

CLIQUE-FACTORS IN SPARSE PSEUDORANDOM GRAPHS

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ABSTRACT. We prove that for any $t \geq 3$ there exist constants $c > 0$ and n_0 such that any d -regular n -vertex graph G with $t \mid n \geq n_0$ and second largest eigenvalue in absolute value λ satisfying $\lambda \leq cd^t/n^{t-1}$ contains a K_t -factor, that is, vertex-disjoint copies of K_t covering every vertex of G .

1. INTRODUCTION

An (n, d, λ) -graph is an n -vertex d -regular graph whose second largest eigenvalue in absolute value is at most λ . These graphs are central objects in extremal, random and algebraic graph theory. The interest in these graphs lies in the fact that various pseudorandom properties can be inferred from the value of λ , in terms of the other parameters. For example, if $\lambda \ll d$ then such a graph has the property that its edges are ‘distributed’ uniformly, which is one of the essential properties exploited in random graphs and the regularity method from extremal graph theory. More precisely, the following inequality, called the *expander mixing lemma* (see e.g. [4]), makes this quantitative:

$$\left| e(A, B) - \frac{d}{n}|A||B| \right| < \lambda \sqrt{|A||B|}, \quad (1)$$

whenever A and B are vertex subsets of an (n, d, λ) -graph G , where $e(A, B)$ denotes the number of edges between A and B (edges in $A \cap B$ are counted twice). An excellent introduction to the study of (n, d, λ) -graphs is given in a survey of Krivelevich and Sudakov [11]. The emphasis there and throughout the field is on the interplay between the parameters n , d and λ and graph properties of interest; more precisely, given some property, one wishes to establish best possible conditions on n , d and λ that ensure that any (n, d, λ) -graph with parameters satisfying such conditions has the property.

In this note we will only be concerned with conditions on the parameters n , d and λ that guarantee the existence of certain spanning structures, i.e., subgraphs that occupy the whole vertex set (and have minimum degree at least one). Thus, we will be somewhat selective in our discussion. In particular, we are interested in whether, for some fixed $t \geq 3$, our (n, d, λ) -graph contains a family of vertex-disjoint copies of K_t covering each vertex exactly once, which we call a K_t -factor.¹ We remark that the case $d = \Theta(n)$ is well-understood since the existence of bounded degree spanning graphs in (n, d, λ) -graphs follows from the celebrated Blow-up lemma of Komlós, Sárközy and Szemerédi [10]; see the discussion in e.g. [8]. Moreover, a sparse blow-up lemma for subgraphs of (n, d, λ) -graphs [2] gives general nontrivial conditions for the existence of a given bounded degree spanning subgraph in the case $d = o(n)$. These conditions

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¹This is also sometimes called a *perfect K_t -matching* or a *perfect K_t -tiling* in the literature.

are stronger than those discussed in what follows (and hence, as expected, the general results in [2] are weaker than the ones below).

While extremal and random graph theory provide tools to answer questions in this line of research positively, one is naturally interested in the asymptotic tightness of the obtained results. This requires constructions of special pseudorandom graphs, most of the known examples of which come from algebraic graph theory or geometry. For our study here, there is essentially one prime example of such a construction, due to Alon [5], who gave K_3 -free (n, d, λ) -graphs with $d = \Theta(n^{2/3})$ and $\lambda = \Theta(n^{1/3})$. Krivelevich, Sudakov and Szabó [12] then extended these to the whole possible range of $d = d(n)$, constructing K_3 -free (n, d, λ) -graphs with $\lambda = \Theta(d^2/n)$. Alon's construction is an important milestone in the study of (n, d, λ) -graphs. It provides a rare example of something that is reminiscent of threshold phenomena in random graphs in the context of (n, d, λ) -graphs: if $\lambda \leq 0.1d^2/n$, then any vertex of an (n, d, λ) -graph is contained in a copy of K_3 , while there are K_3 -free (n, d, λ) -graphs with $\lambda = \Omega(d^2/n)$. Even more is true: as proved by Krivelevich, Sudakov and Szabó [12], (n, d, λ) -graphs with $\lambda \leq 0.1d^2/n$ contain a fractional triangle-factor.² A natural conjecture from the same authors [12] states the following.

Conjecture 1.1 (Conjecture 7.1 in [12]). *There exists an absolute constant $c > 0$ such that every (n, d, λ) -graph G on $n \in 3\mathbb{N}$ vertices with $\lambda \leq cd^2/n$ has a triangle-factor.*

The first result in this direction was given by Krivelevich, Sudakov and Szabó [12], who proved that it suffices to impose $\lambda \leq cd^3/(n^2 \log n)$ for some absolute constant $c > 0$. This was improved by Allen, Böttcher, Hàn and two of the current authors [3] to $\lambda \leq cd^{5/2}/n^{3/2}$ for some $c > 0$. In fact, the result in [3] is that this condition on λ is enough to guarantee the appearance of squares of Hamilton cycles (the square of a Hamilton cycle is obtained by connecting distinct vertices at distance at most 2 in the cycle). Another piece of evidence in support of Conjecture 1.1 is a result in [8, 9] that states that, under the condition $\lambda \leq (1/600)d^2/n$, any (n, d, λ) -graph G with n sufficiently large contains a ‘near-perfect K_3 -factor’; in fact, G contains a family of vertex-disjoint copies of K_3 covering all but at most $n^{647/648}$ vertices of G . Very recently, Nenadov [15] proved that $\lambda \leq cd^2/(n \log n)$ for some constant $c > 0$ yields a K_3 -factor.³

With regard to K_t -factors for general $t \in \mathbb{N}$, Krivelevich, Sudakov and Szabó [12] remark that the condition $\lambda \leq cd^{t-1}/n^{t-2}$ yields, for appropriate $c = c(t)$, a fractional K_t -factor. Although there is, alas, no known suitable generalization of Alon's construction to K_t -free graphs, this condition on λ may be seen as a benchmark in the study of K_t -factors in (n, d, λ) -graphs. The first nontrivial result for this study was the result in [3] showing that $\lambda \leq cd^{3t/2}n^{1-3t/2}$ for some $c = c(t) > 0$ is sufficient to guarantee t -powers of Hamilton cycles (and thus K_{t+1} -factors when $(t+1) \mid n$). The aforementioned result of Nenadov [15] generalizes to K_t -factors [15], giving the condition $\lambda \leq cd^{t-1}/(n^{t-2} \log n)$ for some constant $c = c(t) > 0$. The purpose of this note is to present a proof that, under the condition $\lambda \leq cd^t/n^{t-1}$ for some suitable $c = c(t) > 0$, any (n, d, λ) -graph contains a K_t -factor. More precisely, we prove the following.

Theorem 1.2. *Given an integer $t \geq 3$, there exist $c > 0$ and $n_0 > 0$ such that every (n, d, λ) -graph G with $n \geq n_0$ and $\lambda \leq cd^t/n^{t-1}$ contains a K_t -factor.*

²A fractional K_t -factor in a graph G is a function $f : \mathcal{K}_t \rightarrow \mathbb{R}_+$, where \mathcal{K}_t is the set of copies of K_t in G , such that $\sum_{v \in K \in \mathcal{K}_t} f(K) = 1$ for all $v \in V(G)$.

³Nenadov studied a larger class of graphs, namely, the class of bijumbled graphs with sufficient minimum degree. This class contains (n, d, λ) -graphs as a special case. For details, see concluding remarks at the end of this note.

If $d \geq cn/\log n$ for some suitable $c = c(t) > 0$ and $t \geq 4$, then the condition in Theorem 1.2 is the weakest that is currently known to imply the existence of K_t -factors. Theorem 1.2, first announced in [8], was obtained independently of Nenadov's result. There are however similarities in both approaches, since they both use absorption techniques. However, the structures and arguments that are used are different and Nenadov's method is more effective in the more interesting sparse range.

We use standard notation from graph theory, see e.g. [16]. We will omit floor and ceiling signs in order not to clutter the arguments.

2. TOOLS

2.1. Properties of (n, d, λ) -graphs. We begin by giving some basic properties of (n, d, λ) -graphs. Some of these are well known and used throughout the study of (n, d, λ) -graphs whilst others are specifically catered to our purposes here.

Theorem 2.1 (Expander mixing lemma [4]). *If G is an (n, d, λ) -graph and $A, B \subseteq V(G)$, then*

$$\left| e(A, B) - \frac{d}{n}|A||B| \right| < \lambda\sqrt{|A||B|}. \quad (2)$$

Proposition 2.2 (Proposition 2.3 in [12]). *Let G be an (n, d, λ) -graph with $d \leq n/2$. Then $\lambda \geq \sqrt{d/2}$.*

Fact 2.3. *Let G be an (n, d, λ) -graph with $d \leq n/2$. Suppose $\lambda \leq d^t/n^{t-1}$ for some $t \geq 2$. Then $d \geq n^{1-1/(2t-1)}/2$.*

Proof. Proposition 2.2 tells us that $\lambda \geq \sqrt{d/2}$. Thus $\lambda \leq d^t/n^{t-1}$ implies that $d^{2t-1} \geq n^{2t-2}/2$, whence $d \geq n^{1-1/(2t-1)}/2^{1/(2t-1)}$ follows. \square

Fact 2.4. *Let G be an (n, d, λ) -graph with $\lambda \leq \varepsilon d^t/n^{t-1}$. If U is a set of $m' \geq d/2$ vertices, then there are at most εd vertices u with $|N_G(u) \cap U| < dm'/(2n)$.*

Proof. Let U' be the set of vertices u such that $|N_G(u) \cap U| < dm'/(2n)$. By Theorem 2.1, we have

$$\frac{d}{n}m'|U'| - \lambda\sqrt{m'|U'|} < e(U, U') < |U'| \frac{dm'}{2n}.$$

Together with $\lambda \leq \varepsilon d^t/n^{t-1}$, we obtain that $|U'| \leq 8\varepsilon^2 d^{2t-3}/n^{2t-4} \leq \varepsilon d$. \square

Write $K_t(1, \dots, 1, 40)$ for the graph obtained by replacing one vertex of K_t by an independent set of size 40.

Fact 2.5. *Let G be an (n, d, λ) -graph with $\lambda \leq cd^t/n^{t-1}$ and suppose $cd^t/n^{t-1} \geq 40$. Then any set of $2^t cd$ vertices spans a copy of $K_t(1, \dots, 1, 40)$. Moreover, any set of $2^{t-1} cd^2/n$ vertices spans a copy of K_{t-1} .*

Proof. Let U be a set of at least $2cd^{t-1}/n^{t-2}$ vertices in G . Since $\lambda \leq cd^t/n^{t-1} \leq d|U|/(2n)$, it follows from Theorem 2.1 that

$$2e(U) \geq \frac{d}{n}|U|^2 - \lambda|U| \geq |U| \cdot \frac{d|U|}{2n},$$

which implies that U contains a vertex with degree at least $d|U|/(2n)$.

Thus, given a set of $2^t cd$ vertices, we can iteratively pick vertices with large degree in the common neighborhood, and get a $(t-1)$ -clique whose common neighborhood has size at least

$2^t cd \cdot (d/(2n))^{t-1} = 2cd^t/n^{t-1} \geq 40$ (the smallest set from which we pick a vertex in this process has size $4cd^{t-1}/n^{t-2}$). Therefore, we obtain a copy of $K_t(1, \dots, 1, 40)$. The proof of the ‘moreover’-part is analogous. \square

2.2. Templates and absorbing structures. A *template* T with *flexibility* $m \in \mathbb{N}$ is a bipartite graph on $7m$ vertices with vertex parts X and $Z_1 \cup Z_2$, such that $|X| = 3m$, $|Z_1| = |Z_2| = 2m$, and for any $\bar{Z} \subseteq Z_1$, with $|\bar{Z}| = m$, the induced graph $T[V(T) \setminus \bar{Z}]$ has a perfect matching. We call Z_1 the *flexible* set of vertices for the template. Montgomery first introduced the use of such templates when applying the absorbing method in his work on spanning trees in random graphs [14]. There, he used a sparse template of maximum degree 40, which we will also use. It is not difficult to prove the existence of such templates for large enough m probabilistically; see e.g. [14, Lemma 2.8]. The use of a sparse template as part of the absorbing method was also applied by Kwan in [13], where he generalized the notion to 3-uniform hypergraphs in order to study random Steiner triple systems, and by Ferber and Nenadov [6] in their work on the universality of random graphs.

Here, we will use a sparse template to build an absorbing structure suitable for our purposes. The absorbing structure we will use is defined as follows. Let m be a sufficiently large integer. Let $T = (X, Z_1 \cup Z_2, E)$ be the bipartite template with flexibility m , maximum degree $\Delta(T) \leq 40$ and flexible set Z_1 . Write $X = \{x_1, \dots, x_{3m}\}$, $Z_1 = \{z_1, \dots, z_{2m}\}$, $Z_2 = \{z_{2m+1}, \dots, z_{4m}\}$ and define $Z := Z_1 \cup Z_2$. An *absorbing structure* $(T, \mathcal{K}, A, \mathcal{S}, Z, Z_1)$ with flexibility m contains two sets \mathcal{K} and \mathcal{S} consisting of vertex-disjoint $(t-1)$ -cliques and two vertex sets A and Z such that $V(\mathcal{K})$, $V(\mathcal{S})$, A and Z are pairwise disjoint. Furthermore, with the labelling $\mathcal{K} := \{K^1, K^2, \dots, K^{3m}\}$, $A = \{a_{ij} : x_i z_j \in E(T)\}$ and $\mathcal{S} = \{S_{ij} : x_i z_j \in E(T)\}$, the following holds. For all i and j such that $x_i z_j \in E(T)$,

- each $\{a_{ij}\} \cup K^i$ spans a copy of K_t ,
- each $\{a_{ij}\} \cup S_{ij}$ spans a copy of K_t , and
- each $\{z_j\} \cup S_{ij}$ spans a copy of K_t .

Fact 2.6. *The absorbing structure $(T, \mathcal{K}, A, \mathcal{S}, Z, Z_1)$ has the property that, for any subset $\bar{Z} \subseteq Z_1$ with $|\bar{Z}| = m$, the removal of \bar{Z} leaves a graph with a K_t -factor.*

Proof. By the property of the template $T = (X, Z_1 \cup Z_2, E)$, there is a perfect matching M in T that covers X and $Z \setminus \bar{Z}$. Then for each edge $x_i z_j \in M$, we take the t -cliques on $\{a_{ij}\} \cup K^i$ and $\{z_j\} \cup S_{ij}$; for the pairs $\{i, j\}$ such that $x_i z_j \in E(T) \setminus M$, we take the t -cliques on $\{a_{ij}\} \cup S_{ij}$. This gives the desired K_t -factor. \square

The following lemma asserts that (n, d, λ) -graphs possess the absorbing structures above.

Lemma 2.7. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there is an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Let G be an (n, d, λ) -graph with $\lambda \leq \varepsilon d^t/n^{t-1}$ and suppose $m = \varepsilon d$. Then there exists an absorbing structure $(T, \mathcal{K}, A, \mathcal{S}, Z, Z_1)$ with flexibility m such that, for any vertex v in G , we have $\deg(v, Z_1) \geq d|Z_1|/(2n)$.*

The following concentration result will be used in the proof of Lemma 2.7

Lemma 2.8 (Lemma 2.2 in [2]). *Let Ω be a finite probability space and let $\mathcal{F}_0 \subseteq \dots \subseteq \mathcal{F}_n$ be a filtration on Ω . For each $i \in [n]$ let Y_i be a Bernoulli random variable on Ω that is constant*

on each atom of \mathcal{F}_i , that is, let Y_i be \mathcal{F}_i -measurable. Furthermore, let p_i be a real-valued \mathcal{F}_{i-1} -measurable random variable on Ω . Let x and δ be real numbers with $\delta \in (0, 3/2)$, and let $X = Y_1 + \dots + Y_n$. If $\sum_{i=1}^n p_i \geq x$ holds almost surely and $\mathbb{E}[Y_i | \mathcal{F}_{i-1}] \geq p_i$ holds almost surely for all $i \in [n]$, then

$$\Pr(X < (1 - \delta)x) < e^{-\delta^2 x/3}.$$

Proof of Lemma 2.7. First we choose $\varepsilon_0 = 1/(300t)$ and let $\varepsilon \in (0, \varepsilon_0)$. Then we take n_0 large enough.

Let $T = (X, Z_1 \cup Z_2, E)$ be a bipartite template with flexibility m and flexible set Z_1 such that $\Delta(T) \leq 40$. Pick an arbitrary collection of $3m$ vertex-disjoint copies of $K_t(1, \dots, 1, 40)$ (using Fact 2.5). Label the copies of K_{t-1} as $\mathcal{K} := \{K^1, K^2, \dots, K^{3m}\}$. Then label $A = \{a_{ij} : x_i z_j \in E(T)\}$ as the vertices in the classes of 40 vertices in the copies of $K_t(1, \dots, 1, 40)$ such that each a_{ij} together with K^i forms a copy of K_t (we may then discard some extra vertices, according to the degree of x_i in T).

We will pick $Z = \{z_1, \dots, z_{4m}\}$ and $\mathcal{S} = \{S_{ij} : x_i z_j \in E(T)\}$ satisfying the definition of the absorbing structure as follows. Suppose that we have picked $Z(j-1) = \{z_1, \dots, z_{j-1}\}$ and $\mathcal{S}(j-1) = \{S_{ij'} : j' < j\}$ with the desired properties. At step j , we pick as z_j a uniform random vertex in $V(G) \setminus (V(\mathcal{K}) \cup Z(j-1) \cup V(\mathcal{S}(j-1)) \cup B_j)$, where B_j is the set of vertices z in G such that $|(N_G(a_{ij}) \setminus (V(\mathcal{K}) \cup Z(j-1) \cup V(\mathcal{S}(j-1)))) \cap N_G(z)| < d^2/(4n)$ for some i with $x_i z_j \in E(T)$. Since $|V(\mathcal{K})| + 4m + 120(t-1)m \leq (123t+1)\varepsilon d < d/2$ and $\Delta(T) \leq 40$, Fact 2.4 with $U = N_G(a_{ij}) \setminus (V(\mathcal{K}) \cup Z(j-1) \cup V(\mathcal{S}(j-1)))$ implies that $|B_j| \leq 40\varepsilon d$. Next, for each i such that $x_i \in N_T(z_j)$, we pick a $(t-1)$ -clique S_{ij} in $(N_G(a_{ij}) \setminus (V(\mathcal{K}) \cup Z(j-1) \cup V(\mathcal{S}(j-1)))) \cap N_G(z_j)$, which is possible by Fact 2.5 because this set contains at least $d^2/(4n)$ vertices of G . Moreover, we can choose these at most 40 cliques to be vertex-disjoint, because they only take up $40(t-1)$ vertices and any set in G of size $d^2/(4n) - 40(t-1) > d^2/(5n)$ still contains a $(t-1)$ -clique.

At last we analyse the random process for Z and prove that, with positive probability, all vertices v of G are such that $\deg(v, Z_1)$ is appropriately large. Note that at step j we have fixed $|V(\mathcal{K}) \cup V(\mathcal{S}(j-1)) \cup Z(j-1)| \leq 3(t+39)m + 120(t-1)m + 4m \leq (123t+1)\varepsilon d$ vertices. Since when we pick z_j , we also need to avoid the set B_j of size at most $40\varepsilon d$, in total we need to avoid at most $140t\varepsilon d$ vertices. Let v be a vertex in G . Given a choice of $V(\mathcal{K}) \cup V(\mathcal{S}(j-1)) \cup Z(j-1)$, the probability that $z_j \in N_G(v)$ is at least $(1 - 140t\varepsilon)d/n$. Then, by Lemma 2.8 with $\delta = \varepsilon$, we have

$$\Pr\left(\deg(v, Z_1) < \frac{d}{2n} |Z_1|\right) < \Pr(\deg_G(v, Z_1) < (1 - \varepsilon)(1 - 140t\varepsilon)|Z_1|d/n) < e^{-\varepsilon^3 d^2/n} = o\left(\frac{1}{n}\right).$$

Thus, the union bound over all vertices of G implies that the existence of Z_1 with the desired property in the lemma. \square

2.3. A Hall-type result. Another tool that we will use is the following theorem of Aharoni and Haxell [1, Corollary 1.2].

Theorem 2.9. *Let \mathcal{H} be a family of k -uniform hypergraphs on the same vertex set. A sufficient condition for the existence of a system of disjoint representatives⁴ for \mathcal{H} is that for every $\mathcal{G} \subseteq \mathcal{H}$ there exists a matching in $\cup_{H \in \mathcal{G}} E(H)$ of size greater than $k(|\mathcal{G}| - 1)$.*

⁴By this we mean a selection of edges $e_H \in H$ for all $H \in \mathcal{H}$ such that $e_H \cap e_{H'} = \emptyset$ for all $H \neq H' \in \mathcal{H}$.

3. PROOF OF THEOREM 1.2

We are now ready to prove our main result.

Proof of Theorem 1.2. Let $t \geq 3$ be given. Let ε_0 be given by Lemma 2.7. Choose $\varepsilon := \min\{\varepsilon_0, t^{-2}\}$ and let n_0 be given by Lemma 2.7 on input ε_0 . Finally, set $c = \varepsilon^2/2^{t+1}$. We may assume that

$$\frac{d}{n} \leq \frac{\varepsilon}{2t}, \quad (3)$$

since otherwise the existence of a K_t -factor is guaranteed by the Blow-up Lemma [10] (see the discussion in [8]). We apply Lemma 2.7 to G and obtain an absorbing structure $(T, \mathcal{K}, A, \mathcal{S}, Z, Z_1)$ with flexibility $m = \varepsilon d$ on a set W of at most $124t\varepsilon d$ vertices. Thus $Z_1 \subseteq W$ is such that $|Z_1| = 2\varepsilon d$ and, for any subset $\bar{Z} \subseteq Z_1$ with $|\bar{Z}| = \varepsilon d$, the absorbing structure with \bar{Z} removed has a K_t -factor. Moreover, $\deg(v, Z_1) \geq d|Z_1|/2n$ for any vertex v in G .

Now we greedily find vertex-disjoint copies of K_t in $G \setminus W$ as long as there are at most $\varepsilon^2 d$ vertices left. This is possible by Fact 2.5 because $\varepsilon^2 d > 2^t c d$. We denote the set of uncovered vertices in $V(G) \setminus W$ by U . Thus $|U| \leq \varepsilon^2 d$.

Next we will cover U by vertex-disjoint copies of K_t with one vertex in U and the other vertices from Z_1 by applying Theorem 2.9. To that end, for each vertex $v \in U$, let H_v be the set of $(t-1)$ -element sets of $N(v) \cap Z_1$ that induce copies of K_{t-1} in G and let $\mathcal{H} = \{H_v : v \in U\}$. We claim that \mathcal{H} has a system of disjoint representatives. To verify the assumption of Theorem 2.9, we first consider sets $X \subseteq U$ of size at least d^{2t-3}/n^{2t-4} . Let Z' be any subset of Z_1 of size εd . Note that $|X||Z'| \geq \varepsilon d^{2t-2}/n^{2t-4}$, which implies that $\lambda \leq cd^t/n^{t-1} \leq \varepsilon(d/n)\sqrt{|X||Z'|}$. By Theorem 2.1, we have

$$e(X, Z') \geq \frac{d}{n}|X||Z'| - \lambda\sqrt{|X||Z'|} \geq \frac{d}{n}|X||Z'| - \varepsilon\frac{d}{n}|X||Z'| \geq \frac{d}{2n}|X||Z'|.$$

Hence there exists a vertex $v \in X$ such that $\deg(v, Z') \geq d|Z'|/(2n) = \varepsilon d^2/(2n)$. By Fact 2.5, we can find a copy of K_{t-1} in $N(v) \cap Z'$. Thus in this case we can greedily find vertex-disjoint copies of K_{t-1} which belong to $\cup_{v \in X} H_v$, as long as there are εd vertices in Z_1 left uncovered. This gives a matching in $\cup_{v \in X} H_v$ of size $\varepsilon d/(t-1) > (t-1)\varepsilon^2 d \geq (t-1)|X|$, for ε sufficiently small. It remains to consider sets $X \subseteq U$ of size at most d^{2t-3}/n^{2t-4} . In this case we fix any vertex $v \in X$ and only consider matchings in H_v . Indeed, by the construction of the absorbing structure (see Lemma 2.7), we have $\deg(v, Z_1) \geq d|Z_1|/2n = \varepsilon d^2/n$ for any v . Since by Fact 2.5 there is a copy of K_{t-1} in every set of size $\varepsilon d^2/(2n)$, we can find a set of $\varepsilon d^2/(2(t-1)n)$ vertex-disjoint copies of K_{t-1} in H_v . We are done because $\varepsilon d^2/(2(t-1)n) \geq d^{2t-3}/n^{2t-4}$ holds by our initial assumption (3). Theorem 2.9 then tells us that a system of disjoint representatives does exist for \mathcal{H} , whence we conclude that there are vertex-disjoint copies of K_t which cover all the vertices in $V(G) \setminus W$ and $(t-1)\varepsilon^2 d$ vertices of Z_1 . We can then greedily find vertex-disjoint copies of K_t in the remainder of Z_1 , which exist by Fact 2.5, until exactly εd vertices of Z_1 remain (which will be the case due to the divisibility assumption $t | n$). Then the key property of the absorbing structure completes a full K_t -factor. \square

4. CONCLUDING REMARKS

Bijumbled graphs. A graph G on n vertices is called (λ, p) -bijumbled if for any two vertex subsets $A, B \subseteq V(G)$,

$$|e(A, B) - p|A||B|| < \lambda\sqrt{|A||B|}.$$

Thus by Theorem 2.1, any (n, d, λ) -graph is $(\lambda, d/n)$ -bijumbled. Nenadov’s result [15] asserts that for $p \in (0, 1]$, any $(\varepsilon p^{t-1} n / \log n, p)$ -bijumbled graph of minimum degree at least $pn/2$ contains a K_t -factor if $\varepsilon = \varepsilon_t > 0$ is sufficiently small and $t \mid n$. It is straightforward to generalize Theorem 1.2 to K_t -factors and $(\varepsilon p^t n, p)$ -bijumbled graphs with minimum degree $pn/2$ and $\varepsilon = \varepsilon_t > 0$ sufficiently small.

A condition for arbitrary 2-factors. In his concluding remarks, Nenadov [15] raises the question whether the condition $\lambda = o(p^2 n / \log n)$ is sufficient to force any (λ, p) -bijumbled graph G of minimum degree $\Omega(pn)$ to contain any given 2-factor, i.e., any n -vertex 2-regular graph. Since any 2-factor consists of vertex-disjoint cycles whose lengths add up to n , the problem is thus to find any given collection of such cycles in G . We will return to this question elsewhere [7], with a positive answer to Nenadov’s question.

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