{1}, s_{2}, \ldots, s_{n-l-\left|S_{p}\right|}\right\}$. Then $Q$ satisfies $|Q|=n-l$ and $S_{p} \subset Q \subset S_{p} \cup S^{\prime}$. Let $R=\left(J \backslash S^{\prime} \backslash\right.$ $\left.S_{p}\right) \cup \underbrace{\left\{1,2, \ldots, l+1-\left|J \backslash S^{\prime} \backslash S_{p}\right|\right\}}_{\subseteq S_{p}}$. Then $R$ satisfies $|R|=l+1 \leq u$ and $\left(J \backslash S^{\prime} \backslash S_{p}\right) \cup\{1,2\} \subseteq R \subseteq J \backslash S^{\prime}$ as $n-l<\left|S^{\prime} \cup S_{p}\right|$. Note that $\left|S^{\prime} \cup S_{p-1}\right| \leq n-l$. For $i \in\{1, \ldots, p-1\}$, we let $T_{i}$ denote the first $l$ elements of $J \backslash S^{\prime} \backslash S_{i}$.

Consider the points (using Definition 3) associated with the sets

$$
\begin{gathered}
R \backslash\{j\} \text { for } j \in R, \quad R, \quad(R \backslash\{1\}) \cup\{j\} \text { for } j \in S_{p} \backslash R, \quad\left\{T_{i}\right\}_{i \in\{1, \ldots, p-1\}}, \\
\left(J \backslash Q \backslash\left\{s_{\left|S^{\prime}\right|}\right\}\right) \cup\{j\} \text { for } j \in Q \backslash S_{p}, \quad(J \backslash Q \backslash\{j\}) \cup\left\{s_{1}\right\} \text { for } j \in S^{\prime} \backslash Q \backslash\left\{s_{\left|S^{\prime}\right|}\right\}, \quad J \backslash Q .
\end{gathered}
$$

These $(n+p)$ points are feasible with $\delta_{i}=0, i \in\{p+1, \ldots, m\}$ (as $\left(J \backslash S^{\prime} \backslash S_{p}\right) \cap S_{p+1} \neq \emptyset$ by assumption L3), and lie on the hyperplane

$$
-\sum_{j \in S^{\prime}} z_{j}+\left(\left|S^{\prime} \cup S_{p}\right|-n+l\right) \delta_{p}=0
$$

In the rest of the proof (presented in Appendix), we first show that these points are affinely independent and therefore inequality (44) defines an $(n+p-1)$-dimensional face of $\operatorname{conv}\left(X^{l, u}\right) \cap\left\{(z, \delta): \delta_{i}=0, i \in\right.$ $\{p+1, \ldots, m\}\}$. We then lift the coefficients of $\delta_{p+1}, \ldots, \delta_{m}$ to conclude that inequality (39) is facetdefining.

Remark 19. For the case when $p=m$ and $\left|S_{m}\right|=n-l$, inequality (39) is facet-defining for conv $\left(X^{l, u}\right)$ if and only if $S^{\prime}=\{j\}$ for some $j \in J \backslash S_{m}$.

## 5 Valid inequalities when $\mathcal{S}$ is not nested

In Section 3, we described inequalities (35) and (39) and showed that together with the standard linearization and 2-link inequalities they define $\operatorname{conv}\left(X^{0, u}\right)$ and $\operatorname{conv}\left(X^{l, n}\right)$, respectively. In this section, we extend these inequalities to the general case when the sets in $\mathcal{S}$ are not necessarily nested.

Notice that since we derived inequalities (35) using the mixing procedure, they are still valid for $\operatorname{conv}\left(X^{l, u}\right)$ in the general case, provided that

$$
\begin{equation*}
u-1 \geq\left|S^{\prime} \backslash S_{p}\right| \geq\left|S^{\prime} \backslash S_{p+1}\right| \geq \ldots \geq\left|S^{\prime} \backslash S_{m}\right| \tag{45}
\end{equation*}
$$

hold. We next generalize inequalities (35) to the case when (45) is not satisfied.
Proposition 20. Assume that sets $S_{[1]}, S_{[2]}, \ldots, S_{[t]} \in \mathcal{S}$ are distinct and let $\delta_{[i]}$ denote the $\delta$ variable associated with $S_{[i]}$. For $S^{\prime} \subseteq J$, the following inequality is valid for $\operatorname{conv}\left(X^{l, u}\right)$

$$
\begin{equation*}
\sum_{j \in S^{\prime}} z_{j}+\left(u-\left|S^{\prime} \backslash S_{[1]}\right|\right) \delta_{[1]}+\sum_{i=2}^{t}\left(\left|S^{\prime} \cap S_{[i]} \backslash \bigcup_{k=1}^{i-1} S_{[k]}\right|\right) \delta_{[i]} \leq u \tag{46}
\end{equation*}
$$

provided that $\max _{i=2, \ldots, t}\left|S^{\prime} \backslash\left(S_{[1]} \cap S_{[i]}\right)\right| \leq u$.

Proof. As the indices of the sets in $\mathcal{S}$ are arbitrary, we assume that $S_{[i]}=S_{i}$ for $i=1, \ldots, t$, without loss of generality. First note that the following inequality

$$
\begin{equation*}
\sum_{j \in S^{\prime}} z_{j}+\left(u-\left|S^{\prime} \backslash S_{1}\right|\right) \delta_{1} \leq u \tag{47}
\end{equation*}
$$

is valid for $\operatorname{conv}\left(X^{l, u}\right)$ as it is implied by $\sum_{j \in S^{\prime}} z_{j} \leq u$ when $\delta_{1}=0$ and $\sum_{j \in S^{\prime}} z_{j} \leq\left|S^{\prime} \backslash S_{1}\right|$ when $\delta_{1}=1$ (and therefore $z_{j}=0$ for all $j \in S_{1}$ ). We will derive inequality (46) by sequential lifting, starting with inequality (47) and showing that if (46) with $t$ replaced by $t^{\prime}$ is valid for $X^{l, u}$ for $t^{\prime} \in\{1,2, \ldots, t-1\}$, then

$$
\begin{equation*}
\sum_{j \in S^{\prime}} z_{j}+\left(u-\left|S^{\prime} \backslash S_{1}\right|\right) \delta_{1}+\sum_{i=2}^{t^{\prime}}\left(\left|S^{\prime} \cap S_{i} \backslash \bigcup_{k=1}^{i-1} S_{k}\right|\right) \delta_{i} \leq u-\left|S^{\prime} \cap S_{t^{\prime}+1} \backslash \bigcup_{k=1}^{t^{\prime}} S_{k}\right| \tag{48}
\end{equation*}
$$

holds for all $(z, \delta) \in X^{l, u}$ with $\delta_{t^{\prime}+1}=1$. This would imply that (46) with $t$ replaced by $t^{\prime}+1$ is also valid. Fix $t^{\prime} \in\{1, \ldots, t-1\}$ and note that for all $(z, \delta) \in X^{l, u}$ with $\delta_{t^{\prime}+1}=1$, we have $z_{j}=0$ for $j \in S_{t^{\prime}+1}$. Therefore, given any arbitrary $(z, \delta) \in X^{l, u}$ with $\delta_{t^{\prime}+1}=1$, we have

$$
\begin{aligned}
& \sum_{j \in S^{\prime}} z_{j}+\left(u-\left|S^{\prime} \backslash S_{1}\right|\right) \delta_{1}+\sum_{i=2}^{t^{\prime}}\left(\left|S^{\prime} \cap S_{i} \backslash \bigcup_{k=1}^{i-1} S_{k}\right|\right) \delta_{i} \\
&= \underbrace{\sum_{j \in S^{\prime} \cap S_{t^{\prime}+1}} z_{j}}_{=0}+\sum_{j \in S^{\prime} \backslash\left(\bigcup_{k=1}^{t^{\prime}+1} S_{k}\right)} z_{j}+\sum_{j \in S^{\prime} \cap S_{1} \backslash S_{t^{\prime}+1}}\left(z_{j}+\delta_{1}\right)+(\underbrace{\left.u-\left|S^{\prime} \backslash\left(S_{1} \cap S_{t^{\prime}+1}\right)\right|\right) \delta_{1}}_{\geq 0 \text { by assumption }} \\
& \quad+\sum_{i=2}^{t^{\prime}}\left[\sum_{j \in S^{\prime} \cap S_{i} \backslash\left(\bigcup_{k=1}^{i-1} S_{k}\right) \backslash S_{t^{\prime}+1}}\left(z_{j}+\delta_{i}\right)+\left(\left|S^{\prime} \cap S_{i} \cap S_{t^{\prime}+1} \backslash \bigcup_{k=1}^{i-1} S_{k}\right|\right) \delta_{i}\right] \\
& \leq\left|S^{\prime} \backslash \bigcup_{k=1}^{t^{\prime}+1} S_{k}\right|+u-\left|S^{\prime} \backslash S_{1}\right|+\sum_{i=2}^{t^{\prime}}\left|S^{\prime} \cap S_{i} \backslash \bigcup_{k=1}^{i-1} S_{k}\right| \\
&=\left|S^{\prime} \backslash \bigcup_{k=1}^{t^{\prime}+1} S_{k}\right|+u-\left|S^{\prime} \backslash \bigcup_{i=1}^{t^{\prime}} S_{i}\right|=u-\left|S^{\prime} \cap S_{t^{\prime}+1} \backslash \bigcup_{k=1}^{t^{\prime}} S_{k}\right|
\end{aligned}
$$

We note that inequality (46) reduces to (35) when $\mathcal{S}$ is nested by taking $t=m-p+1$ and $S_{[i]}=S_{p+i-1}$ for $i=1, \ldots, t$.

Similarly, as we have shown that inequalities (39) can also be derived via mixing, they are valid in the general case as long as $n-l+1 \leq\left|S^{\prime} \cup S_{p}\right| \leq\left|S^{\prime} \cup S_{p+1}\right| \leq \ldots \leq\left|S^{\prime} \cup S_{m}\right|$. We next extend (39) to a more general case.

Proposition 21. Assume that sets $S_{[1]}, S_{[2]}, \ldots, S_{[t]} \in \mathcal{S}$ are distinct and let $\delta_{[i]}$ denote the $\delta$ variable associated with $S_{[i]}$. For $S^{\prime} \subseteq J$, the following inequality is valid for $\operatorname{conv}\left(X^{l, u}\right)$

$$
\begin{equation*}
-\sum_{j \in S^{\prime}} z_{j}+\left(\left|S^{\prime} \cup S_{[1]}\right|-n+l\right) \delta_{[1]}+\sum_{i=2}^{t}\left(\left|S_{[i]} \backslash\left(\bigcup_{k=1}^{i-1} S_{[k]}\right) \backslash S^{\prime}\right|\right) \delta_{[i]} \leq 0 \tag{49}
\end{equation*}
$$

provided that $\min _{i=2, \ldots, t}\left|S^{\prime} \cup\left(S_{[1]} \cap S_{[i]}\right)\right| \geq n-l$.
Proof. Without loss of generality, we assume that $S_{[i]}=S_{i}$ for $i=1, \ldots, t$. Note that the following inequality

$$
\begin{equation*}
-\sum_{j \in S^{\prime}} z_{j}+\left(\left|S^{\prime} \cup S_{1}\right|-n+l\right) \delta_{1} \leq 0 \tag{50}
\end{equation*}
$$

is valid for $\operatorname{conv}\left(X^{l, u}\right)$ as it is implied by $z_{j} \geq 0$ for all $j \in S_{1}$ when $\delta_{1}=0$ and $\left|S^{\prime} \cup S_{1}\right|-\sum_{j \in S^{\prime}} z_{j}=$ $\sum_{j \in S^{\prime} \cup S_{1}}\left(1-z_{j}\right) \leq n-l$ when $\delta_{1}=1$ (and therefore $z_{j}=0$ for all $j \in S_{1}$ ). We will derive (49) by sequential lifting, starting with inequality (50) and showing that if (49) with $t$ replaced by $t^{\prime}$ is valid for $X^{l, u}$ for $t^{\prime} \in\{1,2, \ldots, t-1\}$, then

$$
\begin{equation*}
-\sum_{j \in S^{\prime}} z_{j}+\left(\left|S^{\prime} \cup S_{1}\right|-n+l\right) \delta_{1}+\sum_{i=2}^{t^{\prime}}\left(\left|S_{i} \backslash\left(\bigcup_{k=1}^{i-1} S_{k}\right) \backslash S^{\prime}\right|\right) \delta_{i} \leq-\left|S_{t^{\prime}+1} \backslash\left(\bigcup_{k=1}^{t^{\prime}} S_{k}\right) \backslash S^{\prime}\right| \tag{51}
\end{equation*}
$$

holds for all $(z, \delta) \in X^{l, u}$ with $\delta_{t^{\prime}+1}=1$. This would imply that (49) with $t$ replaced by $t^{\prime}+1$ is also valid. Fix $t^{\prime} \in\{1, \ldots, t-1\}$ and note that for all $(z, \delta) \in X^{l, u}$ with $\delta_{t^{\prime}+1}=1$, we have $z_{j}=0$ for $j \in S_{t^{\prime}+1}$. Therefore, given any arbitrary $(z, \delta) \in X^{l, u}$ with $\delta_{t^{\prime}+1}=1$, we have

$$
\begin{aligned}
-\sum_{j \in S^{\prime}} & z_{j}+\left(\left|S^{\prime} \cup S_{1}\right|-n+l\right) \delta_{1}+\sum_{i=2}^{t^{\prime}}\left(\left|S_{i} \backslash\left(\bigcup_{k=1}^{i-1} S_{k}\right) \backslash S^{\prime}\right|\right) \delta_{i} \\
= & \underbrace{\sum_{j \in S_{t^{\prime}+1} \backslash S^{\prime}}}_{=0} z_{j}-\sum_{j \in S^{\prime} \cup S_{t^{\prime}+1}} z_{j}+[(\underbrace{\left|S^{\prime} \cup\left(S_{1} \cap S_{t^{\prime}+1}\right)\right|-n+l}_{\geq 0 \text { by assumption }}) \delta_{1}+\left(\left|S_{1} \backslash S^{\prime} \backslash S_{t^{\prime}+1}\right|\right) \delta_{1}] \\
& +\sum_{i=2}^{t^{\prime}}\left[\left(\left|S_{i} \cap S_{t^{\prime}+1} \backslash\left(\bigcup_{k=1}^{i-1} S_{k}\right) \backslash S^{\prime}\right|\right) \delta_{i}+\left(\left|S_{i} \backslash\left(\bigcup_{k=1}^{i-1} S_{k}\right) \backslash S^{\prime} \backslash S_{t^{\prime}+1}\right|\right) \delta_{i}\right] \\
\leq & -\sum_{j \in S^{\prime} \cup S_{t^{\prime}+1}} z_{j}+\left[\left(\left|S^{\prime} \cup\left(S_{1} \cap S_{t^{\prime}+1}\right)\right|-n+l\right)+\sum_{j \in S_{1} \backslash S^{\prime} \backslash S_{t^{\prime}+1}}\left(1-z_{j}\right)\right] \\
& +\sum_{i=2}^{t^{\prime}}\left[\left(\left|S_{i} \cap S_{t^{\prime}+1} \backslash\left(\bigcup_{k=1}^{i-1} S_{k}\right) \backslash S^{\prime}\right|\right)+\sum_{j \in S_{i} \backslash\left(\cup_{k=1}^{i-1} S_{k}\right) \backslash S^{\prime} \backslash S_{t^{\prime}+1}}\left(1-z_{j}\right)\right] \\
= & -\sum_{j \in S^{\prime} \cup\left(\bigcup_{i=1}^{t^{\prime}+1}\right.} z_{j}+\left|S^{\prime} \cup\left(\bigcup_{i=1}^{t^{\prime}} S_{i}\right)\right|-n+l
\end{aligned}
$$

$$
\begin{aligned}
& =-\sum_{j \in J} z_{j}+\sum_{j \in J \backslash\left[S^{\prime} \cup\left(\cup_{i=1}^{t^{\prime}+1} S_{i}\right)\right]} z_{j}+\left|S^{\prime} \cup\left(\bigcup_{i=1}^{t^{\prime}} S_{i}\right)\right|-n+l \\
& \leq-l+\left[n-\left|S^{\prime} \cup\left(\bigcup_{i=1}^{t^{\prime}+1} S_{i}\right)\right|\right]+\left|S^{\prime} \cup\left(\bigcup_{i=1}^{t^{\prime}} S_{i}\right)\right|-n+l \\
& =-\left|S_{t^{\prime}+1} \backslash\left(\bigcup_{k=1}^{t^{\prime}} S_{k}\right) \backslash S^{\prime}\right| .
\end{aligned}
$$

Inequality (49) reduces to (39) when $\mathcal{S}$ is nested by taking $t=m-p+1$ and $S_{[i]}=S_{p+i-1}$ for $i=1, \ldots, t$.

## 6 Conclusions

In this paper, we study the convex hull of the multilinear set with (two-sided) cardinality constraints and give a polyhedral characterization of it when the sets involved have a nested structure. We first show that the convex hull can be obtained by intersecting the convex hulls of two simpler sets, each with one sided cardinality constraints. Convex hull of one of these sets $\left(\operatorname{conv}\left(X^{l, n}\right)\right)$ has already been characterized earlier in 18]. The description of the second set $\left(\operatorname{conv}\left(X^{0, u}\right)\right)$ is new. The two descriptions bear some resemblance due to the fact that the inequalities involved can be derived using the mixing procedure starting with different base inequalities. To the best of our knowledge, the similarity between the inequality descriptions of the two sets does not imply that one of the sets can be used (via a complementation) to obtain the other. The authors of [18] also agree with this assessment.

For the general (non-nested) case, we are able to derive a family of valid inequalities that generalize the inequalities for the nested case. Derivation of these inequalities do not involve the mixing procedure. These inequalities do not necessarily yield the convex hull as the polyhedral structure of the general case seems to be significantly more complicated even when only two non-nested sets are involved [3].

See also [19], where Fischer, Fischer and McCormick extend their earlier work on matroids by considering multilinear terms defined by all subsets of a fixed subset of the ground set instead of nested sets. Note that all subsets of a set form a proper family by Proposition 2 provided that $\Delta^{l, u}=\Delta^{0, n}$, which is one of the assumptions in (19).

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## Appendix

In this section we present the full proofs of Lemma 7 and Theorems 15 and 18 ,

## Proof of Lemma 7

Proof. Given $z \in\{0,1\}^{n}$ satisfying $l \leq \sum_{j \in J} z_{j} \leq u$, there exists a unique $\delta$ such that $(z, \delta) \in X^{l, u}$. Therefore, given any $U \subseteq J$ with $l \leq|U| \leq u$, we can define the corresponding point $v^{U}$ as follows:

$$
v^{U}=\left(z^{U}, \boldsymbol{\delta}^{U}\right) \in X^{l, u} \quad \text { where } \quad z_{j}^{U}=\left\{\begin{array}{ll}
1, & \text { if } j \in U, \\
0, & \text { otherwise },
\end{array} \quad \text { and } \quad \boldsymbol{\delta}_{i}^{U}=\prod_{j \in S_{i}}\left(1-z_{j}^{U}\right)\right.
$$

For $j=1, \ldots, n$, let $e^{j} \in \mathbb{R}^{n}$ denote the $j$-th unit vector in $\mathbb{R}^{n}$. In addition, let $0_{m} \in \mathbb{R}^{m}$ denote the vector of all zeros, and for $i=1, \ldots, m$, let $d^{i} \in \mathbb{R}^{m}$ denote the vector whose first $i$ components are one and the rest are zero. We now consider 2 cases:

Case 1: Assume $l=0$. In this case, we will argue that the following $m+n+1$ points in $\operatorname{conv}\left(X^{0, u}\right)$ are affinely independent:

$$
v^{\{2\}}=\left[\begin{array}{c}
e^{2} \\
0_{m}
\end{array}\right], \quad v^{\left\{k_{i}+1\right\}}=\left[\begin{array}{c}
e^{k_{i}+1} \\
d^{i}
\end{array}\right] \text { for } i \in I, \quad v^{\{1\}}=\left[\begin{array}{c}
e^{1} \\
0_{m}
\end{array}\right], \quad v^{\{1, j\}}=\left[\begin{array}{c}
e^{1}+e^{j} \\
0_{m}
\end{array}\right] \text { for } j \in J \backslash\{1\} .
$$

Clearly these points are in $X^{0, u}$ and together they form the following matrix $V \in \mathbb{R}^{(m+n) \times(m+n+1)}$ :

$$
V=\left[\begin{array}{c|c|c}
e^{2} & \mathbb{K} & \frac{\mid c}{\mathbb{1}_{n}^{T}}  \tag{52}\\
\hline 0_{m} & \mathbb{D} & \mathbb{0}_{(n-1)} \mid \mathbb{1}_{(n-1)} \\
\hline \mathbb{O}_{m \times n}
\end{array}\right]
$$

where $\mathbb{1}_{*} \in \mathbb{R}^{*}$ is a vector/matrix of all ones, $0_{*} \in \mathbb{R}^{*}$ is a vector of all zeros, and, $\mathbb{O}_{*}$ and $\mathbb{I}_{*}$, respectively, denote the matrix of all zeros and the identity matrix of the specified dimension. The $i$-th column of the matrix $\mathbb{K} \in \mathbb{R}^{n \times m}$ is equal to $e^{k_{i}+1}$, and $i$-th column of $\mathbb{D} \in \mathbb{R}^{m \times m}$ is $d^{i}$. Note that $\mathbb{D}$ is an upper triangular matrix with all ones on and above the diagonal.

To show that the columns of $V$ are affinely independent, we need to argue that the unique solution to the system of equations:

$$
\begin{equation*}
V \lambda=0, \quad \sum_{t=1}^{m+n+1} \lambda_{t}=0 \tag{53}
\end{equation*}
$$

is $\lambda=0$. Note that the first row of $\mathbb{K}$ is all zeros and therefore the first row of $V$ has $m+1$ consecutive zeros followed by $n$ ones. Therefore, the first row of $V \lambda=0$ implies that $\sum_{t=m+2}^{m+n+1} \lambda_{t}=0$ and consequently $\sum_{t=1}^{m+1} \lambda_{t}=0$.

As $\mathbb{D}$ is an upper triangular matrix of ones, the last $m$ rows of $V \lambda=0$ imply that

$$
0=\sum_{t=2}^{m+1} \lambda_{t}=\sum_{t=3}^{m+1} \lambda_{t}=\ldots=\sum_{t=m+1}^{m+1} \lambda_{t}=0
$$

and therefore $\lambda_{t}=0$ for $t=2, \ldots, m+1$. Moreover, $\sum_{t=1}^{m+1} \lambda_{t}=0$, implies that $\lambda_{1}=0$ as well. As the first $m+1$ components of $\lambda$ have to be zero, the first $n$ rows of $V \lambda=0$ now imply that

$$
\sum_{t=m+2}^{m+n+1} \lambda_{t}=0, \quad \text { and } \quad \lambda_{t}=0 \text { for } t=m+3, \ldots, m+n+1
$$

Using the first equation, we have $\lambda_{m+2}=0$ as well and the columns of $V$ are indeed affinely independent.
Case 2: Assume $l \geq 1$. In this case, we let $Q=\{n-l+1, \ldots, n\} \subseteq J \backslash S_{m}$, where $|Q|=l$. We now consider the following $m+n+1$ points in $\operatorname{conv}\left(X^{l, u}\right)$ :

$$
v^{\{2\} \cup Q}, \quad v^{\left\{k_{i}+1\right\} \cup Q} \text { for } i \in I, \quad v^{\{1\} \cup Q}, \quad v^{\{1, j\} \cup Q \backslash\{n\}} \text { for } j \in J \backslash\{1\} \backslash Q, \quad v^{\{1\} \cup Q \backslash\{j\}} \text { for } j \in Q .
$$

These points form the matrix

$$
V^{l}=\left[\right],
$$

where $\mathbb{K}^{\prime}$ is a matrix with all entries of its first row being zero and $\mathbb{D}$ is the upper triangular matrix described in (52).

As in Case 1, we first observe that the first row of $V^{l}$ has $m+1$ consecutive zeros followed by $n$ ones and argue that $\sum_{t=m+2}^{m+n+1} \lambda_{t}=0$ and $\sum_{t=1}^{m+1} \lambda_{t}=0$. In addition, as the last $m$ rows of $V^{l}$ are the same as $V$, we also conclude that the first $m+1$ components of of $\lambda$ have to be zero.

Finally, note that the $n$ by $n$ matrix on the upper right corner of $V^{l}$ is nonsingular as adding rows 2 to $n-l$ of this matrix to the last ( $n$-th) row and then subtracting its first row from each one of the last $l$ rows leads to the upper triangular matrix:


Therefore, we conclude that $\operatorname{conv}\left(X^{l, u}\right)$ is full-dimensional.

## Proof of Theorem 15

Proof. Assume that $S^{\prime} \subseteq J$ and $p \in I$ satisfy the conditions above. As $S^{\prime} \supseteq S_{p} \supseteq S_{1}$, we can assume $S^{\prime}=\left\{s_{1}, s_{2}, s_{3}, s_{4}, \ldots, s_{\left|S^{\prime}\right|}\right\}$ where $s_{1}=1, s_{2}=2$ and $2<s_{3}<s_{4}<\ldots<s_{\left|S^{\prime}\right|}$. We first show that the inequality

$$
\begin{equation*}
\sum_{j \in S^{\prime}} z_{j}+\left(u-\left|S^{\prime} \backslash S_{p}\right|\right) \delta_{p} \leq u \tag{54}
\end{equation*}
$$

defines an $(n+p-1)$-dimensional face of $\operatorname{conv}\left(X^{l, u}\right) \cap\left\{(z, \delta): \delta_{i}=0, i \in\{p+1, \ldots, m\}\right\}$. Let $Q=$ $\left\{s_{\left|S^{\prime}\right|-u+1}, \ldots, s_{\left|S^{\prime}\right|}\right\}$. Then $Q$ satisfies $|Q|=u \geq 2$ and $S^{\prime} \backslash S_{p} \subset Q \subseteq S^{\prime} \backslash S_{p-1}$. Note that $p<m$ or $\left|S_{m}\right|<n-l$ implies that $\left|J \backslash S_{p}\right| \geq l+1$. Let $R$ be a set satisfying $S^{\prime} \backslash S_{p} \subseteq R \subseteq J \backslash S_{p}$ and $|R|=\max \left\{l+1,\left|S^{\prime} \backslash S_{p}\right|\right\} \in[l+1, u]$. Define

$$
R^{\prime}= \begin{cases}R, & \text { if } l+1 \leq\left|S^{\prime} \backslash S_{p}\right|(\leq u-1), \text { i.e., } R=S^{\prime} \backslash S_{p} \\ R \backslash\left\{j_{0}\right\} \text { for some fixed } j_{0} \in R \backslash S^{\prime}, & \text { if } l+1>\left|S^{\prime} \backslash S_{p}\right|, \text { i.e., } R \backslash S^{\prime} \neq \emptyset\end{cases}
$$

Then $R^{\prime}$ satisfies $R^{\prime} \supseteq S^{\prime} \backslash S_{p}$ and $l \leq\left|R^{\prime}\right| \leq u-1$. Let $T$ be a set satisfying $|T|=u-1$ and $T \subseteq S^{\prime} \backslash S_{p-1}$. Consider points (using Definition 3) associated with the sets
$Q, \quad(Q \cup\{1\}) \backslash\{j\}$ for $j \in Q, \quad\left(Q \backslash\left\{s_{\left|S^{\prime}\right|-1}, s_{\left|S^{\prime}\right|}\right\}\right) \cup\{1, j\}$ for $j \in S^{\prime} \backslash Q \backslash\{1\}$,

$$
\begin{gather*}
R, \quad R \backslash\{j\} \text { for } j \in R \backslash S^{\prime}, \quad R^{\prime} \cup\{j\} \text { for } j \in J \backslash R \backslash S_{p}  \tag{56}\\
\left(Q \backslash\left\{s_{\left|S^{\prime}\right|}\right\}\right) \cup\{2\} \text { if } p \geq 2, \quad T \cup\left\{k_{i}\right\} \text { for } i \in\{2, \ldots, p-1\} .
\end{gather*}
$$

Note that some of the index sets used for defining the sets in (57) can be empty, in which case the associated points are not considered but sets in (57) would always contribute $p-1$ points in total. These $(n+p)$ points are feasible and satisfy $\delta_{i}=0$ for $i \in\{p+1, \ldots, m\}$ (as $\left(S^{\prime} \backslash S_{p}\right) \cap S_{p+1} \neq \emptyset$ by assumption U4), and lie on the hyperplane $\sum_{j \in S^{\prime}} z_{j}+\left(u-\left|S^{\prime} \backslash S_{p}\right|\right) \delta_{p}=u$ associated with inequality (54).
We next reorder the $\left(\left\{z_{j}\right\}_{j \in J}, \delta_{1}, \ldots, \delta_{p}\right)$ coordinates of the points (see Figure 1) in the ordering

$$
\left(\left\{z_{j}\right\}_{j \in S_{p} \backslash Q},\left\{z_{j}\right\}_{j \in Q},\left\{z_{j}\right\}_{j \in R \backslash Q},\left\{z_{j}\right\}_{j \in J \backslash\left(S_{p} \cup Q \cup R\right)}, \delta_{1}, \ldots, \delta_{p}\right),
$$

and consider the matrix $V$ formed by these reordered coordinates of the (column) points.


Figure 1: Reordered $z_{j}$ coordinates in the proof of Theorem 15

We would now argue that the unique solution to the system of equations

$$
\begin{equation*}
V \lambda=0, \quad \sum_{t=1}^{n+p} \lambda_{t}=0 \tag{58}
\end{equation*}
$$

is $\lambda=0$. We separately consider two cases, namely $p \geq 2$ and $p=1$.

First consider the case when $p \geq 2$. If this case, $\left|S^{\prime}\right| \geq u+\left|S_{p-1}\right| \geq u+2$. Therefore, $\{1,2\} \cap Q=\emptyset$. We look at the matrix $V_{p}$ formed by the last $p$ rows of $V$ corresponding to the $\left\{\delta_{i}\right\}_{i=1}^{p}$ coordinates:

$$
V_{p}=\left[\begin{array}{l|l|l|l|l|l|l}
d^{p-1} & \mathbb{O}_{p \times\left(\left|S^{\prime}\right|-1\right)} & \mathbb{1}_{p \times\left(n+1-\left|S^{\prime}\right|\right)} & 0_{p} & d^{1} & \ldots & d^{p-2}
\end{array}\right],
$$

where $d^{i}$ is defined in the proof of Lemma 7 Equations $V_{p} \lambda=0$ imply $\lambda_{1}=0$ and $\lambda_{n+3}=\ldots=\lambda_{n+p}=0$.


Figure 2: Matrix $V$ in the proof of Theorem 15

Therefore, (58) reduces to equations

$$
\begin{equation*}
\bar{V} \bar{\lambda}=0, \quad \sum_{t=2}^{n+2} \lambda_{t}=0 \tag{59}
\end{equation*}
$$

where $\bar{V}$ is a matrix formed by columns 2 to $n+2$ of $V$ and $\bar{\lambda}=\left(\lambda_{2}, \ldots, \lambda_{n+2}\right)^{T}$. Note that matrix $\bar{V}$ is of the form

| $\mathbb{1}_{\left\|S^{\prime}\right\|-1}^{T}$ |  | $\mathbb{O}_{\left\|S_{p}\right\| \times\left(n+1-\left\|S^{\prime}\right\|\right)}$ |  |  | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\underline{\mathbb{O}_{\left(\left\|S_{p} \backslash Q\right\|-1\right) \times\|Q\|}}$ | $\mathbb{I}_{\left(\left\|S^{\prime} \backslash Q\right\|-1\right)}$ | $\mathbb{1}_{\|R\|}$ | $\mathbb{1}_{\left\|S^{\prime} \backslash S_{p}\right\| \times\left\|R \backslash S^{\prime}\right\|}$ | * | 1 |
| $\mathbb{1}_{\|Q\| \times\|Q\|}-\mathbb{I}_{\|Q\|}$ | $\mathbb{1}_{(\|Q\|-2) \times\left(\left\|S^{\prime} \backslash Q\right\|-1\right)}$ |  | $\longrightarrow$ |  | $0_{\left\|S^{\prime} \backslash Q\right\|-2}$ |
|  | $\mathbb{O}_{2 \times\left(\left\|S^{\prime} \backslash Q\right\|-1\right)}$ |  | $\mathbb{U}_{\left\|R \backslash S^{\prime}\right\| \times\left\|R \backslash S^{\prime}\right\|}$ <br> II |  | $\mathbb{1}_{\|Q\|-1}$ |
| $\mathbb{O}_{\left(n-\left\|S^{\prime}\right\|\right) \times\left(\left\|S^{\prime}\right\|-1\right)}$ |  | $\mathbb{O}_{1}$ | $\frac{\mathbb{1}_{\left\|R \backslash S^{\prime}\right\|}}{\text { R\S }{ }_{\text {p }} \mid \times\left(\left\|R \backslash S^{\prime}\right\|+1\right)}$ | $\mathbb{I}_{\left\|J \backslash R \backslash S_{p}\right\|}$ | $0_{n+1-\left\|S^{\prime}\right\|}$ |
| $\mathbb{O}_{p \times\left(\left\|S^{\prime}\right\|-1\right)}$ |  | $\mathbb{1}_{p \times\left(n+1-\left\|S^{\prime}\right\|\right)}$ |  |  | $0_{p}$ |

By looking at the first and the last row of $\bar{V} \bar{\lambda}=0$ and $\sum_{t=2}^{n+2} \lambda_{t}=0$, we have $\lambda_{n+2}=0$. It is then easy to verify that $\bar{V} \bar{\lambda}=0$ and $\lambda_{n+2}=0$ imply $\lambda_{|Q|+2}=\ldots=\lambda_{\left|S^{\prime}\right|}=0$ and $\lambda_{n-\left|J \backslash R \backslash S_{p}\right|+2}=\ldots=\lambda_{n+1}=0$ by looking at the $\left\{z_{j}\right\}_{j \in S^{\prime} \backslash Q \backslash\{1\}}$ and $\left\{z_{j}\right\}_{j \in J \backslash R \backslash S_{p}}$ coordinates, respectively. The remaining columns of $\bar{V}$ are of the form:

$$
\left[\begin{array}{c|c|c}
\mathbb{1}_{|Q|}^{T} & \mathbb{O}_{\left|S_{p}\right| \times\left(\left|R \backslash S^{\prime}\right|+1\right)} \\
\hline \mathbb{O}_{\left(\left|S_{p} \backslash Q\right|-1\right) \times|Q|} \\
\hline \frac{\mathbb{1}_{|Q| \times|Q|}-\mathbb{I}_{|Q|}}{} & \mathbb{1}_{|R|} & \frac{\mathbb{1}_{\left|R \backslash S^{\prime} \backslash S_{p}\right| \times\left|R \backslash \backslash \backslash S^{\prime}\right|}-}{\mathbb{O}_{\left(n-\left|S^{\prime}\right|\right) \times|Q|}} \\
\hline \mathbb{O}_{p \times|Q|} & \mathbb{I}_{\left|R \backslash S^{\prime}\right|} \\
\hline \mathbb{O}_{\left|J \backslash R \backslash S_{p}\right| \times\left(\left|R \backslash S^{\prime}\right|+1\right)} \\
\mathbb{1}_{p \times\left(\left|R \backslash S^{\prime}\right|+1\right)}
\end{array}\right] .
$$

By looking at the $z_{1},\left\{z_{j}\right\}_{j \in Q \cup R}, \delta_{p}$ coordinates:

$\left[\right.$| $\mathbb{1}_{\|Q\|}^{T}$ | $0_{\left\|R \backslash S^{\prime}\right\|+1}^{T}$ |  |
| :---: | :---: | :---: |
| $\mathbb{1}_{\|Q\| \times\|Q\|}-\mathbb{1}_{\|Q\|}$ | $\mathbb{O}_{\left\|Q \cap S_{p}\right\| \times\left(\left\|R \backslash S^{\prime}\right\|+1\right)}$ |  |
| $\mathbb{O}_{\left\|R \backslash S^{\prime}\right\| \times\|Q\|}$ | $\mathbb{1}_{\left\|R \backslash \backslash S^{\prime}\right\|}$ | $\mathbb{1}_{\mid R \backslash\left(S^{\prime}\left\|\times\left\|R \backslash S^{\prime}\right\|\right.\right.}-\mathbb{I}_{\left\|R \backslash S^{\prime}\right\|}$ |
| $0_{\|Q\|}^{T}$ |  | $\mathbb{1}_{\left\|R \backslash S^{\prime}\right\|+1}^{T}$ |$]$,

we can finally conclude that the unique solution of (59) is $\bar{\lambda}=0$ as these columns are linearly independent. When $p=1$, sets defined in (57) would disappear and the matrix $V$ is of the form:

| $\mathbb{1}_{\left\|S^{\prime}\right\|}^{T}$ |  | $\mathbb{O}_{\left\|S_{p}\right\| \times\left(n+1-\left\|S^{\prime}\right\|\right)}$ |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $\mathbb{O}_{\left(\left\|S_{p} \backslash Q\right\|-1\right) \times(\|Q\|+1)}$ | $\mathbb{I}_{\left(\left\|S^{\prime} \backslash Q\right\|-1\right)}$ | $\mathbb{1}_{\|R\|}$ |  | * |
| $\mathbb{1}_{\|Q\|} \mid \mathbb{1}_{\|Q\| \times\|Q\|}-\mathbb{I}_{\|Q\|}$ | $\frac{\mathbb{1}_{(\|Q\|-2) \times\left(\left\|S^{\prime} \backslash Q\right\|-1\right)}}{\mathbb{O}_{2 \times\left(\left\|S^{\prime} \backslash Q\right\|-1\right)}}$ |  | $\mathbb{I}_{\left\|R \backslash S^{\prime}\right\|}$ |  |
| $\mathbb{O}_{\left(n-\left\|S^{\prime}\right\|\right) \times\left\|S^{\prime}\right\|}$ |  | $\mathbb{O}_{\left\|J \backslash R \backslash S_{p}\right\| \times\left(\left\|R \backslash S^{\prime}\right\|+1\right)}$ |  | $\mathbb{I}_{\left\|J \backslash R \backslash S_{p}\right\|}$ |
| $0_{\left\|S S^{\prime}\right\|}^{T}$ |  | $\mathbb{1}_{n+1-\left\|S^{\prime}\right\|}^{T}$ |  |  |

We get a matrix of the structure similar to the submatrix of $V$ formed from its first $n+1$ rows and first $n+1$ columns. We can verify that in this case the columns of $V$ are affinely independent based on the discussion for the $p \geq 2$ case.

Therefore, the given points are affinely independent and inequality (54) defines an ( $n+p-1$ )-dimensional face of $\operatorname{conv}\left(X^{l, u}\right) \cap\left\{(z, \delta): \delta_{i}=0, i \in\{p+1, \ldots, m\}\right\}$.

We finish the proof by lifting the coefficients of $\delta_{p+1}, \ldots, \delta_{m}$. By validity of (35), the following inequalities are valid:

$$
\begin{aligned}
\left|S^{\prime} \backslash S_{m^{\prime}}\right|-\left|S^{\prime} \backslash S_{m^{\prime}+1}\right| \leq u-\max \{ & \sum_{j \in S^{\prime}} z_{j}+\left(u-\left|S^{\prime} \backslash S_{p}\right|\right) \delta_{p}+\sum_{i=p+1}^{m^{\prime}}\left(\left|S^{\prime} \backslash S_{i-1}\right|-\left|S^{\prime} \backslash S_{i}\right|\right) \delta_{i}: \\
& \left.(z, \delta) \in X^{l, u}, \delta_{m^{\prime}+1}=1, \delta_{i}=0, i>m^{\prime}+1\right\}, \quad m^{\prime}=p, \ldots, m-1 .
\end{aligned}
$$

And the above inequalities hold at equality for the points of $\left(S^{\prime} \backslash S_{m^{\prime}+1}\right) \cup Q_{m^{\prime}}$ for $m^{\prime}=p, \ldots, m-1$, respectively. Here $Q_{m^{\prime}}=J \backslash S^{\prime} \backslash S_{m^{\prime}+1}$ if $\left|J \backslash S_{m^{\prime}+1}\right| \leq u$. Otherwise, we construct $Q_{m^{\prime}} \subset J \backslash S^{\prime} \backslash S_{m^{\prime}+1}$ (see Figure (3) such that

1. $\left|Q_{m^{\prime}}\right|=\min \left\{u-\left|S^{\prime} \backslash S_{m^{\prime}+1}\right|,\left|J \backslash S^{\prime} \backslash S_{m^{\prime}+1}\right|\right\}$, (this implies $\left|\left(S^{\prime} \backslash S_{m^{\prime}+1}\right) \cup Q_{m^{\prime}}\right| \geq l$ as either

$$
\left.\left|\left(S^{\prime} \backslash S_{m^{\prime}+1}\right) \cup Q_{m^{\prime}}\right|=u, \text { or }\left|\left(S^{\prime} \backslash S_{m^{\prime}+1}\right) \cup Q_{m^{\prime}}\right|=\left|J \backslash S_{m^{\prime}+1}\right| \geq\left|J \backslash S_{m}\right| \geq l\right),
$$

2. $\left(\left(S^{\prime} \backslash S_{m^{\prime}+1}\right) \cup Q_{m^{\prime}}\right) \cap S_{m^{\prime}+2} \neq \emptyset$ if $m^{\prime}<m-1$.


Figure 3: Construction of $Q_{m^{\prime}}$ in the proof of Theorem 15

In total, we find $(n+m)$ affinely independent points lying on the hyperplane

$$
\sum_{j \in S^{\prime}} z_{j}+\left(u-\left|S^{\prime} \backslash S_{p}\right|\right) \delta_{p}+\sum_{i=p+1}^{m}\left(\left|S^{\prime} \backslash S_{i-1}\right|-\left|S^{\prime} \backslash S_{i}\right|\right) \delta_{i}=u
$$

Therefore, inequality (35) is facet-defining.

## Proof of Theorem 18

Proof. Assume that $S^{\prime} \subseteq J$ and $p \in I$ satisfy the conditions above. Then the assumption $p<m$ or $\left|S_{m}\right|<n-l$ implies that $\left|S_{p}\right|<n-l$, and $S^{\prime} \neq \emptyset$ as $\left|S^{\prime} \cup S_{p}\right|>n-l$. Assume $S^{\prime}=\left\{s_{1}, \ldots, s_{\left|S^{\prime}\right|}\right\}$ with $s_{1}<\ldots<s_{\left|S^{\prime}\right|}$. We first show that the inequality

$$
\begin{equation*}
-\sum_{j \in S^{\prime}} z_{j}+\left(\left|S^{\prime} \cup S_{p}\right|-n+l\right) \delta_{p} \leq 0 \tag{60}
\end{equation*}
$$

defines an $(n+p-1)$-dimensional face of $\operatorname{conv}\left(X^{l, u}\right) \cap\left\{(z, \delta): \delta_{i}=0, i \in\{p+1, \ldots, m\}\right\}$.
Let $Q=S_{p} \cup\left\{s_{1}, s_{2}, \ldots, s_{n-l-\left|S_{p}\right|}\right\}$. Then $Q$ satisfies $|Q|=n-l$ and $S_{p} \subset Q \subset S_{p} \cup S^{\prime}$. Let $R=$ $\left(J \backslash S^{\prime} \backslash S_{p}\right) \cup\left\{1,2, \ldots, l+1-\left|J \backslash S^{\prime} \backslash S_{p}\right|\right\}$. Then $R$ satisfies $|R|=l+1 \leq u$ and $\left(J \backslash S^{\prime} \backslash S_{p}\right) \cup\{1,2\} \subseteq R \subseteq J \backslash S^{\prime}$ as $n-l<\left|S^{\prime} \cup S_{p}\right|$. Note that $\left|S^{\prime} \cup S_{p-1}\right| \leq n-l$. For $i \in\{1, \ldots, p-1\}$, we can let $T_{i}$ denote the first $l$ elements of $J \backslash S^{\prime} \backslash S_{i}$. Consider the points (using Definition 3) associated with the sets

$$
\begin{gather*}
R \backslash\{j\} \text { for } j \in R, \quad R, \quad(R \backslash\{1\}) \cup\{j\} \text { for } j \in S_{p} \backslash R,  \tag{61}\\
\left(J \backslash Q \backslash\left\{s_{\left|S^{\prime}\right|}\right\}\right) \cup\{j\} \text { for } j \in Q \backslash S_{p}, \quad(J \backslash Q \backslash\{j\}) \cup\left\{s_{1}\right\} \text { for } j \in S^{\prime} \backslash Q \backslash\left\{s_{\left|S^{\prime}\right|}\right\}, J \backslash Q,  \tag{62}\\
T_{i} \text { for } i \in\{1, \ldots, p-1\} . \tag{63}
\end{gather*}
$$

These $(n+p)$ points are feasible with $\delta_{i}=0, i \in\{p+1, \ldots, m\}$ (as $\left(J \backslash S^{\prime} \backslash S_{p}\right) \cap S_{p+1} \neq \emptyset$ by assumption $\mathrm{L} 3)$, and lie on the hyperplane $-\sum_{j \in S^{\prime}} z_{j}+\left(\left|S^{\prime} \cup S_{p}\right|-n+l\right) \delta_{p}=0$.


Figure 4: Reordered $z_{j}$ coordinates in the proof of Theorem 18

We reorder the $\left(\left\{z_{j}\right\}_{j \in J}, \delta_{1}, \ldots, \delta_{p}\right)$ coordinates of the points (see Figure (4) in the ordering

$$
\left(\left\{z_{j}\right\}_{j \in S_{p}},\left\{z_{j}\right\}_{j \in J \backslash\left(S_{p} \cup S^{\prime}\right)},\left\{z_{j}\right\}_{j \in S^{\prime}}, \delta_{1}, \ldots, \delta_{p}\right)
$$

and consider the matrix $V$ formed by these reordered coordinates of the (column) points.


Figure 5: Matrix $V$ in the proof of Theorem 18

We will argue that the unique solution to the system of equations

$$
\begin{equation*}
V \lambda=0, \quad \sum_{t=1}^{n+p} \lambda_{t}=0 \tag{64}
\end{equation*}
$$

is $\lambda=0$. First consider matrix $V_{p}$ formed by the last $p$ rows of $V$ corresponding to the $\left\{\delta_{i}\right\}_{i=1}^{p}$ coordinates:

$$
V_{p}=\left[\mathbb{O}_{p \times\left(\left|J \backslash S^{\prime}\right|+1\right)}\left|\mathbb{1}_{p \times\left|S^{\prime}\right|}\right| d^{1}|\ldots| d^{p-1}\right],
$$

where $d^{i}$ is defined in the proof of Lemma [7. Equations $V_{p} \lambda=0$ imply $\lambda_{n+2}=\ldots=\lambda_{n+p}=0$ and $\lambda_{\left|J \backslash S^{\prime}\right|+2}+\ldots+\lambda_{n+1}=0$. Therefore, (64) reduces to equations

$$
\bar{V} \bar{\lambda}=0, \quad \sum_{t=1}^{n+1} \lambda_{t}=0
$$

where $\bar{V}$ is the matrix formed by the first $n+1$ columns of $V$ and $\bar{\lambda}=\left(\lambda_{1}, \ldots, \lambda_{n+1}\right)^{T}$. Then we write down
the matrix $\bar{V} S^{\prime}$ formed by rows $\left|J \backslash S^{\prime}\right|+1$ to $|J|$ of $\bar{V}$ :

$$
\bar{V}^{S^{\prime}}=\left[\begin{array}{c|c|c}
\mathbb{O}_{\left|S^{\prime}\right| \times\left(\left|J \backslash S^{\prime}\right|+1\right)} & \mathbb{1}_{\left|S^{\prime} \cap Q\right|} & \frac{\mathbb{1}_{\left|S^{\prime} \backslash Q\right|-1}^{T}}{\mathbb{O}_{\left(\left|S^{\prime} \cap Q\right|-1\right) \times\left(\left|S^{\prime} \backslash Q\right|-1\right)}} \\
\hline \frac{\mathbb{1}_{\left(\left|S^{\prime} \backslash Q\right|-1\right) \times\left|S^{\prime} \cap Q\right|}}{0_{\left|S^{\prime} \cap Q\right|}^{T}} & \frac{\mathbb{1}_{\left(\left|S^{\prime} \backslash Q\right|-1\right) \times\left(\left|S^{\prime} \backslash Q\right|-1\right)}-\mathbb{1}_{\left(\left|S^{\prime} \backslash Q\right|-1\right)}}{} & \mathbb{1}_{\left|S^{\prime} \backslash Q\right|} \\
\mathbb{1}_{\left|S^{\prime} \backslash Q\right|-1}^{T}
\end{array}\right] .
$$

By looking at $\left\{z_{j}\right\}_{j \in S^{\prime} \backslash Q}$ and $\left\{z_{j}\right\}_{j \in S^{\prime} \cap Q}$ coordinates, equations $\bar{V}^{S^{\prime}} \bar{\lambda}=0$ and $\lambda_{\left|J \backslash S^{\prime}\right|+2}+\ldots+\lambda_{n+1}=0$ imply that $\lambda_{n+2-\left|S^{\prime} \backslash Q\right|}=\ldots=\lambda_{n+1}=0$ and $\lambda_{\left|J \backslash S^{\prime}\right|+2}=\ldots=\lambda_{n+1-\left|S^{\prime} \backslash Q\right|}=0$. Therefore, (64) further reduces to equations

$$
\overline{\bar{V}} \overline{\bar{\lambda}}=0, \quad \sum_{t=1}^{\left|J \backslash S^{\prime}\right|+1} \lambda_{t}=0
$$

where $\overline{\bar{V}}$ is the submatrix of $V$ formed from its first $\left|J \backslash S^{\prime}\right|$ rows and first $\left|J \backslash S^{\prime}\right|+1$ columns and $\overline{\bar{\lambda}}=$ $\left(\lambda_{1}, \ldots, \lambda_{\left|J \backslash S^{\prime}\right|+1}\right)^{T}$. The matrix $\overline{\bar{V}}$ is of the form:
$\left[\begin{array}{c|c|c|c}\mathbb{1}_{\left|R \cap S_{p}\right| \times\left|R \cap S_{p}\right|}-\mathbb{I}_{\left|R \cap S_{p}\right|} & \mathbb{1}_{\left|R \cap S_{p}\right| \times\left|J \backslash S^{\prime} \backslash S_{p}\right|} & \mathbb{1}_{\left|R \cap S_{p}\right|} & 0_{\left|S_{p} \backslash R\right|}^{T} \\ \hline \mathbb{O}_{\left|S_{p} \backslash R\right| \times\left|R \cap S_{p}\right|} & \mathbb{O}_{\left|S_{p} \backslash R\right| \times\left|J \backslash S^{\prime} \backslash S_{p}\right|} & 0_{\left|S_{p} \backslash R\right|} & \mathbb{1}_{\left|S_{p} \backslash R\right|} \\ \hline \mathbb{1}_{\left|J \backslash S^{\prime} \backslash S_{p}\right| \times\left|R \cap S_{p}\right|} & \mathbb{1}_{\left|J \backslash S^{\prime} \backslash S_{p}\right| \times\left|J \backslash S^{\prime} \backslash S_{p}\right|}-\mathbb{1}_{\left|J \backslash S^{\prime} \backslash S_{p}\right|} & \mathbb{1}_{\left|J \backslash S^{\prime} \backslash S_{p}\right|} & \mathbb{1}_{\left|J \backslash S^{\prime} \backslash S_{p}\right| \times\left|S_{p} \backslash R\right|}\end{array}\right]$.

Rows of $\overline{\bar{V}}$ with index $j \in J \backslash S^{\prime} \backslash S_{p}$ together with $\sum_{t=1}^{\left|J \backslash S^{\prime}\right|+1} \lambda_{t}=0$ imply $\lambda_{\left|R \cap S_{p}\right|+1}=\ldots=\lambda_{|R|}=0$. Rows of $\overline{\bar{V}}$ with index $j \in S_{p} \backslash R$ imply $\lambda_{|R|+2}=\ldots=\lambda_{\left|J \backslash S^{\prime}\right|+1}=0$. The rest of rows together with $\sum_{t=1}^{\left|J \backslash S^{\prime}\right|+1} \lambda_{t}=0$ imply $\lambda_{1}=\ldots=\lambda_{\left|R \cap S_{p}\right|}=0$ and $\lambda_{|R|+1}=0$. Therefore, the given points are affinely independent and inequality (60) defines an $(n+p-1)$-dimensional face of $\operatorname{conv}\left(X^{l, u}\right) \cap\left\{(z, \delta): \delta_{i}=0, i \in\{p+1, \ldots, m\}\right\}$.
We finish the proof by lifting the coefficients of $\delta_{p+1}, \ldots, \delta_{m}$. Define $S_{m+1}=J$ and $\delta_{m+1}=0$. By the validity of (39), for each $m^{\prime} \in\{p, p+1, \ldots, m-1\}$

$$
\begin{array}{r}
\left|S^{\prime} \cup S_{m^{\prime}+1}\right|-\left|S^{\prime} \cup S_{m^{\prime}}\right| \leq-\max \left\{-\sum_{j \in S^{\prime}} z_{j}+\left(\left|S^{\prime} \cup S_{p}\right|-n+l\right) \delta_{p}+\sum_{i=p+1}^{m^{\prime}}\left(\left|S^{\prime} \cup S_{i}\right|-\left|S^{\prime} \cup S_{i-1}\right|\right) \delta_{i}\right. \\
\left.(z, \delta) \in X^{l, u}, \delta_{m^{\prime}+1}=1, \delta_{i}=0, i>m^{\prime}+1\right\}
\end{array}
$$

Actually the above inequality holds at equality by taking $(z, \delta)$ as the points of $\left(J \backslash S^{\prime} \backslash S_{m^{\prime}+1}\right) \cup L_{m^{\prime}}$ for $m^{\prime}=p, \ldots, m-1$, respectively. Here $L_{m^{\prime}} \subset S^{\prime} \backslash S_{m^{\prime}+1}$ can be constructed by starting with an element in $S_{m^{\prime}+2}$ if $\left(J \backslash S^{\prime} \backslash S_{m^{\prime}+1}\right) \cap S_{m^{\prime}+2}=\emptyset$ and then augmenting it to have cardinality $\left|S^{\prime} \cup S_{m^{\prime}+1}\right|-(n-l)(\geq 1)$. Set $L_{m^{\prime}} \subset S^{\prime} \backslash S_{m^{\prime}+1}$ (see Figure (6) satisfies

1. $\left|L_{m^{\prime}}\right|=\left|S^{\prime} \cup S_{m^{\prime}+1}\right|-(n-l)=l-\left|J \backslash S^{\prime} \backslash S_{m^{\prime}+1}\right|$,
2. $\left(\left(J \backslash S^{\prime} \backslash S_{m^{\prime}+1}\right) \cup L_{m^{\prime}}\right) \cap S_{m^{\prime}+2} \neq \emptyset$ if $m^{\prime}<m-1$.


Figure 6: Construction of $L_{m^{\prime}}$ in the proof of Theorem 18

In total, we find $(n+m)$ affinely independent points lying on the hyperplane

$$
-\sum_{j \in S^{\prime}} z_{j}+\left(\left|S^{\prime} \cup S_{p}\right|-n+l\right) \delta_{p}+\sum_{i=p+1}^{m}\left(\left|S^{\prime} \cup S_{i}\right|-\left|S^{\prime} \cup S_{i-1}\right|\right) \delta_{i}=0
$$

Therefore, inequality (39) is facet-defining.

