# On Ryser's Conjecture for Linear Intersecting Multipartite Hypergraphs* 

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#### Abstract

Ryser conjectured that $\tau \leqslant(r-1) \nu$ for $r$-partite hypergraphs, where $\tau$ is the covering number and $\nu$ is the matching number. We prove this conjecture for $r \leqslant 9$ in the special case of linear intersecting hypergraphs, in other words where every pair of lines meets in exactly one vertex.

Aharoni formulated a stronger version of Ryser's conjecture which specified that each $r$-partite hypergraph should have a cover of size $(r-1) \nu$ of a particular form. We provide a counterexample to Aharoni's conjecture with $r=13$ and $\nu=1$.

We also report a number of computational results. For $r=7$, we find that there is no linear intersecting hypergraph that achieves the equality $\tau=r-1$ in Ryser's conjecture, although non-linear examples are known. We exhibit intersecting non-linear examples achieving equality for $r \in\{9,13,17\}$. Also, we find that $r=8$ is the smallest value of $r$ for which there exists a linear intersecting $r$-partite hypergraph that achieves $\tau=r-1$ and is not isomorphic to a subhypergraph of a projective plane.


## 1 Introduction

A hypergraph $H$ is a set of non-empty subsets, variously called lines, edges or hyperedges, of a finite underlying vertex set $V(H)$. The degree of a vertex $v \in V(H)$, denoted $\operatorname{deg}(v)$, is the number of lines in $H$ that contain $v$. A hypergraph is $r$-uniform if every line contains exactly $r$ vertices. Thus a 2 -uniform hypergraph is simply a graph.

Covers and matchings in hypergraphs are widely studied [6]. A cover of a hypergraph $H$ is a set of vertices $C \subseteq V(H)$ such that every line of $H$ contains at least one vertex of $C$. The covering number of $H$, denoted $\tau(H)$, is the minimum size of a cover of $H$. A matching in $H$ is a set of pairwise disjoint lines of $H$ and the matching number of $H$, denoted $\nu(H)$, is the maximum size of a matching in $H$. Most hypergraphs in this paper are intersecting, meaning that every pair of lines meets in at least one vertex; equivalently $\nu=1$.

The covering number and matching number of a hypergraph are related. First, for every hypergraph, $\nu \leqslant \tau$ since each cover contains at least one vertex from each line in any given matching. Second, for every $r$-uniform hypergraph, $\tau \leqslant r \nu$ since a cover can be obtained from the union of the lines in a maximal matching; this bound is sharp and is achieved, for example, by projective planes of order $r-1$.

A hypergraph is $r$-partite if its vertex set can be partitioned into $r$ sets, called sides, such that every line consists of exactly one vertex from each side. Hence every $r$-partite hypergraph is necessarily $r$-uniform. An $r$-partite hypergraph can be constructed from a projective plane of order $r-1$ by removing a single vertex $v$ and the $r$ lines through $v$. The resulting hypergraph

[^0]$\mathcal{P}_{r}^{\prime}$ is called a truncated projective plane and the $r$ sides of $\mathcal{P}_{r}^{\prime}$ are defined by the sets of vertices on each of the removed lines. Our notation hides the fact that there may be non-isomorphic choices for the structure of $\mathcal{P}_{r}^{\prime}$, but the distinction between the different choices will not matter in our work.

The following famous conjecture is due to Ryser 9].
Conjecture 1.1. Every $r$-partite hypergraph with covering number $\tau$ and matching number $\nu$ satisfies $\tau \leqslant(r-1) \nu$.

Ryser's Conjecture is far from being resolved. When $r=2$, the conjecture is König's Theorem for bipartite graphs, which is also equivalent to Hall's Theorem (see e.g. [12]). Using topological methods, Aharoni and Haxell 4] proved a hypergraph generalisation of Hall's Theorem and Aharoni [2] used this result to prove Conjecture 1.1 for $r=3$. Conjecture 1.1 was proved for intersecting $r$-partite hypergraphs with $r \leqslant 5$ by Tuza [11]. Haxell and Scott [7] built on this result to prove that for $r \leqslant 5$, there exists $\epsilon>0$ such that $\tau<(r-\epsilon) \nu$ for all $r$-partite hypergraphs.

In this paper, we prove another restricted version of Ryser's Conjecture for small r. A hypergraph is linear if each pair of lines meets in at most one vertex. Hence, in a linear intersecting hypergraph, each pair of lines intersects in exactly one vertex. In $\S_{2}$ we prove that Conjecture 1.1 holds for $r \leqslant 9$ when $H$ is a linear intersecting hypergraph. The proof of the $r=9$ case is computational.

In an intersecting $r$-partite hypergraph each side is a cover and each line is also a cover. In [3] the following strenthening of Conjecture 1.1 is recorded. The conjecture is due to Aharoni, as are several generalisations that are also given in [3].

Conjecture 1.2. An intersecting $r$-partite hypergraph $H$ has a side of size at most $r-1$ or a cover of the form $e \backslash\{v\}$ for some $e \in H$ and $v \in e$.

In 33.1 we show that Conjecture 1.2 is false, by providing a counterexample when $r=13$. Furthermore, in 93.3 we describe an infinite family of linear intersecting $r$-partite hypergraphs with $\tau=r-2$, built from mutually orthogonal latin squares. Although this family of hypergraphs have covers consisting of an line with a vertex removed, they have the property that no minimal cover is contained within a line or a side.

In studying Conjecture 1.1 it is natural to investigate hypergraphs that achieve the equality $\tau=(r-1) \nu$. A well-known infinite family of such hypergraphs with $\nu=1$ is the family of truncated projective planes $\mathcal{P}_{r}^{\prime}$ (where $r-1$ is a prime power). Note that $\tau \leqslant r-1$ since each side of $\mathcal{P}_{r}^{\prime}$ has $r-1$ vertices and $\tau \geqslant r-1$ since each vertex lies on $r-1$ lines and the total number of lines is $(r-1)^{2}$. However, $\mathcal{P}_{r}^{\prime}$ has more lines than is necessary in the sense that many of its subhypergraphs achieve $\tau=r-1$. Conversely, we will report in $\S 4$ that for $r \leqslant 7$ the only way to achieve $\tau=r-1$ in a linear intersecting $r$-partite hypergraph is to take a subhypergraph of $\mathcal{P}_{r}^{\prime}$. In particular, there are no linear intersecting 7 -partite hypergraphs with $\tau=6$, since $\mathcal{P}_{7}^{\prime}$ does not exist. By contrast, it was shown in [1, 3] that there are non-linear intersecting 7 -partite hypergraphs with $\tau=6$. We also describe in 43.4 a linear intersecting hypergraph having $r=8$ and $\tau=7$ and which is not a subhypergraph of $\mathcal{P}_{8}^{\prime}$. Moreover, in $\$ 3.2$ we give examples of non-linear intersecting $r$-partite hypergraphs with $r \in\{9,13,17\}$ which have covering number $r-1$.

### 1.1 Notation

We deal with $r$-partite hypergraphs throughout. To avoid degeneracies we always assume that $r \geqslant 2$ and that every vertex has positive degree. The sides of our hypergraphs are always denoted $V_{0}, V_{1}, \ldots, V_{r-1}$ and we have $V=\cup_{i=0}^{r-1} V_{i}$. The covering number, matching
number, minimum degree and maximum degree of a hypergraph are denoted by $\tau, \nu, \delta$ and $\Delta$, respectively. The number of lines in a hypergraph $H$ is denoted $h$ or $|H|$. We often use discrete interval notation $\left[n_{1}, n_{2}\right]=\left\{n_{1}, n_{1}+1, n_{1}+2, \ldots, n_{2}\right\}$, where $n_{1}$ and $n_{2}$ are two integers and $n_{1} \leqslant n_{2}$.

## 2 Ryser's conjecture for linear intersecting hypergraphs

In this section we prove that Conjecture 1.1 holds for all linear intersecting hypergraphs with at most nine sides. We begin by establishing some properties of a hypothetical counterexample.

### 2.1 General properties

Lemma 2.1. Let $H$ be an intersecting $r$-partite hypergraph with $\tau(H)=r$. Then
(i) Each side of $H$ has size at least $r$.
(ii) Each vertex of $H$ has degree at least 2.
(iii) Each line of $H$ contains at most one vertex of degree 2.

Proof. Each side is a cover so it contains at least $\tau(H)$ vertices, hence (i) holds. If a vertex $v$ has degree 1 and $\ell$ is the line containing $v$, then $\ell \backslash\{v\}$ is a cover of size $r-1$, hence (ii) holds. Finally, if there is a line $\ell$ containing distinct vertices $u$ and $v$ of degree 2 , then we get a cover of size $r-1$ by taking $\ell \backslash\{u, v\} \cup\{x\}$, where $x$ is a vertex in the intersection of the (at most) two lines other than $\ell$ that meet $\{u, v\}$. Therefore, (iii) is proved.

Lemma 2.2. Let $H$ be an intersecting $r$-partite hypergraph with $\tau(H)=r$. Then $\Delta \geqslant 4$. Furthermore, if $H$ is linear, then $\Delta \leqslant r-2$.

Proof. By Lemma 2.1, we may assume that $\delta(H) \geqslant 2$. Suppose there are $h$ lines in $H$. If we count the lines which intersect a given line, we find that $h \leqslant(\Delta-1) r+1$ with equality only if $H$ is $\Delta$-regular. If we count the lines incident with a side that contains a vertex of degree $\Delta$, we find that $h \geqslant 2(r-1)+\Delta$, with equality only if all other vertices on that side have degree 2 . The above observations together show that $\Delta \geqslant 4$.

Now suppose that $H$ is linear and let $v$ be a vertex of degree $\Delta$. Without loss of generality, assume that $v$ is on side $V_{0}$ and let $u \in V_{0} \backslash\{v\}$ (such a $u$ exists, since $\tau(H)>1$ ). Any line $\ell$ through $u$ meets the lines through $v$ in $\Delta$ distinct vertices on sides other than $V_{0}$, since $H$ is linear and intersecting. Hence $\Delta \leqslant r-1$. If $\Delta=r-1$ then side $V_{1}$ has at most $r-1$ vertices. Indeed, $\ell$ cannot contain any vertex on side $V_{1}$ other than the vertices on lines through $v$. As this is true for any line $\ell$ which does not contain $v$, there can only be $r-1$ vertices in $V_{1}$, which means that $\tau(H) \leqslant r-1$. We conclude that $\Delta \leqslant r-2$.

Theorem 2.3. Let $H$ be an intersecting r-partite hypergraph with $\tau(H)=r$ and maximum degree $\Delta$. Then

$$
\begin{equation*}
(r+1 / 2)^{2} \Delta^{2}+4 r^{2} \geqslant\left(8 r^{2}-2 r\right) \Delta \tag{2.1}
\end{equation*}
$$

Proof. Suppose that $H$ contains $d_{i}$ vertices of degree $i$ for $2 \leqslant i \leqslant \Delta$. Since each side of $H$ is a cover, there must be $r^{2}+\varepsilon$ vertices in $H$ for some $\varepsilon \geqslant 0$. Hence

$$
\begin{equation*}
r^{2}+\varepsilon=\sum_{i=2}^{\Delta} d_{i} . \tag{2.2}
\end{equation*}
$$

By counting vertex-line incidences we find that

$$
\begin{equation*}
r h=\sum_{i=2}^{\Delta} i d_{i} . \tag{2.3}
\end{equation*}
$$

Also, since $H$ is intersecting, we get the following inequality by counting incidences between lines:

$$
\begin{equation*}
\binom{h}{2} \leqslant \sum_{i=2}^{\Delta}\binom{i}{2} d_{i} . \tag{2.4}
\end{equation*}
$$

Our aim is to show that (2.4) cannot be satisfied unless (2.1) holds. Since, by Lemma 2.1(iii), no line of $H$ may contain more than one vertex of degree $2, d_{2}=h / 2-\varepsilon^{\prime}$ for some $\varepsilon^{\prime} \geqslant 0$. Now solving (2.2) and (2.3) for $d_{3}$ and $d_{\Delta}$ (recall that $\Delta>3$ by Lemma (2.2) and substituting into (2.4), we get

$$
\begin{align*}
h^{2}-(\Delta r+ & \Delta / 2+2 r) h+3 r^{2} \Delta  \tag{2.5}\\
& \leqslant 3 \Delta \sum_{i=4}^{\Delta-1} d_{i}-(\Delta+2) \sum_{i=4}^{\Delta-1} i d_{i}+\sum_{i=4}^{\Delta-1} i(i-1) d_{i}-3 \Delta \varepsilon-(\Delta-2) \varepsilon^{\prime} \\
& \leqslant-\sum_{i=4}^{\Delta-1}(i-3)(\Delta-i) d_{i}-3 \Delta \varepsilon-(\Delta-2) \varepsilon^{\prime}  \tag{2.6}\\
& \leqslant-\sum_{i=4}^{\Delta-1}(i-3)(\Delta-i) d_{i} \leqslant 0
\end{align*}
$$

Hence, the discriminant of the quadratic in $h$ given by (2.5) is non-negative, which implies (2.1).

In a linear intersecting $r$-partite hypergraph with $\tau=r$, we have $4 \leqslant \Delta \leqslant r-2$, by Lemma 2.2. Substituting $\Delta \in\{4,5,6\}$ into (2.1), yields an immediate contradiction. Hence:

Corollary 2.4. If there exists a linear intersecting r-partite hypergraph $H$ such that $\tau(H)=r$, then $\Delta(H) \geqslant 7$.

Corollary 2.5. Conjecture 1.1 holds for all linear intersecting $r$-partite hypergraphs when $r \leqslant 8$.

### 2.2 Linear intersecting 9-partite hypergraphs

In this subsection we prove that a linear intersecting 9-partite hypergraph has covering number at most 8. For a potential counterexample $H$, let $h=|H|$ be the number of lines in $H$ and $\varepsilon:=|V(H)|-r^{2} \geqslant 0$. First, we demonstrate some further properties that such a hypergraph would have, if it were to exist. Note that we may assume that $\Delta=7$, given Corollary 2.4 and Lemma 2.2.

Lemma 2.6. If $H$ is a linear intersecting 9-partite hypergraph with $\Delta=7$ and $\tau=9$, then $h \geqslant 39$.

Proof. Let $u$ be a vertex of degree $\Delta=7$, and without loss of generality, assume that $u \in V_{0}$. Let $A$ be the set of vertices that lie on lines through $u$ in the remaining eight sides. Then $|A|=8 \cdot 7=56$. Let $B=V(H) \backslash\left(V_{0} \cup A\right)$. Then $|B| \geqslant 16$.

Let $\ell$ be a line through a vertex in $B$. Then $\ell$ is incident with a vertex other than $u$ in side $V_{0}$ and it intersects all seven lines incident with $u$. That is, $|\ell \cap A|=7$ and $|\ell \cap B|=1$. Since every vertex in $B$ has degree at least 2 , there are at least $2|B| \geqslant 32$ such lines. Therefore, $h \geqslant 7+32=39$.

By (2.6),

$$
\begin{equation*}
h^{2}-(\Delta r+\Delta / 2+2 r) h+3 r^{2} \Delta \leqslant-3 \Delta \varepsilon . \tag{2.7}
\end{equation*}
$$

This inequality has no integer solution for $h$ when $\varepsilon \geqslant 5$ if $r=9$ and $\Delta=7$. Otherwise, together with Lemma 2.6, we obtain that $h_{\min }(\varepsilon) \leqslant h \leqslant h_{\max }(\varepsilon)$ where $h_{\min }(\varepsilon)$ and $h_{\max }(\varepsilon)$ are as follows:

| $\varepsilon$ | 0 | 1 | 2 | 3 | 4 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $h_{\min }$ | 39 | 39 | 39 | 39 | 42 |
| $h_{\max }$ | 51 | 50 | 48 | 46 | 42 |

Lemma 2.7. If $H$ is a linear intersecting 9-partite hypergraph with $\Delta=7$ and $\tau(H)=9$, then for every pair of degree 7 vertices on different sides there is a line of $H$ that contains both vertices.

Proof. Suppose to the contrary that $v_{0}$ and $v_{1}$ are degree 7 vertices on sides $V_{0}$ and $V_{1}$, respectively, which do not lie on a common line. Let $e_{1}, \ldots, e_{7}$ be the lines which contain $v_{0}$ and $f_{1}, \ldots, f_{7}$ be the lines which contain $v_{1}$. For every $i, j \in[1,7]$, lines $e_{i}$ and $f_{j}$ meet at a vertex in one of the last seven sides $V_{2}, \ldots, V_{8}$.

Define $V_{0}^{\prime}=V_{0} \backslash\left(\left(\cup_{i} e_{i}\right) \cup\left(\cup_{i} f_{i}\right)\right)$ and $V_{1}^{\prime}=V_{1} \backslash\left(\left(\cup_{i} e_{i}\right) \cup\left(\cup_{i} f_{i}\right)\right)$. Since $\left|V_{0}\right| \geqslant 9$, there exists $y \in V_{0}^{\prime}$. Any line $\ell$ through $y$ intersects each of $f_{1}, \ldots, f_{7}$ in the last seven sides and thus also intersects each of $e_{1}, \ldots, e_{7}$ in the last seven sides. Hence $\ell$ contains a vertex in $V_{1}^{\prime}$. By symmetry, all lines through $V_{1}^{\prime}$ contain a vertex in $V_{0}^{\prime}$. Let $G$ be the bipartite graph induced on $V_{0}^{\prime} \cup V_{1}^{\prime}$ by $H$. Then $G$ inherits from $H$ the properties of having minimum degree at least 2 and no line between vertices of degree 2 . Therefore $G$ has at least 5 vertices and at least 6 lines. It follows that $\varepsilon \geqslant 3$ and hence $h \leqslant 46$, from (2.7).

We have already encountered 20 distinct lines of $H$, namely $e_{1}, \ldots e_{7}, f_{1}, \ldots, f_{7}$ and at least 6 lines through $V_{0}^{\prime}$. Let $B$ be the set of vertices in the last seven sides which do not lie on any of these 20 lines. Then $|B| \geqslant 14$, since $B$ includes at least two vertices from each of $V_{2}, \ldots, V_{8}$. Let $x \in B$, and let $g$ be a line through $x$. Since $g$ intersects each line $e_{1}, \ldots, e_{7}$, it follows that $g \cap B=\{x\}$. However $x$ has degree at least 2. Therefore $h \geqslant 20+2|B| \geqslant 48$, which is a contradiction.

Remark. Although the proof of Lemma 2.7 does not completely generalise to larger $r$, it does demonstrate that if $H$ is a linear intersecting $r$-partite hypergraph with $\Delta=r-2, \tau=r$ and $\epsilon \leqslant 2$, then each pair of degree $\Delta$ vertices on different sides lie on a common line.

### 2.2.1 Degree sequences, line types, and side types

Recall that $H$ is assumed to be a linear intersecting 9-partite hypergraph with $\tau(H)=r$ and $\Delta=7$. For each $h$ and $\varepsilon$ within the bounds given by (2.7), we computed the set $\mathcal{D}(h, \varepsilon)$ of all possible degree sequences that satisfy the equations (2.2), (2.3) and (2.4). Note that since $H$ is linear, equality is enforced in (2.4). From here on, we denote a degree sequence in $\mathcal{D}(h, \varepsilon)$ by $\left[d_{2}, d_{3}, \ldots, d_{7}\right]$, where $d_{i}$ is the number of vertices of degree $i$, for $i \in[2,7]$. For each of the possible values of $h$ and $\varepsilon$, we obtained the following number of degree sequences in the set $\mathcal{D}(h, \varepsilon)$. An example souce code for this and other computational results in this section is posted on the arXiv as an ancillary file with the priprint of this article.

| $\varepsilon \backslash h$ | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 |
| ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| 0 | 223 | 297 | 307 | 358 | 323 | 311 | 236 | 181 | 107 | 50 | 16 | 2 | 0 |
| 1 | 86 | 129 | 135 | 164 | 144 | 145 | 102 | 82 | 42 | 24 | 6 | 0 | - |
| 2 | 22 | 39 | 42 | 58 | 48 | 48 | 27 | 20 | 6 | 3 | - | - | - |
| 3 | 1 | 6 | 6 | 11 | 8 | 9 | 3 | 1 | - | - | - | - | - |
| 4 | - | - | - | 1 | - | - | - | - | - | - | - | - | - |

Similarly, for given $h$ and $\varepsilon$, we determined the set $\mathcal{S}(h, \varepsilon)$ of all possible degree sequences of vertices on a side of $H$, and the set $\mathcal{L}(h)$ of all possible degree sequence of vertices on a line of $H$. We arbitrarily order sets $\mathcal{S}(h, \varepsilon)$ and $\mathcal{L}(h)$ to easily index their elements. Then for $t \in\{1,2, \ldots,|\mathcal{S}(h, \varepsilon)|\}$, a side of type $t$ in $\mathcal{S}(h, \varepsilon)$ is a sequence of non-negative integers $\left[s_{2}^{t}, s_{3}^{t}, \ldots, s_{7}^{t}, s_{\varepsilon}^{t}\right]$, in which $s_{i}^{t}$ denotes the number of vertices of degree $i \in[2,7]$, such that the following conditions hold:
(i) $\sum_{i=2}^{7} s_{i}^{t}=r+s_{\varepsilon}^{t} \quad$ which is the length of a side;
(ii) $\sum_{i=2}^{7} i s_{i}^{t}=h$
since every line meets every side; and
(iii) $s_{\varepsilon}^{t} \leqslant \varepsilon$.

A line of type $t$ in $\mathcal{L}(h)$, where $t \in\{1,2, \ldots,|\mathcal{L}(h)|\}$, is a sequence of non-negative integers $\left[\ell_{2}^{t}, \ell_{3}^{t}, \ldots, \ell_{7}^{t}\right]$, in which $\ell_{i}^{t}$ denotes the number of vertices of degree $i \in[2,7]$, such that the following conditions hold:

$$
\begin{array}{ll}
\text { (i) } \sum_{i=2}^{7} \ell_{i}^{t}=r & \text { since every line meets every side; } \\
\text { (ii) } \sum_{i=2}^{7} i \ell_{i}^{t}=h+r-1 & \text { since } H \text { is linear intersecting; and } \\
\text { (iii) } \ell_{2}^{t} \in\{0,1\} & \text { by Lemma [2.1((iii). }
\end{array}
$$

### 2.2.2 Pairwise conditions

Our next goal is to verify which degree sequences are feasible. We assume that $h$ and $\varepsilon$ are given, and that $D=\left[d_{2}, d_{3}, \ldots, d_{7}\right]$ is a degree sequence in $\mathcal{D}(h, \varepsilon)$. Suppose that there exists $H$, a linear intersecting hypergraph with $r=\tau=9, \Delta=7$ on $r^{2}+\varepsilon$ vertices with $h$ lines and the given degree sequence. For every $t \in\{1,2, \ldots,|\mathcal{S}(h, \varepsilon)|\}$, let $x_{t}$ denote the number of sides of type $t$ that are contained in $H$. For every $t \in\{1,2, \ldots,|\mathcal{L}(h)|\}$, let $y_{t}$ denote the number of lines of type $t$ that are contained in $H$. Then $x_{t}$ and $y_{t}$ are non-negative integers which satisfy some obvious necessary conditions listed by equations (2.8)-(2.11). Equation (2.12) is a double count of pairs of vertices of degree $i$ and lines on which these vertices lie, for $i \in[2,7]$. Since $H$ is a linear intersecting hypergraph, no pair of vertices is contained on two lines. Hence, the number of pairs of vertices contained on a line or on a side cannot exceed the total number of pairs given by the degree sequence. This condition is given by equations (2.13)-(2.15), depending on whether we count pairs of vertices having the same degree, or pairs of vertices with different degree. Putting all of these conditions together, we formulate an integer program on variables $x_{t}$ and $y_{t}$.

$$
\begin{array}{ll}
\sum_{t} x_{t}=r & \text { there are } r \text { sides } \\
\sum_{t} x_{t} s_{\varepsilon}^{t}=\varepsilon & \text { there are } \varepsilon \text { extra vertices } \\
\sum_{t} x_{t} s_{i}^{t}=d_{i} & \text { there are } d_{i} \text { vertices of degree } i \in[2,7] \\
\sum_{t} y_{t}=h & \text { there are } h \text { lines } \\
\sum_{t} y_{t} t_{i}^{t}=i d_{i} & i \in[2,7] \\
\sum_{t} x_{t}\binom{s_{i}^{t}}{2}+\sum_{t} y_{t}\binom{\ell_{i}^{t}}{2} \leqslant\binom{ d_{i}}{2} & i \in[2,6] \\
\sum_{t} x_{t}\binom{s_{7}^{t}}{2}+\sum_{t} y_{t}\binom{\ell_{7}^{t}}{2}=\binom{d_{7}}{2} & \text { by Lemma } 2.7 \\
\sum_{t} x_{t} s_{i}^{t} s_{j}^{t}+\sum_{t} y_{t} \ell_{i}^{t} \ell_{j}^{t} \leqslant d_{i} d_{j} & \text { distinct } i, j \in[2,7]
\end{array}
$$

This integer program has a feasible solution only for the following twelve degree sequences $D=\left[d_{2}, d_{3}, \ldots, d_{7}\right]$. For all of these, $\varepsilon=0$.

$$
\begin{array}{lll}
h=45 & D=[22,3,7,2,15,32], & h=46 D=[23,2,1,7,13,35], \\
h=46 & D=[23,1,3,7,11,36], & h=46 D=[23,0,6,4,12,36], \\
h=46 & D=[23,0,5,7,9,37], & h=46 D=[22,2,5,5,10,37], \\
h=46 & D=[23,0,4,10,6,38], & h=46 D=[22,1,7,5,8,38], \\
h=46 & D=[22,0,10,2,9,38], & h=47 D=[23,0,3,5,10,40], \\
h=47 & D=[22,1,4,6,6,42], & h=47 D=[21,2,6,4,5,43] .
\end{array}
$$

### 2.2.3 Assignment of lines to vertices

If a linear intersecting 9-partite hypergraph $H$ with $\tau(H)=9$ exists, then it has 81 vertices $(\varepsilon=0), h \in\{45,46,47\}$, and one of the twelve degree sequence listed at the end of $\$ 2.2 .2$, Let $D$ be one of these degree sequences. Next we generated the set $\mathbf{S}(D)$ of all possible non-negative integer solutions for the system of equations (2.8), (2.9), and (2.10). Let $S=\left[x_{1}, x_{2}, \ldots, x_{|\mathcal{S}(h, 0)|}\right]$ denote a particular solution in $\mathbf{S}(D)$. We formulate another system of linear equations which takes $D$ and $S$ as input values.

Let $V=V(H)$ be the vertex set of $H$ which is partitioned into 9 sides of equal size. Then the set of degree sequences of vertices in the sides of $H$ corresponds to a solution $S \in \mathbf{S}(D)$. We change the notation slightly, to let $s_{j}^{v}$ denote the number of vertices of degree $j$ in the side that contains vertex $v$.

As before, let $y_{t}$ be the number of lines of type $t$ from the set $\mathcal{L}(h)$ present in $H$. Define $z_{t}^{v}$ to be the number of lines of type $t$ incident with a vertex $v \in V$. If a line of type $t$ has no vertices of degree $\operatorname{deg}(v)$ then $z_{t}^{v}=0$. Otherwise, $z_{t}^{v}$ is a non-negative integer. In addition to equations (2.11) -(2.15), the following equations hold for $y_{t}$ and $z_{t}^{v}$, where $t \in\{1,2, \ldots,|\mathcal{L}(h)|\}$ and $v \in V$.

$$
\begin{array}{ll}
\sum_{t} z_{t}^{v}=\operatorname{deg}(v) & \text { for all } v \in V ; \\
\sum_{t} z_{t}^{v}\left(\ell_{i}^{t}-1\right) \leqslant d_{i}-s_{i}^{v} & \text { for all } v \in V \text { where } i=\operatorname{deg}(v) ; \\
\sum_{t} z_{t}^{v}\left(\ell_{7}^{t}-1\right)=d_{7}-s_{7}^{v} & \text { for all } v \in V \text { such that } \operatorname{deg}(v)=7 \\
\sum_{t} z_{t}^{v} \ell_{j}^{t} \leqslant d_{j}-s_{j}^{v} & \text { for all } v \in V \text { and all } j \in[2,7], j \neq \operatorname{deg}(v) ; \\
\sum_{v \in V_{k}} z_{t}^{v}=y_{t} & \text { for all } t \in\{1,2, \ldots,|\mathcal{L}(h)|\} \text { and all } k \in[1, \\
\sum_{v, \operatorname{deg}(v)=i} z_{t}^{v}=y_{t} \ell_{i}^{t} & \text { for all } i \in[2,7] \text { and } t \in\{1,2, \ldots,|\mathcal{L}(h)|\} . \tag{2.21}
\end{array}
$$

Note that since $S \in \mathbf{S}(D)$ is given, now values $x_{t}$ in equations (2.11)-(2.15) are constants. The total number of lines incident with a vertex equals the degree of that vertex, which is given by (2.16). Two vertices are neighbours if there is a line containing both of them. Observe that, for each vertex $v$ in a given side, the number of neighbours of $v$ which have degree $j$ is at most the total number of degree $j$ vertices in the remaining sides, for $j \in[2,7]$. Equations (2.17)(2.18) correspond to counting the number of neighbours of $v$ which are of the same degree as $v$, whereas (2.19) counts the neighbours of $v$ which have degree different from $\operatorname{deg}(v)$. The equality in (2.18) is implied by Lemma [2.7. Since every line contains exactly one vertex in each side, each side is incident with as many lines of a type $t$ as there are lines of type $t$ present in the hypergraph, which is given by (2.20). Finally, (2.21) is a double count of pairs of lines of type $t$ and vertices of degree $i$ incident with these lines.

We found a feasible solution for the integer program given by (2.11)- (2.21) only for two pairs of input values of a degree sequence $D$ and $S \in \mathbf{S}(D)$ which we consider more closely in the following subsection.

### 2.2.4 Remaining cases

The two cases for which the integer program in the previous section has a feasible solution both have $h=46$. Below we give the input degree sequence $D$ and a matrix representation of $S$. Here, a column of $S$ corresponds to a side, and each entry is the degree of a vertex in that side. Each matrix $S$ is given uniquely, up to permutation of sides and permutation of vertices within each side.

## Case 1

$$
\begin{aligned}
& D=[23,1,3,7,11,36] \\
& S=\left(\begin{array}{ccccccccc}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
4 & 4 & 4 & 2 & 2 & 3 & 2 & 2 & 2 \\
5 & 5 & 5 & 6 & 6 & 5 & 5 & 5 & 5 \\
6 & 6 & 6 & 6 & 6 & 6 & 7 & 7 & 7 \\
6 & 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7
\end{array}\right)
\end{aligned}
$$

## Case 2

$$
\begin{aligned}
& D=[23,0,5,7,9,37] \\
& S=\left(\begin{array}{ccccccccc}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
4 & 5 & 2 & 2 & 4 & 4 & 2 & 2 & 2 \\
5 & 5 & 6 & 6 & 4 & 4 & 5 & 5 & 5 \\
6 & 5 & 6 & 6 & 6 & 6 & 7 & 7 & 7 \\
6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7
\end{array}\right)
\end{aligned}
$$

Case 1: Suppose there exists a linear intersecting 9-partite hypergraph $H$ with $\tau(H)=9$, $h=46$ lines and degree sequence $D=[23,1,3,7,11,36]$. First, we consider the set of types of
lines in $H$. Since $H$ has 46 lines and $d_{2}=23$, every line in $H$ contains a vertex of degree 2 . Also, $\ell_{i}^{t} \leqslant d_{i}$ for all $i \in[3,7]$. Hence, the possible line types $\mathcal{L}$ in $H$ are

$$
\mathcal{L}=\{[1,0,0,0,4,4],[1,0,0,1,2,5],[1,0,0,2,0,6],[1,0,1,0,1,6],[1,1,0,0,0,7]\} \subset \mathcal{L}(46) .
$$

Note that $H$ has one vertex $v$ such that $\operatorname{deg}(v)=3$. Let $e_{1}$ and $e_{2}$ be two lines which contain $v$. Then $e_{1}$ and $e_{2}$ have the same line type, namely $[1,1,0,0,0,7]$, since this is the only line type in $\mathcal{L}$ which contains a vertex of degree 3 . Let $u_{1}$ and $u_{2}$ be the vertices of degree 2 on $e_{1}$ and $e_{2}$, respectively. If $u_{1}$ and $u_{2}$ belong to two different sides, then let $w$ be the vertex of degree 7 on $e_{2}$ which is on the same side as $u_{1}$. By Lemma 2.7, there is a distinct line through $w$ and each of the degree 7 vertices on $e_{1}$. Hence, $\operatorname{deg}(w) \geqslant 8$, which is a contradiction. Therefore, $u_{1}$ and $u_{2}$ are on the same side in $H$. Let $f$ be the line other than $e_{1}$ which contains $u_{1}$ and let $w^{\prime}$ be the vertex in which $e_{2}$ and $f$ intersect. Then $\operatorname{deg}\left(w^{\prime}\right)=7$. Again by Lemma 2.7, there is a line through $w^{\prime}$ and every degree 7 vertex on $e_{1}$ which is not on the same side as $w^{\prime}$. Hence, $\operatorname{deg}\left(w^{\prime}\right) \geqslant 2+6=8$, which is a contradiction. Therefore, such a hypergraph does not exist.
Case 2: Suppose there exists a linear intersecting 9-partite hypergraph $H$ with $\tau(H)=9$, $h=46$ lines and degree sequence $D=[23,0,5,7,9,37]$. Moreover, without loss of generality, assume that the degrees of vertices in $V=V(H)$ are given by $S$ for a suitable permutation of sides and permutation of vertices within each side. Let $\mathcal{L} \subseteq \mathcal{L}(46)$ be the set of all possible line types which may occur in $H$. As in the previous case, line types in $\mathcal{L}$ satisfy the obvious constraints on degrees. Hence

$$
\mathcal{L}=\{[1,0,0,0,4,4],[1,0,0,1,2,5],[1,0,0,2,0,6],[1,0,1,0,1,6]\}
$$

For clarity, we index the line types in $\mathcal{L}$ by $A, B, C$ and $D$, respectively. Then for $t \in\{A, B, C, D\}, y_{t}$ is the number of lines of type $t$ in $H$ and $\left(y_{A}, y_{B}, y_{C}, y_{D}\right)$ satisfies the equations (2.11)-(2.15). In this case, many of these equations are dependent and it is enough to consider equations (2.11), (2.12) for $i \in\{4,5\}$, and (2.14) to obtain the unique solution $\left(y_{A}, y_{B}, y_{C}, y_{D}\right)=(1,15,10,20)$.

Moreover, for each $t \in\{A, B, C, D\}$ and each $v \in V, z_{t}^{v}$ denotes the number of lines of type $t$ incident with the vertex $v$ and the set of all values $\left\{z_{t}^{v}: v \in V, t \in\{A, B, C, D\}\right\}$ satisfies equations (2.16) $-(2.21)$. We restrict our attention to vertices of degree 7 and compute all possible solutions to the system of equations given only by (2.16), (2.18) and (2.19) for $j \in[2,6]$. For each $0 \leqslant k \leqslant 8$, the possible solutions $\left(z_{A}^{v}, z_{B}^{v}, z_{C}^{v}, z_{D}^{v}\right)$ for $v \in V_{k}$, where $\operatorname{deg}(v)=7$, are given in the table below.

| $V_{0}$ | $V_{1}$ | $V_{2}, V_{3}, V_{4}, V_{5}$ | $V_{6}, V_{7}, V_{8}$ |
| :---: | :---: | :---: | :---: |
| $(0,1,2,4)$ | $(0,1,1,5)$ | $(1,0,3,3)$ | $(1,1,2,3)$ |
|  |  | $(0,2,2,3)$ | $(0,3,1,3)$ |

Let $e$ be a line of type $C$. Line $e$ intersects each of the 20 lines of type $D$ exactly once. Moreover, line $e$ intersects a line of type $D$ either in its vertex of degree 2 or in one of its 6 vertices of degree 7. Observe that vertices of degree 7 in sides $V_{0}$ and $V_{1}$ are incident with 4 and 5 lines of type $D$, respectively; all other vertices of degree 7 are always incident with exactly 3 lines of type $D$. Depending on whether $e$ intersects a line of type $D$ in its vertex of degree 2 or not, it is easy to see that, in order for $e$ to meet 20 lines of type $D, e$ either contains exactly one vertex of degree 7 on side $V_{0}$ or exactly one vertex of degree 7 on side $V_{1}$, but not both. Since $z_{C}^{v}=2$ if $v \in V_{0}$ and $z_{C}^{v}=1$ if $v \in V_{1}$ when $\operatorname{deg}(v)=7$, there are at most $2 \cdot 3+1 \cdot 3=9$ lines of type $C$, which gives a contradiction. Therefore, such a hypergraph does not exist.

We conclude that there does not exist a linear intersecting 9-partite hypergraph with covering number 9 and $\Delta=7$. Together with Corollary 2.5 we have shown:
Theorem 2.8. For $2 \leqslant r \leqslant 9$ every linear intersecting $r$-partite hypergraph has covering number at most $r-1$.

## 3 Hypergraph constructions

In this section we describe several hypergraph constructions that are of interest. First we define some additional notation. Let $H$ be an $r$-partite hypergraph. In each side $V_{k}$, we label the vertices by $(k, 0),(k, 1), \ldots,\left(k,\left|V_{k}\right|-1\right)$, which we abbreviate to $0,1, \ldots,\left|V_{k}\right|-1$ when the side is clear from context. We say that a vertex $(k, l) \in V_{k}$ is at level $l$ in side $V_{k}$. We denote a line $e$ in $H$ by $\left[l_{0}, l_{1}, l_{2}, \ldots l_{r-1}\right]$, where $e$ contains the vertex at level $l_{k}$ in side $V_{k}$, where $k \in[0, r-1]$. When presenting a specific example, we omit the square brackets and commas to make the notation cleaner. The cyclic 1 -shift of a line $\left[l_{0}, l_{1}, l_{2}, \ldots, l_{r-1}\right]$ is the line $\left[l_{r-1}, l_{0}, l_{1}, \ldots, l_{r-2}\right]$ and the cyclic $t$-shift of a line $e$, denoted $e^{t}$, is the line obtained from $e$ by applying $t$ cyclic 1 -shifts. In particular, $e^{0}=e=e^{r}$. An $r$-partite hypergraph is cyclic if its automorphism group contains a cyclic subgroup of order $r$ acting transitively on the sides. A cyclic $r$-partite hypergraph can be obtained by developing a set of starter lines by cyclic shifts.

### 3.1 A counterexample to Conjecture 1.2

In this subsection we give a counterexample to Conjecture 1.2,
Theorem 3.1. For at least one value of $r$ there is an intersecting $r$-partite hypergraph $H$ such that each of its sides has size $r$ and $e \backslash\{v\}$ is not a cover for any $e \in H$ and any $v \in e$.

Proof. Let $r=13$. We give an example of a cyclic linear intersecting $r$-partite hypergraph $H$ in which every side has size $r$. The $3 r$ lines of $H$ are obtained by taking all possible cyclic shifts of the following three starter lines.

$$
\begin{array}{lccccccccccccc}
e_{1}= & 0 & \mathbf{1} & 4 & \mathbf{2} & \mathbf{3} & 5 & 6 & 6 & 5 & \mathbf{3} & \mathbf{2} & 4 & \mathbf{1} \\
e_{2}= & 0 & \mathbf{3} & 7 & \mathbf{1} & \mathbf{2} & 8 & 9 & 9 & 8 & \mathbf{2} & \mathbf{1} & 7 & \mathbf{3}  \tag{3.1}\\
e_{3}= & 0 & \mathbf{2} & 10 & \mathbf{3} & \mathbf{1} & 11 & 12 & 12 & 11 & \mathbf{1} & \mathbf{3} & 10 & \mathbf{2}
\end{array}
$$

Vertices at level 0 in $H$ have degree 3, vertices at levels 1, 2 and 3 have degree 6, and all other vertices have degree 2 .

We claim that $H$ is a linear intersecting hypergraph. Observe that $\left|e_{i}^{t_{1}} \cap e_{j}^{t_{2}}\right|=\left|e_{i} \cap e_{j}^{t_{2}-t_{1}}\right|$ for any $i, j \in\{1,2,3\}$ and $0 \leqslant t_{1} \leqslant t_{2}<r$. Hence, it suffices to show that for any $i, j \in\{1,2,3\}$ and $0 \leqslant t<r$ where $(j, t) \neq(i, 0)$, lines $e_{i}$ and $e_{j}^{t}$ intersect in exactly one vertex.

First we consider the case when $i=j$. By construction, $e_{i}^{3}$ has the underlying structure of a 4 -extended Skolem sequence of order 6 , namely 6420246531135 . For example, this sequence is obtained from $e_{1}^{3}$ by relabelling the levels using the permutation $(1,2,6)(3,5)$. For definitions and background on Skolem sequences, see [5]. In our case, the result is that for every $t \in$ $\left[1, \frac{r-1}{2}\right]$, there is a unique pair $(k, l)$ and $\left(k^{\prime}, l\right)$ of vertices in $e_{i}$ with $k^{\prime}-k \equiv t(\bmod r)$. Hence $e_{i} \cap e_{i}^{t}=\left\{\left(k^{\prime}, l\right)\right\}$ and $e_{i} \cap e_{i}^{r-t}=\{(k, l)\}$.

Now assume that $i \neq j$. Suppose that there are two distinct vertices $\left(k_{1}, l_{1}\right)$ and $\left(k_{2}, l_{2}\right)$ in $e_{i} \cap e_{j}^{t}$. Since $e_{i} \cap e_{j}=\{(0,0)\}$, we may assume that $t \in[1, r-1]$. Then by inspection, we must have $l_{1}, l_{2} \in\{1,2,3\}$ (the relevant entries are shown in bold in (3.1)). For the different possible pairs $\left(l_{1}, l_{2}\right)$ the following table shows the feasible distances $k_{2}-k_{1}(\bmod r)$.

| $l_{1}, l_{2}$ | distances in $e_{1}$ | distances in $e_{2}$ | distances in $e_{3}$ |
| :---: | :---: | :---: | :---: |
| 1,1 | $\pm 2$ | $\pm 6$ | $\pm 5$ |
| 2,2 | $\pm 6$ | $\pm 5$ | $\pm 2$ |
| 3,3 | $\pm 5$ | $\pm 2$ | $\pm 6$ |
| 1,2 | $\pm 2, \pm 4$ | $\pm 1, \pm 6$ | $\pm 3, \pm 5$ |
| 1,3 | $\pm 3, \pm 5$ | $\pm 2, \pm 4$ | $\pm 1, \pm 6$ |
| 2,3 | $\pm 1, \pm 6$ | $\pm 3, \pm 5$ | $\pm 2, \pm 4$ |

Since there is no row of the table where the same distance occurs in different columns, we conclude that

$$
\begin{equation*}
\left|e_{i} \cap e_{j}^{t}\right| \leqslant 1 \tag{3.2}
\end{equation*}
$$

Since $e_{i}$ and $e_{j}$ intersect in the vertex $(0,0)$, and both have a pair of vertices on each level $l \in\{1,2,3\}$, there are $1+3 \cdot 4=13$ vertices in which $e_{i}$ intersects the set of all cyclic shifts of $e_{j}$. It follows that we must have equality in (3.2) for each $i, j, t$. Thus $H$ is a linear intersecting hypergraph. Since $\delta(H)=2$, it follows that for every line $e \in H$ and each vertex $v \in e$, the set of vertices $e \backslash\{v\}$ is not a cover.

For the linear intersecting 13-partite hypergraph constructed in the proof of Theorem 3.1 we found by computation that $\tau(H)=9$. The following set of vertices is a 9 cover for $H$ :

$$
\{(0,1),(0,2),(0,3),(2,0),(5,0),(8,0),(10,1),(10,2),(10,3)\} .
$$

### 3.2 Cyclic intersecting hypergraphs with $\tau=r-1$

Next we give an example of a cyclic intersecting $r$-partite hypergraph which has covering number $r-1$ for $r \in\{9,13,17\}$. The methodology for building these hypergraphs is similar to the construction given in the proof of Theorem 3.1. However, in each case below the hypergraph we construct is non-linear.

Lemma 3.2. Let $(r, s) \in\{(9,4),(13,5),(17,6)\}$. Then there exists a cyclic intersecting $r$ partite hypergraph $H$ such that $H$ has sr lines and $\tau(H)=r-1$.

Proof. The $s r$ lines of $H$ are obtained by taking all possible cyclic shifts of lines in the starters given below.

| $r=9$ |  |  |  |  |  |  |  |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $e_{1}=$ | 4 | 3 | 2 | 1 | 0 | 1 | 2 | 3 | 4 |
| $e_{2}=$ | 3 | 6 | 5 | 4 | 0 | 4 | 5 | 6 | 3 |
| $e_{3}=$ | 1 | 2 | 4 | 6 | 0 | 6 | 4 | 2 | 1 |
| $e_{4}=$ | 3 | 0 | 2 | 6 | 7 | 6 | 2 | 0 | 3 |


| $r=13$ |
| :--- |
| $e_{1}=$ | 6


| $e_{1}=$ | 3 | 12 | 11 | 4 | 1 | 10 | 2 | 9 | 0 | 9 | 2 | 10 | 1 | 4 | 11 | 12 | 3 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $e_{2}=$ | 1 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 0 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 1 |
| $e_{3}=$ | 14 | 3 | 9 | 7 | 15 | 11 | 6 | 4 | 0 | 4 | 6 | 11 | 15 | 7 | 9 | 3 | 14 |
| $e_{4}=$ | 6 | 2 | 5 | 11 | 14 | 12 | 4 | 10 | 0 | 10 | 4 | 12 | 14 | 11 | 5 | 2 | 6 |
| $e_{5}=$ | 11 | 10 | 14 | 1 | 7 | 3 | 12 | 8 | 0 | 8 | 12 | 3 | 7 | 1 | 14 | 10 | 11 |
| $e_{6}=$ | 7 | 4 | 13 | 5 | 9 | 1 | 11 | 12 | 0 | 12 | 11 | 1 | 9 | 5 | 13 | 4 | 7 |

We found by computation that the covering numbers for these three hypergraphs are 8,12 and 16 , respectively.

Remark. The hypergraphs with $r=9, r=13$ and $r=17$ in Lemma 3.2 are intrinsically non-linear; each has the property that it does not contain a linear subhypergraph with covering number $r-1$.

First, if $H$ is the hypergraph with $r=9$, then $\left|e_{2} \cap e_{4}\right|=2$, but deleting either $e_{2}$ or $e_{4}$ reduces the covering number. Below are covers of size 7 for $H \backslash\left\{e_{2}\right\}$ and $H \backslash\left\{e_{4}\right\}$, respectively.

$$
\begin{aligned}
& \{(5,6),(6,0),(6,1),(6,2),(6,3),(6,4),(6,6)\} \\
& \{(4,0),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\}
\end{aligned}
$$

Similarly, if $H$ is the hypergraph with $r=13$, then $\left|e_{1} \cap e_{5}\right|=2$ and below are covers of size 11 for $H \backslash\left\{e_{1}\right\}$ and $H \backslash\left\{e_{5}\right\}$, respectively.

$$
\begin{aligned}
& \{(0,7),(1,3),(2,3),(3,7),(8,0),(8,1),(8,3),(8,5),(8,6),(8,7),(12,2)\} \\
& \{(1,9),(4,6),(6,0),(6,1),(6,2),(6,3),(6,6),(6,7),(6,9),(8,6),(11,9)\}
\end{aligned}
$$

Finally, consider the hypergraph $H$ in Lemma 3.2 with $r=17$. Let $H_{1}$ be the subhypergraph obtained from all of the cyclic shifts of starter lines $e_{2}, e_{3}, e_{4}, e_{5}, e_{6}$. Then $H_{1}$ is a linear intersecting hypergraph with covering number 15 , where the following set of vertices is a minimal cover:

$$
\begin{array}{r}
\{(4,5),(4,8),(4,11),(5,4),(5,7),(10,0),(10,4),(10,6), \\
(10,7),(10,14),(15,4),(15,7),(16,5),(16,8),(16,11)\} .
\end{array}
$$

Let $E_{1}$ be the set of all cyclic shifts of the starter line $e_{1}$. Observe that $E_{1}$ is linear intersecting. Furthermore, each line $e \in E_{1}$ intersects exactly 12 lines of $H_{1}$ more than once. For every non-empty subset $E^{*} \subseteq E_{1}$, define $H_{1}^{*}$ to be the hypergraph obtained from $H_{1}$ by adding the lines of $E^{*}$ and removing every line of $H_{1}$ that intersects a line of $E^{*}$ more than once. We computationally checked that each of the $2^{17}-1$ such linear intersecting hypergraphs $H_{1}^{*}$ has covering number less than 16 (in fact, they each have covering number between 10 and 14). Therefore, $H$ has no linear intersecting subhypergraph with $\tau=16$.

### 3.3 Linear intersecting hypergraphs built from latin squares

Next we describe a family of linear intersecting $r$-partite hypergraphs which do not have a minimal cover that consists of a single side or that consists of a subset of a line.

A latin square of order $n$ is an $n \times n$ array of $n$ symbols such that each symbol appears exactly once in each row and exactly once in each column. If $L$ is a latin square, $L[r, c]$ denotes the symbol in row $r$ and column $c$ of $L$. A pair of latin squares of order $n$ are orthogonal if, when the two squares are superimposed, each of the $n^{2}$ possible ordered pairs of symbols occurs exactly once. A set of latin squares is called mutually orthogonal if each pair of latin squares in the set are orthogonal. For more on latin squares, see [12].

Lemma 3.3. If there exist $k$ mutually orthogonal latin squares of order $n \geqslant 3$, then there exists a linear intersecting ( $n+2$ )-partite hypergraph $H$ with $\tau(H)=k+1$ such that no side or subset of a line is a minimal cover.

Proof. Let $L_{0}, \ldots, L_{k-1}$ be $k$ mutually orthogonal latin squares of order $n$. Assume that each $L_{i}$ has $[0,1, \ldots, n-1]$ as its first row. For $i=0,1, \ldots, k-1$, let $M_{i}$ be the column inverse of $L_{i}$, that is, $M_{i}[r, c]=s$ if and only if $L_{i}[s, c]=r$. Since each $L_{i}$ has its first row in reduced form, each $M_{i}$ has symbol 0 on the main diagonal. Set $M_{0}^{\prime}=M_{0}$ and for $i=1, \ldots, k-1$ let $M_{i}^{\prime}$ be the latin square obtained from $M_{i}$ by replacing symbol 0 with symbol $n+i-1$.

Next we define the lines of an $(n+2)$-partite hypergraph $H$. Let $V_{0}, \ldots, V_{n+1}$ be the sides of $H$. Sides $V_{0}, \ldots, V_{n}$ each have $n+k-1$ vertices and side $V_{n+1}$ has $n+1$ vertices. If a line $\left[l_{0}, l_{1}, \ldots, l_{r-1}\right]$ is a concatenation of two lists, we use the notation $\left[l_{0}, \ldots, l_{i}\right] \oplus\left[l_{i+1}, \ldots, l_{r-1}\right]$.

For $i=0, \ldots, k-1$ define $E_{i}:=\left\{M_{i}^{\prime}[x] \oplus[i, x]: 0 \leqslant x \leqslant n-1\right\}$, where $M_{i}^{\prime}[x]$ is row $x$ of the latin square $M_{i}^{\prime}$. Further, define $E^{*}:=\{[x, x, \ldots, x, k+x-1, n]: 1 \leqslant x \leqslant n-1\}$.

It is straightforward to check that each line in $E_{i}$ intersects each line in $E^{*}$ exactly once, and that the lines in $E^{*}$ meet each other exactly once. Since $M_{i}^{\prime}$ is a latin square, two distinct lines in $E_{i}$ intersect only at the vertex $(n, i)$. We next show that lines of $E_{i}$ and $E_{j}$ also intersect linearly for $i \neq j$. Consider a line $\ell_{1}=M_{i}^{\prime}[x] \oplus[i, x] \in E_{i}$ and a line $\ell_{2}=M_{j}^{\prime}[y] \oplus[j, y] \in E_{j}$ where $i \neq j$. Since $L_{i}$ and $L_{j}$ are orthogonal latin squares, there is a unique cell $(r, c)$ where
$L_{i}[r, c]=x$ and $L_{j}[r, c]=y$. If $x=y$, then $r=0$ and $M_{i}[x, x]=M_{j}[x, x]=0$, and thus, by the relabelling of symbol 0 in $M_{1}^{\prime}, \ldots, M_{k-1}^{\prime}$, the lines $\ell_{1}$ and $\ell_{2}$ intersect only at the vertex $(n+1, x)$. If $x \neq y$, then $r \neq 0$ and thus $M_{i}^{\prime}[x, c]=r=M_{j}^{\prime}[y, c]$ and therefore the lines $\ell_{1}$ and $\ell_{2}$ intersect only at the vertex $(c, r)$. It follows that $H=E^{*} \cup E_{0} \cup E_{1} \cup \cdots \cup E_{k-1}$ is a linear intersecting $(n+2)$-partite hypergraph with $k n+n-1$ lines.

Since $H$ has maximum degree $n$ and $k n+n-1$ lines, no set of $k$ vertices is a cover. It is straightforward to check that the vertices at levels $0,1, \ldots, k-1$ of side $V_{n}$ together with the vertex $(n+1, n)$ form a cover of size $k+1$. Therefore $\tau(H)=k+1$.

It is well-known that there are at most $n-1$ mutually orthogonal latin squares of order $n$ (see e.g. [12]), so $k+1 \leqslant n$. Observe that for $k \geqslant 2$ each side has size at least $n+1$, whereas $\tau(H)=2<n$ when $k=1$. Hence, no side is a minimal cover. Also, since $H$ is linear and each line contains exactly one vertex of degree 1 , it follows that the only covers which are subsets of a line have size at least $n+1$. Thus, no side or subset of a line is a minimal cover of $H$.

The following corollary follows immediately from the existence of complete sets of mutually orthogonal latin squares of prime power orders [12].

Corollary 3.4. If $n$ is a prime power then there exists a linear intersecting $(n+2)$-partite hypergraph $H$ with $\tau(H)=n$ such that no side or subset of a line is a minimal cover.

Example: Below are the lines of a hypergraph built from 3 mutually orthogonal latin squares of order 4, as described in Lemma 3.3. This hypergraph is also maximal with respect to the property of being linear and intersecting.

| 031200 | 423110 | 512320 | 111134 |
| :--- | :--- | :--- | :--- |
| 302101 | 241311 | 153221 | 222244 |
| 120302 | 314212 | 235122 | 333354 |
| 213003 | 132413 | 321523 |  |

### 3.4 An 8-partite linear hypergraph with $\tau=7$

We close this section by giving a construction for an interesting 8-partite linear hypergraph $\mathcal{H}_{38}$ that achieves equality in Conjecture 1.1.

Let $F$ denote the Fano plane constructed by developing the triple $\{0,1,3\}$ modulo 7 . Let $G$ denote the stabiliser of the point 0 in $F$. Note that $|G|=24$. Let $\mathcal{C}$ denote the set of 7 -cycles obtained by conjugating the cycle (0123456) by elements of $G$.

For each permutation $p \in \mathcal{C}$ we add one line to $\mathcal{H}_{38}$ which includes the vertex (i,p[i]) for $i \in[0,6]$. If $p_{1}, p_{2} \in \mathcal{C}$ then $p_{1}^{-1} p_{2}$ has at most one fixed point. We can make the lines corresponding to $p_{1}$ and $p_{2}$ meet on side $V_{7}$ if and only if $p_{1}^{-1} p_{2}$ has no fixed points. This produces 8 vertices of degree 3 in $V_{7}$, and completes the description of the lines corresponding to the cycles in $\mathcal{C}$. Next we add two new vertices $v_{1}, v_{2}$ to $V_{7}$. For each $i \in[0,6]$ we put a line through $v_{1}$ and the vertices $(j, i)$ for $j \in[0,6]$. For each $i \in[0,6]$ we put a line through $v_{2},(i, i)$ and all vertices $(a, b)$ for which $\{i, a, b\}$ is a triple of $F$.

The construction just described results in $\mathcal{H}_{38}$, which has 38 lines and an automorphism group isomorphic to $\operatorname{PSL}(2,7)$. For $i \in[0,6]$ the vertex $(i, i)$ has degree 2. All other vertices on sides $V_{0}, \ldots, V_{6}$ have degree 6 . On $V_{7}$ the vertices $v_{1}$ and $v_{2}$ have degree 7 and the other 8 vertices have degree 3. Since $V_{7}$ has 10 vertices it is clear that $\mathcal{H}_{38}$ is not isomorphic to a subhypergraph of $\mathcal{P}_{8}^{\prime}$. Nevertheless, it is routine to check that $\mathcal{H}_{38}$ is a linear intersecting 8 -partite hypergraph. Suppose that $X$ is a 6 -cover of $\mathcal{H}_{38}$. Then $X$ must include $v_{1}$ and $v_{2}$ since otherwise it cannot cover the lines through those vertices. The 24 lines that avoid $v_{1}$ and $v_{2}$ induce a subhypergraph with maximum degree 4 , which thus cannot be covered by fewer
than 6 vertices. This contradiction shows that $X$ does not exist. Since $V_{0}$ is a 7 -cover, we must have $\tau\left(\mathcal{H}_{38}\right)=7$.

Some further properties of $\mathcal{H}_{38}$ are discussed in the next section.

## 4 Computational results

In this section we describe a computational proof of the following result.

## Theorem 4.1.

1. For $r \leqslant 7$ the only linear intersecting $r$-partite hypergraphs to achieve equality in Ryser's conjecture are subhypergraphs of $\mathcal{P}_{r}^{\prime}$. In particular, there are none for $r=7$.
2. No subhypergraph of $\mathcal{H}_{38}$ has $\tau=7$ and is isomorphic to a subhypergraph of $\mathcal{P}_{8}^{\prime}$.
3. The smallest subhypergraph of $\mathcal{H}_{38}$ with $\tau=7$ has 22 lines. The smallest subhypergraph of $P_{8}^{\prime}$ with $\tau=7$ also has 22 lines.

Clearly, by part (1), every 7 -partite linear intersecting hypergraph satisfies $\tau \leqslant 5$. There are a number of non-isomorphic ways to achieve $\tau=5$, including by the construction in Lemma 3.3. For 7-partite intersecting non-linear hypergraphs with $\tau=6$, see [1, 3].

Let $H$ be a 7 -partite linear intersecting hypergraph with $h=|H|$ lines and $\tau \geqslant 6$. By an argument similar to the proof of Lemma [2.2 we know that $\Delta(H) \leqslant 6$. We next argue that $\Delta(H) \geqslant 4$. Let $H$ have $x_{i}$ vertices of degree $i=1, \ldots, \Delta$. Note that no line of $H$ can include two vertices of degree 1 , otherwise the remaining vertices on the line would provide a 5 -cover. Hence $x_{1} \leqslant h$. Together with [3, Lem 2.1] and [3, Thm 2.7], we can then deduce that if $\Delta \leqslant 4$ and $x_{4} \leqslant 7$ then $x_{1}=h=17, x_{3} \geqslant 38$ and hence $x_{2}<0$. It follows that $\Delta \geqslant 4$. Also if $\Delta=4$ then $x_{4}>7$ so some side of $H$ has at least 2 vertices of degree 4 on it.

We next describe the computation that established Part 1 of Theorem 4.1. By the above comments, we can split the problem for $r=7$ into three subcases $\Delta=4, \Delta=5$ and $\Delta=6$. We started with a vertex of degree $\Delta$ on side $V_{0}$. (In the $\Delta=4$ case, we then added a second vertex of degree $\Delta$ to $V_{0}$ in all possible ways up to isomorphism.) Subsequent lines were added one at a time, ensuring that all pairs of lines intersected in a single point and that the assumed maximum degree was not violated. After each line was added, we tested for isomorphism and kept only one representative of each isomorphism class. For isomorphism checking we converted the hypergraphs into vertex-coloured graphs and applied the software nauty [8]. For $\Delta=4,5,6$ the largest hypergraphs we obtained had $16,25,18$ lines respectively. All hypergraphs that we built had a 5 -cover, proving the claim that no linear 7 -partite intersecting hypergraph achieves $\tau=6$.

For $r \leqslant 6$, we performed computations as just described, except that there was no need to split the problem into subcases according to the maximum degree. Every hypergraph that we encountered could be extended to $\mathcal{P}_{r}^{\prime}$.

For $r \geqslant 8$ the above method is not practical for a complete enumeration. However, we did a partial enumeration and found a number of linear intersecting 8-partite hypergraphs that are maximal (no lines can be added), have $\tau=7$ and yet are not isomorphic to $\mathcal{P}_{8}^{\prime}$. Most of these have the property that a few lines can be removed to get something isomorphic to a subhypergraph of $\mathcal{P}_{8}^{\prime}$. However, the hypergraph $\mathcal{H}_{38}$ described in 3.4 seems to be of a very different nature, which is why we tested its properties more thoroughly.

In Table 1 and Table 2 all 2098796663 isomorphism classes of subhypergraphs of $\mathcal{P}_{8}^{\prime}$ with $\tau=7$, and all 17892655 isomorphism classes of subhypergraphs of $\mathcal{H}_{38}$ with $\tau=7$ are classified

| $\|H\|$ | Number | \|H | Number | $\|H\|$ | Number | $\|H\|$ | Number |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 22 | 833 | 29 | 268297692 | 36 | 2660309 | 43 | 179 |
| 23 | 2168877 | 30 | 183765292 | 37 | 936491 | 44 | 32 |
| 24 | 58227758 | 31 | 114391098 | 38 | 296473 | 45 | 8 |
| 25 | 224055209 | 32 | 64949914 | 39 | 84035 | 46 | 3 |
| 26 | 368614512 | 33 | 33653522 | 40 | 21221 | 47 | 1 |
| 27 | 401984117 | 34 | 15894680 | 41 | 4757 | 48 | 1 |
| 28 | 351960321 | 35 | 6828374 | 42 | 953 | 49 | 1 |

Table 1: Number of isomorphism classes of subhypergraphs $H$ of $\mathcal{P}_{8}^{\prime}$, with $\tau(H)=7$.

| $\|H\|$ | Number | $\|H\|$ | Number | $\|H\|$ | Number | \| $H$ \| | Number |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 22 | 5 | 27 | 3376797 | 32 | 14308 | 37 | 3 |
| 23 | 42310 | 28 | 1625274 | 33 | 2803 | 38 | 1 |
| 24 | 1550265 | 29 | 644482 | 34 | 462 |  |  |
| 25 | 5027821 | 30 | 215066 | 35 | 67 |  |  |
| 26 | 5332373 | 31 | 60609 | 36 | 9 |  |  |

Table 2: Number of isomorphism classes of subhypergraphs $H$ of $\mathcal{H}_{38}$, with $\tau(H)=7$.
by their size. The tables were prepared by exhaustive enumeration, using a heuristic upper bound for the covering number to quickly eliminate most subhypergraphs with $\tau(H) \leqslant 6$, and employing nauty to remove isomorphs.

We end with an example of a subhypergraph of $\mathcal{P}_{8}^{\prime}$ that has 22 lines and $\tau=7$ :

| 03426434 | 04505645 | 06663521 | 11264344 | 15636215 | 16055456 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 22642443 | 24366152 | 25550564 | 32133654 | 34624066 | 35345331 |
| 43331546 | 44453313 | 46246660 | 51313465 | 55462606 | 56534133 |
| 60444555 | 61651632 | 62516326 | 63165263 |  |  |

and an example of a subhypergraph of $\mathcal{H}_{38}$ that has 22 lines and $\tau=7$ :

| 00000008 | 03615429 | 10536249 | 20361454 | 22222228 | 24510635 |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 25043169 | 26105341 | 31402659 | 33333338 | 36541026 | 42016356 |
| 43562101 | 44444448 | 46320519 | 54306123 | 56413204 | 60425136 |
| 62503412 | 63140253 | 64251309 | 65312047 |  |  |

Both these hypergraphs have an automorphism group of order 3, which is the largest achieved by subhypergraphs with $h=22$ and $\tau=7$ within $\mathcal{P}_{8}^{\prime}$ and $\mathcal{H}_{38}$, respectively.

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