

A Fast Iterative Algorithm to design phase only sequences by minimizing the ISL metric

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Abstract

Unimodular/Phase only sequence having impulse like aperiodic auto-correlation function plays a central role in the applications of RADAR, SONAR, Cryptography, and Wireless (CDMA) Communication Systems. In this paper, we propose a fast iterative algorithm to design phase only sequences of arbitrary lengths by minimizing the Integrated Side-lobe Level (ISL) metric, which is very closely related to the auto-correlation property of a sequence. The ISL minimization problem is solved iteratively by using the Majorization-Minimization (MM) technique, which ensures a monotonic convergence to the stationary minimum point. To highlight the performance of a proposed algorithm, we conduct the numerical experiments for different sequence lengths using different initializations and also compare them with the existing algorithms. Numerical simulations show that irrespective of the sequence length and initialization, the proposed algorithm is performing better than the state-of-the-art algorithms in terms of speed of convergence. We also show a computationally efficient way to implement our proposed algorithm by using the FFT and IFFT operations.

Index Terms– Majorization-Minimization, Integrated Side-lobe Level, Peak Side-lobe Level, Phase only sequence, aperiodic auto-correlation, Cryptography, Communication Systems, RADAR, SONAR, MIMO RADAR, CDMA.

I.INTRODUCTION

The target detection capability of an active sensing system will solely depend on the accuracy of the estimated underlying parameters. So, to increase the detection performance in applications like active sensing systems (SONAR, RADAR) [1], [2], [3], [4], [5], Cryptography [6], CDMA communication systems [7], [8], [9], and MIMO RADAR [10-17], finite-length transmit sequences with impulse like aperiodic auto-correlation function is a necessity. However, in real life, along with good correlation property, the aforementioned applications also pose different constraints on the transmit sequence like the power and spectral (range of operating frequencies) constraints. The power constraint is mainly due to the limited budget of transmitter power available in the system. Hence, the design of phase only sequences of arbitrary lengths having unit magnitude and impulse like aperiodic auto-correlation function is always desired [18], [19], [7].

Earlier, design of phase only sequences is done mainly by algebraic approaches and some of the sequences designed through algebraic approaches are Barker sequence [20], [21], Frank sequence [22], Golomb sequence [23], Chu sequence [23], [3] and P4 sequence [24], [25]. But all the above-mentioned sequences exist only for the shorter lengths and have limited degrees of freedom. Hence, algebraic approaches are not viable to generate sequences of large lengths. To overcome this issue, recently, computational approaches [7], [26], [27], [28], came into existence and enabled a way to design sequences of arbitrary lengths at a slighter computational cost. Some

of the developed computational approaches are the stochastic search methods [29], exhaustive search methods [30], which are heuristic in nature with no guarantee for convergence to a stationary point of ISL function. To overcome all such issues, very recently several optimization methods [31], [8], [4], [32] came into the existence and some of the approaches are CAN [33], MISL [34], ISL-NEW [35], MM-Corr [36], ADMM approach [37], MWISL, MWISL-Diag, MM-PSL [38], and CPM [39]- a detailed review of some of the methods will follow shortly. The following mathematical notations are used hereafter: boldface lowercase letters denote column vectors, boldface uppercase letters denote matrices and italics denote scalars. The superscripts $()^*$, $()^T$, $()^H$ denote complex conjugate, transpose, and conjugate transpose, respectively. $\text{Tr}()$ denote the trace of a matrix. z_m denote the m^{th} element of a vector \mathbf{z} . $\text{Re}()$ and $\text{Im}()$ denote the real and imaginary parts, respectively. \mathbf{I}_a denote the $a \times a$ identity matrix. $\|\cdot\|_2$ denote the l_2 norm. $\text{vec}(\mathbf{S})$ is a column vector that consists of all the columns of a matrix- \mathbf{S} stacked. $|\cdot|^2$ denote the absolute squared value. $\text{Diag}(\mathbf{z})$ is a diagonal matrix formed with a vector \mathbf{z} as its diagonal. \mathbb{R} and \mathbb{C} represent the real and complex fields. $\lambda_{\max}(\mathbf{R})$ denote the maximum eigenvalue of \mathbf{R} . $\nabla g(\cdot)$ denote the gradient of a function $g(\cdot)$. $\mathbf{b}_{1:N}$ represents the first N elements of a vector \mathbf{b} .

A. SIGNAL MODEL AND PROBLEM FORMULATION

Let $\{z_n\}_{n=1}^P$ be a phase only sequence of length ‘ P ’ to be designed. The m^{th} element of a sequence is denoted as $e^{j\varphi(m)}$, where $\varphi(m)$ is an arbitrary phase angle that varies between 0 and 2π radians. The aperiodic auto-correlation function of a sequence $\{z_n\}_{n=1}^P$ at any lag ‘ l ’ is defined as:

$$r(l) = \sum_{n=1}^{P-l} z_{n+l} z_n^* = r^*(-l), \quad l = 0, \dots, P-1. \quad (1)$$

The Integrated Side-lobe Level (ISL) metric, which is a direct measure of the designed sequence is defined as:

$$\text{ISL} = \sum_{l=1}^{P-1} |r(l)|^2. \quad (2)$$

The Peak Side-lobe Level (PSL) metric is defined as:

$$\text{PSL} = \max \{|r(l)|\}_{l=1}^{P-1} \quad (3)$$

So, the problem to design a phase only sequence that minimizes the ISL metric is formulated as:

$$\begin{aligned} \underset{\mathbf{z}}{\text{minimize}} \quad & \text{ISL} = \sum_{l=1}^{P-1} |r(l)|^2 \\ \text{subject to} \quad & |z_n| = 1, \quad n = 1, \dots, P. \end{aligned} \quad (4)$$

where $\mathbf{z} = [z_1 z_2 \dots z_P]_{1 \times P}^T$.

Apart from the unimodular constraint, there are interests in imposing constraints like binary constraint [28], [26], [40], spectral constraint [41], [42], [43], similarity constraint [44], [45], Peak to Average Power Ratio (PAPR) constraint [46] to name a few.

In the next subsection, we will discuss the general framework of majorization-minimization, which would play a central role in the development of our algorithm.

B. Majorization-Minimization Method

Majorization-Minimization (MM) is a two-step technique, which is used to solve the hard (non-convex or even convex) problems very efficiently [47], [48]. The first step of the MM method is to construct a majorization (upper bound) function $u()$ to the original objective function $g()$ at any point \mathbf{z}^k (\mathbf{z} at k^{th} iteration) and then second step is to minimize the upper-bound function $u()$ to generate a next update \mathbf{z}^{k+1} . So, at every newly generated point, the above mentioned two steps will be applied repeatedly until it reaches the optimum minimum point of an original function $g()$. For any given problem, the construction of a majorization function is not unique and for the same problem, different types of majorization functions will exist. So, the performance will depend solely on the chosen majorization function and the different ways to construct a majorization function are shown in [48], [31].

The majorization function $u(\mathbf{z}|\mathbf{z}^k)$, which is constructed in the first step of the MM method has to satisfy the following properties:

$$u(\mathbf{z}^k|\mathbf{z}^k) = g(\mathbf{z}^k), \quad \forall \mathbf{z} \in \mathbf{Z}. \quad (5)$$

$$u(\mathbf{z}|\mathbf{z}^k) \geq g(\mathbf{z}), \quad \forall \mathbf{z} \in \mathbf{Z}. \quad (6)$$

where \mathbf{Z} is the set consists of all the possible values of \mathbf{z} . As the MM technique is an iterative process, it will generate the sequence of points $\{\mathbf{z}\} = \mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3, \dots, \mathbf{z}^m$ according to the following update rule:

$$\mathbf{z}^{k+1} \triangleq \arg \min_{\mathbf{z} \in \mathbf{Z}} u(\mathbf{z}|\mathbf{z}^k). \quad (7)$$

The cost function value evaluated at every point generated by (7) will satisfy the descent property, i.e.

$$g(\mathbf{z}^{k+1}) \leq u(\mathbf{z}^{k+1}|\mathbf{z}^k) \leq u(\mathbf{z}^k|\mathbf{z}^k) = g(\mathbf{z}^k). \quad (8)$$

C. Related work and our Contributions

The existing algorithms which are developed by solving the same ISL minimization problem (4) are CAN [33], MISL [34], ISL-NEW [35], MM-Corr [36], ADMM approach [37], MWISL, MWISL-Diag [38], CPM [39]. In the following, we will discuss them briefly and highlight their potentials and drawbacks.

Stoica et.al proposed the CAN algorithm [33], which works on the principle of alternating minimization technique. They solved the problem by transforming the objective function in (4) to the frequency domain as:

$$\sum_{l=1}^{P-1} |r(l)|^2 = \frac{1}{4P} \sum_{a=1}^{2P} \left[\left| \sum_{n=1}^P z_n e^{-j\omega_a n} \right|^2 - P \right]^2, \quad (9)$$

where $\omega_a = \frac{2\pi}{2P}a$, $a = 1, \dots, 2P$. are the Fourier grid frequencies. Then the problem (4) is converted into:

$$\begin{aligned} \underset{\mathbf{z}}{\text{minimize}} \quad & \frac{1}{4P} \sum_{a=1}^{2P} \left[\left| \sum_{n=1}^P z_n e^{-j\omega_a n} \right|^2 - P \right]^2 \\ \text{subject to} \quad & |z_n| = 1, n = 1, \dots, P. \end{aligned} \quad (10)$$

The cost function of the problem in (10) is quartic in \mathbf{z} and it is very hard to solve further. So, instead of solving (10) directly, they solved an approximate problem, which is quadratic in \mathbf{z} as shown below:

$$\begin{aligned} \underset{\mathbf{z}, \phi_a}{\text{minimize}} \quad & \sum_{a=1}^{2P} \left[\left| \sum_{n=1}^P z_n e^{-j\omega_a n} - \sqrt{P} e^{j\phi_a} \right|^2 \right] \\ \text{subject to} \quad & |z_n| = 1, n = 1, \dots, P, \end{aligned} \quad (11)$$

where ϕ_a , $a = 1, 2, \dots, 2P$. are the auxiliary phase variables. The resulting problem can be rewritten more compactly as follows:

$$\begin{aligned} \underset{\mathbf{z}, \mathbf{y}}{\text{minimize}} \quad & \left\| \hat{\mathbf{E}}^H \hat{\mathbf{z}} - \sqrt{P} \mathbf{y} \right\|^2 \\ \text{subject to} \quad & |z_n| = 1, n = 1, \dots, P \end{aligned} \quad (12)$$

where $\hat{\mathbf{E}} \triangleq [\mathbf{e}_1, \dots, \mathbf{e}_{2P}]$ be a $2P \times 2P$ matrix with $\mathbf{e}_a \triangleq [e^{j\omega_a(1)}, e^{j\omega_a(2)}, \dots, e^{j\omega_a(2P)}]^T$, $\hat{\mathbf{z}} \triangleq [z_1, z_2, \dots, z_P, 0, \dots, 0]_{1 \times 2P}^T$ and $\mathbf{y} \triangleq [e^{j\phi_1}, \dots, e^{j\phi_{2P}}]^T$. They solved the problem in (12) by alternatively minimizing between the variables \mathbf{z} and \mathbf{y} . For a given \mathbf{z} , minimization of (12) with respect to \mathbf{y} is given by:

$$\mathbf{y} = \frac{\mathbf{v}}{\|\mathbf{v}\|_2}, \quad (13)$$

where $\mathbf{v} \triangleq \hat{\mathbf{E}}^H \hat{\mathbf{z}}$ ($\hat{\mathbf{E}}^H$ is a $2N \times 2N$ FFT matrix) and for a fixed \mathbf{y} , minimizer over \mathbf{z} would be:

$$\mathbf{z} = \frac{\mathbf{b}}{\|\mathbf{b}\|_2}, \quad (14)$$

where $\mathbf{b} \triangleq \hat{\mathbf{E}} \mathbf{y}$ ($\hat{\mathbf{E}}$ is a $2N \times 2N$ IFFT matrix). The pseudocode of the CAN algorithm is summarized in Algorithm 1.

We would like to point out that, instead of solving the original problem (4), the CAN algorithm had solved an approximately equivalent problem (11). So, there is no guarantee for an obtained minimum of (11) is also a minimum of the original problem in (4).

To overcome this issue, Song et.al. proposed the MISL algorithm [34] by solving the original problem (4) directly via the MM approach. So, from (10) we have,

$$\begin{aligned} \underset{\mathbf{z}}{\text{minimize}} \quad & \frac{1}{4P} \sum_{a=1}^{2P} \left[\left| \sum_{n=1}^P z_n e^{-j\omega_a n} \right|^2 - P \right]^2 \\ \text{subject to} \quad & |z_n| = 1, n = 1, \dots, P. \end{aligned}$$

Algorithm 1 :The CAN algorithm proposed in [33]

Require: sequence length ‘ P ’

1: set $k = 0$, initialize z^0

2: **repeat**

3: $v = \hat{E}^H z^k$

4: $y = \frac{v}{\|v\|_2}$

5: $b = \hat{E}y$

6: $z^{k+1} = \frac{b_{1:P}}{\|b_{1:P}\|_2}$

7: $k \leftarrow k + 1$

8: **until** convergence

By expanding the cost function and ignoring the constant and multiplication terms, the above problem can be rewritten more compactly as:

$$\begin{aligned} & \underset{z}{\text{minimize}} && \sum_{a=1}^{2P} \left[e_a^H z z^H e_a \right]^2 \\ & \text{subject to} && |z_n| = 1, n = 1, \dots, P. \end{aligned} \quad (15)$$

In terms of z , the problem in (15) is quartic and very hard to solve further. So, by defining $Z = z z^H$ and $C_a = e_a e_a^H$, problem in (15) can be rewritten as:

$$\begin{aligned} & \underset{z, Z}{\text{minimize}} && \text{vec}(Z)^H \Phi \text{vec}(Z) \\ & \text{subject to} && |z_n| = 1, n = 1, \dots, P, \\ & && Z = z z^H, \end{aligned} \quad (16)$$

where $\Phi = \sum_{a=1}^{2P} \text{vec}(C_a) \text{vec}(C_a)^H$. The cost function in (16) is quadratic in Z . So, they constructed a majorization function for it by using second-order Taylor series method [48], [31], and by neglecting the constant terms, the surrogate problem can be rewritten more compactly as:

$$\begin{aligned} & \underset{z}{\text{minimize}} && z^H \left[\hat{E} \text{Diag}(b^k) \hat{E}^H - 2P^2 z^k (z^k)^H \right] z \\ & \text{subject to} && |z_n| = 1, n = 1, \dots, P, \end{aligned} \quad (17)$$

where $b^k = \left| \hat{E}^H z^k \right|$. The resultant problem in (17) is quadratic in z , and they have majorized the cost function in the above problem once again as mentioned above, to obtain a simple closed-form solution. After majorizing for the second time and by ignoring the constant terms, the final surrogate minimization problem becomes:

$$\begin{aligned} & \underset{z}{\text{minimize}} && \text{Re} \left(z^H \left[\tilde{C} - 2P^2 z^k (z^k)^H \right] z^k \right) \\ & \text{subject to} && |z_n| = 1, n = 1, \dots, P, \end{aligned} \quad (18)$$

where $\tilde{C} = \hat{E} \left(\text{Diag}(\mathbf{b}^{2k}) - b_{max}^k \mathbf{I} \right) \hat{E}^H$ and $b_{max}^k = \max_a \{(b_a^k)^2, a = 1, 2, \dots, 2P\}$. Problem in (18) can be rewritten more compactly as:

$$\begin{aligned} & \underset{\mathbf{z}}{\text{minimize}} && \|\mathbf{z} - \mathbf{d}\|_2^2 \\ & \text{subject to} && |z_n| = 1, n = 1, \dots, P, \end{aligned} \quad (19)$$

where $\mathbf{d} = -\hat{E} \left(\text{Diag}(\mathbf{b}^{2k}) - b_{max}^k \mathbf{I} - N^2 \mathbf{I} \right) \hat{E}^H \mathbf{z}^k$. The problem in (19) has a closed-form solution:

$$\mathbf{z} = \frac{\mathbf{d}}{\|\mathbf{d}\|_2}. \quad (20)$$

The pseudocode of the MISL algorithm is summarized in Algorithm 2.

Algorithm 2 :The MISL algorithm proposed in [34]

Require: sequence length ‘ P ’

- 1: set $k = 0$, initialize \mathbf{z}^k
 - 2: **repeat**
 - 3: $\mathbf{b}^k = |\hat{E}^H \mathbf{z}^k|$
 - 4: $b_{max}^k = \max_a \{(b_a^k)^2, a = 1, \dots, 2P\}$
 - 5: $\mathbf{d} = -\hat{E} \left(\text{Diag}(\mathbf{b}^{2k}) - b_{max}^k \mathbf{I} - N^2 \mathbf{I} \right) \hat{E}^H \mathbf{z}^k$
 - 6: $\mathbf{z}^{k+1} = \frac{\mathbf{d}_{1:P}}{\|\mathbf{d}_{1:P}\|_2}$
 - 7: $k \leftarrow k + 1$
 - 8: **until** convergence
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Compared to the CAN algorithm, the MISL algorithm solves the original problem in (4). So, there is an assurance of obtaining an original optimum minimum point. But, the MISL algorithm faces a drawback of slower convergence due to twice the majorization of the original objective function. To deal with the convergence issue, they have proposed acceleration schemes to accelerate the MISL algorithm.

In [35], Y. Li et.al proposed an algorithm named ISL-NEW using the MM method to design sequence-set. By particularizing it for single sequence, we observe that the only difference between the MISL and ISL-NEW algorithms is in the way they arrive at their majorizing functions. After majorizing the objective function in (16) and removing the constant terms, the final surrogate problem they solve is given by:

$$\begin{aligned} & \underset{\mathbf{z}}{\text{minimize}} && \mathbf{z}^H \left[\hat{E} \text{Diag}(\mathbf{b}^k) \hat{E}^H - P^2 \mathbf{z}^k (\mathbf{z}^k)^H \right] \mathbf{z} \\ & \text{subject to} && |z_n| = 1, n = 1, \dots, P. \end{aligned} \quad (21)$$

The resultant problem in (21) is quadratic in \mathbf{z} . So, they majorized the cost function in (21) once again and arrive at the following problem:

$$\begin{aligned} & \underset{\mathbf{z}}{\text{minimize}} \quad \text{Re} \left(\mathbf{z}^H \left[\bar{\mathbf{C}} - P^2 \mathbf{z}^k (\mathbf{z}^k)^H \right] \mathbf{z}^k \right) \\ & \text{subject to} \quad |z_n| = 1, n = 1, \dots, P, \end{aligned} \quad (22)$$

where $\bar{\mathbf{C}} = \hat{\mathbf{E}} \left(\text{Diag}(\mathbf{b}^{2k}) - 0.5b_{\max}^k \mathbf{I} \right) \hat{\mathbf{E}}^H$. The problem in (22) can be rewritten as:

$$\begin{aligned} & \underset{\mathbf{z}}{\text{minimize}} \quad \|\mathbf{z} - \hat{\mathbf{d}}\|_2^2 \\ & \text{subject to} \quad |z_n| = 1, n = 1, \dots, P, \end{aligned} \quad (23)$$

where $\hat{\mathbf{d}} = -\hat{\mathbf{E}} \left(\text{Diag}(\mathbf{b}^{2k}) - 0.5b_{\max}^k \mathbf{I} - 0.5N^2 \mathbf{I} \right) \hat{\mathbf{E}}^H \mathbf{z}^k$. The problem in (23) has a closed-form solution

$$\mathbf{z} = \frac{\hat{\mathbf{d}}}{\|\hat{\mathbf{d}}\|_2}. \quad (24)$$

The pseudocode of the ISL-NEW algorithm is summarized in Algorithm 3.

Algorithm 3 :The ISL-NEW algorithm proposed in [35]

Require: sequence length ‘ P ’

1: set $k = 0$, initialize \mathbf{z}^k

2: **repeat**

3: $\mathbf{b}^k = \left| \hat{\mathbf{E}}^H \mathbf{z}^k \right|$

4: $b_{\max}^k = \max_a \{ (b_a^k)^2 : a = 1, \dots, 2P \}$

5: $\hat{\mathbf{d}} = -\hat{\mathbf{E}} \left(\text{Diag}(\mathbf{b}^{2k}) - 0.5b_{\max}^k \mathbf{I} - 0.5N^2 \mathbf{I} \right) \hat{\mathbf{E}}^H \mathbf{z}^k$

6: $\mathbf{z}^{k+1} = \frac{\hat{\mathbf{d}}_{1:P}}{\|\hat{\mathbf{d}}_{1:P}\|_2}$

7: $k \leftarrow k + 1$

8: **until** convergence

Y. Li et.al has also solved the problem (4) directly and concluded it as a fast algorithm in terms of the convergence. However, due to similarity in the update step of ISL-NEW and MISL (with very little difference), ISL-NEW also suffers from slow convergence and they have also proposed acceleration schemes to accelerate the ISL-NEW algorithm. The above mentioned three algorithms CAN, MISL and ISL-NEW can be implemented via FFT and IFFT operations. Hence, they are computationally efficient for generating sequences of large lengths.

In [36], J. Song et.al proposed an algorithm named as MM-Corr to design the sequence set using the MM method. In [37], J.Liang et.al proposed a new algorithm by solving the approximately equivalent problem to problem (4) (i.e, same as CAN algorithm) by using the ADMM method and they concluded that its performance is worse than the MISL algorithm in terms of the PSL metric value in its aperiodic autocorrelation function. In [38], J.song et.al proposed three different algorithms named MWISL, MWISL-Diag, and MM-PSL by using the MM method. We observe that out of three algorithms MWISL and MWISL-Diag are variants of the MISL algorithm [34] and MM-PSL algorithm is derived by solving the l_p -norm, $2 < p < \infty$, which is different from the ISL metric. In

[39], Mohammad et.al has proposed an algorithm named as CPM based on the coordinate descent framework and concluded that CPM performs well only in the case of binary and finite discrete phase constraints. Some more algorithms, which are derived based on different metrics like PSL [42], [38], [49], ambiguity function shaping [50], [51], [52], [53], SINR [54], beam pattern synthesis [55], [12], are used to design sequences.

So, the summary of the related literature is as follows:

- CAN algorithm has solved the approximate problem in (11) and there is no guarantee for an obtained minimum to be also the minimum of the original problem in (4).
- Even though the MISL and ISL-NEW algorithms has solved the original problem in (4), they face a drawback of slower convergence due to two times the majorization of the original objective function.
- In comparison to the CAN and MISL algorithms, the authors in [35] claimed that the ISL-NEW algorithm is fast but it is only a marginal improvement.
- ADMM algorithm solves the approximate problem (same as CAN algorithm) and it is a non-monotonic and does not minimize the ISL function.
- The CPM algorithm is derived based on the coordinate descent method, as the length of the sequence increases its computational complexity will also increase and convergence to a minimizer will also get slower.

As all the above mentioned state-of-the-art algorithms have either slower convergence or do not solve the original ISL minimization problem. This motivated us to solve the original ISL minimization problem (4) with a faster algorithm and we named our algorithm as FISL (Faster ISL minimization algorithm).

The major contributions of this paper are as follows:

- An algorithm based on the MM framework is proposed, to design phase only sequences of arbitrary length P by minimizing the ISL metric.
- To obtain faster convergence speed, we constructed a majorization function that acts like a tighter global upper bound to the original ISL function.
- Through MATLAB simulations we compare different ways of constructing a majorization function and pick out the best approach to implement our algorithm.
- By using FFT and IFFT operations, we show a computationally efficient way of implementing our proposed algorithm.
- We prove that the proposed algorithm converges to a stationary point of a problem in (4).
- Numerical experiments were conducted to prove that our proposed algorithm performs better than the state-of-the-art algorithms in terms of the speed of convergence.

The rest of the paper is organized as follows. In section II, we propose our algorithm and discuss its convergence analysis, computational & space complexities. Section III consists of numerical experiments and finally, section IV concludes the paper.

II. FISL-FASTER ISL MINIMIZATION ALGORITHM

A. ISL minimization via MM method

From (4), we have

$$\begin{aligned} \underset{\mathbf{z}}{\text{minimize}} \quad \text{ISL} &= \sum_{l=1}^{P-1} |r(l)|^2. \\ \text{subject to} \quad &|z_n| = 1, \quad n = 1, \dots, P. \end{aligned}$$

During the problem formulation, we considered only the positive lags, but now we will reframe it to make the problem of interest consists of both the positive and negative lags along with the zeroth lag (due to the unimodular property always equal to the length of a sequence P , which is a constant value).

So, the problem of interest becomes as:

$$\begin{aligned} \underset{\mathbf{z}}{\text{minimize}} \quad g(\mathbf{z}) &= \sum_{l=-(P-1)}^{P-1} |r(l)|^2 \\ \text{subject to} \quad &|z_n| = 1, \quad n = 1, \dots, P. \end{aligned} \quad (25)$$

We can write $r(l) = \mathbf{z}^H \mathbf{W}_l \mathbf{z}$, where \mathbf{W}_l is a Toeplitz matrix of dimension $P \times P$, with entries given by:

$$\mathbf{W}_l = \begin{cases} 1 & ; j - i = l \\ 0 & ; \text{else} \end{cases} \quad (26)$$

i, j denote the row and column indexes of \mathbf{W}_l respectively.

So, the objective function of a problem in (25) can be rewritten as $g(\mathbf{z}) = \mathbf{z}^H \mathbf{R}(\mathbf{z}) \mathbf{z}$, where

$$\mathbf{R}(\mathbf{z}) = \sum_{l=1}^{P-1} r^*(l) \mathbf{W}_l + \sum_{l=1}^{P-1} r(l) \mathbf{W}_l^H + \text{Diag}(\mathbf{r}_c). \quad (27)$$

where $\mathbf{r}_c = [r(0), r(0), \dots, r(0)]_{1 \times P}^T$. So,

$$\mathbf{R}(\mathbf{z}) = \begin{bmatrix} r(0) & r^*(1) & \cdot & \cdot & r^*(P-2) & r^*(P-1) \\ r(1) & r(0) & r^*(1) & \cdot & \cdot & r^*(P-2) \\ \cdot & r(1) & r(0) & r^*(1) & \cdot & \cdot \\ \cdot & \cdot & r(1) & \cdot & \cdot & \cdot \\ r(P-2) & \cdot & \cdot & \cdot & \cdot & r^*(1) \\ r(P-1) & r(P-2) & \cdot & \cdot & r(1) & r(0) \end{bmatrix} \quad (28)$$

is a Hermitian Toeplitz matrix and to implement it, one can find autocorrelation of \mathbf{z} using FFT and IFFT operations as:

$$\mathbf{r} = \hat{\mathbf{E}} | \hat{\mathbf{E}}^H \mathbf{z} |^2. \quad (29)$$

Here $| \cdot |^2$ is element wise operation. Then the problem of interest (25) becomes as:

$$\begin{aligned} \underset{\mathbf{z}}{\text{minimize}} \quad g(\mathbf{z}) &= \mathbf{z}^H \mathbf{R}(\mathbf{z}) \mathbf{z} \\ \text{subject to} \quad &|z_n| = 1, \quad n = 1, \dots, P. \end{aligned} \quad (30)$$

In the following, we will introduce a lemma which will be useful in deriving a majorizing function for the objective in (30).

Lemma-1: Let $f : \mathbb{C}^N \rightarrow \mathbb{R}$ be a continuously twice differentiable function and if $f(\mathbf{x})$ has a bounded curvature, then there exists a matrix $\mathbf{M} \succeq \nabla^2 f(\mathbf{x})$, such that by using the second-order Taylor series expansion, at any fixed point \mathbf{x}^k , $f(\mathbf{x})$ can be upper bounded (majorized) as,

$$f(\mathbf{x}) = f(\mathbf{x}^k) + \nabla f(\mathbf{x}^k)^H (\mathbf{x} - \mathbf{x}^k) + \frac{1}{2} (\mathbf{x} - \mathbf{x}^k)^H \nabla^2 f(\mathbf{x}^k) (\mathbf{x} - \mathbf{x}^k) \quad (31)$$

$$f(\mathbf{x}) \leq f(\mathbf{x}^k) + \nabla f(\mathbf{x}^k)^H (\mathbf{x} - \mathbf{x}^k) + \frac{1}{2} (\mathbf{x} - \mathbf{x}^k)^H \mathbf{M} (\mathbf{x} - \mathbf{x}^k) \quad (32)$$

Proof: The proof can be found in [48] ■

So, according to the lemma-1, by using the second-order Taylor series expansion, at any fixed point \mathbf{z}^k , the objective function of the problem in (30) can be majorized as,

$$\mathbf{z}^H \mathbf{R}(\mathbf{z}) \mathbf{z} = (\mathbf{z}^k)^H \mathbf{R}(\mathbf{z}^k) \mathbf{z}^k + \text{Re}((4\mathbf{R}(\mathbf{z}^k) \mathbf{z}^k)^H (\mathbf{z} - \mathbf{z}^k)) + \frac{1}{2} (\mathbf{z} - \mathbf{z}^k)^H (8\mathbf{R}(\mathbf{z}^k)) (\mathbf{z} - \mathbf{z}^k)$$

$$\mathbf{z}^H \mathbf{R}(\mathbf{z}) \mathbf{z} \leq (\mathbf{z}^k)^H \mathbf{R}(\mathbf{z}^k) \mathbf{z}^k + \text{Re}((4\mathbf{R}(\mathbf{z}^k) \mathbf{z}^k)^H (\mathbf{z} - \mathbf{z}^k)) + \frac{1}{2} (\mathbf{z} - \mathbf{z}^k)^H (\mathbf{M}) (\mathbf{z} - \mathbf{z}^k) \quad (33)$$

There are more than one way to construct a matrix \mathbf{M} , such that (33) holds, some simple ways would be to choose:

$$\mathbf{M} = \text{Tr}(8\mathbf{R}(\mathbf{z}^k)) \mathbf{I}_P = 8P^2 \mathbf{I}_P. \quad (34)$$

or

$$\mathbf{M} = \lambda_{\max}(8\mathbf{R}(\mathbf{z}^k)) \mathbf{I}_P. \quad (35)$$

But in practice, for large dimension sequences, calculating the maximum eigenvalue is a computationally demanding procedure. So, in the following we try to explore the tighter upper bounds on maximum eigenvalue of the Hessian matrix.

Theorem-1 [Theorem 2.1 [56]]: Let \mathbf{A} be a $P \times P$ matrix with complex entries having real eigenvalues and let

$$m = \frac{1}{P} \text{Tr}(\mathbf{A}), \quad s^2 = \left(\frac{1}{P} \text{Tr}(\mathbf{A}^2) \right) - m^2 \quad (36)$$

Then

$$m - s(P-1)^{1/2} \leq \lambda_{\min}(\mathbf{A}) \leq m - \frac{s}{(P-1)^{1/2}} \quad (37)$$

$$m + \frac{s}{(P-1)^{1/2}} \leq \lambda_{\max}(\mathbf{A}) \leq m + s(P-1)^{1/2} \quad (38)$$

So, by using the result from Theorem-1 one can find an upper bound on the maximum eigenvalue of $\mathbf{R}(z^k)$ and form \mathbf{M} as:

$$\mathbf{M} = (m + s(P-1)^{1/2})\mathbf{I}_P \quad (39)$$

where $m = \frac{8}{P}\text{Tr}(\mathbf{R}(z^k))$, $s^2 = (\frac{64}{P}\text{Tr}(\mathbf{R}(z^k)^2)) - m^2$. Here on, we name the three approaches of obtaining \mathbf{M} as TR (using TRace), EI (using EIgen value), BEI (using Bound on the EIgen value). In the following we will explore another approach to arrive at \mathbf{M} .

Lemma-2 [Lemma-3 and Lemma-4 [38]]: Let \mathbf{A} be an $P \times P$ Hermitian Toeplitz matrix defined as follows

$$\mathbf{A} = \begin{bmatrix} a(0) & a^*(1) & \cdot & \cdot & a^*(P-2) & a^*(P-1) \\ a(1) & a(0) & a^*(1) & \cdot & \cdot & a^*(P-2) \\ \cdot & a(1) & a(0) & a^*(1) & \cdot & \cdot \\ \cdot & \cdot & a(1) & \cdot & \cdot & \cdot \\ a(P-2) & \cdot & \cdot & \cdot & \cdot & a^*(1) \\ a(P-1) & a(P-2) & \cdot & \cdot & a(1) & a(0) \end{bmatrix}$$

and $\hat{\mathbf{E}}^H$ be a $2P \times 2P$ FFT matrix with $\hat{\mathbf{E}}(m, n) = e^{j\frac{2\pi}{2P}mn}$, $0 \leq m, n \leq 2P$. Let $\mathbf{d} = [a_0, a_1, \dots, a_{P-1}, 0, a_{P-1}^*, \dots, a_1^*]^T$ and $\mathbf{s} = \hat{\mathbf{E}}^H \mathbf{d}$ be the discrete fourier transform of \mathbf{d} .

(a) Then the maximum eigenvalue of the Hermitian Toeplitz matrix \mathbf{A} can be bounded as

$$\lambda_{max}(\mathbf{A}) \leq \frac{1}{2} \left(\max_{1 \leq i \leq P} s_{2i} + \max_{1 \leq i \leq P} s_{2i-1} \right) \quad (40)$$

(b) The Hermitian Toeplitz matrix \mathbf{A} can be decomposed as

$$\mathbf{A} = \frac{1}{2P} \hat{\mathbf{E}}_{:,1:P} \text{Diag}(\mathbf{d}) \hat{\mathbf{E}}_{:,1:P}^H \quad (41)$$

Proof: The proof can be find in [38] ■

Using Lemma-2, one can also find the bound on maximum eigenvalue of a Hermitian Toeplitz matrix $\mathbf{R}(z^k)$ using FFT and IFFT operations as:

$$\mathbf{M} = 4 \left(\max_{1 \leq i \leq P} s_{2i} + \max_{1 \leq i \leq P} s_{2i-1} \right) \mathbf{I}_P. \quad (42)$$

where $\mathbf{d} = [r(0), r(1), \dots, r(P-1), 0, r(P-1)^*, \dots, r(1)^*]^T$ and $\mathbf{s} = \hat{\mathbf{E}}^H \mathbf{d}$.

We will name this approach as BEFFT (Bound on Eigenvalue using FFT).

So, from (33) we have the upper bound (majorization) function of the original objective function $g(z)$ at any fixed point z^k as:

$$u(z|z^k) = z^H(0.5\mathbf{M})z + 4\text{Re}((z^k)^H(\mathbf{R}(z^k) - 0.25\mathbf{M})z) + (z^k)^H(0.5\mathbf{M} - 3\mathbf{R}(z^k))z^k \quad (43)$$

As the M (obtained by all four approaches described above) is a constant times diagonal matrix and $\mathbf{z}^H \mathbf{z}$ being a constant, the first and last terms in the (43) are constants. So, after ignoring the constant terms, the surrogate minimization problem can be rewritten as:

$$\begin{aligned} & \underset{\mathbf{z}}{\text{minimize}} \quad u(\mathbf{z}|\mathbf{z}^k) = 4\text{Re}((\mathbf{z}^k)^H (\mathbf{R}(\mathbf{z}^k) - 0.25\mathbf{M})\mathbf{z}) \\ & \text{subject to} \quad |z_n| = 1, \quad n = 1, \dots, P. \end{aligned} \quad (44)$$

The problem in (44) can be rewritten more compactly as:

$$\begin{aligned} & \underset{\mathbf{z}}{\text{minimize}} \quad u(\mathbf{z}|\mathbf{z}^k) = \|\mathbf{z} - \tilde{\mathbf{a}}\|_2^2 \\ & \text{subject to} \quad |z_n| = 1, \quad n = 1, \dots, P, \end{aligned} \quad (45)$$

where $\tilde{\mathbf{a}} = -(\mathbf{R}(\mathbf{z}^k) - 0.25\mathbf{M})\mathbf{z}^k$, which involves computing Hermitian Toeplitz matrix-vector multiplication. By using decomposition of a Toeplitz matrix (41), one can easily implement it using FFT and IFFT operations.

The problem in (45) has a closed-form solution of

$$\mathbf{z}^{k+1} = \frac{\tilde{\mathbf{a}}}{\|\tilde{\mathbf{a}}\|_2}. \quad (46)$$

The pseudocode of the proposed algorithm-FISL is given below

Algorithm 4 :FISL -Faster ISL minimization

Require: sequence length ‘ P ’

- 1: set $k = 0$, initialize \mathbf{z}^0
 - 2: **repeat**
 - 3: compute $\mathbf{R}(\mathbf{z}^k)$ using (28)
 - 4: compute \mathbf{M} using (42)
 - 5: $\tilde{\mathbf{a}} = -(\mathbf{R}(\mathbf{z}^k) - 0.25\mathbf{M})\mathbf{z}^k$
 - 6: $\mathbf{z}^{k+1} = \frac{\tilde{\mathbf{a}}}{\|\tilde{\mathbf{a}}\|_2}$
 - 7: $k \leftarrow k + 1$
 - 8: **until** convergence
-

B. Convergence analysis

The proposed algorithm (FISL) is derived based on the MM technique. The working principle of the MM technique is explained in the section I-B. From (8), we have

$$g(\mathbf{z}^{k+1}) \leq u(\mathbf{z}^{k+1}|\mathbf{z}^k) \leq u(\mathbf{z}^k|\mathbf{z}^k) = g(\mathbf{z}^k)$$

So, MM technique is ensuring that the cost function value evaluated at every point $\{\mathbf{z}^k\}$ generated by the FISL algorithm will be monotonically decreasing and by the nature of the cost function of the problem in (25), one can

observe that it is always bounded below by zero. So, the sequence of cost function values is guaranteed to converge to a finite value.

Now, we will discuss the convergence of points $\{z^k\}$ generated by the FISL algorithm to a stationary point. So, starting with the definition of a stationary point.

Proposition 1: Let $f : \mathbf{R}^n \rightarrow \mathbf{R}$ be any smooth function and let x^* be a local minimum of f over a subset χ of \mathbf{R}^n [57]. Then

$$\nabla f(x^*)\mathbf{y} \geq 0, \forall \mathbf{y} \in T_\chi(x^*) \quad (47)$$

where $T_\chi(x^*)$ denotes the tangent cone of χ at x^* . Such any point x^* , which satisfies (47) is called as a stationary point.

Now, the convergence property of the FISL algorithm is explained as follows.

Theorem 2: Let $\{z^k\}$ be the sequence of points generated by the FISL algorithm. Then every point $\{z^k\}$ is a stationary point of the problem in (25).

Proof: Assume that there exists a converging subsequence $z^{l_j} \rightarrow z^*$, then from the theory of MM technique, we have

$$u(z^{l_{j+1}}|z^{l_{j+1}}) = g(z^{l_{j+1}}) \leq g(z^{l_j}) \leq u(z^{l_j+1}|z^{l_j}) \leq u(z|z^{l_j})$$

$$u(z^{l_{j+1}}|z^{l_{j+1}}) \leq u(z|z^{l_j})$$

Letting $j \rightarrow +\infty$, we obtain

$$u(z^\infty|z^\infty) \leq u(z|z^\infty) \quad (48)$$

Replacing z^∞ with z^* , we have

$$u(z^*|z^*) \leq u(z|z^*) \quad (49)$$

So, (49) conveys that z^* is a stationary point and also a global minimizer of $u(\cdot)$ i.e.,

$$\nabla u(z^*)\mathbf{d} \geq 0, \forall \mathbf{d} \in T_\chi(z^*) \quad (50)$$

From the majorization step, we know that the first-order behavior of majorized function $u(z|z^k)$ is equal to the original cost function $g(z)$. So, we can show

$$u(z^*|z^*) \leq u(z|z^*) \Leftrightarrow g(z^*) \leq g(z) \quad (51)$$

and it leads to

$$\nabla g(\mathbf{z}^*)\mathbf{y} \geq 0, \forall \mathbf{y} \in T_{\mathbf{Z}}(\mathbf{z}^*) \quad (52)$$

So, the set of points generated by the FISL algorithm are stationary points and \mathbf{z}^* is the minimizer of $g(\mathbf{z})$. This concludes the proof. \blacksquare

C. Computational & Space Complexity

The per iteration computational complexity of the proposed algorithm (FISL) is dominated in forming a Hermitian Toeplitz matrix $\mathbf{R}(\mathbf{z}^k)$, Diagonal matrix \mathbf{M} and Hermitian Toeplitz matrix-vector multiplication to form $\tilde{\mathbf{a}}$. But by using the Lemma-2, we replaced all of them using FFT and IFFT operations, and to implement our algorithm we require only 3-FFT and 2-IFFT operations and the computational complexity would be $\mathcal{O}(P \log P)$. In each iteration of our algorithm, the space complexity is dominated by the three different vectors of sizes $P \times 1$, $(2P-1) \times 1$, $2P \times 1$, respectively and the space complexity would be $\mathcal{O}(P)$. The computational & space complexity of state-of-the-art algorithms are given as: CAN- $\mathcal{O}(P \log P)$, $\mathcal{O}(P)$, MISL- $\mathcal{O}(P \log P)$, $\mathcal{O}(P)$, ISL-NEW- $\mathcal{O}(P \log P)$, $\mathcal{O}(P^2)$, ADMM- $\mathcal{O}(P^3)$, $\mathcal{O}(P^2)$, CPM- $\mathcal{O}(K \log P)$, $\mathcal{O}(P^2)$ where $K \in$ number of iterations in the bisection method. Hence, our proposed algorithm has either same or better computational & space complexity than the state-of-the-art algorithms.

III. NUMERICAL EXPERIMENTS

In this section, we will show the potential of our proposed algorithm Faster ISL minimization (FISL) through some numerical simulations. All simulations were performed in MATLAB on a laptop with a 2.50GHz i7 processor. Experiments has been conducted for different sequence lengths of $P = 100, 225, 400, 625, 900, 1225$ using different initializations like Golomb sequence [23], Frank sequence [22], random sequence, and to stop all the algorithms, we use the following convergence criterion:

$$\left| \frac{(\text{ISL}(k+1) - \text{ISL}(k))}{\max(1, \text{ISL}(k))} \right| \leq 10^{-5}, \quad (53)$$

where $\text{ISL}(k)$ is the ISL metric value at k^{th} iteration. In the case of random initialization, for every length each experiment is repeated for 30 Monte Carlo trials and for each trial different random initial sequence is used i.e., \mathbf{z}^0 is chosen as $\{e^{j2\pi\theta_i}\}_{i=1}^P$, where $\{\theta_i\}$ are drawn randomly from the uniform distribution $[0, 1]$.

In each experiment, the performance of the designed sequence such as ISL metric value, auto-correlation side-lobe levels and algorithm performance in terms of convergence speed to reach the stationary point is observed and compared with the state-of-the-art algorithms like CAN [33], MISL [34], ISL-NEW [35], ADMM approach [37], and CPM [39]. First, we will show the comparison of different approaches to construct the matrix \mathbf{M} , which plays a major role in the majorization step of our algorithm.

Figures. 1, 2 shows the normal and zoomed version (where ever it is necessary) plots of ISL value vs time, auto-correlation value vs lag for different sequence lengths $P = 100, 1225$ using three different initializations, respectively. From the simulation plots, we observe that, for all the initializations, all the approaches to construct a matrix \mathbf{M} will give the same auto-correlation function but their convergence times to reach minimum ISL value

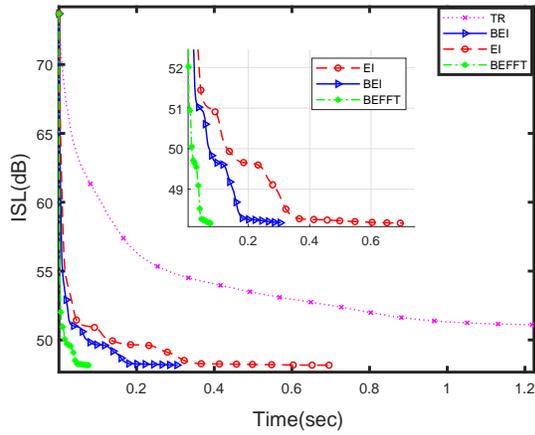
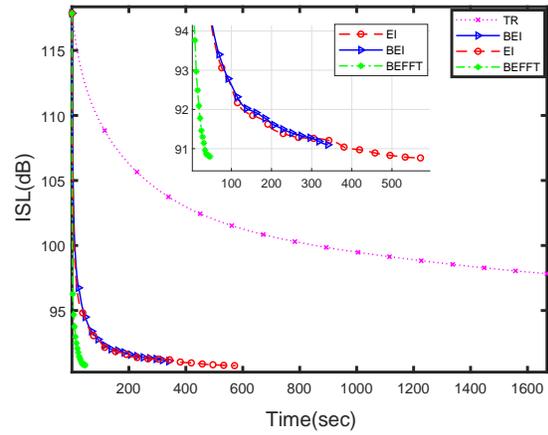
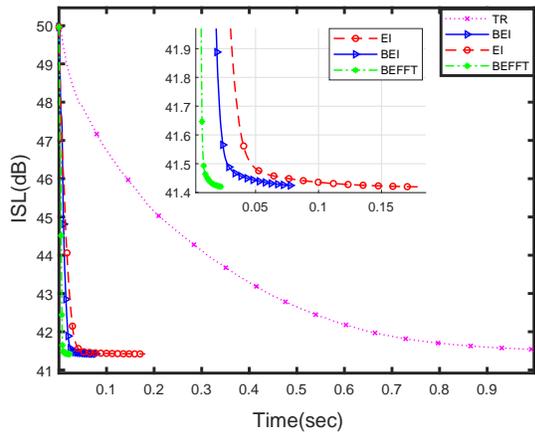
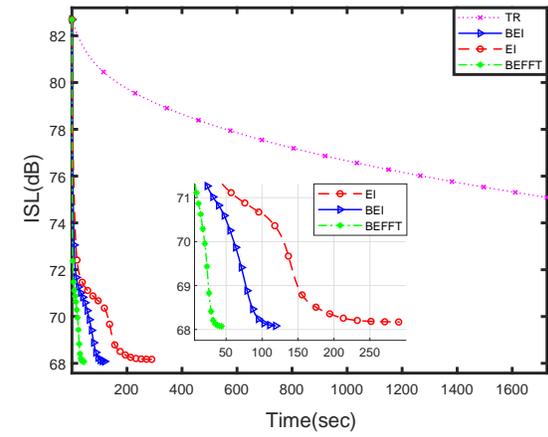
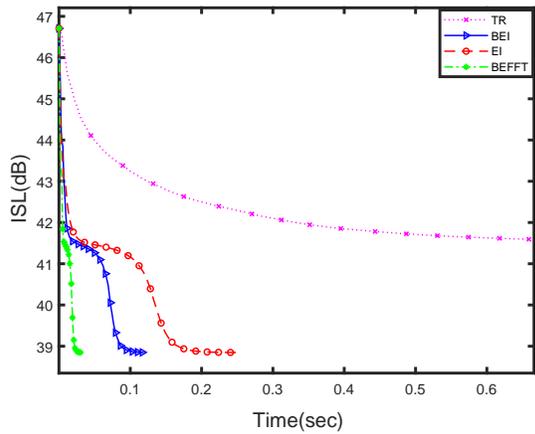
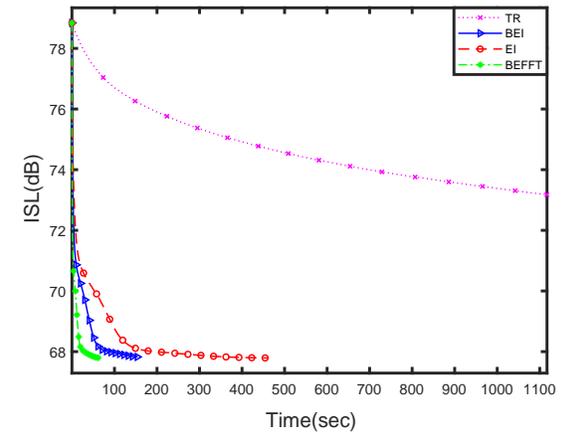
(a) $P = 100$ (b) $P = 1225$ (c) $P = 100$ (d) $P = 1225$ (e) $P = 100$ (f) $P = 1225$

Figure 1: ISL with respect to time for sequence lengths $P = 100, 1225$. (a) and (b) are for initialization via Random sequence. (c) and (d) are for initialization via Golomb sequence. (e) and (f) are for initialization via Frank sequence.

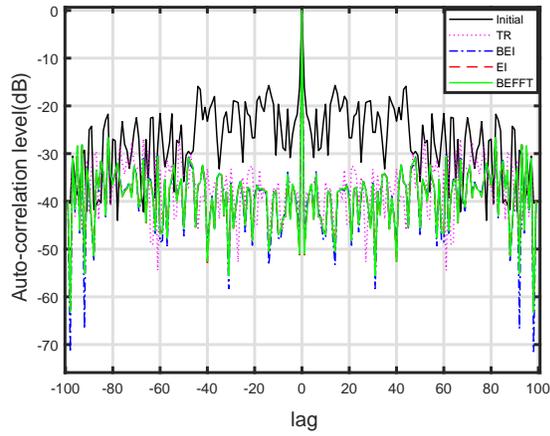
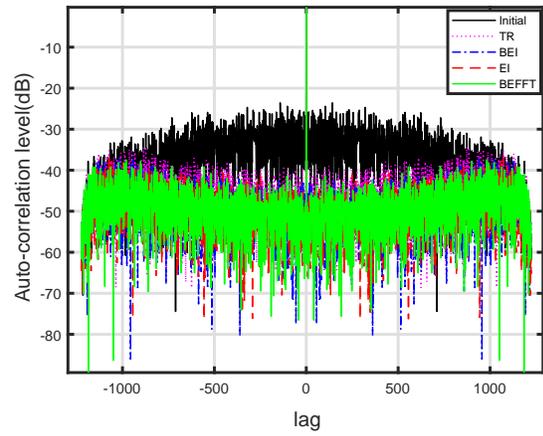
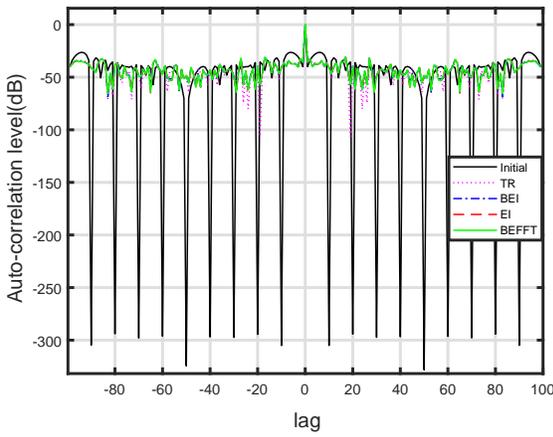
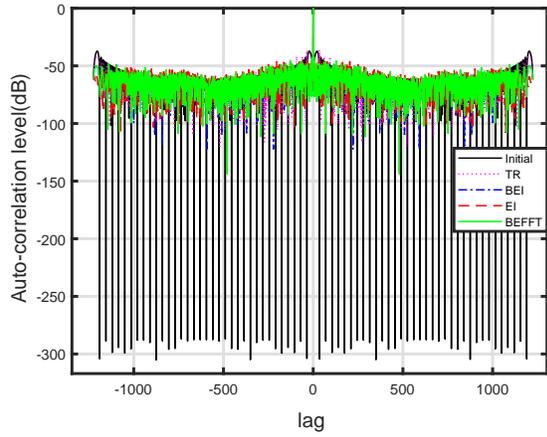
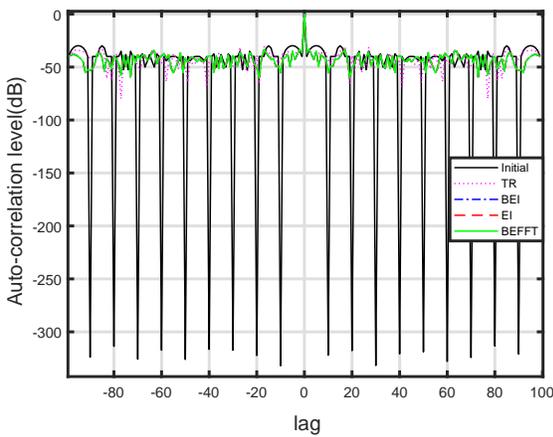
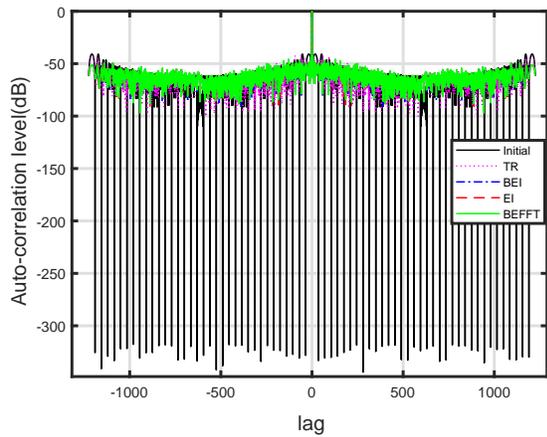
(a) $P = 100$ (b) $P = 1225$ (c) $P = 100$ (d) $P = 1225$ (e) $P = 100$ (f) $P = 1225$

Figure 2: Auto-correlation value with respect to lag for sequence lengths $P = 100, 1225$. (a) and (b) are for initialization via Random sequence. (c) and (d) are for initialization via Golomb sequence. (e) and (f) are for initialization via Frank sequence.

are different. In plots, we have shown results of four different ways to construct M namely TR (i.e, by using an approach of TRace of a matrix), EI (i.e, by using maximum ElGenvalue), BEI (i.e, by using Bound on the maximum ElGen value), and BEFFT (i.e, by using Bound on the maximum Eigenvalue using FFT operations). Among the four approaches, irrespective of length and initialization, the BEFFT approach seems to have faster convergence. From figure-1(b), one can observe that the BEFFT approach is faster than TR, EI, BEI approaches by 38, 13, 8 times respectively. So, in the following, we have used only the BEFFT approach in the update steps of our FISL algorithm.

Now, we will compare the performance of our FISL algorithm with the state-of-the-art algorithms in terms of the ISL metric value, convergence time, and auto-correlation side-lobe levels. For better comparison, for each experiment, all the algorithms are initialized with the same sequence and stopped using the same convergence criterion.

Figures. 3, 4 shows the normal and zoomed versions of the comparison plots of ISL value vs time, ISL value vs the number of iterations for different lengths and different initializations, respectively. We have considered the squared iterative method (SQUAREM) [34] accelerating scheme to implement the accelerated MISL (ACC-MISL) and accelerated ISL-NEW (ACC-ISL-NEW) algorithms. From simulation plots, one can observe that all the algorithms are starting at the same objective value, except the CAN and ADMM method all the methods are converging to the same minimum value but with different converging rates. From figures-3(b) and 4(b), for a sequence length of $P = 1225$, FISL algorithm is faster than the MISL, ACC-MISL, ISL-NEW, ACC-ISL-NEW and CPM algorithms by 125, 34, 42, 20, 43 times (with respect to the convergence time), 123, 14, 119, 10, 9 times (with respect to the number of iterations) respectively.

Now in Figure. 5, we are comparing all the algorithms in terms of auto-correlation side-lobe levels vs different lags, for different sequence lengths and different initializations. From simulation plots, we observe that except the ADMM approach, all the other algorithms are performing well in terms of the PSL metric value.

Figure. 6 consists of the comparison plots of average running time vs different sequence lengths for two different initializations. From simulation plots, one can observe that, irrespective of the sequence length and initialization, FISL algorithm is always taking less time when compared to the state-of-the-art algorithms. From figure-6(a), one can observe that the FISL algorithm is better than the MISL, ACC-MISL, ISL-NEW, ACC-ISL-NEW, and CPM algorithms by 126, 34, 42, 16, 46 times respectively.

IV.CONCLUSION

In this paper, we address the problem, design of phase only sequences of arbitrary lengths by directly minimizing the ISL metric. We proposed a fast iterative algorithm by using the Majorization-Minimization method. Numerical simulations of the proposed algorithm were conducted for different sequence lengths using different initializations that confirm our algorithm performs better than the state-of-the-art algorithms in terms of the speed of convergence.

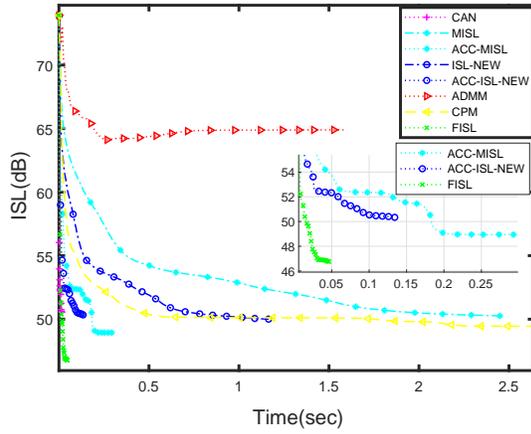
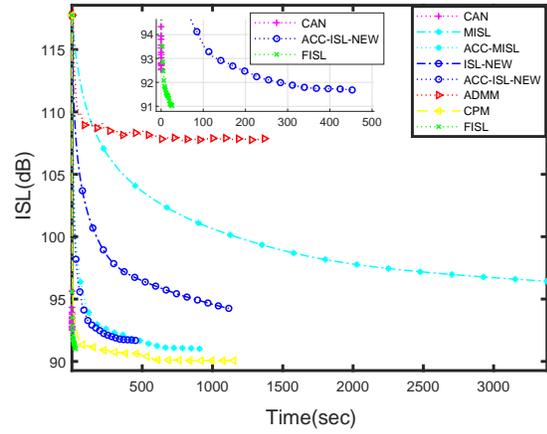
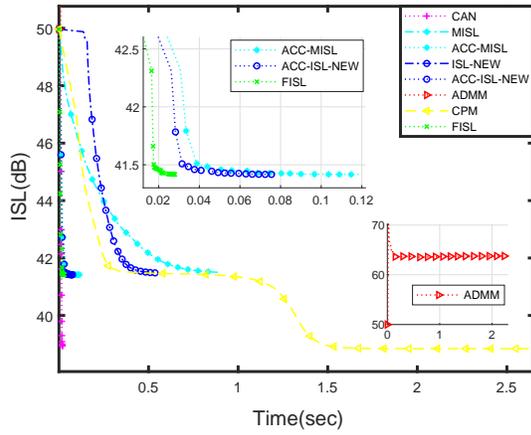
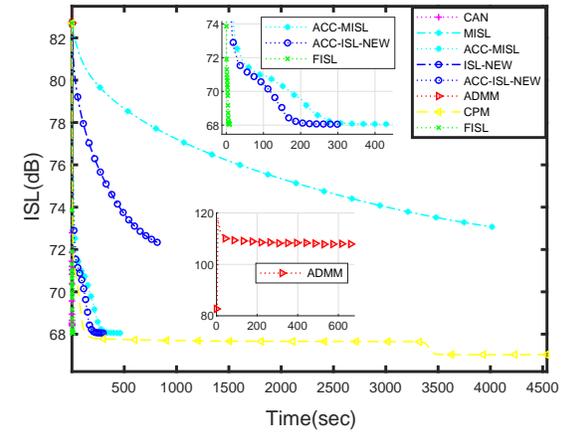
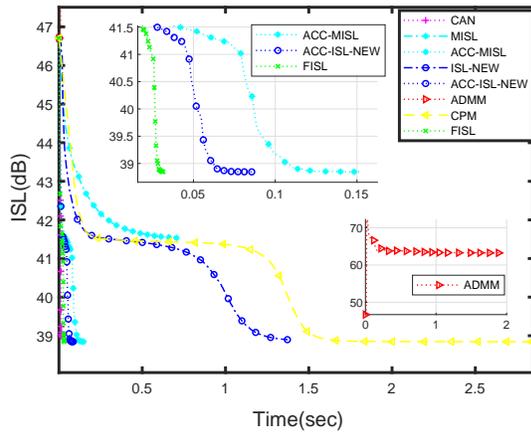
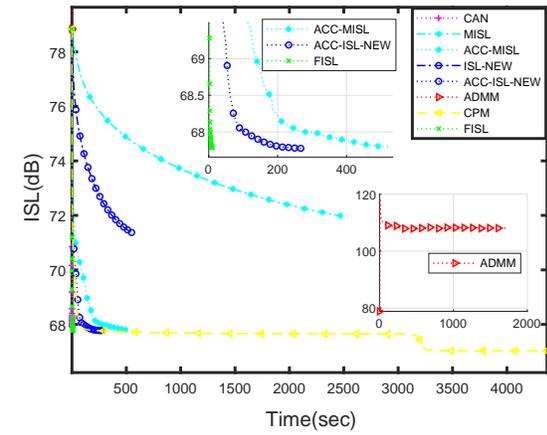
(a) $P = 100$ (b) $P = 1225$ (c) $P = 100$ (d) $P = 1225$ (e) $P = 100$ (f) $P = 1225$

Figure 3: ISL with respect to time for a sequence length $P = 100, 1225$. (a) and (b) are for initialization via Random sequence. (c) and (d) are for initialization via Golomb sequence. (e) and (f) are for initialization via Frank sequence.

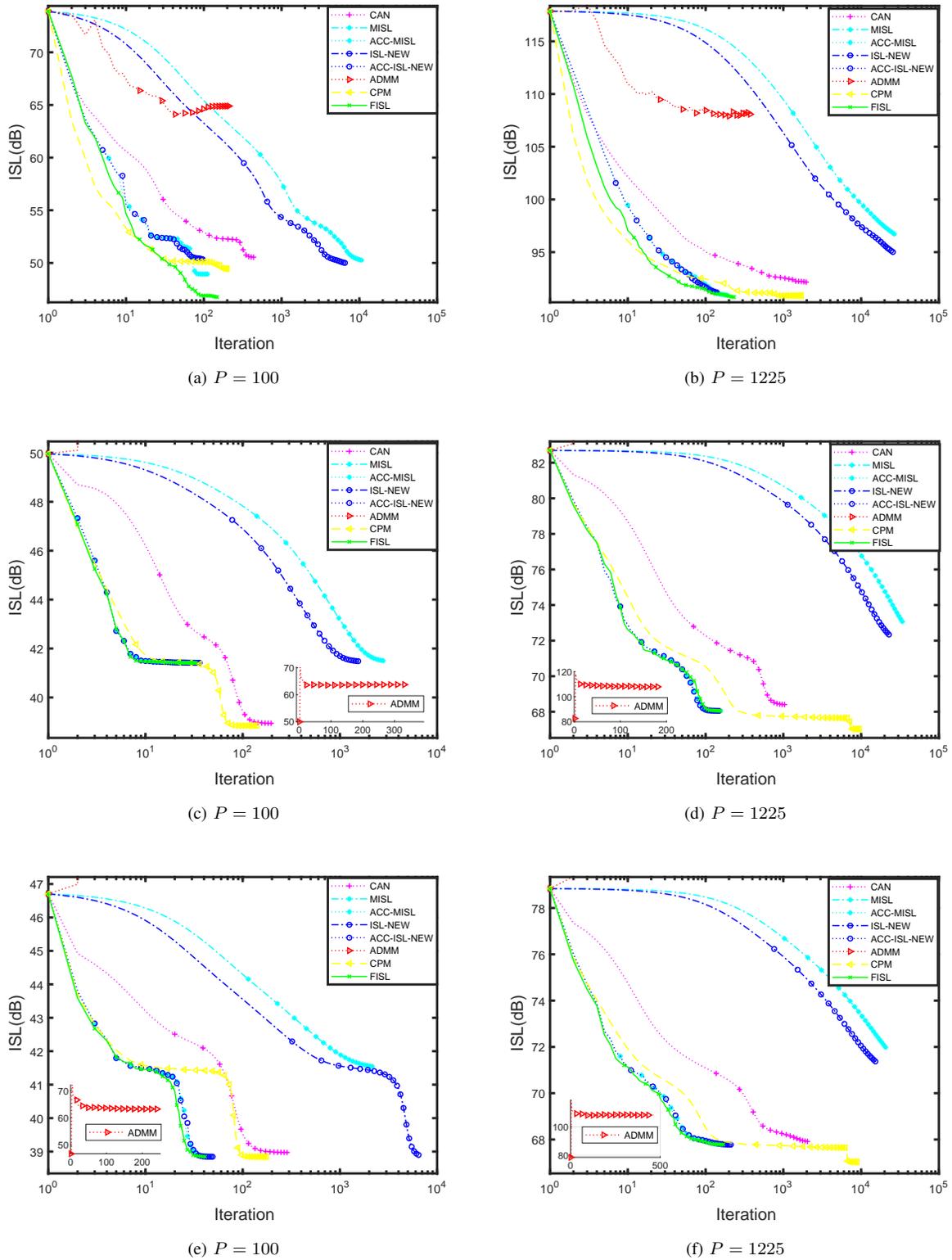


Figure 4: ISL with respect to iteration for a sequence length $P = 100, 1225$. (a) and (b) are for initialization via Random sequence. (c) and (d) are for initialization via Golomb sequence. (e) and (f) are for initialization via Frank sequence.

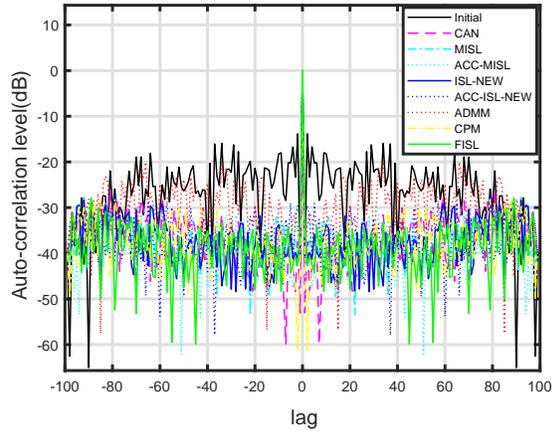
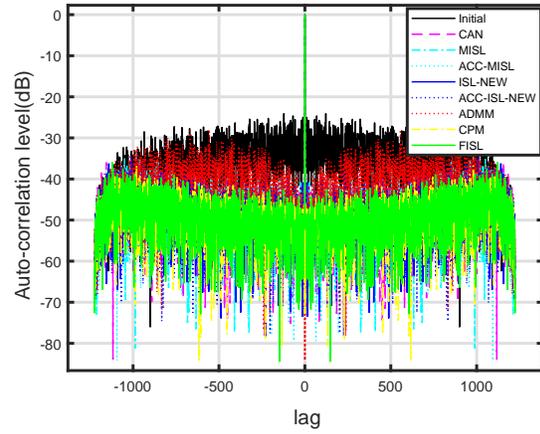
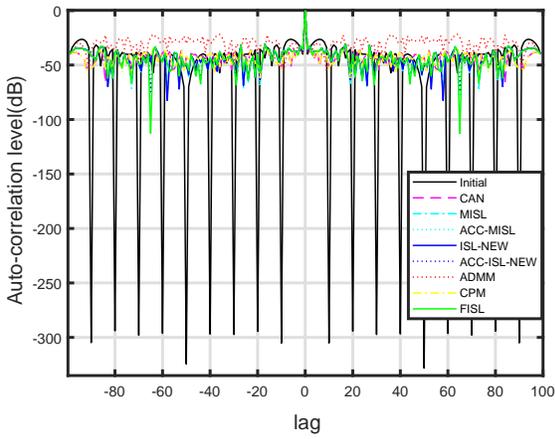
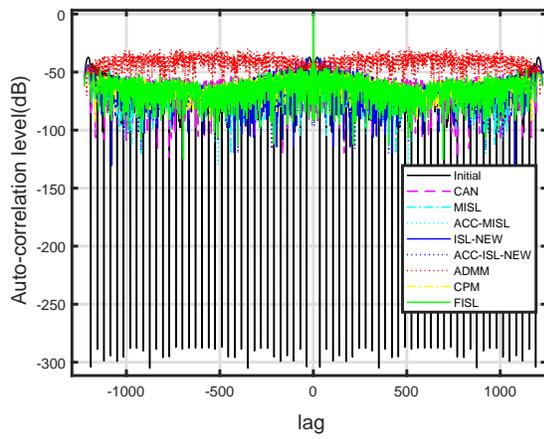
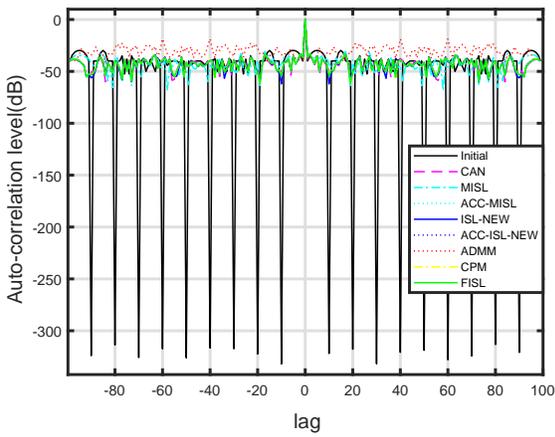
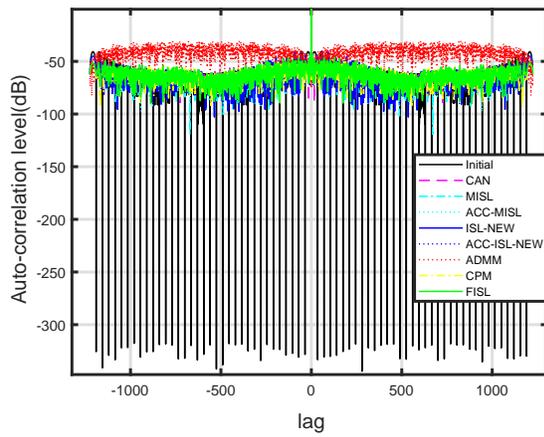
(a) $P = 100$ (b) $P = 1225$ (c) $P = 100$ (d) $P = 1225$ (e) $P = 100$ (f) $P = 1225$

Figure 5: Auto-correlation value with respect to lag for a sequence length $P = 100, 1225$. (a) and (b) are for initialization via Random sequence. (c) and (d) are for initialization via Golomb sequence. (e) and (f) are for initialization via Frank sequence.

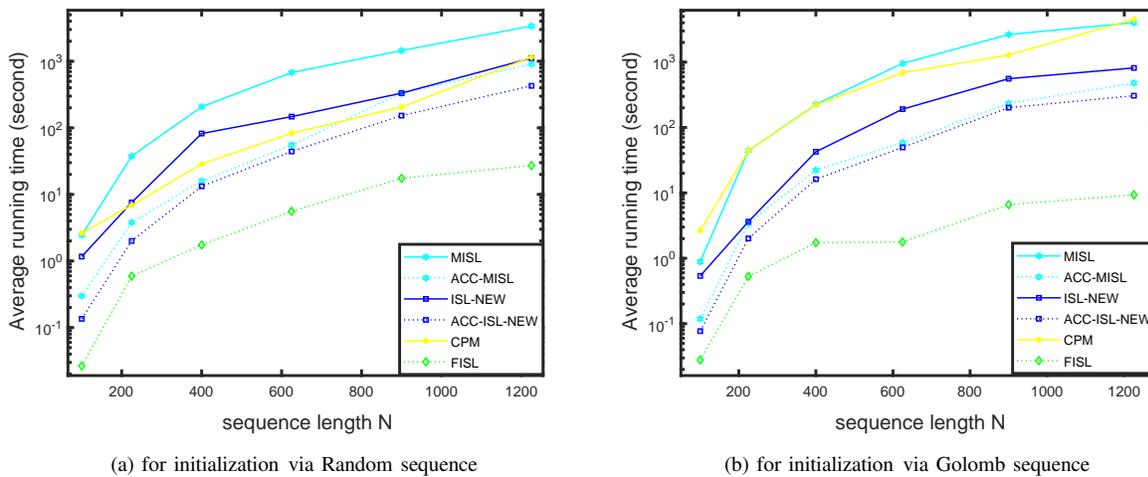


Figure 6: Average time with respect to sequence length

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