

Every planar graph without adjacent cycles of length at most 8 is 3-choosable

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Abstract

DP-coloring as a generalization of list coloring was introduced by Dvořák and Postle in 2017, who proved that every planar graph without cycles from 4 to 8 is 3-choosable, which was conjectured by Borodin *et al.* in 2007. In this paper, we prove that every planar graph without adjacent cycles of length at most 8 is 3-choosable, which extends this result of Dvořák and Postle.

1 Introduction

Coloring is one of the main topics in graph theory. A *proper k -coloring* of a graph G is a mapping $f : V(G) \rightarrow [k]$ such that $f(u) \neq f(v)$ whenever $uv \in E(G)$, where $[k] = \{1, 2, \dots, k\}$. The smallest k such that G has a k -coloring is called the *chromatic number* of G and is denoted by $\chi(G)$. List coloring was introduced by Vizing [18], and independently Erdős, Rubin, and Taylor [10]. A *list assignment* of a graph $G = (V, E)$ is a function L that assigns to each vertex $v \in V$ a list $L(v)$ of colors. An *L -coloring* of G is a function $\lambda : V \rightarrow \cup_{v \in V} L(v)$ such that $\lambda(v) \in L(v)$ for every $v \in V$ and $\lambda(u) \neq \lambda(v)$ whenever $uv \in E$. A graph G is *k -choosable* if G has an L -coloring for every assignment L with $|L(v)| \geq k$ for each $v \in V(G)$. The *choice number*, denoted by $\chi_l(G)$, is the minimum k such that G is k -choosable.

The techniques to approach the list problems are less than those used in ordinary coloring. For ordinary coloring, identifications of vertices are involved in the reduction configurations. In list coloring, since different vertices have different lists, it is not possible for one to use identification of vertices. With this motivation, Dvořák and Postle [9] introduced correspondence coloring (or DP-coloring) as a generalization of list-coloring.

A *k -correspondence assignment* for G consists of a list assignment L on vertices in $V(G)$ and a function C that assigns every edge $e = uv \in E(G)$ a matching C_e between $\{u\} \times L(u)$ and $\{v\} \times L(v)$.

A *C -coloring* of G is a function ϕ that assigns each vertex $v \in V(G)$ a color $\phi(v) \in L(v)$, such that for every $e = uv \in E(G)$, the vertices $(u, \phi(u))$ and $(v, \phi(v))$ are not adjacent in C_e . We say that G is *C -colorable* if such a C -coloring exists.

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The *correspondence chromatic number* $\chi_{DP}(G)$ of G is the smallest integer k such that G is C -colorable for every k -correspondence assignment (L, C) .

Let (L, C) be a k -correspondence assignment for a graph G , and let $W = v_1v_2 \dots v_m$ with $v_m = v_1$ be a closed walk of length m in G . We say that the assignment (L, C) is *inconsistent* on W if there exist $c_i \in L(v_i)$ for $i \in [m]$ such that $(v_i, c_i)(v_{i+1}, c_{i+1})$ is an edge of $C_{v_iv_{i+1}}$ for $i \in [m-1]$, and $c_1 \neq c_m$. The (L, C) is *consistent* if (L, C) is inconsistent on none of the closed walks in G .

Recently, DP-coloring is studied widely. On the one hand, some results of list coloring were generalized to DP-coloring. For this literature, the readers can see [13, 14, 15, 16]. On the other hand, some new results on DP-coloring are obtained and can be found in [1, 2, 3]. We here pay more attention to the result of Dvořák and Postle [9], who solved a conjecture of Borodin (2007) by the technique developed in DP-coloring, as follows.

Theorem 1.1 ([9]) *Every planar graph G without cycles of lengths 4 to 8 is 3-choosable.*

The well-known Four Color Theorem states that all planar graphs can be colored with four colors such that no adjacent vertices share a common color. There was much effort to find sufficient conditions for a graph to be 3-colorable. The classic theorem by Grötzsch ([11], 1959) shows that planar graphs without 3-cycles are 3-colorable.

In 1969, Havel ([12]) posed a problem: Does there exist a constant C such that every planar graph with the minimum distance between triangles at least C is 3-colorable? Recently, Dvořák, Král' and Thomas [8] proved that the distance is at least 10^{100} . In 1976, Steinberg ([17]) conjectured that every planar graph without 4-cycles and 5-cycles is 3-colorable. Erdős relaxed Steinberg's conjecture and suggested to determine the smallest integer k , if it exists, such that every planar graph without cycles of length from 4 to k is 3-colorable. The best bound for such k is 7, and it is proved by Borodin, Glebov, Raspaud, and Salavatipour [4]. In 2003, Strong Bordeaux Conjecture ([5]) by Borodin and Raspaud was posed as follows: Every planar graph without 5-cycle and adjacent triangles is 3-colorable.

Recently, Cohen-Addad, Hebdige, Kral, Li and Salgado [7] presented a counterexample to the Steinberg's Conjecture, as well as to the Strong Bordeaux Conjecture. Borodin, Montassier and Raspaud [6] asked to determine the smallest integer k such that every planar graph without adjacent cycles of length at most k is 3-colorable and proved the following result.

Theorem 1.2 *Every planar graph without adjacent cycles of length at most 7 is 3-colorable.*

Motivated by Theorems 1.1 and 1.2, we present the following result in this paper.

Theorem 1.3 *Every planar graph without adjacent cycles of length at most 8 is 3-choosable.*

In the end of this section, we introduce some notations used in the paper. Graphs mentioned in this paper are all simple. For a cycle K of a plane graph G , we use $int(K)$ and $ext(K)$ to denote the sets of vertices located inside and outside K , respectively. The cycle K is called a *separating cycle* if $int(K) \neq \emptyset \neq ext(K)$. Let V and F be the set of vertices and faces of G , respectively. For a face $f \in F$, if the vertices on f in a cyclic order are v_1, v_2, \dots, v_k , then we write $f = [v_1v_2 \dots v_k]$. Let $b(f)$ be the vertex set of f . A k -vertex (k^+ -vertex, k^- -vertex) is a

vertex of degree k (at least k , at most k). A k -face (k^+ -face, k^- -face) is a face whose boundary walk has length of k (at least k , at most k). The same notation will be applied to walks and cycles.

2 Lemmas

The C -coloring was recently introduced by Dvořák and Postle [9], who proved the following nice relationship between choosability and correspondence coloring.

Theorem 2.1 ([9]) *A graph G is k -choosable if and only if G is C -colorable for every consistent k -correspondence assignment C .*

Utilizing Theorem 2.1, the technique used by Dvořák and Postle for solving Borodin's conjecture is C -coloring. We follow this Dvořák and Postle's idea and prove the following result.

Theorem 2.2 *Every planar graph G without adjacent cycles of length at most 8 is C -colorable for every 3-correspondence assignment C that is consistent on every closed walk of length 3 in G .*

We actually prove the following result which is a little stronger than Theorem 2.2.

Theorem 2.3 *Let G be a plane graph without adjacent cycles of length at most 8. Let S be a set of vertices of G such that either $|S| = 1$, or S consists of all vertices on a face of G . Let C be a 3-correspondence assignment for G such that C is consistent on every closed walk of length 3 in G . If $|S| \leq 12$, then for every C -coloring ϕ_0 of $G[S]$, there exists a C -coloring ϕ of G whose restriction to S is ϕ_0 .*

Let (L, C) be a k -correspondence assignment on G . An edge $uv \in E(G)$ is *straight* if every $(u, c_1)(v, c_2) \in E(C_{uv})$ satisfies $c_1 = c_2$. An edge $uv \in E(G)$ is *full* if C_{uv} is a perfect matching. The following lemma from [9] says that an C -coloring in certain subgraph H of G is the same as a list coloring, which plays the crucial role in proving results of choosability using DP-coloring.

Lemma 2.4 ([9]) *Let G be a graph with a k -correspondence assignment C . Let H be a subgraph of G such that for every cycle D in H , the assignment C is consistent on D and all edges of D are full. Then we may rename $L(u)$ for $u \in H$ to obtain a k -correspondence assignment C' for G such that all edges of H are straight in C' .*

Proof of Theorem 2.2 By Theorem 2.1, we need to prove that G is C -colorable for arbitrary 3-correspondence assignment C such that G is consistent on every closed walk of length 3 in G . Take S to be an arbitrary vertex in G . By Theorem 2.3, G is C -colorable.

3 Proof of Theorem 2.3

From now on, we always let C be a k -correspondence assignment on G that is consistent on every closed walk of length 3. Assume that Theorem 2.3 fails, and let G be a minimal counterexample, that is, there exists no C -coloring ϕ of G whose restriction to S is equal to ϕ_0 such that

$$|V(G)| \text{ is minimized.} \tag{1}$$

Subject to (1), the number of edges of G that do not join the vertices of S

$$|E(G)| - |E(G[S])| \text{ is minimized.} \quad (2)$$

Subject to (1) and (2), the total number of edges in the matchings of the 3-correspondence assignment C

$$\sum_{uv \in E(G)} |E(C_{uv})| \text{ is maximized.} \quad (3)$$

When S consists of the vertices of a face, we will always assume that D is the outer face of the embedding of plane graph G . And we call a vertex v or a face f *internal* if $v \notin D$ or $f \neq D$.

The following Lemma 3.1 to Corollary 3.5 about some crucial properties of the minimal counterexample and the correspondence assignment are basically from [9]. For completeness, we include the proofs here.

Lemma 3.1 *Each of the following holds:*

- (a) $V(G) \neq S$;
- (b) G is 2-connected;
- (c) each vertex not in S has degree at least 3;
- (d) G does not contain separating k -cycle for $3 \leq k \leq 12$;
- (e) $S = V(D)$ and D is an induced cycle.
- (f) If P is a path of length 2 or 3 with both ends in S and no internal vertex in S , then no edge of P is contained in a triangle that shares at most one vertex with S .

Proof. (a) Suppose otherwise that $V(G) = S$. In this case, ϕ_0 is a C -coloring of G , a contradiction.

(b) By the condition (1), G is connected. Suppose otherwise that v is a cut-vertex of G . Thus, we may assume that $G = G_1 \cup G_2$ such that $V(G_1) \cap V(G_2) = \{v\}$. If $v \in S$, then by the condition (1) G_1 and G_2 have C -coloring extending ϕ_0 such that these C -colorings have the same color at v . Thus, G has a C -coloring, a contradiction. Thus, assume that $v \notin S$. We assume, without loss of generality, that $S \subseteq V(G_1)$. By the condition (1), ϕ_0 can be extended to ϕ_1 of G_1 . Then, $\phi_1(v)$ can be extended to ϕ_2 of G_2 . Now ϕ_1 and ϕ_2 together give an extension of ϕ_0 to G , a contradiction.

(c) Let v be a 2^- -vertex in $G - S$. By the condition (1), ϕ_0 can be extended to a C -coloring ϕ of $G - v$. Then we can extend ϕ to G by selecting a color $\phi(v)$ for v such that for each neighbor u of v , $(u, \phi(u))(v, \phi(v)) \notin E(C_{uv})$, a contradiction.

(d) Let K be a separating k -cycle with $3 \leq k \leq 12$. By the condition (1), ϕ_0 can be extended to a C -coloring ϕ_1 of $ext(K) \cup K$, and the restriction of ϕ_1 to K extends to a C -coloring ϕ_2 of $int(K)$. Thus, ϕ_1 and ϕ_2 together give a C -coloring of G that extends ϕ_0 , a contradiction.

(e) Suppose otherwise that $S = \{v\}$ for some vertex $v \in V(G)$. If v is incident with a 12^- -cycle f_1 , we may assume that v is incident with a 12^- -face by (d). We now redraw G such that f_1 is the outer cycle of G and choose a C -coloring ϕ on the boundary of f_1 . Let $S_1 = V(f_1)$. In

this case, $|E(G)| - |E(G[S_1])| < |E(G)| - |E(G[S])|$. By the condition (2), G has a C -coloring that extends the colors of S_1 , thus G has a C -coloring that extends ϕ_0 , a contradiction. Thus, we may assume that all cycles incident with v are 13^+ -cycles. Let f_2 be a 13^+ -face incident with v . Let v_1 and v_2 be the neighbors of v on f_2 . Let $G_2 = G \cup \{v_1v_2\}$. We redraw G such that f_2 is the outer cycle of G_2 . Let $S_2 = \{v, v_1, v_2\}$ and C_2 be obtained from C by letting the matching between v_1 and v_2 be edgeless. It is easy to verify that $|E(G_2)| - |E(G[S_2])| < |E(G)| - |E(G[S])|$. By the condition (2), G_2 has a C_2 -coloring that extends the colors of S_2 . This implies that G has a C -coloring that extends ϕ_0 , a contradiction again. So $S = V(D)$.

We may assume that D contains a chord uv . By (a) $V(G) \neq S$. Thus D together with the chord uv forms two cycles with common edge uv , each of which has length less than 12 by our assumption that $|S| \leq 12$. By (d), such two cycles are the boundaries of two faces. This means that $S = V(G)$, a contradiction to (a).

(f) Let $P = x_1x_2 \dots x_k$, where $k = 3, 4$, x_1, x_k are in D and no internal vertex in D . Suppose otherwise that one edge $x_i x_{i+1}$ of P is contained in a triangle $f = [x_i x_{i+1} x]$ which has at most one common vertex with D , where $1 \leq i \leq k - 1$. Let P_1 and P_2 be two paths on D between x_1 and x_k . Then $D_i = P \cup P_i$ is a cycle for $i = 1, 2$. We assume, without loss of generality, that f is inside of D_1 . Note that G contains no adjacent cycles of length at most 8. By our assumption, f and P_1 have at most one common vertex. Since D_1 and f are adjacent, D_1 is a 9^+ -cycle. Similarly, D_2 is also 9^+ -cycle. Since $|S| \leq 12$, $9 + 9 \leq |D_1| + |D_2| = |D| + 2(k - 1) \leq 18$, which implies that $k = 4$, $|D_1| = |D_2| = 9$, $|S| = 12$ and P is a path of length 3. By (d), D_1 is not a separating 9-cycle and x is in D . This implies that G has adjacent 8^- -cycles, a contradiction. ■

Lemma 3.2 *If $e = uv \notin E(D)$, then $|E(C_{uv})| \geq 2$. Moreover, if e is not contained in a triangle, then $|E(C_{uv})| = 3$.*

Proof. First we show that if e is not contained in a triangle, then $|E(C_{uv})| = 3$. Suppose otherwise that uv is not full. Let C' be the 3-correspondence assignment obtained from C by adding edges to C_{uv} so that uv is full and keeping C_e unchanged for every edge $e \neq uv$. In this case, $\sum_e |E(C'_e)| > \sum_e |E(C_e)|$, contrary to the condition (3) of G .

We now assume that uv is contained in a triangle $f = [uvw]$. If $C_{uv} = \emptyset$, then let $G' = G - e$. Thus, $|E(G')| - |E(S)| < |E(G)| - |E(S)|$, contrary to the condition (2) of G . So we may assume that $|E(C_{uv})| = 1$. By Lemma 2.4, we may assume that $C_{uv} = \{(u, 1)(v, 1)\}$. Define $C^{a,b}$ by adding an edge $(u, a)(v, b)$ to C_{uv} and keeping C_e unchanged for every edge $e \neq uv$, where $a, b \in \{2, 3\}$. In this case, $\sum_e |E(C_e^{a,b})| > \sum_e |E(C_e)|$. On the other hand, if G has a $C^{a,b}$ -coloring, then it has a C -coloring. Since G has no C -coloring, by the condition (3) of G , there is a closed 3-walk bound f which is inconsistent in $C^{a,b}$.

Assume that C_{uv} has no edge incident with $(u, 2)$. Then the only closed 3-walk in $C^{2,b}$ may be inconsistent is $uvwu$ for $b \in \{2, 3\}$, and thus $C_{vw} \cup C_{wu}$ contains a path $(v, b)(w, d_b)(u, c_b)$ for some colors d_b and c_b such that $c_b \neq 2$. By symmetry, we may assume that $c_b = 1$. This implies that C has a path $(v, 1)(u, 1)(w, d_b)(v, b)$. This means that C is inconsistent on $vwuv$, contrary to our assumption that C is consistent on every closed walk of length 3. So far, we have proved that C_{uv} has an edge incident with $(u, 2)$.

By symmetry, we may assume that C_{uv} has an edge incident with $(u, 3)$, C_{vw} has two independent edges incident with $(v, 2)$ and $(v, 3)$, respectively. By the pigeonhole principle, there

exist colors $c_u, c_v \in \{2, 3\}$ and c_w such that $C_{vw} \cup C_{wu}$ contains a path $(u, c_u)(w, c_w)(v, c_v)$. This means that C^{c_u, c_v} is consistent on all closed 3-walks, which is a contradiction. Therefore, add an edge $(u, c_u)(v, c_v)$ to C and denote by C'' . This leads to that $\sum_e |E(C''_e)| > \sum_e |E(C_e)|$, contrary to the condition (3) of G . ■

Lemma 3.3 *Let $f = [uvw]$ be a 3-face in G with $d(u) = d(w) = 3$ and $u, w \notin S$. Then each edge of f is full.*

Proof. Suppose to the contrary that at least one of all edges of f is not full. Let u_1 (w_1) be the neighbor of u (w) rather than v and w (v and u). Applying Lemma 2.4 to subgraph induced by the edge set $\{uu_1, uv, uw, ww_1\}$, these edges are straight in C .

Suppose that C' is the 3-correspondence assignment for G such that $C'_e = C_e$ for each $e \in E(G) \setminus E(f)$ and that all edges of f are straight and full in C' . Since each edge of f is full in C' but not in C , $\sum_{e \in E(G)} |E(C'_e)| > \sum_{e \in E(G)} |E(C_e)|$. By the condition (3) of G , φ_0 can be extended to a C' -coloring φ' of G . On the other hand, by our assumption, G is not C -colorable. This contradiction is produced since C' differs from C only on the edges of f . Note that all edges of f other than vw are straight in C . By symmetry, we may assume that $\varphi'(v) = 1, \varphi'(w) = 2$ and $(v, 1)(w, 2) \in E(C_{vw})$. If C_{uv} has no edge incident with $(v, 1)$, then we can modify φ' to a C -coloring of G by recoloring w by a color c in $\{1, 3\} \setminus \{\varphi'(w_1)\}$ and by recoloring u by a color from $\{1, 2, 3\} \setminus \{c, \varphi'(u_1)\}$. Thus, we may assume that C_{uv} has one edge incident with $(v, 1)$. Since the edge uv is straight in C , $(v, 1)(u, 1) \in E(C_{uv})$. By Lemma 3.2, $|E(C_{uw})| \geq 2$. So $(u, 2)(w, 2) \in E(C_{uw})$ or $(u, 1)(w, 1) \in E(C_{uw})$, which implies that C is not consistent on all closed 3-walks in f , a contradiction. ■

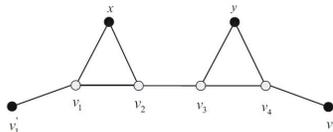


Figure 1: A tetrad.

A *tetrad* in a plane graph is a path $v_1v_2v_3v_4$ of vertices of degree three contained in the boundary of a face, such that both v_1v_2 and v_3v_4 are edges of triangles. See Fig.1.

Lemma 3.4 *Every tetrad in G contains a vertex of S .*

Proof. Let $v_1v_2v_3v_4$ be a tetrad in G . Let $N(v_1) = \{x, v_2, v'_1\}$ and $N(v_4) = \{y, v_3, v'_4\}$, where v'_1, v'_4 are not in the tetrad. Since G has no adjacent 8^- -cycles, the vertices in $\{v_1, v_2, v_3, v_4, v'_1, v'_4, x, y\}$ are distinct. Suppose otherwise that none of the vertices in the tetrad is in S .

Since two edges of the path xv_2v_3y are contained in two triangles, at least one of x and y is not in S by Lemma 3.1(f). Applying Lemma 3.1(f) to paths $xv_1v'_1$ and $yv_4v'_4$, respectively, $|\{v'_1, x\} \cap S| \leq 1$ and $|\{v'_4, y\} \cap S| \leq 1$. By symmetry, we may assume that

$$y \notin S \text{ and either } v'_1 \notin S \text{ or } y \text{ has no neighbors in } S. \quad (4)$$

By Lemmas 3.2 and 3.3, all edges incident with the vertices in the tetrad are full. By Lemma 2.4, they are straight. Let G' be the graph obtained by identifying y and v'_1 of $G - \{v_1, v_2, v_3, v_4\}$ and let C' be the restriction of C to $E(G')$. By our assumption (4), the identification does not create an edge between vertices of S , and thus ϕ_0 is also a C' -coloring of the subgraph of G' induced by S . If there were a path Q of length at most 8 between y and v'_1 , then G would have a cycle K of length at most 12 which is obtained from the path $yv_3v_2v_1v'_1$ and Q . Note that K is a separating cycle, contrary to Lemma 3.1(d). Thus, no new 8^- -cycles are created in G' . This implies that G' contains no adjacent cycles of length at most 8. By the argument above, the identifying y and v'_1 does not create any new triangle. Thus, C' is consistent on all closed walks of length three in G' . Since $|V(G')| < |V(G)|$, ϕ_0 can be extended to a C' -coloring ϕ of G' by (1). We extend ϕ to a C -coloring of G by coloring v'_1 and y with the color of the identifying vertex and then color v_4 and v_3 in order with colors distinct from the colors used by their neighbors. Finally, we color v_1 and v_2 as follows. Note that v'_1 and v_3 have different colors and all edges incident with v_1 and v_2 are straight and full. If the color of x is the same color of v'_1 , color v_2 and then color v_1 ; if the color of x is the same color of v_3 , color v_1 and then color v_2 ; if the colors of v_3, x and v'_1 are distinct, assign the colors of v_3 and v'_1 to v_1 and v_2 , respectively, a contradiction. ■

From now on, let $F_k = \{f : f \text{ is a } k\text{-face and } b(f) \cap D = \emptyset\}$, $F'_k = \{f : f \text{ is a } k\text{-face and } |b(f) \cap D| = 1\}$. A k -face f is *special* if $f \in F'_k$, where $3 \leq k \leq 8$. An internal 3-vertex is *bad* if it is incident with a 3-face $f \notin F'_3$, *light* if it is either incident with a 3-face $f \in F'_3$ or with a 4-face from F_4 or 5-face from F_5 , *good* if it is neither bad nor light.

Corollary 3.5 *No face of G is incident with five consecutive bad vertices. Furthermore, if a face of G is incident with consecutive vertices v_0, v_1, \dots, v_5 and the vertices v_1, \dots, v_4 are bad, then the edges v_0v_1, v_2v_3 and v_4v_5 are incident with triangles.*

Proof. Suppose otherwise that f is incident with five consecutive bad vertices v_1, \dots, v_5 . Since v_3 is on a 3-face and G contains no adjacent 3-faces, either v_2v_3 or v_3v_4 is an edge on a 3-face. In the former case, v_4v_5 is an edge on a 3-face; in the latter case, v_1v_2 must be an edge of a 3-face. This implies that either $v_2v_3v_4v_5$ or $v_1v_2v_3v_4$ is a tetrad. On the other hand, none of v_1, \dots, v_5 is in S since each of them is a bad vertex. This contradicts Lemma 3.4.

Similarly, if a face of G is incident with consecutive vertices v_0, v_1, \dots, v_5 and the vertices v_1, \dots, v_4 are bad, then either each of v_1v_2, v_3v_4 is an edge of a 3-face or each of v_0v_1, v_2v_3, v_4v_5 is an edge incident with a triangle. In the former case, the path $v_1v_2v_3v_4$ is a tetrad and each of v_1, \dots, v_4 is a bad vertex, contrary to Lemma 3.4. ■

The following two lemmas play key roles in the proof of Theorem 2.3.

Lemma 3.6 *Each internal 4-face is incident with at most two vertices of the outer face D . Furthermore, if it is indeed incident with two vertices u, v of D , then $uv \in E(D)$.*

Proof. Suppose otherwise that an internal 4-face $f = [uvw x]$ is incident with at least three vertices of the outer face D . By Lemma 3.1(e), D has no chords. Thus, f and D have exactly three common vertices and share a common 2-vertex u and w is an internal vertex. By Lemma 3.1(c) $d(w) \geq 3$. Suppose that w has a neighbor on D other than v and x . Let f_1 and f_2 be two cycles sharing $w x$ and $w v$ with f , respectively. Since D is a k -face with $9 \leq k \leq 12$, at least one of

f_1 and f_2 is a 7^- -face which is adjacent to f , contrary to our assumption that G contains no adjacent cycles of length at most 8. Thus, w has no other neighbors on D other than v and x . Let P be the longer path on D joining v and x . Then $xPvwx$ is a separating 12^- -cycle in G , contrary to Lemma 3.1(d).

Next, suppose otherwise that f and D share exactly two non-consecutive vertices. Since D is a 12^- -face, f must be adjacent a 8^- -cycle in G , a contradiction. ■

A 9-face $f = [v_1v_2 \dots v_9]$ in F_9 is *special* if each vertex in $\{v_1, v_2\}$ is a light 3-vertex and each vertex in $\{v_3, v_4, v_5, v_7, v_8, v_9\}$ is a bad 3-vertex.

Lemma 3.7 *G does not contain any special 9-face.*

Proof. Suppose otherwise that $f = [v_1v_2 \dots v_9]$ is a special 9-face in G . We claim that v_1 and v_2 are not on a 3-face. Suppose otherwise. Since each of $\{v_3, v_4, v_5, v_7, v_8, v_9\}$ is a bad 3-vertex, v_3 and v_4 are on a 3-face, v_8 and v_9 are on a 3-face, v_5 and v_6 are on a 3-face and v_6 and v_7 are on a 3-face. This implies that the subgraph induced by v_1, v_2, v_3 and v_4 is a tetrad. By Lemma 3.4, one of v_1, v_2, v_3 and v_4 is in D , contrary to our assumption that f is a special 9-face. Thus, v_1 and v_2 must be on a truly internal 4-or 5-face, and each of $\{v_3v_4, v_5v_6, v_6v_7, v_8v_9\}$ is on a 3-face. Let v'_2 be the neighbor of v_2 not on f .

By Lemmas 3.2 and 3.3, all edges incident with v_1, v_2, v_3 and v_4 are full. By Lemma 2.4, they are straight. Let G' be the graph by identifying v'_2 and v_5 of $G - \{v_1, v_2, v_3, v_4\}$ and let C' be the restriction of C to $E(G')$. Note that all the neighbors of each vertex on f are internal. So the identification does not create an edge between vertices of S , and thus ϕ_0 is a C' -coloring of the subgraph of G' induced by S . If there were a path Q of length at most 8 between v'_2 and v_5 , then G would have a cycle K of length at most 12 which is obtained from the path $v'_2v_2v_3v_4v_5$ and Q . Note that D is a separating cycle, contrary to Lemma 3.1(d). Thus, no new 8^- -cycles are created in G' . This implies that G' contains no adjacent cycles of length at most 8 and C' is consistent on all closed walks of length three in G' . Since $|V(G')| < |V(G)|$, ϕ_0 can be extended to a C' -coloring ϕ of G' by (1). We extend ϕ to a C -coloring of G by coloring v'_2 and v_5 with the color of the identifying vertex and then coloring v_1 and v_2 in order with colors distinct from the colors used by their neighbors. Finally, we can color v_3 and v_4 . Note that v_2 and v_5 have different colors and all edges incident with v_3 and v_4 are straight and full. Let v_{34} be the common neighbor of v_3 and v_4 . If the color of v_{34} is the same color of v_2 , color v_4 and then color v_3 ; if the color of v_{34} is the same color of v_5 , color v_3 and then color v_4 ; if the colors of v_{34}, v_2 and v_5 are distinct, assign the colors of v_2 and v_5 to v_4 and v_3 , respectively, a contradiction. ■

We are now ready to present a discharging procedure that will complete the proof of the Theorem 2.3. Let each vertex $v \in V(G)$ have an initial charge of $\mu(v) = d(v) - 4$, and each face $f \neq D$ in our fixed plane drawing of G have an initial charge of $\mu(f) = d(f) - 4$. Let $\mu(D) = d(D) + 4$. By Euler's Formula, $\sum_{x \in V \cup F} \mu(x) = 0$.

Let $\mu^*(x)$ be the charge of $x \in V \cup F$ after the discharge procedure. To lead to a contradiction, we shall prove that $\mu^*(x) \geq 0$ for all $x \in V \cup F$ and $\mu^*(D)$ is positive.

The discharging rules:

(R1) Each non-special 3-face receives $\frac{1}{3}$ from each incident internal vertex.

- (R2) Each 2-vertex receives $\frac{2}{3}$ from each incident internal face.
- (R3) Each internal 3-vertex receives $\frac{2}{3}$ and $\frac{1}{2}$ from each incident 9^+ -face if it is bad and light 3-vertex, respectively; each good 3-vertex receives $\frac{1}{3}$ from each incident face.
- (R4) Each internal 4-vertex v receives $\frac{1}{3}$ from the incident 9^+ -face which is not adjacent to the non-special 3-face if v is incident with one non-special 3-face and three 9^+ -faces, $\frac{1}{3}$ from each incident 9^+ -face if v is incident with two non-special 3-faces and two 9^+ faces, $\frac{1}{6}$ from each incident 9^+ -face if v is incident with one non-special 3-face, one 8^- -face and two 9^+ -faces.
- (R5) After (R1) to (R4) each face sends its remaining positive charge to the outer face D .
- (R6) Each 3^+ -vertex on D sends 1 to each incident special 8^- -face, $\frac{1}{3}$ to each other incident 8^- -face other than D .
- (R7) The outer face D sends $\frac{4}{3}$ to each incident 2-vertex or 3-vertex incident with an internal 8^- -face, and 1 to each other incident vertex.

Lemma 3.8 *Every vertex v in G has nonnegative final charge.*

Proof. We first consider the final charge of the vertices on D . If $d(v) = 2$, then v receives $\frac{2}{3}$ from the incident internal face and $\frac{4}{3}$ from D by (R2) and (R7). So $\mu^*(v) = -2 + \frac{2}{3} + \frac{4}{3} = 0$. Let $d(v) = 3$. By our assumption, v is at most incident with one non-special internal 8^- -face. If v is incident with an internal 8^- -face, then v receives $\frac{4}{3}$ from D and sends out $\frac{1}{3}$ by (R6) and (R7). If v is not incident with any internal 8^- -face, then v receives 1 from D by (R7). Thus, $\mu^*(v) = -1 + \min\{\frac{4}{3} - \frac{1}{3}, 1\} = 0$. If $d(v) = 4$, then v gives 1 to each incident special 8^- -face, $\frac{1}{3}$ to each non-special 8^- -face and receives 1 from D by (R6) and (R7). So $\mu^*(v) \geq 4 - 4 + 1 - \max\{1, \frac{1}{3} \times 2\} = 0$. If $d(v) \geq 5$, then v gives at most 1 to each of the $\lfloor \frac{d(v)}{2} \rfloor$ incident internal 8^- -faces and receives 1 from D by (R6) and (R7). So $\mu^*(v) \geq d(v) - 4 + 1 - \lfloor \frac{d(v)}{2} \rfloor \geq 0$.

Now we consider the final charge of the vertices not on D . By Lemma 3.1(c) $d(v) \geq 3$. If $d(v) = 3$, then v is incident with at least two 9^+ -faces. If v is not incident with any 3-face, then v gets $\frac{1}{3}$ from each incident face by (R3). If v is incident non-special 3-face, then v receives either $\frac{2}{3}$ from each 9^+ -face and sends $\frac{1}{3}$ to the non-special 3-face by (R3) and (R1). If v is incident with special 3-face, then v receives $\frac{1}{2}$ from each incident 9^+ -face by (R3). Thus, $\mu^*(v) = -1 + \min\{\frac{2}{3} \times 2 - \frac{1}{3}, \frac{1}{2} \times 2, \frac{1}{3} \times 3\} = 0$. If $d(v) = 4$, then let f_i be the four incident faces of v in clockwise order, where $1 \leq i \leq 4$. If v is incident with no non-special 3-faces, then $\mu^*(v) = \mu(v) = 0$ by (R4). If v is incident with exactly one non-special 3-face, say f_1 , then by (R4) v receives either $\frac{1}{6}$ from each of f_2 and f_4 if f_3 is a 8^- -face or $\frac{1}{3}$ from f_3 if f_3 is a 9^+ -face. If v is incident with two non-special 3-faces, say f_1 and f_3 , then v receives $\frac{1}{3}$ from each of f_2 and f_4 by (R4). By (R1) v gives $\frac{1}{3}$ to each incident non-special 3-face. Thus, $\mu^*(v) = 4 - 4 + \min\{\frac{1}{6} \times 2 - \frac{1}{3}, \frac{1}{3} - \frac{1}{3}, \frac{1}{3} \times 2 - \frac{1}{3} \times 2\} = 0$. If $d(v) \geq 5$, then v gives at most $\frac{1}{3}$ to each incident 3-face by (R1). Thus, $\mu^*(v) \geq d(v) - 4 - \frac{1}{3} \times \lfloor \frac{d(v)}{2} \rfloor > 0$. ■

Lemma 3.9 *Each face f other than D has nonnegative final charge.*

Proof. Let $\mu'(f)$ be the charge of f after (R1) to (R4). By (R5) we only need to show that $\mu'(f) \geq 0$. Assume that $d(f) = 3$. By Lemma 3.1(e), f is incident with no 2-vertices. If $f \in F'_3$,

then f receives 1 from the incident vertex from D by (R6). If $f \notin F'_3$, then f receives $\frac{1}{3}$ from each incident vertex by (R1) and (R6). Thus, $\mu'(f) = 3 - 4 + \min\{1, \frac{1}{3} \times 3\} = 0$.

Assume that $4 \leq d(f) \leq 8$. Since G contains no adjacent faces of length at most 8, f is not adjacent to any 3-faces. Assume first that $b(f) \cap D = \emptyset$. Let $d(f) \in \{4, 5\}$. If f sends out the charge, then it is incident with 3-vertices by (R1) and (R3). If f is incident with an internal 3-vertex v , then v is light by the definition, and so f can not send charge to v by (R3). Thus, $\mu'(f) \geq d(f) - 4 \geq 0$. If $d(f) \in \{6, 7, 8\}$, then f sends out at most $\frac{1}{3}$ to each incident vertex by (R3). So $\mu'(f) \geq d(f) - 4 - \frac{1}{3}d(f) \geq 0$. Thus, we may assume that $b(f) \cap D \neq \emptyset$. By (R6) f receives 1 from the incident vertex on D if it is special, $\frac{1}{3}$ from each incident 3^+ -vertex on D if it is not special. On the other hand, by (R3) f sends at most $\frac{1}{3}$ to each incident internal vertex. If $d(f) = 4$, then by Lemma 3.6 f shares either exactly one vertex or one edge with D . So $\mu'(f) \geq 4 - 4 + \min\{1 - \frac{1}{3} \times 3, \frac{1}{3} \times 2 - \frac{1}{3} \times 2\} = 0$. Let t be the number of 2-vertices on f . Let $5 \leq d(f) \leq 8$. If $t = 0$, then $\mu^*(f) \geq d(f) - 4 + \min\{1 - \frac{1}{3}(d(f) - 1), 2 \times \frac{1}{3} - \frac{1}{3}(d(f) - 2)\} > 0$. If $t \geq 1$, then f is incident with at least two 3^+ -vertices from D , each of them gives $\frac{1}{3}$ to f by (R6). So $\mu^*(f) \geq d(f) - 4 + \frac{1}{3} \times 2 - \frac{2}{3}t - \frac{1}{3}(d(f) - t - 2) \geq \frac{1}{3}(d(f) - 5) \geq 0$ since $t \leq d(f) - 3$.

Next assume that $d(f) \geq 9$. If f is incident with a 2-vertex, then f is incident with at least two 3^+ -vertices from D . By (R6), each of these two 3^+ -vertices receives nothing from f . By (R2),(R3) and (R4), each 2-vertex on D receives $\frac{2}{3}$ from f and each internal vertex of f receives at most $\frac{2}{3}$ from f . Thus, $\mu'(f) \geq d(f) - 4 - \frac{2}{3}(d(f) - 2) = \frac{1}{3}(d(f) - 8) > 0$. It remains for us to consider that f is incident with no 2-vertices.

By Corollary 3.5, f is incident with at most $(d(f) - 2)$ bad 3-vertices. By (R3) and (R4), each 9^+ -face sends $\frac{2}{3}$ to each incident bad 3-vertex, $\frac{1}{2}$ to each incident light 3-vertex and at most $\frac{1}{3}$ to each other incident vertex. Let s and t be the number of bad 3-vertices and light 3-vertices of f , respectively. If $d(f) \geq 11$, then $\mu'(f) \geq d(f) - 4 - \frac{2}{3}(d(f) - 2) - \frac{1}{2} \times 2 \geq 0$.

Assume that $d(f) = 10$. If $s \leq 6$, then $\mu'(f) \geq d(f) - 4 - \frac{2}{3} \times 6 - \frac{1}{2} \times 4 = 0$ by (R3). Let $s = 7$. Since a light 3-vertex is either in 3-face from F'_3 or on a 4-face from F_4 or a 5-face from F_5 , $t \leq 2$. Thus, $\mu'(f) \geq 10 - 4 - \frac{2}{3} \times 7 - \frac{1}{2} \times 2 - \frac{1}{3} = 0$ by (R3) and (R4). If $s = 8$, then the eight bad 3-vertices must be divided into two parts each of which consists of consecutive 4 bad 3-vertices by the other vertices of f by Corollary 3.5. Thus, the two non-bad vertices cannot be light 3-vertices. By (R3) and (R4), $\mu'(f) \geq 10 - 4 - \frac{2}{3} \times 8 - \frac{1}{3} \times 2 = 0$.

We are left to consider that $d(f) = 9$. If $s \leq 3$, then $\mu'(f) \geq 9 - 4 - \frac{2}{3} \times 3 - \frac{1}{2} \times 6 = 0$. If $s = 4$, then $t \leq 4$. So $\mu'(f) \geq 9 - 4 - \frac{2}{3} \times 4 - \frac{1}{2} \times 4 - \frac{1}{3} = 0$. If $s = 5$, then $t \leq 3$. In fact, if $t = 4$, then each vertex of f is of degree 3. This implies that a bad 3-vertex and a light 3-vertex are in a 3-face, contrary to the definition of bad 3-vertex. If $t \leq 2$, then $\mu'(f) \geq 9 - 4 - \frac{2}{3} \times 5 - \frac{1}{2} \times 2 - \frac{1}{3} \times 2 = 0$ by (R3) and (R4). If $t = 3$, then f is incident with a 4^+ -vertex which is on a non-special 3-face and a 5^- -face which is not a non-special 3-face, thus gets at most $\frac{1}{6}$ from f by (R4). So $\mu'(f) \geq 9 - 4 - \frac{2}{3} \times 5 - \frac{1}{2} \times 3 - \frac{1}{6} = 0$ by (R3) and (R4). Let $s = 6$. If any two of the three non-bad vertices on f are not adjacent, then none of them is a light 3-vertex. So $\mu'(f) \geq 9 - 4 - \frac{2}{3} \times 6 - \frac{1}{3} \times 3 = 0$ by (R3) and (R4). Let $f = [v_1 v_2 \dots v_9]$. So we assume that there are two consecutive non-bad vertices v_1 and v_2 on f . By symmetry and Corollary 3.5, either v_7 or v_6 is a non-bad vertex. In the former case, each of $\{v_2 v_3, v_4 v_5, v_6 v_7\}$ is on a 3-face by Corollary 3.5. If $v_8 v_9$ is on a 3-face, then v_7 is a not light 3-vertex. In this case, none of v_1 and v_2 is a light 3-vertex. By (R3) and (R4), $\mu'(f) \geq 9 - 4 - \frac{2}{3} \times 6 - \frac{1}{3} \times 3 = 0$. Thus, assume that $v_8 v_9$ is not an

edge of a 3-face. In this case, v_7v_8 and v_1v_9 are on 3-faces. This implies that none of v_1, v_2, v_7 are light 3-vertices. By (R3) and (R4), we conclude similarly that $\mu'(f) \geq 9 - 4 - \frac{2}{3} \times 6 - \frac{1}{3} \times 3 = 0$. In the latter case, since each of v_5 and v_7 is a bad 3-vertex, v_6 cannot be a light 3-vertex. If none of v_1 and v_2 is light 3-vertices, then $\mu'(f) \geq 9 - 4 - \frac{2}{3} \times 6 - \frac{1}{3} \times 3 = 0$ by (R3) and (R4). If exactly one, say v_1 , of them is a light 3-vertex, then v_1v_2 is an edge of a 3-face or a 4- or 5-face. Since v_2 is not a light 3-vertex, v_2 is a 4^+ -vertex. Thus, by (R3) and (R4), f sends at most $\frac{1}{2}, \frac{1}{6}, \frac{1}{3}$ to v_1, v_2, v_6 respectively. Then $\mu'(f) \geq 9 - 4 - \frac{2}{3} \times 6 - \frac{1}{2} - \frac{1}{6} - \frac{1}{3} = 0$. If both of them are light 3-vertices, then f is a special 9-face which contradicts Lemma 3.7. The other possibility is that $v_6 \in D$ in which case v_6 does not receive charge from f . In this case, $\mu'(f) \geq 9 - 4 - \frac{2}{3} \times 6 - \frac{1}{2} \times 2 = 0$. If $s = 7$, then the seven bad vertices must be divided into two parts, one consisting of consecutive 3 bad 3-vertices; the other consisting of consecutive 4 bad 3-vertices by the other vertices of f by Corollary 3.5. We assume, without loss of generality, that none of v_1 and v_5 is a bad 3-vertex and each of v_1v_9, v_8v_7, v_5v_6 is an edge on a 3-face. By symmetry, assume that v_1v_2 and v_3v_4 are edges on two different 3-faces. In this case, v_5 is incident with at least three 9^+ -faces. Since v_1 is a 4^+ -vertex, it is not a light 3-vertex. Thus, v_5 gets nothing from f and $\mu'(f) \geq 9 - 4 - \frac{2}{3} \times 7 - \frac{1}{3} = 0$ by (R3) and (R4). ■

Proof of Theorem 2.3. By Lemmas 3.8 and 3.9, it is sufficient for us to check that the outer face D has positive final charge. By (R7) D sends each incident vertex at most $\frac{4}{3}$. Thus, $\mu^*(D) \geq d(D) + 4 - \frac{4}{3}d(D) = \frac{1}{3}(12 - d(D)) \geq 0$. On the other hand, $\mu^*(D) = 0$ if and only if $d(D) = 12$ and each vertex on f receives $\frac{4}{3}$ from D and D does not get charge by (R5). Therefore, each vertex on D is either a 2-vertex or a 3-vertex incident with an internal 8^- -face. Since $D \neq G$, D is incident with at least one 3-vertex. In this case, v is incident with a 9^+ -face f_1 other than D . Furthermore, f_1 is incident with at least two 3-vertices on D , which receive no charge from f_1 by (R6). Thus, f_1 remains at least $d(f_1) - 4 - \frac{2}{3}(d(f_1) - 2) = \frac{1}{3}(d(f_1) - 8) \geq \frac{1}{3}$ after sending its charge to all vertices on f_1 . By (R5), D gets at least $\frac{1}{3}$ from f_1 , which implies that D has a positive final charge.

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