

# A characterization of tightly triangulated 3-manifolds

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## Abstract

For a field  $\mathbb{F}$ , the notion of  $\mathbb{F}$ -tightness of simplicial complexes was introduced by Kühnel. Kühnel and Lutz conjectured that any  $\mathbb{F}$ -tight triangulation of a closed manifold is the most economic of all possible triangulations of the manifold. The boundary of a triangle is the only  $\mathbb{F}$ -tight triangulation of a closed 1-manifold. A triangulation of a closed 2-manifold is  $\mathbb{F}$ -tight if and only if it is  $\mathbb{F}$ -orientable and neighbourly. In this paper we prove that a triangulation of a closed 3-manifold is  $\mathbb{F}$ -tight if and only if it is  $\mathbb{F}$ -orientable, neighbourly and stacked. In consequence, the Kühnel-Lutz conjecture is valid in dimension  $\leq 3$ .

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## 1 Introduction

All simplicial complexes considered in this paper are finite and abstract. The vertex set of a simplicial complex  $X$  will be denoted by  $V(X)$ . For  $A \subseteq V(X)$ , the induced subcomplex  $X[A]$  of  $X$  on the vertex set  $A$  is defined by  $X[A] := \{\alpha \in X : \alpha \subseteq A\}$ . For  $x \in V(X)$ , the subcomplexes  $\{\alpha \in X : x \notin \alpha\} = X[V(X) \setminus \{x\}]$  and  $\{\alpha \in X : x \notin \alpha, \alpha \sqcup \{x\} \in X\}$  are called the *antistar* and the *link* of  $x$  in  $X$ , respectively. A simplicial complex  $X$  is said to be a *triangulated (closed) manifold* if it triangulates a (closed) manifold, i.e., if the geometric carrier  $|X|$  of  $X$  is a (closed) topological manifold. A triangulated closed  $d$ -manifold  $X$  is said to be  $\mathbb{F}$ -orientable if  $H_d(X; \mathbb{F}) \neq 0$ . If two triangulated  $d$ -manifolds  $X$  and  $Y$  intersect precisely in a common  $d$ -face  $\alpha$  then  $X \# Y := (X \cup Y) \setminus \{\alpha\}$  triangulates the connected sum  $|X| \# |Y|$  and is called the *connected sum of  $X$  and  $Y$  along  $\alpha$* .

For our purpose, a *graph* may be defined as a simplicial complex of dimension  $\leq 1$ . For  $n \geq 3$ , the  *$n$ -cycle*  $C_n$  is the unique  $n$ -vertex connected graph in which each vertex lies on exactly two edges. For  $n \geq 1$ , the *complete graph*  $K_n$  is the  $n$ -vertex graph in which any two vertices form an edge. For  $m, n \geq 1$ , the *complete bipartite graph*  $K_{m,n}$  is the graph with  $m+n$  vertices and  $mn$  edges in which each of the first  $m$  vertices forms an edge with each of the last  $n$  vertices. Two graphs are said to be *homeomorphic* if their geometric carriers are homeomorphic. A graph is said to be *planar* if it is a subcomplex of a triangulation of the 2-sphere  $S^2$ . In this paper, we shall have an occasion to use the easy half of Kuratowski's

famous characterization of planar graphs [5]: A graph is planar if and only if it has no homeomorph of  $K_5$  or  $K_{3,3}$  as a subgraph.

If  $\mathbb{F}$  is a field and  $X$  is a simplicial complex then, following Kühnel [9], we say that  $X$  is  $\mathbb{F}$ -tight if (a)  $X$  is connected, and (b) the  $\mathbb{F}$ -linear map  $H_*(Y; \mathbb{F}) \rightarrow H_*(X; \mathbb{F})$ , induced by the inclusion map  $Y \hookrightarrow X$ , is injective for every induced subcomplex  $Y$  of  $X$ .

If  $X$  is a simplicial complex of dimension  $d$ , then its *face vector*  $(f_0, \dots, f_d)$  is defined by  $f_i = f_i(X) := \#\{\alpha \in X : \dim(\alpha) = i\}$ ,  $0 \leq i \leq d$ . A simplicial complex  $X$  is said to be *neighbourly* if any two of its vertices form an edge, i.e., if  $f_1(X) = \binom{f_0(X)}{2}$ .

A simplicial complex  $X$  is said to be *strongly minimal* if, for every triangulation  $Y$  of the geometric carrier  $|X|$  of  $X$ , we have  $f_i(X) \leq f_i(Y)$  for all  $i$ ,  $0 \leq i \leq \dim(X)$ . Our interest in the notion of  $\mathbb{F}$ -tightness mainly stems from the following famous conjecture [10].

**Conjecture 1.1** (Kühnel-Lutz). *For any field  $\mathbb{F}$ , every  $\mathbb{F}$ -tight triangulated closed manifold is strongly minimal.*

Following Walkup [16] and McMullen-Walkup [12], a triangulated ball  $B$  is said to be *stacked* if all the faces of  $B$  of codimension 2 are contained in the boundary  $\partial B$  of  $B$ . A triangulated sphere  $S$  is said to be *stacked* if there is a stacked ball  $B$  such that  $S = \partial B$ . This notion was extended to triangulated manifolds by Murai and Nevo [14]. Thus, a triangulated manifold  $\Delta$  with boundary is said to be *stacked* if all its faces of codimension 2 are contained in the boundary  $\partial\Delta$  of  $\Delta$ . A triangulated closed manifold  $M$  is said to be *stacked* if there is a stacked triangulated manifold  $\Delta$  such that  $M = \partial\Delta$ . A triangulated manifold is said to be *locally stacked* if all its vertex links are stacked spheres or stacked balls. The main result of this paper is the following characterization of  $\mathbb{F}$ -tight triangulated closed 3-manifolds, for all fields  $\mathbb{F}$ .

**Theorem 1.2.** *A triangulated closed 3-manifold  $M$  is  $\mathbb{F}$ -tight if and only if  $M$  is  $\mathbb{F}$ -orientable, neighbourly and stacked.*

The special case of Theorem 1.2, where  $\text{char}(\mathbb{F}) \neq 2$ , was proved in our previous paper [4]. In this paper we conjectured [4, Conjecture 1.12] the validity of Theorem 1.2 in general.

As a consequence of Theorem 1.2, we show that the Kühnel-Lutz conjecture (Conjecture 1.1) is valid up to dimension 3. Thus,

**Corollary 1.3.** *If  $M$  is an  $\mathbb{F}$ -tight triangulated closed manifold of dimension  $\leq 3$ , then  $M$  is strongly minimal.*

As a second consequence of Theorem 1.2, we show:

**Corollary 1.4.** *The only closed topological 3-manifolds which may possibly have  $\mathbb{F}$ -tight triangulations are  $S^3$ ,  $(S^2 \times S^1)^{\#k}$  and  $(S^2 \times S^1)^{\#k}$ , where  $k$  is a positive integer such that  $80k + 1$  is a perfect square.*

Kühnel conjectured that any triangulated closed 3-manifold  $M$  satisfies  $(f_0(M) - 4) \times (f_0(M) - 5) \geq 20\beta_1(M; \mathbb{F})$ . (This is a part of his Pascal-like triangle of conjectures reported in [11].) This bound was proved by Novic and Swartz in [15]. Burton et al proved in [6] that if the equality holds in this inequality then  $M$  is neighbourly and locally stacked. (Actually, these authors stated this result for  $\mathbb{F} = \mathbb{Z}_2$ , but their argument goes through for all fields  $\mathbb{F}$ .) In [1], the first author proved that the equality holds in this inequality if and only if  $M$  is neighbourly and stacked. In [13], Murai generalized this to all dimensions  $\geq 3$ . Another consequence of Theorem 1.2 is:

**Corollary 1.5.** *A triangulated closed 3-manifold  $M$  is  $\mathbb{F}$ -tight if and only if  $M$  is  $\mathbb{F}$ -orientable and  $(f_0(M) - 4)(f_0(M) - 5) = 20\beta_1(M; \mathbb{F})$ .*

In [7],  $\mathbb{Z}_2$ -tight triangulations of  $(S^2 \times S^1)^{\#k}$  were constructed for  $k = 1, 30, 99, 208, 357$  and 546. However, we do not know any  $\mathbb{F}$ -tight triangulations of  $(S^2 \times S^1)^{\#k}$ .

**Question 1.6.** *Is there any positive integer  $k$  for which  $(S^2 \times S^1)^{\#k}$  has an  $\mathbb{F}$ -tight triangulation?*

## 2 Proofs

The following result is Theorem 3.5 of [4].

**Theorem 2.1.** *Let  $C$  be an induced cycle in the link  $S$  of a vertex  $x$  in an  $\mathbb{F}$ -tight simplicial complex  $X$ . Then the induced subcomplex of  $X$  on the vertex set of the cone  $x * C$  is a neighbourly triangulated closed 2-manifold.*

If, in Theorem 2.1,  $C$  is an  $n$ -cycle then the triangulated 2-manifold guaranteed by this theorem has  $n + 1$  vertices,  $n(n + 1)/2$  edges and hence  $n(n + 1)/3$  triangles. Thus 3 divides  $n(n + 1)$ , i.e.,  $n \not\equiv 1 \pmod{3}$ . Therefore, Theorem 2.1 has the following immediate consequence.

**Corollary 2.2.** *Let  $X$  be an  $\mathbb{F}$ -tight simplicial complex. Let  $S$  be the link of a vertex in  $X$ . Then  $S$  has no induced  $n$ -cycle for  $n \equiv 1 \pmod{3}$ .*

We recall that the Möbius band has a unique 5-vertex triangulation  $\mathcal{M}$ . The boundary of  $\mathcal{M}$  is a 5-cycle  $C_5$ . The simplicial complex  $\mathcal{M}$  may be uniquely recovered from  $C_5$  as follows. The triangles of  $\mathcal{M}$  are  $\{x\} \cup e_x$ , where, for each vertex  $x$  of  $C_5$ ,  $e_x$  is the edge of  $C_5$  opposite to  $x$ . We also note the following consequence of Theorem 2.1.

**Corollary 2.3.** *Let  $S$  be the link of a vertex  $x$  in an  $\mathbb{F}$ -tight simplicial complex  $X$ . Let  $C$  be an induced cycle in  $S$ .*

- (a) *If  $C$  is a 3-cycle, then it bounds a triangle of  $X$ .*
- (b) *If  $C$  is a 5-cycle then it bounds an induced subcomplex of  $X$  isomorphic to the 5-vertex Möbius band.*

*Proof.* If  $C$  is a 3-cycle, then the induced subcomplex of  $X$  on the vertex set of  $x * C$  is a neighbourly, 4-vertex, triangulated closed 2-manifold, which must be the boundary complex  $\mathcal{T}$  of the tetrahedron. But all four possible triangles occur in  $\mathcal{T}$ , and  $C$  bounds one of them. If  $C$  is a 5-cycle then the induced subcomplex  $X[V(x * C)]$  of  $X$  is a neighbourly, 6-vertex, triangulated closed 2-manifold, which must be the unique 6-vertex triangulation  $\mathbb{RP}_6^2$  of the real projective plane. Therefore, the induced subcomplex  $X[V(C)]$  of  $X$  is the antistar of the vertex  $x$  in  $\mathbb{RP}_6^2$ , which is the 5-vertex Möbius band.  $\square$

Let  $\mathcal{T}$  and  $\mathcal{I}$  denote the boundary complexes of the tetrahedron and the icosahedron, respectively. Thus the faces of  $\mathcal{T}$  are all the proper subsets of a set of four vertices. Up to isomorphism, the 20 triangles of  $\mathcal{I}$  are as follows:

$$\begin{aligned} &012, 015, 023, 034, 045, 124', 153', 13'4', 235', 24'5', 341', \\ &31'5', 452', 41'2', 52'3', 0'1'2', 0'1'5', 0'2'3', 0'3'4', 0'4'5'. \end{aligned} \tag{1}$$

The following is Corollary 5.5 of [4].

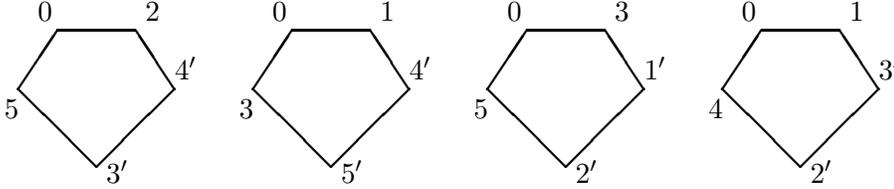
**Theorem 2.4.** *Let  $S$  be a triangulated 2-sphere which has no induced  $n$ -cycle for any  $n \equiv 1 \pmod{3}$ . Then  $S$  is a connected sum of finitely many copies of  $\mathcal{T}$  and  $\mathcal{I}$  (in some order).*

As an immediate consequence of Corollary 2.2 and Theorem 2.4, we have:

**Corollary 2.5.** *Let  $S$  be the link of a vertex in an  $\mathbb{F}$ -tight triangulated closed 3-manifold  $M$ . Then  $S$  is a connected sum of finitely many copies of  $\mathcal{T}$  and  $\mathcal{I}$  (in some order).*

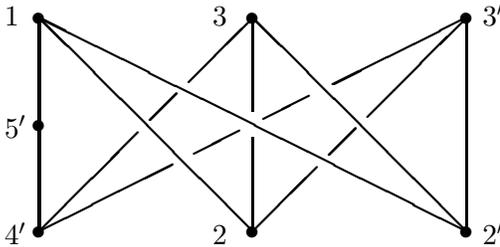
*Proof of Theorem 1.2.* Let  $M$  be an  $\mathbb{F}$ -orientable, neighbourly, stacked, triangulated closed 3-manifold. Then  $M$  is  $\mathbb{F}$ -tight by the case  $k = 1$  of Theorem 2.24 in [2]. This proves the “if part”. Conversely, let  $M$  be  $\mathbb{F}$ -tight. Since any  $\mathbb{F}$ -tight triangulated closed manifold is neighbourly and  $\mathbb{F}$ -orientable (Lemmas 2.2 and 2.5 in [4]), it follows that  $M$  is  $\mathbb{F}$ -orientable and neighbourly. To complete the proof of the “only if” part, it suffices to show that  $M$  must be stacked. By [2, Theorem 2.24], every locally stacked,  $\mathbb{F}$ -tight, triangulated closed 3-manifold is automatically stacked. So, it is enough to show that if  $S$  is the link of an arbitrary vertex of  $M$ , then  $S$  is a stacked 2-sphere. By Corollary 2.5,  $S = S_1 \# \cdots \# S_m$ , where each  $S_i$  is either  $\mathcal{T}$  or  $\mathcal{I}$ . Since any connected sum of copies of  $\mathcal{T}$  is stacked (as may be seen by an easy induction on the number of summands), it suffices to show that no  $S_i$  can be  $\mathcal{I}$ .

Suppose, on the contrary, that  $S_i = \mathcal{I}$  for some index  $i$ . We may take the triangles of  $\mathcal{I}$  to be as given in (1). Note that each triangle of  $S_i$  is either a triangle of  $S$  or its boundary is an induced 3-cycle of  $S$ . Since  $S \subseteq M$ , Corollary 2.3 (a) implies that each triangle of  $S_i$  is a triangle of  $M$ . Thus,  $\mathcal{I} \subseteq M$ . In particular, 012 and 023 are triangles of  $M$ . Also, we have the following induced 5-cycles (among others) in  $\mathcal{I}$ , and hence in  $S$ .



**Figure 1 : Some induced 5-cycles in  $\mathcal{I}$**

Hence Corollary 2.3 (b) gives us eight more triangles of  $M$  through the vertex 0, namely, 023', 03'4', 015', 034', 04'5', 032', 012', 02'3'. Thus, if  $S'$  is the link of the vertex 0 in  $M$ , then we have the graph of Fig. 2 as a subcomplex of  $S'$ .



**Figure 2 : A homeomorph of  $K_{3,3}$**

So we have a homeomorph of  $K_{3,3}$  as a subcomplex of the triangulated 2-sphere  $S'$ . This is a contradiction since  $K_{3,3}$  is not a planar graph.  $\square$

*Proof of Corollary 1.3.* Let  $M$  be an  $\mathbb{F}$ -tight triangulated closed  $d$ -manifold,  $d \leq 3$ . By Lemma 2.2 in [4],  $M$  is neighbourly. But the boundary complex of the triangle is the only

neighbourly triangulated closed 1-manifold. This is trivially the strongly minimal triangulation of  $S^1$ . So, we have the result for  $d = 1$ . Next let  $d = 2$ . Let  $N$  be another triangulation of  $|M|$ . Let  $(f_0, f_1, f_2)$  be the face vector of  $N$ . Let  $\chi$  be the Euler characteristic of  $M$  (hence also of  $N$ ). Then  $f_0 - f_1 + f_2 = \chi$  and  $2f_1 = 3f_2$ . Therefore we get  $f_1 = 3(f_0 - \chi)$  and  $f_2 = 2(f_0 - \chi)$ . Thus,  $f_1$  and  $f_2$  are strictly increasing functions of  $f_0$ . So, it is sufficient to show that  $f_0 \geq f_0(M)$ . Now, trivially,  $f_1 \leq \binom{f_0}{2}$ , with equality if and only if  $N$  is neighbourly. Substituting  $f_1 = 3(f_0 - \chi)$  in this inequality, we get that  $f_0(f_0 - 7) \geq -6\chi = f_0(M)(f_0(M) - 7)$ . This implies that  $f_0 \geq f_0(M)$ . Thus,  $M$  is strongly minimal. So we have the result for  $d = 2$ . If  $d = 3$  then, by Theorem 1.2,  $M$  is stacked and hence is locally stacked. But any locally stacked,  $\mathbb{F}$ -tight triangulated closed manifold is strongly minimal by Corollary 3.13 in [3]. So, we are done when  $d = 3$ .  $\square$

*Proof of Corollary 1.4.* Let  $M$  be a closed 3-manifold which has an  $\mathbb{F}$ -tight triangulation  $X$ . By Theorem 1.2,  $X$  is stacked. But, by Corollary 3.13 (case  $d = 3$ ) of [8], any stacked triangulation of a closed 3-manifold can be obtained from a stacked 3-sphere by a finite sequence of elementary handle additions. It is easy to see by an induction on the number  $k$  of handles added that  $X$  triangulates either  $S^3$  ( $k = 0$ ) or  $(S^2 \times S^1)^{\#k}$  or  $(S^2 \times S^1)^{\#k}$  ( $k \geq 1$ ). Let  $X$  be obtained from the stacked 3-sphere  $S$  by  $k$  elementary handle additions. It follows by induction on  $k$  that  $f_0(S) = f_0(X) + 4k$  and  $f_1(S) = f_1(X) + 6k = \binom{f_0(X)}{2} + 6k$ . Since  $S$  is a stacked 3-sphere,  $f_1(S) = 4f_0(S) - 10$ . Thus,  $\binom{f_0(X)}{2} + 6k = 4(f_0(X) + 4k) - 10$ . This implies  $(f_0(X) - 4)(f_0(X) - 5) = 20k$  and hence  $f_0(X) = \frac{1}{2}(9 + \sqrt{80k + 1})$ . So,  $80k + 1$  must be a perfect square.  $\square$

*Proof of Corollary 1.5.* If  $(f_0(M) - 4)(f_0(M) - 5) = 20\beta_1(M; \mathbb{F})$  then Theorem 1.3 of [6] says that  $M$  must be neighbourly and locally stacked. Therefore, the ‘if part’ follows from Theorem 2.24 of [2]. The ‘if part’ also follows from Theorem 1.12 of [1] and Theorem 2.24 of [2]. The ‘only if’ part follows from the proof of Corollary 1.4 above.  $\square$

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