On Ryser's Conjecture for Linear Intersecting Multipartite Hypergraphs^{*}

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Abstract

Ryser conjectured that $\tau \leq (r-1)\nu$ for r-partite hypergraphs, where τ is the covering number and ν is the matching number. We prove this conjecture for $r \leq 9$ in the special case of *linear intersecting* hypergraphs, in other words where every pair of lines meets in exactly one vertex.

Aharoni formulated a stronger version of Ryser's conjecture which specified that each r-partite hypergraph should have a cover of size $(r-1)\nu$ of a particular form. We provide a counterexample to Aharoni's conjecture with r = 13 and $\nu = 1$.

We also report a number of computational results. For r = 7, we find that there is no linear intersecting hypergraph that achieves the equality $\tau = r - 1$ in Ryser's conjecture, although non-linear examples are known. We exhibit intersecting non-linear examples achieving equality for $r \in \{9, 13, 17\}$. Also, we find that r = 8 is the smallest value of rfor which there exists a linear intersecting r-partite hypergraph that achieves $\tau = r - 1$ and is not isomorphic to a subhypergraph of a projective plane.

1 Introduction

A hypergraph H is a set of non-empty subsets, variously called *lines, edges* or hyperedges, of a finite underlying vertex set V(H). The *degree* of a vertex $v \in V(H)$, denoted deg(v), is the number of lines in H that contain v. A hypergraph is *r*-uniform if every line contains exactly r vertices. Thus a 2-uniform hypergraph is simply a graph.

Covers and matchings in hypergraphs are widely studied [6]. A cover of a hypergraph H is a set of vertices $C \subseteq V(H)$ such that every line of H contains at least one vertex of C. The covering number of H, denoted $\tau(H)$, is the minimum size of a cover of H. A matching in His a set of pairwise disjoint lines of H and the matching number of H, denoted $\nu(H)$, is the maximum size of a matching in H. Most hypergraphs in this paper are intersecting, meaning that every pair of lines meets in at least one vertex; equivalently $\nu = 1$.

The covering number and matching number of a hypergraph are related. First, for every hypergraph, $\nu \leq \tau$ since each cover contains at least one vertex from each line in any given matching. Second, for every *r*-uniform hypergraph, $\tau \leq r\nu$ since a cover can be obtained from the union of the lines in a maximal matching; this bound is sharp and is achieved, for example, by projective planes of order r - 1.

A hypergraph is *r*-partite if its vertex set can be partitioned into r sets, called *sides*, such that every line consists of exactly one vertex from each side. Hence every r-partite hypergraph is necessarily r-uniform. An r-partite hypergraph can be constructed from a projective plane of order r - 1 by removing a single vertex v and the r lines through v. The resulting hypergraph

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 \mathcal{P}'_r is called a *truncated projective plane* and the *r* sides of \mathcal{P}'_r are defined by the sets of vertices on each of the removed lines. Our notation hides the fact that there may be non-isomorphic choices for the structure of \mathcal{P}'_r , but the distinction between the different choices will not matter in our work.

The following famous conjecture is due to Ryser [9].

Conjecture 1.1. Every *r*-partite hypergraph with covering number τ and matching number ν satisfies $\tau \leq (r-1)\nu$.

Ryser's Conjecture is far from being resolved. When r = 2, the conjecture is König's Theorem for bipartite graphs, which is also equivalent to Hall's Theorem (see e.g. [12]). Using topological methods, Aharoni and Haxell [4] proved a hypergraph generalisation of Hall's Theorem and Aharoni [2] used this result to prove Conjecture 1.1 for r = 3. Conjecture 1.1 was proved for intersecting *r*-partite hypergraphs with $r \leq 5$ by Tuza [11]. Haxell and Scott [7] built on this result to prove that for $r \leq 5$, there exists $\epsilon > 0$ such that $\tau < (r - \epsilon)\nu$ for all *r*-partite hypergraphs.

In this paper, we prove another restricted version of Ryser's Conjecture for small r. A hypergraph is *linear* if each pair of lines meets in at most one vertex. Hence, in a *linear* intersecting hypergraph, each pair of lines intersects in exactly one vertex. In §2 we prove that Conjecture 1.1 holds for $r \leq 9$ when H is a linear intersecting hypergraph. The proof of the r = 9 case is computational.

In an intersecting r-partite hypergraph each side is a cover and each line is also a cover. In [3] the following strenthening of Conjecture 1.1 is recorded. The conjecture is due to Aharoni, as are several generalisations that are also given in [3].

Conjecture 1.2. An intersecting *r*-partite hypergraph *H* has a side of size at most r - 1 or a cover of the form $e \setminus \{v\}$ for some $e \in H$ and $v \in e$.

In §3.1 we show that Conjecture 1.2 is false, by providing a counterexample when r = 13. Furthermore, in §3.3 we describe an infinite family of linear intersecting *r*-partite hypergraphs with $\tau = r - 2$, built from mutually orthogonal latin squares. Although this family of hypergraphs have covers consisting of an line with a vertex removed, they have the property that no minimal cover is contained within a line or a side.

In studying Conjecture 1.1 it is natural to investigate hypergraphs that achieve the equality $\tau = (r-1)\nu$. A well-known infinite family of such hypergraphs with $\nu = 1$ is the family of truncated projective planes \mathcal{P}'_r (where r-1 is a prime power). Note that $\tau \leq r-1$ since each side of \mathcal{P}'_r has r-1 vertices and $\tau \geq r-1$ since each vertex lies on r-1 lines and the total number of lines is $(r-1)^2$. However, \mathcal{P}'_r has more lines than is necessary in the sense that many of its subhypergraphs achieve $\tau = r-1$. Conversely, we will report in §4 that for $r \leq 7$ the only way to achieve $\tau = r-1$ in a linear intersecting r-partite hypergraph is to take a subhypergraph of \mathcal{P}'_r . In particular, there are no linear intersecting 7-partite hypergraphs with $\tau = 6$, since \mathcal{P}'_7 does not exist. By contrast, it was shown in [1, 3] that there are non-linear intersecting 7-partite hypergraphs with $\tau = 6$. We also describe in §3.4 a linear intersecting hypergraph having r = 8 and $\tau = 7$ and which is not a subhypergraph of \mathcal{P}'_8 . Moreover, in §3.2 we give examples of non-linear intersecting r-partite hypergraphs with $r \in \{9, 13, 17\}$ which have covering number r-1.

1.1 Notation

We deal with r-partite hypergraphs throughout. To avoid degeneracies we always assume that $r \ge 2$ and that every vertex has positive degree. The sides of our hypergraphs are always denoted $V_0, V_1, \ldots, V_{r-1}$ and we have $V = \bigcup_{i=0}^{r-1} V_i$. The covering number, matching number, minimum degree and maximum degree of a hypergraph are denoted by τ , ν , δ and Δ , respectively. The number of lines in a hypergraph H is denoted h or |H|. We often use discrete interval notation $[n_1, n_2] = \{n_1, n_1 + 1, n_1 + 2, \ldots, n_2\}$, where n_1 and n_2 are two integers and $n_1 \leq n_2$.

2 Ryser's conjecture for linear intersecting hypergraphs

In this section we prove that Conjecture 1.1 holds for all linear intersecting hypergraphs with at most nine sides. We begin by establishing some properties of a hypothetical counterexample.

2.1 General properties

Lemma 2.1. Let H be an intersecting r-partite hypergraph with $\tau(H) = r$. Then

- (i) Each side of H has size at least r.
- (ii) Each vertex of H has degree at least 2.
- (iii) Each line of H contains at most one vertex of degree 2.

Proof. Each side is a cover so it contains at least $\tau(H)$ vertices, hence (i) holds. If a vertex v has degree 1 and ℓ is the line containing v, then $\ell \setminus \{v\}$ is a cover of size r - 1, hence (ii) holds. Finally, if there is a line ℓ containing distinct vertices u and v of degree 2, then we get a cover of size r - 1 by taking $\ell \setminus \{u, v\} \cup \{x\}$, where x is a vertex in the intersection of the (at most) two lines other than ℓ that meet $\{u, v\}$. Therefore, (iii) is proved.

Lemma 2.2. Let H be an intersecting r-partite hypergraph with $\tau(H) = r$. Then $\Delta \ge 4$. Furthermore, if H is linear, then $\Delta \le r - 2$.

Proof. By Lemma 2.1, we may assume that $\delta(H) \ge 2$. Suppose there are h lines in H. If we count the lines which intersect a given line, we find that $h \le (\Delta - 1)r + 1$ with equality only if H is Δ -regular. If we count the lines incident with a side that contains a vertex of degree Δ , we find that $h \ge 2(r-1) + \Delta$, with equality only if all other vertices on that side have degree 2. The above observations together show that $\Delta \ge 4$.

Now suppose that H is linear and let v be a vertex of degree Δ . Without loss of generality, assume that v is on side V_0 and let $u \in V_0 \setminus \{v\}$ (such a u exists, since $\tau(H) > 1$). Any line ℓ through u meets the lines through v in Δ distinct vertices on sides other than V_0 , since H is linear and intersecting. Hence $\Delta \leq r - 1$. If $\Delta = r - 1$ then side V_1 has at most r - 1 vertices. Indeed, ℓ cannot contain any vertex on side V_1 other than the vertices on lines through v. As this is true for any line ℓ which does not contain v, there can only be r - 1 vertices in V_1 , which means that $\tau(H) \leq r - 1$. We conclude that $\Delta \leq r - 2$.

Theorem 2.3. Let H be an intersecting r-partite hypergraph with $\tau(H) = r$ and maximum degree Δ . Then

$$(r+1/2)^2 \Delta^2 + 4r^2 \ge (8r^2 - 2r)\Delta.$$
 (2.1)

Proof. Suppose that H contains d_i vertices of degree i for $2 \leq i \leq \Delta$. Since each side of H is a cover, there must be $r^2 + \varepsilon$ vertices in H for some $\varepsilon \geq 0$. Hence

$$r^2 + \varepsilon = \sum_{i=2}^{\Delta} d_i.$$
(2.2)

By counting vertex-line incidences we find that

$$rh = \sum_{i=2}^{\Delta} id_i.$$
(2.3)

Also, since H is intersecting, we get the following inequality by counting incidences between lines:

$$\binom{h}{2} \leqslant \sum_{i=2}^{\Delta} \binom{i}{2} d_i.$$
(2.4)

Our aim is to show that (2.4) cannot be satisfied unless (2.1) holds. Since, by Lemma 2.1(iii), no line of H may contain more than one vertex of degree 2, $d_2 = h/2 - \varepsilon'$ for some $\varepsilon' \ge 0$. Now solving (2.2) and (2.3) for d_3 and d_{Δ} (recall that $\Delta > 3$ by Lemma 2.2) and substituting into (2.4), we get

$$h^{2} - (\Delta r + \Delta/2 + 2r)h + 3r^{2}\Delta$$

$$\leq 3\Delta \sum_{i=4}^{\Delta -1} d_{i} - (\Delta + 2) \sum_{i=4}^{\Delta -1} i d_{i} + \sum_{i=4}^{\Delta -1} i(i-1)d_{i} - 3\Delta\varepsilon - (\Delta - 2)\varepsilon'$$

$$\leq -\sum_{i=4}^{\Delta -1} (i-3)(\Delta - i)d_{i} - 3\Delta\varepsilon - (\Delta - 2)\varepsilon'$$

$$\leq -\sum_{i=4}^{\Delta -1} (i-3)(\Delta - i)d_{i} \leq 0.$$

$$(2.5)$$

$$(2.5)$$

Hence, the discriminant of the quadratic in h given by (2.5) is non-negative, which implies (2.1).

In a linear intersecting r-partite hypergraph with $\tau = r$, we have $4 \leq \Delta \leq r - 2$, by Lemma 2.2. Substituting $\Delta \in \{4, 5, 6\}$ into (2.1), yields an immediate contradiction. Hence:

Corollary 2.4. If there exists a linear intersecting r-partite hypergraph H such that $\tau(H) = r$, then $\Delta(H) \ge 7$.

Corollary 2.5. Conjecture 1.1 holds for all linear intersecting r-partite hypergraphs when $r \leq 8$.

2.2 Linear intersecting 9-partite hypergraphs

In this subsection we prove that a linear intersecting 9-partite hypergraph has covering number at most 8. For a potential counterexample H, let h = |H| be the number of lines in H and $\varepsilon := |V(H)| - r^2 \ge 0$. First, we demonstrate some further properties that such a hypergraph would have, if it were to exist. Note that we may assume that $\Delta = 7$, given Corollary 2.4 and Lemma 2.2.

Lemma 2.6. If H is a linear intersecting 9-partite hypergraph with $\Delta = 7$ and $\tau = 9$, then $h \ge 39$.

Proof. Let u be a vertex of degree $\Delta = 7$, and without loss of generality, assume that $u \in V_0$. Let A be the set of vertices that lie on lines through u in the remaining eight sides. Then $|A| = 8 \cdot 7 = 56$. Let $B = V(H) \setminus (V_0 \cup A)$. Then $|B| \ge 16$.

Let ℓ be a line through a vertex in B. Then ℓ is incident with a vertex other than u in side V_0 and it intersects all seven lines incident with u. That is, $|\ell \cap A| = 7$ and $|\ell \cap B| = 1$. Since every vertex in B has degree at least 2, there are at least $2|B| \ge 32$ such lines. Therefore, $h \ge 7 + 32 = 39$.

By (2.6),

$$h^{2} - (\Delta r + \Delta/2 + 2r)h + 3r^{2}\Delta \leqslant -3\Delta\varepsilon.$$
(2.7)

This inequality has no integer solution for h when $\varepsilon \ge 5$ if r = 9 and $\Delta = 7$. Otherwise, together with Lemma 2.6, we obtain that $h_{min}(\varepsilon) \le h \le h_{max}(\varepsilon)$ where $h_{min}(\varepsilon)$ and $h_{max}(\varepsilon)$ are as follows:

Lemma 2.7. If H is a linear intersecting 9-partite hypergraph with $\Delta = 7$ and $\tau(H) = 9$, then for every pair of degree 7 vertices on different sides there is a line of H that contains both vertices.

Proof. Suppose to the contrary that v_0 and v_1 are degree 7 vertices on sides V_0 and V_1 , respectively, which do not lie on a common line. Let e_1, \ldots, e_7 be the lines which contain v_0 and f_1, \ldots, f_7 be the lines which contain v_1 . For every $i, j \in [1, 7]$, lines e_i and f_j meet at a vertex in one of the last seven sides V_2, \ldots, V_8 .

Define $V'_0 = V_0 \setminus ((\cup_i e_i) \cup (\cup_i f_i))$ and $V'_1 = V_1 \setminus ((\cup_i e_i) \cup (\cup_i f_i))$. Since $|V_0| \ge 9$, there exists $y \in V'_0$. Any line ℓ through y intersects each of f_1, \ldots, f_7 in the last seven sides and thus also intersects each of e_1, \ldots, e_7 in the last seven sides. Hence ℓ contains a vertex in V'_1 . By symmetry, all lines through V'_1 contain a vertex in V'_0 . Let G be the bipartite graph induced on $V'_0 \cup V'_1$ by H. Then G inherits from H the properties of having minimum degree at least 2 and no line between vertices of degree 2. Therefore G has at least 5 vertices and at least 6 lines. It follows that $\varepsilon \ge 3$ and hence $h \le 46$, from (2.7).

We have already encountered 20 distinct lines of H, namely $e_1, \ldots, e_7, f_1, \ldots, f_7$ and at least 6 lines through V'_0 . Let B be the set of vertices in the last seven sides which do not lie on any of these 20 lines. Then $|B| \ge 14$, since B includes at least two vertices from each of V_2, \ldots, V_8 . Let $x \in B$, and let g be a line through x. Since g intersects each line e_1, \ldots, e_7 , it follows that $g \cap B = \{x\}$. However x has degree at least 2. Therefore $h \ge 20 + 2|B| \ge 48$, which is a contradiction.

Remark. Although the proof of Lemma 2.7 does not completely generalise to larger r, it does demonstrate that if H is a linear intersecting r-partite hypergraph with $\Delta = r - 2$, $\tau = r$ and $\epsilon \leq 2$, then each pair of degree Δ vertices on different sides lie on a common line.

2.2.1 Degree sequences, line types, and side types

Recall that H is assumed to be a linear intersecting 9-partite hypergraph with $\tau(H) = r$ and $\Delta = 7$. For each h and ε within the bounds given by (2.7), we computed the set $\mathcal{D}(h, \varepsilon)$ of all possible degree sequences that satisfy the equations (2.2), (2.3) and (2.4). Note that since H is linear, equality is enforced in (2.4). From here on, we denote a degree sequence in $\mathcal{D}(h, \varepsilon)$ by $[d_2, d_3, \ldots, d_7]$, where d_i is the number of vertices of degree i, for $i \in [2, 7]$. For each of the possible values of h and ε , we obtained the following number of degree sequences in the set $\mathcal{D}(h, \varepsilon)$. An example souce code for this and other computational results in this section is posted on the arXiv as an ancillary file with the priprint of this article.

$\varepsilon \backslash h$	39	40	41	42	43	44	45	46	47	48	49	50	51
0	223	297	307	358	323	311	236	181	107	50	16	2	0
1	86	129	135	164	144	145	102	82	42	24	6	0	-
2	22	39	42	58	48	48	27	20	6	3	-	-	-
3	1	6	6	11	8	9	3	1	-	-	-	-	-
4	-	-	_	1	_	_	_	-	_	-	-	_	-

Similarly, for given h and ε , we determined the set $\mathcal{S}(h, \varepsilon)$ of all possible degree sequences of vertices on a side of H, and the set $\mathcal{L}(h)$ of all possible degree sequence of vertices on a line of H. We arbitrarily order sets $\mathcal{S}(h, \varepsilon)$ and $\mathcal{L}(h)$ to easily index their elements. Then for $t \in \{1, 2, \ldots, |\mathcal{S}(h, \varepsilon)|\}$, a side of type t in $\mathcal{S}(h, \varepsilon)$ is a sequence of non-negative integers $[s_2^t, s_3^t, \ldots, s_7^t, s_{\varepsilon}^t]$, in which s_i^t denotes the number of vertices of degree $i \in [2, 7]$, such that the following conditions hold:

(i)
$$\sum_{i=2}^{7} s_i^t = r + s_{\varepsilon}^t$$
 which is the length of a side;
(ii) $\sum_{i=2}^{7} i s_i^t = h$ since every line meets every side; and
(iii) $s_{\varepsilon}^t \leq \varepsilon$.

A line of type t in $\mathcal{L}(h)$, where $t \in \{1, 2, ..., |\mathcal{L}(h)|\}$, is a sequence of non-negative integers $[\ell_2^t, \ell_3^t, \ldots, \ell_7^t]$, in which ℓ_i^t denotes the number of vertices of degree $i \in [2, 7]$, such that the following conditions hold:

(i)
$$\sum_{i=2}^{7} \ell_i^t = r$$
 since every line meets every side;
(ii) $\sum_{i=2}^{7} i\ell_i^t = h + r - 1$ since *H* is linear intersecting; and
(iii) $\ell_2^t \in \{0, 1\}$ by Lemma 2.1(iii).

2.2.2 Pairwise conditions

Our next goal is to verify which degree sequences are feasible. We assume that h and ε are given, and that $D = [d_2, d_3, \ldots, d_7]$ is a degree sequence in $\mathcal{D}(h, \varepsilon)$. Suppose that there exists H, a linear intersecting hypergraph with $r = \tau = 9$, $\Delta = 7$ on $r^2 + \varepsilon$ vertices with h lines and the given degree sequence. For every $t \in \{1, 2, \ldots, |\mathcal{S}(h, \varepsilon)|\}$, let x_t denote the number of sides of type t that are contained in H. For every $t \in \{1, 2, \ldots, |\mathcal{L}(h)|\}$, let y_t denote the number of lines of type t that are contained in H. Then x_t and y_t are non-negative integers which satisfy some obvious necessary conditions listed by equations (2.8)–(2.11). Equation (2.12) is a double count of pairs of vertices of degree i and lines on which these vertices lie, for $i \in [2, 7]$. Since H is a linear intersecting hypergraph, no pair of vertices is contained on two lines. Hence, the number of pairs of vertices contained on a line or on a side cannot exceed the total number of pairs given by the degree sequence. This condition is given by equations (2.13)–(2.15), depending on whether we count pairs of vertices having the same degree, or pairs of vertices with different degree. Putting all of these conditions together, we formulate an integer program on variables x_t and y_t .

$$\sum_{t}^{t} x_{t} = r \qquad \text{there are } r \text{ sides} \qquad (2.8)$$

$$\sum_{t}^{t} x_{t} s_{\varepsilon}^{t} = \varepsilon \qquad \text{there are } \varepsilon \text{ extra vertices} \qquad (2.9)$$

$$\sum_{t}^{t} x_{t} s_{i}^{t} = d_{i} \qquad \text{there are } d_{i} \text{ vertices of degree } i \in [2,7] \qquad (2.10)$$

$$\sum_{t}^{t} y_{t} = h \qquad \text{there are } h \text{ lines} \qquad (2.11)$$

$$\sum_{t}^{t} y_{t} \ell_{i}^{t} = id_{i} \qquad i \in [2,7] \qquad (2.12)$$

$$\sum_{t}^{t} x_{t} \binom{s_{t}^{t}}{2} + \sum_{t} y_{t} \binom{\ell_{t}^{t}}{2} \leqslant \binom{d_{i}}{2} \qquad i \in [2,6] \qquad (2.13)$$

$$\sum_{t}^{t} x_{t} \binom{s_{t}^{t}}{2} + \sum_{t} y_{t} \binom{\ell_{t}^{t}}{2} = \binom{d_{7}}{2} \qquad \text{by Lemma } 2.7 \qquad (2.14)$$

$$\sum_{t}^{t} x_{t} s_{i}^{t} s_{j}^{t} + \sum_{t} y_{t} \ell_{i}^{t} \ell_{j}^{t} \leqslant d_{i} d_{j} \qquad \text{distinct } i, j \in [2,7] \qquad (2.15)$$

This integer program has a feasible solution only for the following twelve degree sequences $D = [d_2, d_3, \ldots, d_7]$. For all of these, $\varepsilon = 0$.

 $\begin{array}{ll} h=45 \quad D=[22,3,7,2,15,32], & h=46 \quad D=[23,2,1,7,13,35], \\ h=46 \quad D=[23,1,3,7,11,36], & h=46 \quad D=[23,0,6,4,12,36], \\ h=46 \quad D=[23,0,5,7,9,37], & h=46 \quad D=[22,2,5,5,10,37], \\ h=46 \quad D=[23,0,4,10,6,38], & h=46 \quad D=[22,1,7,5,8,38], \\ h=46 \quad D=[22,0,10,2,9,38], & h=47 \quad D=[23,0,3,5,10,40], \\ h=47 \quad D=[21,2,6,4,5,43]. \end{array}$

2.2.3 Assignment of lines to vertices

If a linear intersecting 9-partite hypergraph H with $\tau(H) = 9$ exists, then it has 81 vertices $(\varepsilon = 0), h \in \{45, 46, 47\}$, and one of the twelve degree sequence listed at the end of §2.2.2. Let D be one of these degree sequences. Next we generated the set $\mathbf{S}(D)$ of all possible non-negative integer solutions for the system of equations (2.8), (2.9), and (2.10). Let $S = [x_1, x_2, \ldots, x_{|\mathcal{S}(h,0)|}]$ denote a particular solution in $\mathbf{S}(D)$. We formulate another system of linear equations which takes D and S as input values.

Let V = V(H) be the vertex set of H which is partitioned into 9 sides of equal size. Then the set of degree sequences of vertices in the sides of H corresponds to a solution $S \in \mathbf{S}(D)$. We change the notation slightly, to let s_j^v denote the number of vertices of degree j in the side that contains vertex v.

As before, let y_t be the number of lines of type t from the set $\mathcal{L}(h)$ present in H. Define z_t^v to be the number of lines of type t incident with a vertex $v \in V$. If a line of type t has no vertices of degree deg(v) then $z_t^v = 0$. Otherwise, z_t^v is a non-negative integer. In addition to equations (2.11)–(2.15), the following equations hold for y_t and z_t^v , where $t \in \{1, 2, \ldots, |\mathcal{L}(h)|\}$ and $v \in V$.

$$\sum_{t} z_{t}^{v} = \deg(v) \qquad \text{for all } v \in V; \qquad (2.16)$$

$$\sum_{t} z_{t}^{v}(\ell_{t}^{t}-1) \leqslant d_{i} - s_{i}^{v} \qquad \text{for all } v \in V \text{ where } i = \deg(v); \qquad (2.17)$$

$$\sum_{t} z_{t}^{v}(\ell_{7}^{t}-1) = d_{7} - s_{7}^{v} \qquad \text{for all } v \in V \text{ such that } \deg(v) = 7; \qquad (2.18)$$

$$\sum_{t} z_{t}^{v}\ell_{j}^{t} \leqslant d_{j} - s_{j}^{v} \qquad \text{for all } v \in V \text{ and all } j \in [2,7], \ j \neq \deg(v); \qquad (2.19)$$

$$\sum_{v \in V_{k}} z_{t}^{v} = y_{t} \qquad \text{for all } t \in \{1, 2, \dots, |\mathcal{L}(h)|\} \text{ and all } k \in [1, r]; \qquad (2.20)$$

$$\sum_{v, \deg(v)=i} z_{t}^{v} = y_{t}\ell_{i}^{t} \qquad \text{for all } i \in [2,7] \text{ and } t \in \{1, 2, \dots, |\mathcal{L}(h)|\}. \qquad (2.21)$$

Note that since $S \in \mathbf{S}(D)$ is given, now values x_t in equations (2.11)-(2.15) are constants. The total number of lines incident with a vertex equals the degree of that vertex, which is given by (2.16). Two vertices are *neighbours* if there is a line containing both of them. Observe that, for each vertex v in a given side, the number of neighbours of v which have degree j is at most the total number of degree j vertices in the remaining sides, for $j \in [2,7]$. Equations (2.17)-(2.18) correspond to counting the number of neighbours of v which are of the same degree as v, whereas (2.19) counts the neighbours of v which have degree different from deg(v). The equality in (2.18) is implied by Lemma 2.7. Since every line contains exactly one vertex in each side, each side is incident with as many lines of a type t as there are lines of type t present in the hypergraph, which is given by (2.20). Finally, (2.21) is a double count of pairs of lines of type t and vertices of degree i incident with these lines.

We found a feasible solution for the integer program given by (2.11)-(2.21) only for two pairs of input values of a degree sequence D and $S \in \mathbf{S}(D)$ which we consider more closely in the following subsection.

2.2.4 Remaining cases

The two cases for which the integer program in the previous section has a feasible solution both have h = 46. Below we give the input degree sequence D and a matrix representation of S. Here, a column of S corresponds to a side, and each entry is the degree of a vertex in that side. Each matrix S is given uniquely, up to permutation of sides and permutation of vertices within each side.

Case	1									Case 2							
D =	[23,	1,3	8, 7,	11	, 36]		D = [23, 0, 5, 7, 9, 37]	D = [23, 0, 5, 7, 9, 37]								
	$\begin{pmatrix} 2\\ 2\\ 4 \end{pmatrix}$	$2 \\ 2 \\ 4$	$2 \\ 2 \\ 4$	$2 \\ 2 \\ 2$	$ \begin{array}{c} 2 \\ 2 \\ 2 \end{array} $	$ \begin{array}{c} 2 \\ 2 \\ 3 \end{array} $	$ \begin{array}{c} 2 \\ 2 \\ 2 \end{array} $	$ \begin{array}{c} 2 \\ 2 \\ 2 \end{array} $	2 2 2 2	$\left(\begin{array}{rrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrrr$	2 2 2						
S =	5 6	5 6 6	5 6 6	$\frac{2}{6}$	$\frac{2}{6}$	5 6 7	2 5 7 7	2 5 7 7	5 7 7	$S = \begin{bmatrix} 1 & 5 & 2 & 2 & 1 & 1 & 2 & 2 \\ 5 & 5 & 6 & 6 & 4 & 4 & 5 & 5 \\ 6 & 5 & 6 & 6 & 6 & 6 & 7 & 7 \\ 6 & 6 & 7 & 7 & 7 & 7 & 7 & 7 \end{bmatrix}$	$\frac{2}{5}$ 7						
	0 7 7	0 7 7	6 7 7	7 7 7	7 7 7	7 7 7	7 7 7	7 7 7	7 7 7	0 0 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7	7 7 7						
	17	1	1	1	1	1	1	1	· ()	$\mathbf{\chi}_{1} 1 1 1 1 1 1 1$	- 'C)						

Case 1: Suppose there exists a linear intersecting 9-partite hypergraph H with $\tau(H) = 9$, h = 46 lines and degree sequence D = [23, 1, 3, 7, 11, 36]. First, we consider the set of types of

lines in H. Since H has 46 lines and $d_2 = 23$, every line in H contains a vertex of degree 2. Also, $\ell_i^t \leq d_i$ for all $i \in [3, 7]$. Hence, the possible line types \mathcal{L} in H are

$$\mathcal{L} = \{ [1, 0, 0, 0, 4, 4], [1, 0, 0, 1, 2, 5], [1, 0, 0, 2, 0, 6], [1, 0, 1, 0, 1, 6], [1, 1, 0, 0, 0, 7] \} \subset \mathcal{L}(46)$$

Note that H has one vertex v such that $\deg(v) = 3$. Let e_1 and e_2 be two lines which contain v. Then e_1 and e_2 have the same line type, namely [1, 1, 0, 0, 0, 7], since this is the only line type in \mathcal{L} which contains a vertex of degree 3. Let u_1 and u_2 be the vertices of degree 2 on e_1 and e_2 , respectively. If u_1 and u_2 belong to two different sides, then let w be the vertex of degree 7 on e_2 which is on the same side as u_1 . By Lemma 2.7, there is a distinct line through w and each of the degree 7 vertices on e_1 . Hence, $\deg(w) \ge 8$, which is a contradiction. Therefore, u_1 and u_2 are on the same side in H. Let f be the line other than e_1 which contains u_1 and let w' be the vertex in which e_2 and f intersect. Then $\deg(w') = 7$. Again by Lemma 2.7, there is a line through w' and every degree 7 vertex on e_1 which is not on the same side as w'. Hence, $\deg(w') \ge 2 + 6 = 8$, which is a contradiction. Therefore, such a hypergraph does not exist.

Case 2: Suppose there exists a linear intersecting 9-partite hypergraph H with $\tau(H) = 9$, h = 46 lines and degree sequence D = [23, 0, 5, 7, 9, 37]. Moreover, without loss of generality, assume that the degrees of vertices in V = V(H) are given by S for a suitable permutation of sides and permutation of vertices within each side. Let $\mathcal{L} \subseteq \mathcal{L}(46)$ be the set of all possible line types which may occur in H. As in the previous case, line types in \mathcal{L} satisfy the obvious constraints on degrees. Hence

$$\mathcal{L} = \{ [1, 0, 0, 0, 4, 4], [1, 0, 0, 1, 2, 5], [1, 0, 0, 2, 0, 6], [1, 0, 1, 0, 1, 6] \}.$$

For clarity, we index the line types in \mathcal{L} by A, B, C and D, respectively. Then for $t \in \{A, B, C, D\}$, y_t is the number of lines of type t in H and (y_A, y_B, y_C, y_D) satisfies the equations (2.11)–(2.15). In this case, many of these equations are dependent and it is enough to consider equations (2.11), (2.12) for $i \in \{4, 5\}$, and (2.14) to obtain the unique solution $(y_A, y_B, y_C, y_D) = (1, 15, 10, 20).$

Moreover, for each $t \in \{A, B, C, D\}$ and each $v \in V$, z_t^v denotes the number of lines of type t incident with the vertex v and the set of all values $\{z_t^v : v \in V, t \in \{A, B, C, D\}\}$ satisfies equations (2.16)–(2.21). We restrict our attention to vertices of degree 7 and compute all possible solutions to the system of equations given only by (2.16), (2.18) and (2.19) for $j \in [2, 6]$. For each $0 \leq k \leq 8$, the possible solutions $(z_A^v, z_B^v, z_C^v, z_D^v)$ for $v \in V_k$, where $\deg(v) = 7$, are given in the table below.

V_0	V_1	V_2, V_3, V_4, V_5	V_{6}, V_{7}, V_{8}
(0, 1, 2, 4)	(0, 1, 1, 5)	(1, 0, 3, 3)	(1, 1, 2, 3)
		(0, 2, 2, 3)	(0, 3, 1, 3)

Let e be a line of type C. Line e intersects each of the 20 lines of type D exactly once. Moreover, line e intersects a line of type D either in its vertex of degree 2 or in one of its 6 vertices of degree 7. Observe that vertices of degree 7 in sides V_0 and V_1 are incident with 4 and 5 lines of type D, respectively; all other vertices of degree 7 are always incident with exactly 3 lines of type D. Depending on whether e intersects a line of type D in its vertex of degree 2 or not, it is easy to see that, in order for e to meet 20 lines of type D, e either contains exactly one vertex of degree 7 on side V_0 or exactly one vertex of degree 7 on side V_1 , but not both. Since $z_C^v = 2$ if $v \in V_0$ and $z_C^v = 1$ if $v \in V_1$ when $\deg(v) = 7$, there are at most $2 \cdot 3 + 1 \cdot 3 = 9$ lines of type C, which gives a contradiction. Therefore, such a hypergraph does not exist.

We conclude that there does not exist a linear intersecting 9-partite hypergraph with covering number 9 and $\Delta = 7$. Together with Corollary 2.5 we have shown:

Theorem 2.8. For $2 \leq r \leq 9$ every linear intersecting r-partite hypergraph has covering number at most r - 1.

3 Hypergraph constructions

In this section we describe several hypergraph constructions that are of interest. First we define some additional notation. Let H be an r-partite hypergraph. In each side V_k , we label the vertices by $(k, 0), (k, 1), \ldots, (k, |V_k| - 1)$, which we abbreviate to $0, 1, \ldots, |V_k| - 1$ when the side is clear from context. We say that a vertex $(k, l) \in V_k$ is at *level* l in side V_k . We denote a line ein H by $[l_0, l_1, l_2, \ldots, l_{r-1}]$, where e contains the vertex at level l_k in side V_k , where $k \in [0, r-1]$. When presenting a specific example, we omit the square brackets and commas to make the notation cleaner. The *cyclic* 1-*shift* of a line $[l_0, l_1, l_2, \ldots, l_{r-1}]$ is the line $[l_{r-1}, l_0, l_1, \ldots, l_{r-2}]$ and the *cyclic* t-*shift* of a line e, denoted e^t , is the line obtained from e by applying t cyclic 1-shifts. In particular, $e^0 = e = e^r$. An r-partite hypergraph is *cyclic* if its automorphism group contains a cyclic subgroup of order r acting transitively on the sides. A cyclic r-partite hypergraph can be obtained by developing a set of *starter lines* by cyclic shifts.

3.1 A counterexample to Conjecture 1.2

In this subsection we give a counterexample to Conjecture 1.2.

Theorem 3.1. For at least one value of r there is an intersecting r-partite hypergraph H such that each of its sides has size r and $e \setminus \{v\}$ is not a cover for any $e \in H$ and any $v \in e$.

Proof. Let r = 13. We give an example of a cyclic linear intersecting *r*-partite hypergraph H in which every side has size r. The 3r lines of H are obtained by taking all possible cyclic shifts of the following three starter lines.

Vertices at level 0 in H have degree 3, vertices at levels 1, 2 and 3 have degree 6, and all other vertices have degree 2.

We claim that H is a linear intersecting hypergraph. Observe that $|e_i^{t_1} \cap e_j^{t_2}| = |e_i \cap e_j^{t_2-t_1}|$ for any $i, j \in \{1, 2, 3\}$ and $0 \leq t_1 \leq t_2 < r$. Hence, it suffices to show that for any $i, j \in \{1, 2, 3\}$ and $0 \leq t < r$ where $(j, t) \neq (i, 0)$, lines e_i and e_j^t intersect in exactly one vertex.

First we consider the case when i = j. By construction, e_i^3 has the underlying structure of a 4-extended Skolem sequence of order 6, namely 6420246531135. For example, this sequence is obtained from e_1^3 by relabelling the levels using the permutation (1, 2, 6)(3, 5). For definitions and background on Skolem sequences, see [5]. In our case, the result is that for every $t \in [1, \frac{r-1}{2}]$, there is a unique pair (k, l) and (k', l) of vertices in e_i with $k' - k \equiv t \pmod{r}$. Hence $e_i \cap e_i^t = \{(k', l)\}$ and $e_i \cap e_i^{r-t} = \{(k, l)\}$.

Now assume that $i \neq j$. Suppose that there are two distinct vertices (k_1, l_1) and (k_2, l_2) in $e_i \cap e_j^t$. Since $e_i \cap e_j = \{(0, 0)\}$, we may assume that $t \in [1, r - 1]$. Then by inspection, we must have $l_1, l_2 \in \{1, 2, 3\}$ (the relevant entries are shown in **bold** in (3.1)). For the different possible pairs (l_1, l_2) the following table shows the feasible distances $k_2 - k_1 \pmod{r}$.

l_1, l_2	distances in e_1	distances in e_2	distances in e_3
1, 1	± 2	± 6	± 5
2, 2	± 6	± 5	± 2
3, 3	± 5	± 2	± 6
1, 2	$\pm 2, \pm 4$	$\pm 1, \pm 6$	$\pm 3, \pm 5$
1, 3	$\pm 3, \pm 5$	$\pm 2, \pm 4$	$\pm 1, \pm 6$
2,3	$\pm 1, \pm 6$	$\pm 3, \pm 5$	$\pm 2, \pm 4$

Since there is no row of the table where the same distance occurs in different columns, we conclude that

$$|e_i \cap e_j^t| \leqslant 1. \tag{3.2}$$

Since e_i and e_j intersect in the vertex (0,0), and both have a pair of vertices on each level $l \in \{1,2,3\}$, there are $1+3 \cdot 4 = 13$ vertices in which e_i intersects the set of all cyclic shifts of e_j . It follows that we must have equality in (3.2) for each i, j, t. Thus H is a linear intersecting hypergraph. Since $\delta(H) = 2$, it follows that for every line $e \in H$ and each vertex $v \in e$, the set of vertices $e \setminus \{v\}$ is not a cover.

For the linear intersecting 13-partite hypergraph constructed in the proof of Theorem 3.1 we found by computation that $\tau(H) = 9$. The following set of vertices is a 9 cover for H:

 $\{(0,1), (0,2), (0,3), (2,0), (5,0), (8,0), (10,1), (10,2), (10,3)\}.$

3.2 Cyclic intersecting hypergraphs with $\tau = r - 1$

Next we give an example of a cyclic intersecting r-partite hypergraph which has covering number r-1 for $r \in \{9, 13, 17\}$. The methodology for building these hypergraphs is similar to the construction given in the proof of Theorem 3.1. However, in each case below the hypergraph we construct is non-linear.

Lemma 3.2. Let $(r,s) \in \{(9,4), (13,5), (17,6)\}$. Then there exists a cyclic intersecting *r*-partite hypergraph H such that H has sr lines and $\tau(H) = r - 1$.

Proof. The sr lines of H are obtained by taking all possible cyclic shifts of lines in the starters given below.

r =	9							<i>r</i> =	= 13												
$e_1 =$	= 4 3	3 2	1	0 1	2	3 4	1	e_1	= (5 5	5 4	3	2	1	0	1	2	3	4	5	6
$e_2 =$	= 3 (5 5	4	0 4	5	6 3	3	e_2	= 9) (51	8	$\overline{7}$	5	0	5	7	8	1	6	9
$e_3 =$	= 1 2	2 4	6	0 6	4	2 1	L	e_3	= ′	7 2	2 9	1	3	10	0	10	3	1	9	2	7
$e_4 =$	= 3 () 2	6	7 6	2	0 3	3	e_4	= ;	5 3	3 2	6	9	7	0	7	9	6	2	3	5
								e_5	= 9) ;	5 2	6	3	7 1	11	7	3	6	2	5	9
	r = 1	17	10		4	- 1	10	0	0	0			10	1	4	11	1	0	0	=	
	$e_1 =$	3	12	11	4	1	10	2	9	0	9	2	10	1	4	11	1	.2	3		
	$e_2 =$	1	8	7	6	5	4	3	2	0	2	3	4	5	6	7	8	8	1		
	$e_3 =$	14	3	9	7	15	11	6	4	0	4	6	11	15	7	9	•	3	14		
	$e_4 =$	6	2	5	11	14	12	4	10	0	10	4	12	14	11	5	4	2	6		
	$e_5 =$	11	10	14	1	7	3	12	8	0	8	12	3	7	1	14	1	0	11		
	$e_6 =$	7	4	13	5	9	1	11	12	0	12	11	1	9	5	13	4	4	7		

We found by computation that the covering numbers for these three hypergraphs are 8, 12 and 16, respectively. $\hfill \Box$

Remark. The hypergraphs with r = 9, r = 13 and r = 17 in Lemma 3.2 are intrinsically non-linear; each has the property that it does not contain a linear subhypergraph with covering number r - 1.

First, if H is the hypergraph with r = 9, then $|e_2 \cap e_4| = 2$, but deleting either e_2 or e_4 reduces the covering number. Below are covers of size 7 for $H \setminus \{e_2\}$ and $H \setminus \{e_4\}$, respectively.

 $\{(5,6), (6,0), (6,1), (6,2), (6,3), (6,4), (6,6)\}$ $\{(4,0), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$ Similarly, if H is the hypergraph with r = 13, then $|e_1 \cap e_5| = 2$ and below are covers of size 11 for $H \setminus \{e_1\}$ and $H \setminus \{e_5\}$, respectively.

 $\{(0,7), (1,3), (2,3), (3,7), (8,0), (8,1), (8,3), (8,5), (8,6), (8,7), (12,2)\}$ $\{(1,9), (4,6), (6,0), (6,1), (6,2), (6,3), (6,6), (6,7), (6,9), (8,6), (11,9)\}$

Finally, consider the hypergraph H in Lemma 3.2 with r = 17. Let H_1 be the subhypergraph obtained from all of the cyclic shifts of starter lines e_2, e_3, e_4, e_5, e_6 . Then H_1 is a linear intersecting hypergraph with covering number 15, where the following set of vertices is a minimal cover:

 $\{(4,5), (4,8), (4,11), (5,4), (5,7), (10,0), (10,4), (10,6), (10,7), (10,14), (15,4), (15,7), (16,5), (16,8), (16,11)\}.$

Let E_1 be the set of all cyclic shifts of the starter line e_1 . Observe that E_1 is linear intersecting. Furthermore, each line $e \in E_1$ intersects exactly 12 lines of H_1 more than once. For every non-empty subset $E^* \subseteq E_1$, define H_1^* to be the hypergraph obtained from H_1 by adding the lines of E^* and removing every line of H_1 that intersects a line of E^* more than once. We computationally checked that each of the $2^{17} - 1$ such linear intersecting hypergraphs H_1^* has covering number less than 16 (in fact, they each have covering number between 10 and 14). Therefore, H has no linear intersecting subhypergraph with $\tau = 16$.

3.3 Linear intersecting hypergraphs built from latin squares

Next we describe a family of linear intersecting r-partite hypergraphs which do not have a minimal cover that consists of a single side or that consists of a subset of a line.

A latin square of order n is an $n \times n$ array of n symbols such that each symbol appears exactly once in each row and exactly once in each column. If L is a latin square, L[r, c] denotes the symbol in row r and column c of L. A pair of latin squares of order n are orthogonal if, when the two squares are superimposed, each of the n^2 possible ordered pairs of symbols occurs exactly once. A set of latin squares is called mutually orthogonal if each pair of latin squares in the set are orthogonal. For more on latin squares, see [12].

Lemma 3.3. If there exist k mutually orthogonal latin squares of order $n \ge 3$, then there exists a linear intersecting (n+2)-partite hypergraph H with $\tau(H) = k+1$ such that no side or subset of a line is a minimal cover.

Proof. Let L_0, \ldots, L_{k-1} be k mutually orthogonal latin squares of order n. Assume that each L_i has $[0, 1, \ldots, n-1]$ as its first row. For $i = 0, 1, \ldots, k-1$, let M_i be the column inverse of L_i , that is, $M_i[r, c] = s$ if and only if $L_i[s, c] = r$. Since each L_i has its first row in reduced form, each M_i has symbol 0 on the main diagonal. Set $M'_0 = M_0$ and for $i = 1, \ldots, k-1$ let M'_i be the latin square obtained from M_i by replacing symbol 0 with symbol n + i - 1.

Next we define the lines of an (n + 2)-partite hypergraph H. Let V_0, \ldots, V_{n+1} be the sides of H. Sides V_0, \ldots, V_n each have n + k - 1 vertices and side V_{n+1} has n + 1 vertices. If a line $[l_0, l_1, \ldots, l_{r-1}]$ is a concatenation of two lists, we use the notation $[l_0, \ldots, l_i] \oplus [l_{i+1}, \ldots, l_{r-1}]$.

For i = 0, ..., k - 1 define $E_i := \{M'_i[x] \oplus [i, x] : 0 \le x \le n - 1\}$, where $M'_i[x]$ is row x of the latin square M'_i . Further, define $E^* := \{[x, x, ..., x, k + x - 1, n] : 1 \le x \le n - 1\}$.

It is straightforward to check that each line in E_i intersects each line in E^* exactly once, and that the lines in E^* meet each other exactly once. Since M'_i is a latin square, two distinct lines in E_i intersect only at the vertex (n, i). We next show that lines of E_i and E_j also intersect linearly for $i \neq j$. Consider a line $\ell_1 = M'_i[x] \oplus [i, x] \in E_i$ and a line $\ell_2 = M'_j[y] \oplus [j, y] \in E_j$ where $i \neq j$. Since L_i and L_j are orthogonal latin squares, there is a unique cell (r, c) where $L_i[r,c] = x$ and $L_j[r,c] = y$. If x = y, then r = 0 and $M_i[x,x] = M_j[x,x] = 0$, and thus, by the relabelling of symbol 0 in M'_1, \ldots, M'_{k-1} , the lines ℓ_1 and ℓ_2 intersect only at the vertex (n+1,x). If $x \neq y$, then $r \neq 0$ and thus $M'_i[x,c] = r = M'_j[y,c]$ and therefore the lines ℓ_1 and ℓ_2 intersect only at the vertex (c,r). It follows that $H = E^* \cup E_0 \cup E_1 \cup \cdots \cup E_{k-1}$ is a linear intersecting (n+2)-partite hypergraph with kn + n - 1 lines.

Since *H* has maximum degree *n* and kn + n - 1 lines, no set of *k* vertices is a cover. It is straightforward to check that the vertices at levels $0, 1, \ldots, k - 1$ of side V_n together with the vertex (n + 1, n) form a cover of size k + 1. Therefore $\tau(H) = k + 1$.

It is well-known that there are at most n-1 mutually orthogonal latin squares of order n (see e.g. [12]), so $k+1 \leq n$. Observe that for $k \geq 2$ each side has size at least n+1, whereas $\tau(H) = 2 < n$ when k = 1. Hence, no side is a minimal cover. Also, since H is linear and each line contains exactly one vertex of degree 1, it follows that the only covers which are subsets of a line have size at least n+1. Thus, no side or subset of a line is a minimal cover of H.

The following corollary follows immediately from the existence of complete sets of mutually orthogonal latin squares of prime power orders [12].

Corollary 3.4. If n is a prime power then there exists a linear intersecting (n + 2)-partite hypergraph H with $\tau(H) = n$ such that no side or subset of a line is a minimal cover.

Example: Below are the lines of a hypergraph built from 3 mutually orthogonal latin squares of order 4, as described in Lemma 3.3. This hypergraph is also maximal with respect to the property of being linear and intersecting.

031200	423110	512320	111134
302101	241311	153221	222244
120302	314212	235122	333354
213003	132413	321523	

3.4 An 8-partite linear hypergraph with $\tau = 7$

We close this section by giving a construction for an interesting 8-partite linear hypergraph \mathcal{H}_{38} that achieves equality in Conjecture 1.1.

Let F denote the Fano plane constructed by developing the triple $\{0, 1, 3\}$ modulo 7. Let G denote the stabiliser of the point 0 in F. Note that |G| = 24. Let C denote the set of 7-cycles obtained by conjugating the cycle (0123456) by elements of G.

For each permutation $p \in C$ we add one line to \mathcal{H}_{38} which includes the vertex (i, p[i])for $i \in [0, 6]$. If $p_1, p_2 \in C$ then $p_1^{-1}p_2$ has at most one fixed point. We can make the lines corresponding to p_1 and p_2 meet on side V_7 if and only if $p_1^{-1}p_2$ has no fixed points. This produces 8 vertices of degree 3 in V_7 , and completes the description of the lines corresponding to the cycles in C. Next we add two new vertices v_1, v_2 to V_7 . For each $i \in [0, 6]$ we put a line through v_1 and the vertices (j, i) for $j \in [0, 6]$. For each $i \in [0, 6]$ we put a line through v_2 , (i, i)and all vertices (a, b) for which $\{i, a, b\}$ is a triple of F.

The construction just described results in \mathcal{H}_{38} , which has 38 lines and an automorphism group isomorphic to PSL(2,7). For $i \in [0,6]$ the vertex (i,i) has degree 2. All other vertices on sides V_0, \ldots, V_6 have degree 6. On V_7 the vertices v_1 and v_2 have degree 7 and the other 8 vertices have degree 3. Since V_7 has 10 vertices it is clear that \mathcal{H}_{38} is not isomorphic to a subhypergraph of \mathcal{P}'_8 . Nevertheless, it is routine to check that \mathcal{H}_{38} is a linear intersecting 8-partite hypergraph. Suppose that X is a 6-cover of \mathcal{H}_{38} . Then X must include v_1 and v_2 since otherwise it cannot cover the lines through those vertices. The 24 lines that avoid v_1 and v_2 induce a subhypergraph with maximum degree 4, which thus cannot be covered by fewer than 6 vertices. This contradiction shows that X does not exist. Since V_0 is a 7-cover, we must have $\tau(\mathcal{H}_{38}) = 7$.

Some further properties of \mathcal{H}_{38} are discussed in the next section.

4 Computational results

In this section we describe a computational proof of the following result.

Theorem 4.1.

- 1. For $r \leq 7$ the only linear intersecting r-partite hypergraphs to achieve equality in Ryser's conjecture are subhypergraphs of \mathcal{P}'_r . In particular, there are none for r = 7.
- 2. No subhypergraph of \mathcal{H}_{38} has $\tau = 7$ and is isomorphic to a subhypergraph of \mathcal{P}'_8 .
- 3. The smallest subhypergraph of \mathcal{H}_{38} with $\tau = 7$ has 22 lines. The smallest subhypergraph of P'_8 with $\tau = 7$ also has 22 lines.

Clearly, by part (1), every 7-partite linear intersecting hypergraph satisfies $\tau \leq 5$. There are a number of non-isomorphic ways to achieve $\tau = 5$, including by the construction in Lemma 3.3. For 7-partite intersecting non-linear hypergraphs with $\tau = 6$, see [1, 3].

Let H be a 7-partite linear intersecting hypergraph with h = |H| lines and $\tau \ge 6$. By an argument similar to the proof of Lemma 2.2 we know that $\Delta(H) \le 6$. We next argue that $\Delta(H) \ge 4$. Let H have x_i vertices of degree $i = 1, ..., \Delta$. Note that no line of H can include two vertices of degree 1, otherwise the remaining vertices on the line would provide a 5-cover. Hence $x_1 \le h$. Together with [3, Lem 2.1] and [3, Thm 2.7], we can then deduce that if $\Delta \le 4$ and $x_4 \le 7$ then $x_1 = h = 17, x_3 \ge 38$ and hence $x_2 < 0$. It follows that $\Delta \ge 4$. Also if $\Delta = 4$ then $x_4 > 7$ so some side of H has at least 2 vertices of degree 4 on it.

We next describe the computation that established Part 1 of Theorem 4.1. By the above comments, we can split the problem for r = 7 into three subcases $\Delta = 4$, $\Delta = 5$ and $\Delta = 6$. We started with a vertex of degree Δ on side V_0 . (In the $\Delta = 4$ case, we then added a second vertex of degree Δ to V_0 in all possible ways up to isomorphism.) Subsequent lines were added one at a time, ensuring that all pairs of lines intersected in a single point and that the assumed maximum degree was not violated. After each line was added, we tested for isomorphism and kept only one representative of each isomorphism class. For isomorphism checking we converted the hypergraphs into vertex-coloured graphs and applied the software *nauty* [8]. For $\Delta = 4, 5, 6$ the largest hypergraphs we obtained had 16, 25, 18 lines respectively. All hypergraphs that we built had a 5-cover, proving the claim that no linear 7-partite intersecting hypergraph achieves $\tau = 6$.

For $r \leq 6$, we performed computations as just described, except that there was no need to split the problem into subcases according to the maximum degree. Every hypergraph that we encountered could be extended to \mathcal{P}'_r .

For $r \ge 8$ the above method is not practical for a complete enumeration. However, we did a partial enumeration and found a number of linear intersecting 8-partite hypergraphs that are maximal (no lines can be added), have $\tau = 7$ and yet are not isomorphic to \mathcal{P}'_8 . Most of these have the property that a few lines can be removed to get something isomorphic to a subhypergraph of \mathcal{P}'_8 . However, the hypergraph \mathcal{H}_{38} described in §3.4 seems to be of a very different nature, which is why we tested its properties more thoroughly.

In Table 1 and Table 2 all 2098796663 isomorphism classes of subhypergraphs of \mathcal{P}'_8 with $\tau = 7$, and all 17892655 isomorphism classes of subhypergraphs of \mathcal{H}_{38} with $\tau = 7$ are classified

H	Number	H	Number	H	Number	H	Number
22	833	29	268297692	36	2660309	43	179
23	2168877	30	183765292	37	936491	44	32
24	58227758	31	114391098	38	296473	45	8
25	224055209	32	64949914	39	84035	46	3
26	368614512	33	33653522	40	21221	47	1
27	401984117	34	15894680	41	4757	48	1
28	351960321	35	6828374	42	953	49	1

Table 1: Number of isomorphism classes of subhypergraphs H of \mathcal{P}'_8 , with $\tau(H) = 7$.

H	Number	H	Number	H	Number	H	Number
22	5	27	3376797	32	14308	37	3
23	42310	28	1625274	33	2803	38	1
24	1550265	29	644482	34	462		
25	5027821	30	215066	35	67		
26	5332373	31	60609	36	9		

Table 2: Number of isomorphism classes of subhypergraphs H of \mathcal{H}_{38} , with $\tau(H) = 7$.

by their size. The tables were prepared by exhaustive enumeration, using a heuristic upper bound for the covering number to quickly eliminate most subhypergraphs with $\tau(H) \leq 6$, and employing *nauty* to remove isomorphs.

We end with an example of a subhypergraph of \mathcal{P}'_8 that has 22 lines and $\tau = 7$:

03426434	04505645	06663521	11264344	15636215	16055456
22642443	24366152	25550564	32133654	34624066	35345331
43331546	44453313	46246660	51313465	55462606	56534133
60444555	61651632	62516326	63165263		

and an example of a subhypergraph of \mathcal{H}_{38} that has 22 lines and $\tau = 7$:

00000008	03615429	10536249	20361454	22222228	24510635
25043169	26105341	31402659	33333338	36541026	42016356
43562101	4444448	46320519	54306123	56413204	60425136
62503412	63140253	64251309	65312047		

Both these hypergraphs have an automorphism group of order 3, which is the largest achieved by subhypergraphs with h = 22 and $\tau = 7$ within \mathcal{P}'_8 and \mathcal{H}_{38} , respectively.

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