On some properties of representation functions related to the Erdős-Turán conjecture

Csaba Sándor[1*] Quan-Hui Yang[2†]

1. Institute of Mathematics, Budapest University of Technology and Economics, H-1529 B.O. Box, Hungary

2. School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

Abstract

For a set $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let $R_A(n)$ denote the number of ordered pairs $(a, a') \in A \times A$ such that $a + a' = n$. The celebrated Erdős-Turán conjecture says that, if $R_A(n) \geq 1$ for all sufficiently large integers n, then the representation function $R_A(n)$ cannot be bounded. For any positive integer m, Ruzsa's number R_m is defined to be the least positive integer r such that there exists a set $A \subseteq \mathbb{Z}_m$ with $1 \leq R_A(n) \leq r$ for all $n \in \mathbb{Z}_m$. In 2008, Chen proved that $R_m \leq 288$ for all positive integers m. Recently the authors proved that $R_m \geq 6$ for all integers $m \geq 36$. In this paper, for an abelian group G, we prove that if $A \subseteq G$ satisfies $R_A(g) \leq 5$ for all $g \in G$, then $|\{g : g \in G, R_A(g) = 0\}| \geq \frac{1}{4}m - \sqrt{5m}$. This improves a recent result of Li and Chen. We also give upper bounds of $|\{g : g \in G, R_A(g) = i\}|$ for $i = 2, 4$.

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1 Introduction

Let G be an abelian group. For any set $A, B \subseteq G$, let

$$R_{A,B}(g) = \sharp\{(a, b) : \ a \in A, \ b \in B, \ a + b = g\}.$$

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Let $R_A(g) = R_{A,A}(g)$. If $A \subseteq \mathbb{N}$ and $R_A(n) \geq 1$ for all sufficiently large integers n, then we say that A is a basis of $\mathbb{N}$. The celebrated Erdős-Turán conjecture [7] states that if A is a basis of $\mathbb{N}$, then $R_A(n)$ cannot be bounded. Erdős [6] proved that there exists a basis A and two constants $c_1, c_2 > 0$ such that $c_1 \log n \leq R_A(n) \leq c_2 \log n$ for all sufficiently large integers n. Recently, Dubickas [5] gave the explicit values of c_1 and c_2. In 2003, Nathanson [15] proved that the Erdős-Turán conjecture does not hold on $\mathbb{Z}$. In fact, he proved that there exists a set $A \subseteq \mathbb{Z}$ such that $1 \leq R_A(n) \leq 2$ for all integers n. In the same year, Grekos et al. [8] proved that if $R_A(n) \geq 1$ for all n, then $\limsup_{n\to\infty} R_A(n) \geq 6$. Later, Borwein et al. [1] improved 6 to 8. In 2013, Konstantoulas [11] proved that if the upper density $\overline{d}(\mathbb{N} \setminus (A + A))$ of the set of numbers not represented as sums of two numbers of A is less than $1/10$, then $R_A(n) > 5$ for infinitely many natural numbers n. Chen [3] proved that there exists a basis A of $\mathbb{N}$ such that the set of n with $R_A(n) = 2$ has density one. Later, the second author [20] and Tang [19] generalized Chen's result. For the analogue of Erdős-Turán conjecture in groups, one can refer to [9], [10] and [12].

For a positive integer m, let $\mathbb{Z}_m$ be the set of residue classes mod m. If $R_A(n) \geq 1$ for all $n \in \mathbb{Z}_m$, then A is called an additive basis of $\mathbb{Z}_m$.

In 1990, Ruzsa [16] found a basis A of $\mathbb{N}$ for which $R_A(n)$ is bounded in the square mean. Ruzsa's method implies that there exists a constant C such that for any positive integer m, there exists an additive basis A of $\mathbb{Z}_m$ with $R_A(n) \leq C$ for all $n \in \mathbb{Z}_m$. For each positive integer m, Chen [2] defined Ruzsa's number R_m to be the least positive integer r such that there exists an additive basis A of $\mathbb{Z}_m$ with $R_A(n) \leq r$ for all $n \in \mathbb{Z}_m$. In the same paper, Chen proved that $R_m \leq 288$ for all positive integer m and $R_{2p^2} \leq 48$ for all primes p.

In 2016, the authors [17] proved that if $m \geq 36$, then $R_m \geq 6$. That is, if $m \geq 36$ and $A \subseteq \mathbb{Z}_m$ satisfies $R_A(n) \leq 5$ for all integers n, then there exists a $n_0 \in \mathbb{Z}_m$ such that $R_A(n_0) = 0$. Recently, Li and Chen (see [14, Corollary 1.3]) gave a quantitative version of this result.

Li & Chen's Theorem. Let G be a finite abelian group with $|G| = m$ and $A \subseteq G$. If $R_A(g) \leq 5$ for all $g \in G$, then

$$|\{g : g \in G, R_A(g) = 0\}| \geq \frac{7}{32}m - \frac{1}{2}\sqrt{10m} - 1.$$

In this paper, we improve Li and Chen's theorem and also give an example on the other hand. For convenience, for a fixed nonnegative integer i, we denote the set $\{g : g \in G, R_A(g) = i\}$ by S_i.

Theorem 1. *(a) Let G be a finite abelian group with $|G| = m$ and $A \subseteq G$. If $R_A(g) \leq 5$ for all $g \in G$, then $|S_0| \geq \frac{1}{4}m - \sqrt{5m}$.*

(b) Let p be a prime and $m = 2(p^2 + p + 1)$. Then there exists a subset $A \subseteq \mathbb{Z}_m$ such that $R_A(n) \leq 5$ for all $n \in \mathbb{Z}_m$ and $|S_0| < \frac{3}{8}m$.

If $R_A(g) \le 5$ for all $g \in G$, then by $|S_0| + |S_2| + |S_4| \le m$ and Theorem 1 (a), we see that $|S_2| + |S_4| \le \frac{3}{4}m + \sqrt{5m}$. In the next two theorems, we give upper bounds for $|S_2|$ and $|S_4|$ respectively.

Theorem 2. *(a) Let $A \subseteq G$ satisfy $R_A(g) \le 5$ for all $g \in G$. Then $|S_2| \le \frac{1}{2}m + 3\sqrt{5m}$.*

(b) Let p be a prime and $m = p^2 + p + 1$. Then there exists a subset $A \subseteq \mathbb{Z}_m$ such that $R_A(n) \le 2$ for all $n \in \mathbb{Z}_m$ and $|S_2| = \frac{1}{2}m - \frac{1}{2}$.

Remark 1. *The example in Theorem 2 (b) shows that Theorem 2 (a) is nearly best possible.*

If $R_A(g) \le 5$ for all $g \in G$, by the statement before Theorem 2, we have $|S_2| + |S_4| \le \frac{3}{4}m + \sqrt{5m}$, and so $|S_4| \le \frac{3}{4}m + O(\sqrt{m})$. It seems difficult to improve this upper bound. In the following, we will prove this result by a weak condition $R_A(g) \le 7$ for all $g \in G$.

Theorem 3. *(a) Let $A \subseteq G$ satisfy $R_A(g) \le 7$ for all $g \in G$. Then $|S_4| \le \frac{3}{4}m + O(\sqrt{m})$.*

(b) Let p be a prime and $m = 2(p^2 + p + 1)$. Then there exists a subset $A \subseteq \mathbb{Z}_m$ such that $R_A(n) \le 4$ for all $n \in \mathbb{Z}_m$ and $|S_4| = \frac{1}{2}m - 1$.

2 Preliminary Lemmas

Lemma 1. *(See [17, Lemma 3].) Let $A \subseteq G$ and c be a positive integer. If $R_A(g) \le c$ for all $g \in G$, then $|A| \le \sqrt{cm}$.*

Lemma 2. *(See [18, Singer's Theorem].) If l is a prime power, then there exists $A \subseteq \mathbb{Z}_{l^2+l+1}$ such that $R_{A,-A}(n) = 1$ for all $n \in \mathbb{Z}_{l^2+l+1},\ n \ne \overline{0}$.*

Lemma 3. *If A is a subset of G, then for any positive integer k we have*

$$\sum_{g\in G}(R_A(g) - k)^2 \ge km - (2k-1)|A| + k^2 - k.$$

Proof. We use Lev and Sárközy's argument (see [13]) in the following.

$$\begin{aligned}
\sum_{g\in G}(R_A(g)-k)^2 &= \sum_{g\in G} R_A(g)^2 - 2k\sum_{g\in G} R_A(g) + k^2 m \\
&= \sum_{g\in G} R_{A,-A}(g)^2 - 2k|A|^2 + k^2 m \\
&= \sum_{g\in G\setminus\{0\}} R_{A,-A}(g)^2 - (2k-1)|A|^2 + k^2 m \\
&\ge \frac{1}{m-1}\left(\sum_{g\in G\setminus\{0\}} R_{A,-A}(g)\right)^2 - (2k-1)|A|^2 + k^2 m \\
&= \frac{(|A|^2-|A|)^2}{m-1} - (2k-1)(|A|^2-|A|) - (2k-1)|A| + k^2 m \\
&= (m-1)\left(\left(\frac{|A|^2-|A|}{m-1} - \left(k-\frac{1}{2}\right)\right)^2 + k - \frac{1}{4}\right) - (2k-1)|A| + k^2.
\end{aligned}$$

If $\frac{|A|^2-|A|}{m-1} \geq k$ or $\frac{|A|^2-|A|}{m-1} \leq k-1$, then we have

$$\sum_{g \in G}(R_A(g)-k)^2 \geq k(m-1)-(2k-1)|A|+k^2 = km-(2k-1)|A|+k^2-k,$$

and the result is true.

If $k-1 < \frac{|A|^2-|A|}{m-1} < k$, then

$$\sum_{g \in G \setminus \{0\}} R_{A,-A}(g)^2 \geq \min_{\substack{k_1,k_2,\ldots,k_{m-1} \in \mathbb{N} \\ \sum_{i=1}^{m-1} k_i = |A|^2-|A|}} \sum_{i=1}^{m-1} k_i^2.$$

It is known that if $\sum_{i=1}^{m-1} k_i$ is fixed, where $k_i \in \mathbb{N}$, then $\sum_{i=1}^{m-1} k_i^2$ gets the minimal value when $|k_i - k_j| \leq 1$ for all $1 \leq i, j \leq m-1$. Let $|A|^2 - |A| = q(m-1)+r$, where q, r are nonnegative integers and $0 \leq r < m-1$. Then $q = \left\lfloor \frac{|A|^2-|A|}{m-1} \right\rfloor$ and $r = \left\{ \frac{|A|^2-|A|}{m-1} \right\}(m-1)$. Hence

$$\begin{aligned}
\sum_{g \in G \setminus \{0\}} R_{A,-A}(g)^2 &\geq \min_{\substack{k_1,k_2,\ldots,k_{m-1} \in \mathbb{N} \\ \sum_{i=1}^{m-1} k_i = |A|^2-|A|}} \sum_{i=1}^{m-1} k_i^2 = rk + (m-1-r)(k-1) \\
&= \left\{ \frac{|A|^2-|A|}{m-1} \right\}(m-1)k^2 + \left(1 - \left\{ \frac{|A|^2-|A|}{m-1} \right\}\right)(m-1)(k-1)^2 \\
&= (k-1)^2(m-1) + (2k-1)\left\{ \frac{|A|^2-|A|}{m-1} \right\}(m-1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{g \in G}(R_A(g)-k)^2 &= \sum_{g \in G \setminus \{0\}} R_{A,-A}(g)^2 - (2k-1)|A|^2 + k^2 m \\
&\geq (k-1)^2(m-1) + (2k-1)\left\{ \frac{|A|^2-|A|}{m-1} \right\}(m-1) \\
&\quad -(2k-1)(|A|^2-|A|) - (2k-1)|A| + k^2 m \\
&= (k-1)^2(m-1) - (2k-1)(m-1)(k-1) - (2k-1)|A| + k^2 m \\
&= km - (2k-1)|A| + k^2 - k.
\end{aligned}$$

□

3 Proofs

Proof of Theorem 1. Let A be a given subset of G such that $R_A(g) \leq 5$ for all $g \in G$. Then

$$\begin{aligned}
\sum_{g \in G}(R_A(g)-3)^2 &= 9|S_0| + 4|S_1| + |S_2| + |S_4| + 4|S_5| \\
&\leq 8|S_0| + 3(|S_1|+|S_3|+|S_5|) + (|S_0|+|S_1|+|S_2|+|S_3|+|S_4|+|S_5|).
\end{aligned}$$

It is clear that

$$|S_0|+|S_1|+|S_2|+|S_3|+|S_4|+|S_5| = |\{g : \ g \in G, 0 \leq R_A(g) \leq 5\}| = m,$$

$$|S_1|+|S_3|+|S_5| = |\{g : g \in G, 2 \nmid R_A(g)\}| = |\{2a : a \in A\}| \le |A|.$$

Hence we have

$$\sum_{g\in G}(R_A(g)-3)^2 \le 8|S_0|+3|A|+m. \tag{1}$$

On the other hand, by Lemma 3, taking $k=3$, we have

$$\sum_{g\in G}(R_A(g)-3)^2 \ge 3m-5|A|+6. \tag{2}$$

Therefore, by (1),(2) and Lemma 1, it follows that

$$|S_0| \ge \frac{1}{4}m - |A| + \frac{3}{4} \ge \frac{1}{4}m - \sqrt{5m}.$$

Now we prove part (b). Let p be a prime number and $m = 2(p^2+p+1)$. By Lemma 2, there is a set $B \subseteq \mathbb{Z}_{p^2+p+1}$ such that $R_{B,-B}(n) = 1$ for all $n \in \mathbb{Z}_{p^2+p+1}$ and $n \ne \overline{0}$. Then for any integer l with $0 \le l \le p^2+p$, we define

$$A_l = 2B \cup (2B+2l+1) \bmod m, \quad \text{where } 2B = \{2b : \ b \in B\}.$$

Now we first prove that $R_{A_l}(n) \le 4$ for all $n \in \mathbb{Z}_m$.

If $2 \mid n$, then $n = a_1+a_2$ with $a_1, a_2 \in 2B$ or $a_1, a_2 \in 2B+2l+1$. Hence $R_{A_l}(n) = R_B(\frac{n}{2}) + R_B(\frac{n}{2} - (2l+1)) \le 2+2 = 4$.

If $2 \nmid n$, then $n = a_1+a_2$ with $a_1 \in 2B, a_2 \in 2B+2l+1$ or $a_1 \in 2B+2l+1, a_2 \in 2B$. Hence, $R_{A_l}(n) = R_B(\frac{n-2l-1}{2}) \times 2 \le 4$.

Therefore, $R_{A_l}(n) \le 4$ for all $n \in \mathbb{Z}_m$.

Let P be a statement and we define

$$I(P) = \begin{cases} 1, & \text{if the statement } P \text{ is true;} \\ 0, & \text{if the statement } P \text{ is false.} \end{cases}$$

Let

$$X^l_{\text{odd}} = \{2k+1 : \ 2k+1 \in \mathbb{Z}_m \text{ and } R_{A_l}(2k+1) = 0\},$$
$$X^l_{\text{even}} = \{2k : \ 2k \in \mathbb{Z}_m \text{ and } R_{A_l}(2k) = 0\}.$$

Then $S_0 = X^l_{\text{odd}} \cup X^l_{\text{even}}$. It is clear that $R_{A_l}(2n+1) = 0$ if and only if $R_B(n-l) = 0$. Then

$$\begin{aligned} |X^l_{odd}| &= p^2+p+1 - \sharp\{n : n \in \mathbb{Z}_{p^2+p+1}, R_B(n) = 2\} - \sharp\{n : n \in \mathbb{Z}_{p^2+p+1}, R_B(n) = 1\} \\ &= p^2+p+1 - \binom{p+1}{2} - (p+1) = \frac{1}{2}p^2 - \frac{1}{2}p < \frac{1}{4}m. \end{aligned}$$

and

$$\begin{aligned}
\sum_{l=0}^{p^2+p} |X_{\text{even}}^l| &= \sum_{l=0}^{p^2+p} |\{n:\ n \in \mathbb{Z}_{p^2+p+1}, R_B(n) = 0 \text{ and } R_B(n-2l-1) = 0\}| \\
&= \sum_{l=0}^{p^2+p} \sum_{n=0}^{p^2+p} I(R_B(n) = 0 \text{ and } R_B(n-2l-1) = 0) \\
&= \sum_{l=0}^{p^2+p} \sum_{n=0}^{p^2+p} I(R_B(n) = 0) I(R_B(n-2l-1) = 0) \\
&= \sum_{n=0}^{p^2+p} I(R_B(n) = 0) \sum_{l=0}^{p^2+p} I(R_B(n-2l-1) = 0) \\
&= \left(\frac{p^2}{2} - \frac{p}{2}\right) \sum_{n=0}^{p^2+p} I(R_B(n) = 0) = \left(\frac{p^2}{2} - \frac{p}{2}\right)^2.
\end{aligned}$$

Hence there is an integer l such that

$$|X_{even}^l| \le \frac{1}{4} \cdot \frac{(p^2-p)^2}{p^2+p+1} < \frac{1}{4}(p^2+p+1) = \frac{1}{8}m.$$

Therefore, for this integer l,

$$|S_0| = |X_{odd}^l| + |X_{even}^l| < \frac{1}{4}m + \frac{1}{8}m = \frac{3}{8}m.$$

□

Proof of Theorem 2. By Lemma 3, taking $k = 2$, we have

$$\sum_{g \in G} (R_A(g) - 2)^2 \ge 2m - 3|A| + 2. \tag{3}$$

On the other hand,

$$\begin{aligned}
\sum_{g \in G} (R_A(g) - 2)^2 &= 4|S_0| + |S_1| + |S_3| + 4|S_4| + 9|S_5| \\
&\le 4(|S_0| + |S_4|) + 9(|S_1| + |S_3| + |S_5|) \\
&\le 4(|S_0| + |S_4|) + 9|A|.
\end{aligned} \tag{4}$$

Hence, by (3) and (4), we have $|S_0| + |S_4| \ge \frac{1}{2}m - 3|A| + \frac{1}{2} \ge \frac{1}{2}m - 3\sqrt{5m}$. Since

$$|S_0| + |S_1| + |S_2| + |S_3| + |S_4| + |S_5| = m,$$

it follows that

$$|S_2| \le \frac{1}{2}m + 3\sqrt{5m}.$$

Now we prove the part (b). By Lemma 2, there exists a subset $A \subseteq \mathbb{Z}_m$ such that $R_{A,-A}(n) = 1$ for all $n \in \mathbb{Z}_m$, $n \ne \overline{0}$. It is easy to see that $|A| = p+1$ and $R_A(n) \le 2$ for all $n \in \mathbb{Z}_m$. Hence

$$|S_2| = \binom{|A|}{2} = \frac{1}{2}(p^2+p+1) - \frac{1}{2} = \frac{1}{2}m - \frac{1}{2}.$$

$\square$

Proof of Theorem 3. By Lemma 3, taking $k=4$, we have

$$\sum_{g\in G}(R_A(g)-4)^2 \ge 4m-7|A|+12.$$

On the other hand, by $|S_1|+|S_3|+|S_5|+|S_7|\le |A|$, we have

$$\begin{aligned}\sum_{g\in G}(R_A(g)-4)^2 &= 16|S_0|+9|S_1|+4|S_2|+|S_3|+|S_5|+4|S_6|+9|S_7|\\ &\le 4(|S_0|+|S_2|+|S_4|+|S_6|)+9|A|+12|S_0|-4|S_4|\\ &\le 4m+9|A|+12|S_0|-4|S_4|.\end{aligned}$$

Hence $|S_4|\le 4|A|+3|S_0|+3$. Since $\sum_{i=0}^{7}|S_i|=m$, it follows that

$$m\ge |S_0|+|S_4|\ge \frac{|S_4|-4|A|-3}{3}+|S_4|=\frac{4}{3}|S_4|-\frac{4}{3}|A|-1.$$

By Lemma 1, we have

$$|S_4|\le \frac{3}{4}m+|A|+\frac{3}{4}\le \frac{3}{4}m+\sqrt{7m}+\frac{3}{4}.$$

Now we prove the part (b). Let p be a prime and $m=2(p^2+p+1)$. By Lemma 2, there exists a subset $A_p\subseteq \mathbb{Z}_{p^2+p+1}$ such that $R_{A_p,-A_p}(n)=1$ for all $n\ne \overline{0}$. Let $A=2A_p\cup(p^2+p+1+2A_p)\subseteq \mathbb{Z}_m$.

If $2\mid n$, then $R_A(n)=R_{A_p}(\frac{n}{2})+R_{A_p}(\frac{n}{2})\le 2+2=4$. If $2\nmid n$, then $R_A(n)=2R_{A_p}(\frac{n-(p^2+p+1)}{2})\le 4$. Hence $R_A(n)\le 4$ for all $n\in\mathbb{Z}_m$.

$$\begin{aligned}|S_4| &= |\{n:\ 2\mid n, n\in\mathbb{Z}_m \text{ and } R_{A_p}(\frac{n}{2})=2\}|\\ &+ |\{n:\ 2\nmid n,\ n\in\mathbb{Z}_m \text{ and } R_{A_p}(\frac{n-(p^2+p+1)}{2})=2\}|\\ &= |\{n: n\in\mathbb{Z}_{p^2+p+1} \text{ and } R_{A_p}(n)=2\}|+|\{n:\ n\in\mathbb{Z}_{p^2+p+1} \text{ and } R_{A_p}(n-\frac{p^2+p}{2})=2\}|\\ &= 2\binom{p+1}{2}=p^2+p=\frac{1}{2}m-1.\end{aligned}$$

$\square$

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