Existence of non-Cayley Haar graphs

Yan-Quan Feng[a], István Kovács[b], Jie Wang[c], Da-Wei Yang*[d]

[a] *Department of Mathematics, Beijing Jiaotong University, Beijing, 100044, P. R. China*

[b] *IAM and FAMNIT, University of Primorska, Glakoljaška 8, 6000 Koper, Slovenia*

[c] *School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China*

[d] *School of Sciences, Beijing University of Posts and Telecommunications, Beijing, 100876, P.R. China*

Abstract

A Cayley graph of a group H is a finite simple graph Γ such that its automorphism group $\mathrm{Aut}(\Gamma)$ contains a subgroup isomorphic to H acting regularly on $V(\Gamma)$, while a Haar graph of H is a finite simple bipartite graph Σ such that $\mathrm{Aut}(\Sigma)$ contains a subgroup isomorphic to H acting semiregularly on $V(\Sigma)$ and the H-orbits are equal to the partite sets of Σ. It is well-known that every Haar graph of finite abelian groups is a Cayley graph. In this paper, we prove that every finite non-abelian group admits a non-Cayley Haar graph except the dihedral groups D_6, D_8, D_{10}, the quaternion group Q_8 and the group $Q_8 \times \mathbb{Z}_2$. This answers an open problem proposed by Estélyi and Pisanski in 2016.

Keywords: Haar graph, Cayley graph, vertex-transitive graph.

MSC 2010: 05E18 (primary), 20B25 (secondary).

1 Introduction

All groups in this paper are finite and all graphs are finite and undirected. Let H be a group, and let R, L and S be three subsets of H such that $R^{-1} = R$, $L^{-1} = L$, and $R \cup L$ does not contain the identity element 1 of H. The *Cayley graph* of H relative to the subset R, denoted by $\mathrm{Cay}(H, R)$, is the graph having vertex set H, and edge set $\{\{h, xh\} : x \in R, h \in H\}$, and the *bi-Cayley graph* of H relative to the triple (R, L, S), denoted by $\mathrm{BiCay}(H, R, L, S)$, is the graph having vertex set the union of the right part

The first author was partially supported by the National Natural Science Foundation of China (11731002, 11571035) and by the 111 Project of China (B16002). The second author was partially supported by the Slovenian Research Agency (research program P1-0285, and research projects N1-0062, J1-9108 and J1-1695). The third author was partially supported by the National Natural Science Foundation of China (11731002). The fourth author was partially supported by the Fundamental Research Funds for the Central Universities and Innovation Foundation of BUPT for Youth (500419775).

*Corresponding author. E-mails: yqfeng@bjtu.edu.cn (Y.-Q. Feng), istvan.kovacs@upr.si (I. Kovács), wangj@pku.edu.cn (J. Wang), dwyang@bjtu.edu.cn (D.-W. Yang).

$H_0 = \{h_0 : h \in H\}$ and the left part $H_1 = \{h_1 : h \in H\}$, and edge set being the union of the following three sets

- $\big\{\{h_0, (xh)_0\} : x \in R,\, h \in H\big\}$ (right edges),
- $\big\{\{h_1, (xh)_1\} : x \in L,\, h \in H\big\}$ (left edges),
- $\big\{\{h_0, (xh)_1\} : x \in S,\, h \in H\big\}$ (spokes).

In the special case when $R = L = \emptyset$, the bi-Cayley graph $\mathrm{BiCay}(H, \emptyset, \emptyset, S)$ is called a *Haar graph of H relative to the set S*, denoted by $\mathrm{H}(H, S)$. A Haar graph $H(H, S)$ of a finite group H was first defined as a voltage graph of a dipole with no loops and $|S|$ parallel edges (see [15]), and the name *Haar graph* comes from the fact that, when H is an abelian group the Schur norm of the corresponding adjacency matrix can be easily evaluated via the so called Haar integral on H (see [14]).

Symmetries of Cayley graphs have always been an active topic among algebraic combinatorics, and lately, the symmetries of bi-Cayley graphs received considerable attention. For various results and constructions in connection with bi-Cayley graphs and their automorphisms, we refer the reader to [1, 2, 6, 9, 20, 23, 28, 29] and all the references therein. In particular, Estélyi and Pisanski [9] initiated the investigation for the relationship between Cayley graphs and Haar graphs. A Cayley graph is a Haar graph exactly when it is bipartite, but no simple condition is known for a Haar graph to be a Cayley graph. An elementary argument shows that every Haar graph of abelian groups is a Cayley graph (this also follows from Proposition 2.1). On the other hand, Lu et al. [22] constructed cubic semi-symmetric graphs, that is, edge- but not vertex-transitive graphs, as Haar graphs of alternating groups. Clearly, as these graphs are not vertex-transitive, they are examples of Haar graphs which are not Cayley graphs. It is natural to ask which non-abelian groups admit a Haar graph that is not a Cayley graph, or putting it another way, we have the following problem, which was first posed by Estélyi and Pisanski [9, Problem 1].

Problem 1.1. ([9]) *Determine the finite non-abelian groups H for which all Haar graphs $\mathrm{H}(H, S)$ are Cayley graphs.*

We denote by $\mathbb{Z}_n$ the cyclic group of order n, by D_{2n} the dihedral group of order $2n$, and by Q_8 the quaternion group. Estélyi and Pisanski [9, Theorem 8] solved Problem 1.1 for dihedral groups.

Proposition 1.2. ([9]) *Each Haar graph of the dihedral group D_{2n} is a Cayley graph if and only if $n = 2, 3, 4, 5$.*

A group H is called *inner abelian* if H is non-abelian, and all proper subgroups of H are abelian. Recently, Feng et al. [10, Theorem 1.2] solved Problem 1.1 for the class of inner abelian groups.

Proposition 1.3. ([10]) *Each Haar graph of an inner abelian group H is a Cayley graph if and only if $H \cong D_6$, D_8, D_{10} or Q_8.*

In this paper we solve Problem 1.1 completely.

Theorem 1.4. *Let H be a non-abelian group with the property that every Haar graph of H is a Cayley graph. Then H is isomorphic to D_6, D_8, D_{10}, Q_8 or $Q_8 \times \mathbb{Z}_2$.*

The main idea of the proof of Theorem 1.4 is to construct non-Cayley Haar graphs. It is worth mentioning that all non-Cayley Haar graphs of non-abelian groups, constructed in [9, 10] and this paper, are not vertex-transitive. It seems difficulty to construct vertex-transitive non-Cayley Haar graphs. Estélyi and Pisanski [9] raised a question whether there exists a vertex-transitive non-Cayley Haar graph. Later, infinitely many vertex-transitive non-Cayley Haar graphs were constructed by Conder et al. [6] and Feng et al. [12], and this prompts us to consider the following problem.

Problem 1.5. *Determine the finite non-abelian groups H for which all vertex-transitive Haar graphs $\mathrm{H}(H, S)$ are Cayley graphs.*

Note that Problem 1.5 is closely related to the so called non-Cayley numbers. A positive integer n is called a *Cayley number* if every vertex-transitive graph of order n is a Cayley graph, and otherwise it is a *non-Cayley number*. In 1983, Marušič [24] posed the problem of determining Cayley numbers, and this question has generated a fair amount of interests. For some works about Cayley numbers and vertex-transitive non-Cayley graphs, one may refer to [7, 21, 30].

By a *graphical regular representation* (GRR for short) for a group H we mean a Cayley graph Γ of H such that $\mathrm{Aut}(\Gamma) \cong H$. When studying a Cayley graph Γ of a finite group H, a very important question is to determine whether H is in fact the full automorphism group of Γ. For this reason, GRRs have been widely studied. The most natural question is classifying finite groups admitting a GRR, and the solution was derived in several papers (see, for instance, [4, 11, 17, 18, 19, 25, 26, 27]). A bi-Cayley graph Σ of a group H is called a *bi-graphical regular representation* (bi-GRR for short) if $\mathrm{Aut}(\Sigma) \cong H$. The problem of classifying finite groups admitting a bi-GRR was posed by Zhou [31] (also see [16]), and it was solved by Du et al. [8] recently. Motivated by GRR and bi-GRR, a *GHRR* of a group H is a Haar graph Γ of H with $\mathrm{Aut}(\Gamma) \cong H$. Since every Haar graph of abelian groups is a Cayley graph, abelian groups have no GHRR. However, many non-abelian groups have GHRRs, for example, see [9, 10] and Section 3 of this paper. Moreover, Theorem 1.4 implies that the non-abelian groups D_6, D_8, D_{10}, Q_8 and $Q_8 \times \mathbb{Z}_2$ have no GHRRs, and to the best of our knowledge, they are the only known non-abelian groups that have no GHRRs. In the end of this section, we would like to pose the following problem.

Problem 1.6. *Determine the finite non-abelian groups that have no GHRRs.*

The rest of the paper is organized as follows. In the next section we collect all concepts and results that will be used later. In Section 3, we introduce some Haar graphs that are not vertex-transitive, and prove Theorem 1.4 in Section 4.

2 Preliminaries

For a graph Γ, we denote by $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ the vertex set, the edge set and the group of all automorphisms of Γ. Given a vertex $v \in V(\Gamma)$, we denote by $\Gamma(v)$ the set of vertices adjacent to v. For a subgroup G of $\mathrm{Aut}(\Gamma)$, denote by G_v the stabilizer of the vertex v in G, that is, the subgroup of G fixing v. We say that G is *semiregular* on $V(\Gamma)$ if $G_v = 1$ for every $v \in V(\Gamma)$, and *regular* if G is transitive and semiregular.

Let $\Gamma = \mathrm{H}(H, S)$ be a Haar graph of a group H with identity element 1. By [28, Lemma 3.1(2)], up to graph isomorphism, we may always assume that $1 \in S$. The graph Γ is then connected exactly when $H = \langle S \rangle$. For $g \in H$, the *right translation* $R(g)$ is the permutation of H defined by $R(g) : h \mapsto hg$ for $h \in H$, and the *left translation* $L(g)$ is the permutation of H defined by $L(g) : h \mapsto g^{-1}h$ for $h \in H$. Set $R(H) = \{R(h) : h \in H\}$. Recall that $V(\Gamma) = H_0 \cup H_1$. It is easy to see that $R(H)$ can be regarded as a group of automorphisms of $\mathrm{H}(H, S)$ acting on $V(\Gamma)$ by the rule

$$R(g) : \ h_i \mapsto (hg)_i, \ \forall i \in \{0, 1\}, \ \forall h, g \in H.$$

Furthermore, $R(H)$ acts semiregularly on $V(\Gamma)$ with two orbits H_0 and H_1.

For an automorphism $\alpha \in \mathrm{Aut}(H)$ and $x, y, g \in H$, define two permutations on $V(\Gamma) = H_0 \cup H_1$ as follows

$$\delta_{\alpha,x,y} : \ h_0 \mapsto (xh^\alpha)_1, \ h_1 \mapsto (yh^\alpha)_0, \quad \forall h \in H; \tag{1}$$

$$\sigma_{\alpha,g} : \ h_0 \mapsto (h^\alpha)_0, \ h_1 \mapsto (gh^\alpha)_1, \quad \forall h \in H. \tag{2}$$

Set

$$\begin{aligned} \mathrm{I} &= \{\delta_{\alpha,x,y} : \alpha \in \mathrm{Aut}(H), \ S^\alpha = y^{-1}S^{-1}x\}, \\ \mathrm{F} &= \{\sigma_{\alpha,g} : \alpha \in \mathrm{Aut}(H), \ S^\alpha = g^{-1}S\}. \end{aligned}$$

By [28, Lemma 3.3], $\mathrm{F} \leq \mathrm{Aut}(\Gamma)_{1_0}$, and if Γ is connected, then F acts on the set $\Gamma(1_0)$ consisting of all neighbours of 1_0 faithfully. By [28, Theorem 1.1 and Lemma 3.2], we have the following proposition.

Proposition 2.1. *Let $\Gamma = \mathrm{H}(H, S)$ be a connected Haar graph, and let $A = \mathrm{Aut}(\Gamma)$.*

(i) *If $\mathrm{I} = \emptyset$, then the normalizer $N_A(R(H)) = R(H) \rtimes \mathrm{F}$.*

(ii) *If $\mathrm{I} \neq \emptyset$, then $N_A(R(H)) = R(H)\langle \mathrm{F}, \delta_{\alpha,x,y}\rangle$ for some $\delta_{\alpha,x,y} \in \mathrm{I}$.*

Moreover, $\langle R(H), \delta_{\alpha,x,y}\rangle$ acts transitively on $V(\Gamma)$ for any $\delta_{\alpha,x,y} \in \mathrm{I}$.

Throughout the paper we follow the notation defined in [10]:

$$\mathcal{BC} = \big\{H \text{ is a finite group} : \mathrm{H}(H, S) \text{ is a Cayley graph for any } S \subseteq H\big\}.$$

The following proposition was given by [10, Lemma 3.1].

Proposition 2.2. *The class $\mathcal{BC}$ is closed under taking subgroups.*

In view of [10, Theorem 1.3 and Corollary 4.6], we have the following proposition.

Proposition 2.3. *Let H be a group belonging to the class $\mathcal{BC}$. Then the following hold.*

(i) *The group H is solvable.*

(ii) *Each Sylow p-subgroup of H with a prime $p \geq 3$ is abelian.*

(iii) *If H is non-abelian, then H has a subgroup isomorphic to D_6, D_8, D_{10} or Q_8.*

The following proposition is well-known, and one may see [3, (1.12)].

Proposition 2.4. *Let P be a finite abelian p-group. Then $P = \mathbb{Z}_{p^{e_1}} \times \mathbb{Z}_{p^{e_2}} \times \cdots \times \mathbb{Z}_{p^{e_n}}$, where $1 \leq e_1 \leq e_2 \leq \cdots \leq e_n$. Moreover, the integers n and e_i with $1 \leq i \leq n$ are uniquely determined by P.*

3 Haar graphs that are not vertex-transitive

In this section, we introduce some Haar graphs that are not vertex-transitive, which will be used in the proof of Theorem 1.4. First we describe two infinite families of Haar graphs that are not vertex-transitive.

Lemma 3.1. *Let n be an integer with $n \geq 3$, and let p be an odd prime. Let*

$$H = D_{2n} \times \mathbb{Z}_p = \langle a, b, c \mid a^n = b^2 = c^p = [a, c] = [b, c] = 1, a^b = a^{-1} \rangle,$$

and $S = \{1, a, b, c, abc\}$. Then $\mathrm{Aut}(\mathrm{H}(H, S)) = R(H)$ and $H \notin \mathcal{BC}$.

Proof. Let $\Gamma = \mathrm{H}(H, S)$ and let $A = \mathrm{Aut}(\Gamma)$. Note that $R(H) \leq A$ has exactly two orbits on $V(\Gamma)$. Then A is vertex-transitive or has two orbits, that is, H_0 and H_1. For the former, A_{1_0} and A_{1_1} are conjugate in A, and for the latter, the Frattini argument implies that $A = R(H)A_{1_0} = R(H)A_{1_1}$. In the both cases, $|A_{1_0}| = |A_{1_1}|$, and hence $|A_{1_0}| = |A_{h_0}| = |A_{k_1}|$ for any $h, k \in H$. To finish the proof, it suffices to show that $A_{1_0} = 1$ and Γ is not vertex-transitive.

We depicted the subgraph of Γ induced by the vertices at distance at most 2 from 1_0 in Figure 1.

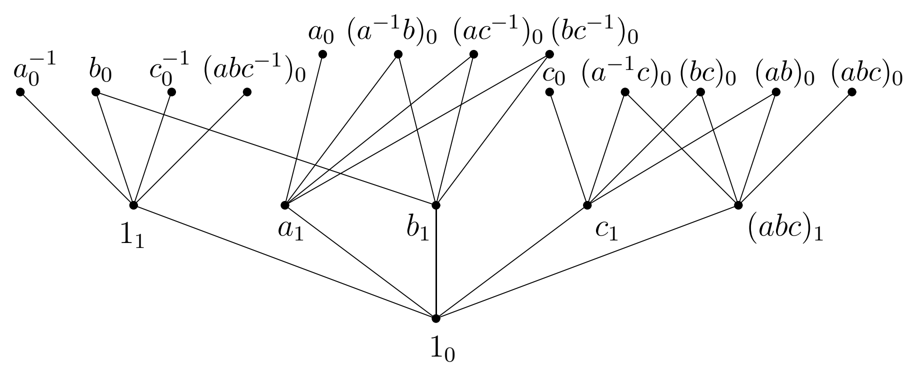

Figure 1: The subgraph of Γ induced by the vertices at distance at most 2 from 1_0.

Consider the 4-cycles of Γ passing through the vertex 1_0. For each $h \in H$, denote by $\Gamma(h_0)$ and $\Gamma(h_1)$ the neighborhoods of h_0 and h_1 in Γ respectively, that is, $\Gamma(h_0) =$

$\{(sh)_1 \mid s \in S\}$ and $\Gamma(h_1) = \{(s^{-1}h)_0 \mid s \in S\}$. By Figure 1, the numbers of 4-cycles passing through the edges $\{1_0, 1_1\}$ and $\{1_0, b_1\}$ are 1 and 4, respectively, while there are exactly three 4-cycles passing through the edge $\{1_0, u_1\}$ for each $u_1 = a_1, c_1$ or $(abc)_1$. This implies that A_{1_0} fixes 1_1 and b_1, and $\{a_1, c_1, (abc)_1\}$ setwise. It follows that $A_{1_0} \leq A_{1_1}$ and $A_{1_0} \leq A_{b_1}$, and since there is a unique 4-cycle passing through 1_0 and 1_1, we have $A_{1_0} \leq A_{b_0}$. Since $|A_{1_0}| = |A_{h_0}| = |A_{k_1}|$ for any $h, k \in H$, we have $A_{1_0} = A_{1_1} = A_{b_1} = A_{b_0}$.

By Figure 1, there are 4-cycles passing through $(a_1, 1_0, b_1)$ but no 4-cycles passing through $(c_1, 1_0, b_1)$ or $((abc)_1, 1_0, b_1)$, and since A_{1_0} fixes b_1 and $\{a_1, c_1, (abc)_1\}$ setwise, A_{1_0} fixes a_1, and $\{c_1, (abc)_1\}$ setwise. Thus, A_{1_0} fixes $\Gamma(a_1)$ setwise, and since there exist 4-cycles passing through 1_0, a_1 and a vertex in $\Gamma(a_1)$ except a_0, we have $A_{1_0} \leq A_{a_0}$. It follows that $A_{1_0} = A_{a_0} = A_{a_1}$.

Now we claim that A_{1_0} fixes c_1 and $(abc)_1$. Note that A_{1_0} fixes $\{c_1, (abc)_1\}$ setwise. Suppose that $\alpha \in A_{1_0}$ interchanges c_1 and $(abc)_1$. By Figure 1, there exist 4-cycles passing through 1_0, c_1 (resp. $(abc)_1$) and a vertex in $\Gamma(c_1)$ (resp. $\Gamma((abc)_1)$) except c_0 (resp. $(abc)_0$), and hence α interchanges c_0 and $(abc)_0$. Since A_{1_0} fixes a_0, we have $(\Gamma(a_0) \cap \Gamma(c_0))^\alpha = \Gamma(a_0) \cap \Gamma((abc)_0)$. Clearly, $(ac)_1 \in \Gamma(a_0) \cap \Gamma(c_0)$. Then $|\Gamma(a_0) \cap \Gamma((abc)_0)| \neq 0$, and hence there exist $s, t \in S$ such that $sa = tabc$, that is, $t^{-1}s = a^2bc \in S^{-1}S$. This is impossible as $S = \{1, a, b, c, abc\}$. Thus A_{1_0} fixes c_1 and $(abc)_1$, and hence c_0 and $(abc)_0$. It follows that $A_{1_0} = A_{c_0} = A_{c_1}$.

Now we have that $A_{1_0} = A_{x_0}$ for each $x \in T := \{a, b, c\}$. For any $y \in T$, we have $A_{1_0}^{R(y)} = A_{x_0}^{R(y)}$, that is, $A_{y_0} = A_{(xy)_0}$. It follows that $A_{1_0} = A_{(xy)_0}$, and an easy inductive argument implies that $A_{1_0} = A_{(x_1x_2\cdots x_n)_0}$ for any $x_1, \cdots, x_n \in T$. Since $\langle T\rangle = H$, A_{1_0} fixes H_0 pointwise. Since $A_{1_0} = A_{1_1}$, we have $A_{h_0} = A_{h_1}$ for any $h \in H$, and hence A_{1_0} fixes H_1 pointwise. Thus, $A_{1_0} = 1$.

To finish the proof, we are left with showing that A is not vertex-transitive. Suppose to the contrary that A is vertex-transitive. Since $A_{1_0} = 1$, we have $|A| = |V(\Gamma)| = 2|R(H)|$ and hence $R(H) \trianglelefteq A$. By Proposition 2.1, there exists $\delta_{\beta,x,y} \in A$ for some $\beta \in \mathrm{Aut}(H)$ and $x, y \in H$ such that $S^\beta = y^{-1}S^{-1}x$. Since $R(H)$ acts transitively on H_1, we may further assume that $1_0^{\delta_{\beta,x,y}} = 1_1$. By Eq. (1), $1_0^{\delta_{\beta,x,y}} = (x1^\beta)_1 = 1_1$, forcing $x = 1$. Thus $S^\beta = y^{-1}S^{-1}$, that is,

$$S^\beta = \{1^\beta, a^\beta, b^\beta, c^\beta, (abc)^\beta\} = y^{-1}\{1, a^{-1}, b, c^{-1}, abc^{-1}\}. \tag{3}$$

Since $1 \in S$, we have $1 \in S^\beta$ and so $y^{-1} = 1, a, b^{-1}, c$ or abc.

Note that $H = D_{2n} \times \mathbb{Z}_p = \langle a, b, c \mid a^n = b^2 = c^p = [a, c] = [b, c] = 1, a^b = a^{-1}\rangle$. If n is odd then the center $Z(H) = \mathbb{Z}_p$, and if $n = 2m$ is even then $Z(H) = \langle a^m\rangle \times \mathbb{Z}_p \cong \mathbb{Z}_2 \times \mathbb{Z}_p$, where $\mathbb{Z}_p$ is characteristic in $\langle a^m\rangle \times \mathbb{Z}_p$. It follows that $\mathbb{Z}_p = \langle c\rangle$ is characteristic in H, and since $\beta \in \mathrm{Aut}(H)$, we have $c^\beta \in \langle c\rangle$.

If $y^{-1} = a, b^{-1}, c$ or abc, we have from Eq. (3) that $S^\beta = \{a, 1, ab, ac^{-1}, a^2bc^{-1}\}$, $\{b, ba^{-1}, 1, bc^{-1}, a^{-1}c^{-1}\}$, $\{c, ca^{-1}, cb, 1, ab\}$ or $\{abc, a^2bc, ac, ab, 1\}$, respectively. This is impossible because $c^\beta \in \langle c\rangle$. Thus, $y = 1$ and $S^\beta = \{1, a^{-1}, b, c^{-1}, abc^{-1}\}$. This implies $c^\beta = c^{-1}$ because $c^\beta \in \langle c\rangle$. Since all involutions of H generate the dihedral subgroup $\langle a, b\rangle$, $\langle a, b\rangle$ is characteristic in H, and since $\langle a, b\rangle$ is dihedral, $\langle a\rangle$ is characteristic in H. Thus, $a^\beta \in \langle a\rangle$ and $b^\beta \in \langle a, b\rangle$, and since $S^\beta = \{1, a^{-1}, b, c^{-1}, abc^{-1}\}$, we have $a^\beta = a^{-1}$, $b^\beta = b$ and

$(abc)^\beta = abc^{-1}$. However, $abc^{-1} = (abc)^\beta = a^\beta b^\beta c^\beta = a^{-1}bc^{-1}$, that is, $a^2 = a$, contrary the hypothesis $n \geq 3$. This completes the proof. $\square$

Lemma 3.2. *Let p be an odd prime, and let*

$$H = Q_8 \times \mathbb{Z}_p = \langle a, b, c \mid a^4 = b^4 = c^p = [a, c] = [b, c] = 1, a^2 = b^2, a^b = a^{-1} \rangle,$$

and $S = \{1, a, c, abc^{-1}, bc\}$. Then $\mathrm{Aut}(\mathrm{H}(H, S)) = R(H)$ and $H \notin \mathcal{BC}$.

Proof. Let $\Gamma = \mathrm{H}(H, S)$ and let $A = \mathrm{Aut}(\Gamma)$. The lemma holds for $p = 3$ and 5 by Magma [5], and we assume that $p \geq 7$ in the rest of the proof. Since A is transitive or has the two orbits H_0 and H_1 as same as $R(H)$, we have $|A_{h_0}| = |A_{k_1}|$ for any $h, k \in H$.

For each $h \in H$, the neighborhood of the vertices h_0 and h_1 in Γ are $\{(sh)_1 \mid s \in S\}$ and $\{(s^{-1}h)_0 \mid s \in S\}$, respectively. From this, it is easy to list the vertices in Γ having distance at most 2 from 1_0 or c_1 in Table 1.

v	neighbors of v	vertices having distance 2 from v
1_0	1_1	$(a^{-1})_0$, $(c^{-1})_0$, $(a^{-1}bc)_0$, $(b^{-1}c^{-1})_0$
	a_1	a_0, $(ac^{-1})_0$, $(b^{-1}c)_0$, $(abc^{-1})_0$
	c_1	c_0, $(a^{-1}c)_0$, $(a^{-1}bc^2)_0$, $(b^{-1})_0$
	$(abc^{-1})_1$	$(abc^{-1})_0$, $(bc^{-1})_0$, $(abc^{-2})_0$, $(a^{-1}c^{-2})_0$
	$(bc)_1$	$(bc)_0$, $(a^{-1}bc)_0$, b_0, $(ac^2)_0$
c_1	c_0	$(ac)_1$, $(c^2)_1$, $(ab)_1$, $(bc^2)_1$
	$(a^{-1}c)_0$	$(a^{-1}c)_1$, $(a^{-1}c^2)_1$, $(b^{-1})_1$, $(abc^2)_1$
	1_0	1_1, a_1, $(abc^{-1})_1$, $(bc)_1$
	$(a^{-1}bc^2)_0$	$(a^{-1}bc^2)_1$, $(bc^2)_1$, $(a^{-1}bc^3)_1$, $(a^{-1}c^3)_1$
	$(b^{-1})_0$	$(b^{-1})_1$, $(a^{-1}b)_1$, $(b^{-1}c)_1$, $(ac^{-1})_1$

Table 1: The vertices in Γ having distance at most 2 from 1_0 or c_1.

Furthermore, we have the following equations:

$$\Gamma((b^{-1})_1) = \{(b^{-1})_0, (ab)_0, (b^{-1}c^{-1})_0, (a^{-1}c)_0, (b^2c^{-1})_0\}, \tag{4}$$
$$\Gamma((bc^2)_1) = \{(bc^2)_0, (a^{-1}bc^2)_0, (bc)_0, (ac^3)_0, c_0\}. \tag{5}$$

By Table 1, there are exactly two 4-cycles C_1 and C_2 passing through 1_0:

$$C_1 = (1_0, 1_1, (a^{-1}bc)_0, (bc)_1), \quad C_2 = (1_0, a_1, (abc^{-1})_0, (abc^{-1})_1),$$

and there are exactly two 4-cycles C_3 and C_4 passing through c_1:

$$C_3 = (c_1, c_0, (bc^2)_1, (a^{-1}bc^2)_0), \quad C_4 = (c_1, (b^{-1})_0, (b^{-1})_1, (a^{-1}c)_0).$$

We depicted, using Table 1, Eqs. (4) and (5), an induced subgraph of Γ in Figure 2.

There exist 4-cycles passing through 1_0 and any given vertex in $\Gamma(1_0)$ except c_1. Then $A_{1_0} \leq A_{c_1}$ and hence A_{1_0} fixes $\{C_1, C_2\}$ and $\{C_3, C_4\}$ setwise. Furthermore, A_{1_0} fixes

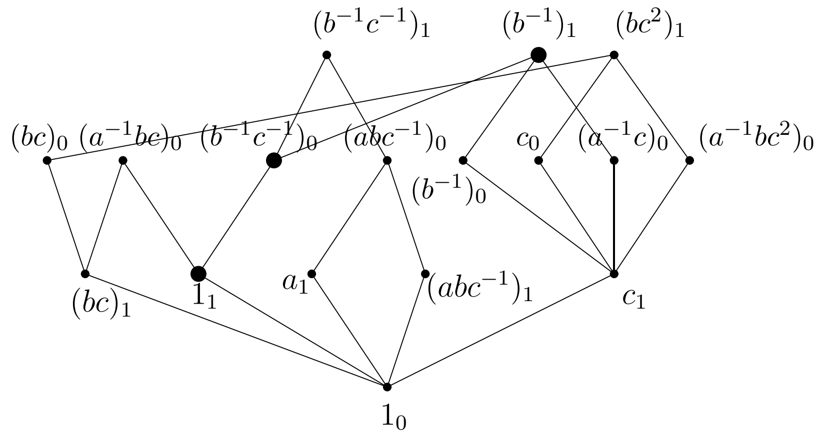

Figure 2: An induced subgraph of Γ.

$\{(b^{-1})_1, (bc^2)_1\}$ setwise because these two vertices are antipodal to c_1 in C_3 and C_4 respectively, and since $|A_{h_0}| = |A_{k_1}|$ for any $h, k \in H$, we have $A_{1_0} = A_{c_1}$.

We first prove that A_{1_0} fixes the 4-cycle C_1 setwise. Recall that A_{1_0} fixes $\{C_1, C_2\}$ setwise. Suppose to the contrary that $\alpha \in A_{1_0}$ interchanges C_1 and C_2. Then $\{1_1, (bc)_1\}^\alpha = \{a_1, (abc^{-1})_1\}$, and since A_{1_0} fixes $\{(b^{-1})_1, (bc^2)_1\}$ setwise, we have $\{[\Gamma(1_1) \cup \Gamma((bc)_1)] \cap [\Gamma((b^{-1})_1) \cup \Gamma((bc^2)_1)]\}^\alpha = [\Gamma(a_1) \cup \Gamma((abc^{-1})_1)] \cap [\Gamma((b^{-1})_1) \cup \Gamma((bc^2)_1)]$, which is impossible because $[\Gamma(a_1) \cup \Gamma((abc^{-1})_1)] \cap [\Gamma((b^{-1})_1) \cup \Gamma((bc^2)_1)] = \emptyset$ and $(bc)_0 \in \Gamma((bc)_1) \cap \Gamma((bc^2)_1)$ by Table 1, Eqs. (4) and (5). Thus, A_{1_0} fixes C_1 setwise.

Now we prove that A_{1_0} fixe C_1 pointwise. Since A_{1_0} fixes C_1 setwise, it fixes C_2 setwise, implying A_{1_0} fixes $(abc^{-1})_0$. Suppose to the contrary that $\beta \in A_{1_0}$ interchanges 1_1 and $(bc)_1$. By Table 1, $\Gamma((bc)_1) \cap [\Gamma((b^{-1})_1) \cup \Gamma((bc^2)_1)] = \{(bc)_0\}$ and $\Gamma(1_1) \cap [\Gamma((b^{-1})_1) \cup \Gamma((bc^2)_1)] = \{(b^{-1}c^{-1})_0\}$. Since A_{1_0} fixes $\{(b^{-1})_1, (bc^2)_1\}$ setwise, β interchanges $(bc)_0$ and $(b^{-1}c^{-1})_0$, implying $\Gamma((bc)_0)^\beta = \Gamma((b^{-1}c^{-1})_0)$. Since A_{1_0} fixes $(abc^{-1})_0$, we have $[\Gamma((abc^{-1})_0) \cap \Gamma((bc)_0)]^\beta = \Gamma((abc^{-1})_0) \cap \Gamma((b^{-1}c^{-1})_0)$. It is easy to see that $(b^{-1}c^{-1})_1 \in \Gamma((abc^{-1})_0) \cap \Gamma((b^{-1}c^{-1})_0)$. Then $\Gamma((abc^{-1})_0) \cap \Gamma((bc)_0) \neq \emptyset$, and there is $s, t \in S$ such that $sabc^{-1} = tbc$, that is, $s^{-1}t = ac^{-2} \in S^{-1}S$, which is impossible because $S^{-1}S = \{1, a, c, abc^{-1}, bc, a^{-1}, a^{-1}c, bc^{-1}, a^{-1}bc, c^{-1}, ac^{-1}, abc^{-2}, b, b^{-1}c, a^{-1}bc^2, ac^2, b^{-1}c^{-1}, b^{-1}, a^{-1}c^{-2}\}$. Thus, A_{1_0} fixes C_1 pointwise, and hence $A_{1_0} = A_{1_1} = A_{(bc)_1} = A_{(a^{-1}bc)_0}$.

Since $A_{1_0} = A_{c_1}$, we have $A_{1_0}^{R(c^{-1})} = A_{c_1}^{R(c^{-1})}$, and so $A_{c_0^{-1}} = A_{1_1} = A_{1_0}$. Similarly, we have $A_{1_0} = A_{(b^{-1}c^{-1})_0}$ because $A_{1_0} = A_{(bc)_1}$. It follows that $A_{1_0} = A_{x_0}$ for any $x \in T := \{c^{-1}, b^{-1}c^{-1}, a^{-1}bc\}$, and an easy inductive argument implies that $A_{1_0} = A_{x_0}$ for any $x \in \langle T \rangle = H$. Thus, A_{1_0} fixes H_0 pointwise. Also, since $A_{1_0} = A_{1_1}$ implies $A_{h_0} = A_{h_1}$ for any $h \in H$, it follows that A_{1_0} fixes H_1 pointwise too. Thus, $A_{1_0} = 1$.

To finish the proof, we are left with showing that A is not vertex-transitive. Suppose to the contrary that A is vertex-transitive. Since $A_{1_0} = 1$, we have $|A| = |V(\Gamma)| = 2|R(H)|$ and hence $R(H) \trianglelefteq A$. By Proposition 2.1, there exists $\delta_{\beta,x,y} \in A$ for some $\beta \in \mathrm{Aut}(H)$ and $x, y \in H$ such that $S^\beta = y^{-1}S^{-1}x$. By the transitivity of $R(H)$ on H_1, we may assume $1_0^{\delta_{\beta,x,y}} = 1_1$, anb by Eq. (1), $1_0^{\delta_{\beta,x,y}} = (x1^\beta)_1 = 1_1$, forcing $x = 1$. Recall that

$S = \{1, a, c, abc^{-1}, bc\}$, and $S^\beta = y^{-1}S^{-1}$, that is,

$$S^\beta = \{1^\beta, a^\beta, c^\beta, (abc^{-1})^\beta, (bc)^\beta\} = y^{-1}\{1, a^{-1}, c^{-1}, a^{-1}bc, b^{-1}c^{-1}\}. \tag{6}$$

Since $1 \in S$, we have $1 \in S^\beta$ and so $y^{-1} = 1, a, c, abc^{-1}$ or bc.

Note that $H = Q_8 \times \mathbb{Z}_p = \langle a, b, c \mid a^4 = b^4 = c^p = [a, c] = [b, c] = 1, a^2 = b^2, a^b = a^{-1}\rangle$. Then Q_8 and $\mathbb{Z}_p$ are characteristic in H, and hence $c^\beta \in \langle c\rangle$ and $\langle a, b\rangle^\beta = \langle a, b\rangle$.

Let $y^{-1} = a, abc^{-1}$ or bc. By Eq (6), $S^\beta = \{a, 1, ac^{-1}, bc, ab^{-1}c^{-1}\}$, $\{abc^{-1}, b^{-1}c^{-1}, abc^{-2}, 1, ac^{-2}\}$ or $\{bc, ba^{-1}c, b, a^{-1}c^2, 1\}$, which is impossible because $c^\beta \in \langle c\rangle$.

Let $y = 1$. Then $S^\beta = \{1, a^{-1}, c^{-1}, a^{-1}bc, b^{-1}c^{-1}\}$ by Eq. (6). Since $a^\beta \in \langle a, b\rangle$ and $c^\beta \in \langle c\rangle$, we have $a^\beta = a^{-1}$, $c^\beta = c^{-1}$ and $\{(abc^{-1})^\beta, (bc)^\beta\} = \{a^{-1}bc, b^{-1}c^{-1}\}$, and since $(bc)^\beta = b^\beta c^{-1} \in \langle a, b\rangle c^{-1}$, we have $(bc)^\beta = b^{-1}c^{-1}$. Hence $b^\beta = b^{-1}$ and $(abc^{-1})^\beta = a^{-1}bc$. However, $a^{-1}bc = (abc^{-1})^\beta = a^\beta b^\beta (c^{-1})^\beta = a^{-1}b^{-1}c$, forcing $b^2 = 1$, a contradiction.

Let $y^{-1} = c$. Then $S^\beta = \{c, a^{-1}c, 1, a^{-1}bc^2, b^{-1}\}$, yielding that $a^\beta = b^{-1}$ and $c^\beta = c$. Since $(abc^{-1})^\beta = (ab)^\beta c^{-1} \in S^\beta$, we have $(ab)^\beta c^{-1} = a^{-1}c$ or $a^{-1}bc^2$, forcing $c^2 = 1$ or $c^3 = 1$, contradicting $p \geq 7$. This completes the proof. □

To end this section, we describe some Haar graphs of small orders that are not vertex-transitive, and this can be checked easily by the computer software Magma [5].

Lemma 3.3. *Let $G = H_i$ and S be given in the following table for each $1 \leq i \leq 9$:*

i	H_i	S
1	$\langle a, b, c \mid a^8, b^2, c^2, [a, c], [b, c], a^b = a^{-1}\rangle \cong D_8 \times \mathbb{Z}_2$	$\{1, a, b, c, ab, abc\}$
2	$\langle a, b, c \mid a^4, b^2, c^2, [a, b], [a, c], [b, c] = a^2\rangle$	$\{1, a, b, ab, ac, abc\}$
3	$\langle a, b \mid a^8, b^2 = a^4, a^b = a^{-1}\rangle$	$\{1, a, b, a^5, ab, a^5b\}$
4	$\langle a, b \mid a^8, b^2, a^b = a^3\rangle$	$\{1, a, b, ab\}$
5	$\langle a, b, c, d \mid a^4, b^4, c^2, d^2, a^2 = b^2, a^b = a^{-1}, [a, c], [a, d],$ $[b, c], [b, d], [c, d]\rangle \cong Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\{1, a, b, b^{-1}, ab, ac,$ $bd, abd\}$
6	$\langle a, b, c \mid a^4, b^4, c^3, a^2 = b^2, a^b = a^{-1}, a^c = b^{\pm1}, b^c = a^{\pm1}b\rangle$	$\{1, a, bc, abc\}$
7	$\langle a, b, c \mid a^2, b^2, c^3, [a, b], a^c = b, b^c = ab\rangle \cong A_4$	$\{1, a, c, abc\}$
8	$\langle a, g \mid a^5, g^4, a^g = a^2\rangle \cong F_{20}$	$\{1, a, g\}$
9	$\langle a, c, b \mid a^p, c^p, b^2, [a, c], a^b = a^{-1}, c^b = c^{-1}\rangle \cong \mathbb{Z}_p^2 \rtimes \mathbb{Z}_2$, $p = 3, 5$	$\{1, a, c, b, ab, cb\}$

Then $\mathrm{Aut}(\mathrm{H}(G, S))$ *is not vertex-transitive and* $G \notin \mathcal{BC}$.

4 Proof of Theorem 1.4

In this section, we aim to prove Theorem 1.4. First, we need two lemmas.

Lemma 4.1. *A non-abelian 2-group belongs to the class $\mathcal{BC}$ if and only if it is isomorphic to D_8, Q_8 or $Q_8 \times \mathbb{Z}_2$.*

Proof. By Proposition 1.3, $D_8 \in \mathcal{BC}$ and $Q_8 \in \mathcal{BC}$. For $Q_8 \times \mathbb{Z}_2$, let $\Gamma = \mathrm{H}(Q_8 \times \mathbb{Z}_2, S)$ be a Haar graph with $1 \in S$. If Γ is not connected, then $\langle S\rangle < Q_8 \times \mathbb{Z}_2$ (see [9, Lemma 1 (i)]), and either $\langle S\rangle$ is abelian or $\langle S\rangle \cong Q_8$. This implies that the Haar graph $\mathrm{H}(\langle S\rangle, S)$ is a Cayley graph, and since Γ is a union of components with each isomorphic to $\mathrm{H}(\langle S\rangle, S)$, Γ is a Cayley graph. If Γ is connected, a computation by MAGMA [5] shows that all connected Haar graphs of $Q_8 \times \mathbb{Z}_2$ are Cayley graphs. Thus, $Q_8 \times \mathbb{Z}_2 \in \mathcal{BC}$.

Let H be a non-abelian 2-group and $H \in \mathcal{BC}$. To prove the necessity, it suffices to show that $H \cong D_8, Q_8$ or $Q_8 \times \mathbb{Z}_2$.

Case 1: $|H| \leq 8$.

Since H is a non-abelian 2-group, we have $H \cong D_8$ or Q_8.

Case 2: $|H| = 16$.

Note that all non-abelian groups of order 16 can be found in [13] (this can also be obtained by the computer software MAGMA [5]). By Proposition 2.3 (iii), H has a subgroup isomorphic to D_8 or Q_8, and hence $H \cong H_i$ for some $1 \leq i \leq 6$:

$$\begin{aligned}
&H_1 = \langle a, b, c \mid a^8 = b^2 = c^2 = [a, c] = [b, c] = 1, a^b = a^{-1}\rangle \ (\cong D_8 \times \mathbb{Z}_2);\\
&H_2 = \langle a, b, c \mid a^4 = b^2 = c^2 = [a, b] = [a, c] = 1, [b, c] = a^2\rangle;\\
&H_3 = \langle a, b \mid a^8 = 1, b^2 = a^4, a^b = a^{-1}\rangle;\\
&H_4 = \langle a, b \mid a^8 = b^2 = 1, a^b = a^3\rangle; \quad H_5 = D_{16}; \quad H_6 = Q_8 \times \mathbb{Z}_2.
\end{aligned}$$

By Proposition 1.2, $H_5 \notin \mathcal{BC}$, and by Lemma 3.3, $H_i \notin \mathcal{BC}$ for each $1 \leq i \leq 4$. It follows that $H \cong H_6 = Q_8 \times \mathbb{Z}_2$.

Case 3: $|H| \geq 32$.

Since $H \in \mathcal{BC}$, Proposition 2.2 implies that each subgroup of H belongs to $\mathcal{BC}$. If each subgroup of H of order 32 is abelian, then H has an inner abelian subgroup of order at least 64, which is impossible by Proposition 1.3. Thus, H has a non-abelian subgroup of order 32, say L. Then $L \in \mathcal{BC}$. Similarly, L has a non-abelian subgroup of order 16, and by the proof of Case 2, each non-abelian subgroup of L of order 16 is isomorphic to $Q_8 \times \mathbb{Z}_2$. By checking the non-abelian groups of order 32 listed in [13], we have $L \cong Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and by Lemma 3.3, $L \notin \mathcal{BC}$, a contradiction. $\square$

Lemma 4.2. *Let p be an odd prime, and let H be a non-abelian $\{2, p\}$-group with $p \mid |H|$. Then $H \in \mathcal{BC}$ if and only if $H \cong D_6$ or D_{10}.*

Proof. By Proposition 1.2, $D_6 \in \mathcal{BC}$ and $D_{10} \in \mathcal{BC}$. Let H be a non-abelian $\{2, p\}$-group with $p \mid |H|$. To prove the necessity, suppose to the contrary that H is a minimal counterexample, that is, $H \in \mathcal{BC}$ has the smallest order such that $H \ncong D_6$ or D_{10}.

Denote by P and P_2 a Sylow p-subgroup and a Sylow 2-subgroup of H, respectively. Then $H = PP_2$, and by Proposition 2.2, $P \in \mathcal{BC}$ and $P_2 \in \mathcal{BC}$. By Proposition 2.3 (ii), P is abelian, and by Lemma 4.1, either P_2 is abelian or $P_2 \cong D_8$, Q_8, or $Q_8 \times \mathbb{Z}_2$. Now we consider the two cases depending whether P_2 is normal in H.

Case 1: $P_2 \trianglelefteq H$.

Suppose that P_2 is abelian. It follows from Proposition 2.3 (iii) that H has a subgroup D_{2p} with $p = 3$ or 5. Since P_2 is the unique Sylow 2-subgroup of H, all involutions of D_{2p} are contained in P_2, and since D_{2p} can be generated by its two involutions, we have $D_{2p} \leq P_2$, which is impossible. Hence P_2 is non-abelain. By Proposition 2.2, $P_2 \in \mathcal{BC}$, and by Lemma 4.1, $P_2 \cong D_8$, Q_8 or $Q_8 \times \mathbb{Z}_2$. It follows $H = P_2 \rtimes P$, and by the minimality of H, $|P| = p$. Thus, $H = P_2 \rtimes P \cong D_8 \rtimes \mathbb{Z}_p$, $Q_8 \rtimes \mathbb{Z}_p$ or $(Q_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_p$.

Consider the centralizer $C_H(P_2)$ of P_2 in H. If $P \leq C_H(P_2)$, then $H = P_2 \times P \cong D_8 \times \mathbb{Z}_p$, $Q_8 \times \mathbb{Z}_p$, or $Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_p$, and by Lemmas 3.1 and 3.2, $D_8 \times \mathbb{Z}_p \notin \mathcal{BC}$ and $Q_8 \times \mathbb{Z}_p \notin \mathcal{BC}$. Also we have $Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_p \notin \mathcal{BC}$ because otherwise $Q_8 \times \mathbb{Z}_2 \times \mathbb{Z}_p \in \mathcal{BC}$ implies $Q_8 \times \mathbb{Z}_p \in \mathcal{BC}$ by Proposition 2.2. Thus, $H \notin \mathcal{BC}$, a contradiction. Hence $P \nleq C_H(P_2)$, and since $|P| = p$, we have $P \cap C_H(P_2) = 1$ and so $C_H(P_2) \leq P_2$. Note that $\mathrm{Aut}(D_8) \cong D_8$, $\mathrm{Aut}(Q_8) \cong S_4$, and $\mathrm{Aut}(Q_8 \times \mathbb{Z}_2) \cong \mathbb{Z}_2^3 \rtimes S_4$. Since $N_H(P_2)/C_H(P_2) = H/C_H(P_2) \lesssim \mathrm{Aut}(P_2)$, we have $P_2 \cong Q_8$ or $Q_8 \times \mathbb{Z}_2$, and $P \cong \mathbb{Z}_3$, which implies that a generator of P induces (by conjugacy) an automorphism of P_2 of order 3. For $P_2 \cong Q_8$, let $P_2 = \langle a, b \mid a^4 = b^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle$, and let α be the automorphism of P_2 of order 3 induced by the map $a \mapsto b$ and $b \mapsto ab$. Since all the automorphisms of P_2 of order 3 are conjugate in $\mathrm{Aut}(P_2)$, we have $H \cong P_2 \rtimes \langle \alpha \rangle = \langle a, b, \alpha \mid a^4 = b^4 = \alpha^3 = 1, a^2 = b^2, a^b = a^{-1}, a^\alpha = b, b^\alpha = ab \rangle$. By Lemma 3.3, $H \cong H_6$ and $H \notin \mathcal{BC}$, a contradiction. Similarly, for $P_2 \cong Q_8 \times \mathbb{Z}_2$, let $P_2 = \langle a, b \mid a^4 = b^4 = 1, a^2 = b^2, a^b = a^{-1} \rangle \times \langle c \rangle$ with $\langle c \rangle \cong \mathbb{Z}_2$, and let α be the automorphism of P_2 of order 3 induced by the map $a \mapsto b$, $b \mapsto ab$ and $c \mapsto c$. It follows that $H \cong (\langle a, b \rangle \times \langle c \rangle) \rtimes \langle \alpha \rangle \cong H_6 \times \mathbb{Z}_2$ and hence $H \notin \mathcal{BC}$ as $H_6 \notin \mathcal{BC}$, a contradiction.

Case 2: $P_2 \ntrianglelefteq H$.

Since P is abelian, we have $P \leq C_H(P) \leq N_H(P)$. If $C_H(P) = N_H(P)$, then the Burnside's p-nilpotency criterion implies that P has a normal complement N, that is, N is a normal Sylow 2-subgroup of H, which is impossible by Case 1. Thus, we may assume that $P \leq C_H(P) < N_H(P)$, and hence there exists a 2-element g such that $g \in N_H(P)$ and $g \notin C_H(P)$. In particular, $P \rtimes \langle g \rangle$ is a non-abelian subgroup of H and $p \mid |P \rtimes \langle g \rangle|$. By the minimality of H, either $H = P \rtimes \langle g \rangle$, or $H > P \rtimes \langle g \rangle \cong D_6$ or D_{10}.

Subcase 2.1: $H = P \rtimes \langle g \rangle$.

In this case, $P_2 = \langle g \rangle$ and $P \trianglelefteq H$. By Proposition 1.3, H contains a proper subgroup K such that $K \cong D_6$ or D_{10}. Then $p = 3$ or 5. Let $K = \langle a, b \mid a^p = b^2 = 1, a^b = a^{-1} \rangle$. We may assume $b \in P_2$, and since $P_2 = \langle g \rangle$, b is the unique involution in P_2. Furthermore, $PK \leq H$ is non-abelian. Let $|P| = p^s$ for some $s \geq 1$. Then $|PK| = 2p^s$.

If $s = 1$ then $P \cong \mathbb{Z}_p$ and hence $P \leq K$. Since $K \cong D_6$ or D_{10}, we have $K < H$, and since $K/P < H/P \cong P_2$, H contains a subgroup of order $4p$ with K as a subgroup of index 2. By the minimality of H, we have $P_2 = \langle g \rangle \cong \mathbb{Z}_4$. It follows $b = g^2$, and since $a^b = a^{-1}$, we have $p = 5$ and $a^g = a^2$ or a^3. This implies that $H \cong F_{20}$ and by Lemma 3.3, $H \notin \mathcal{BC}$, a contradiction. Thus, $s \geq 2$.

Suppose $s \geq 3$. Note that $P \trianglelefteq H$ and $K = \langle a, b \rangle \cong D_{2p} < H$ with $o(a) = p$. By the minimality of H, we have $H = PK$. Then $|H| = 2p^s$ and $P_2 = \langle g \rangle = \langle b \rangle \cong \mathbb{Z}_2$. If P is cyclic, it is easy to see that H is dihedral, which is impossible by Proposition 1.2. Thus, P is not cyclic, and since P is abelian, Proposition 2.4 implies that P has an element c of order p with $\langle c \rangle \cap \langle a \rangle = 1$. If $c^b \notin \langle c \rangle$ then $\langle c, c^b, b \rangle$ is a non-abelian subgroup of order $2p^2$, and by the minimality of H, we have $H = \langle c, c^b, b \rangle$, which is impossible because $|H| = 2p^s$ with $s \geq 3$. Similarly, if $c^b \in \langle c \rangle$ then $\langle a, c, b \rangle$ is a non-abelian subgroup of order $2p^2$, which is also impossible.

Thus, $s = 2$ and $|H| = 2p^2$. From the elementary group theory we know that up to isomorphism there are three non-abelian groups of order $2p^2$ defined as:

$$\begin{array}{rcl} H_1(p) & = & \langle a,b \mid a^{p^2}=b^2=1, b^{-1}ab=a^{-1}\rangle; \\ H_2(p) & = & \langle a,b,c \mid a^p=b^p=c^2=[a,b]=1, c^{-1}ac=a^{-1}, c^{-1}bc=b^{-1}\rangle; \\ H_3(p) & = & \langle a,b,c \mid a^p=b^p=c^2=1, [a,b]=[a,c]=1, c^{-1}bc=b^{-1}\rangle. \end{array}$$

Thus, $H \cong H_1(p), H_2(p)$ or $H_3(p)$. Note that $H_1(p) \cong D_{2p^2}$ and $H_3(p) \cong D_{2p} \times \mathbb{Z}_p$. By Proposition 1.2 and Lemma 3.1, $H_1(p) \notin \mathcal{BC}$ and $H_3(p) \notin \mathcal{BC}$. Recall that $p=3$ or 5. By Lemma 3.3, $H_2(p) \notin \mathcal{BC}$. It follows that $H \notin \mathcal{BC}$, a contradiction.

Subcase 2.2: $H > P \rtimes \langle g\rangle \cong D_6$ or D_{10}.

Clearly, $\langle g\rangle \cong \mathbb{Z}_2$, $p=3$ or 5, and $P \cong \mathbb{Z}_3$ or $\mathbb{Z}_5$. Let $P=\langle a\rangle \cong \mathbb{Z}_p$. Without any loss of generality, we may assume that $g \in P_2$. First we prove two claims.

Claim 1: $P_2 \cong D_8$, Q_8, or $Q_8 \times \mathbb{Z}_2$.

Recall that either P_2 is abelian, or $P_2 \cong D_8, Q_8$ or $Q_8 \times \mathbb{Z}_2$.

Suppose that P_2 is abelian. Then $P_2 \leq C_H(P_2) \leq N_H(P_2)$. Since$|H : P_2| = p$ and $P_2 \ntrianglelefteq H$, we have $P_2 = C_H(P_2) = N_H(P_2)$. By the Burnside's p-nilpotency criterion, P is the normal complement of P_2 in H, that is, $P \trianglelefteq H$ and $H = P \rtimes P_2$. Note that $\mathbb{Z}_2 \cong \langle g\rangle \cong (P \rtimes \langle g\rangle)/P < H/P \cong P_2$. Then H/P contains a subgroup of order 4, and hence H has a non-abelian subgroup L with $P \rtimes \langle g\rangle$ as a subgroup of index 2. By the minimality of H, $H = L$ and so $|P_2| = 4$. In particular, $P_2 \cong \mathbb{Z}_4$ or $\mathbb{Z}_2^2$.

Let $P_2 = \langle b\rangle \cong \mathbb{Z}_4$. Then $H = P \rtimes P_2 = \langle a,b,g \mid a^p=b^4=1, g=b^2, a^b=a^i, a^g=a^{-1}\rangle$ with $i \in \mathbb{Z}_p^*$. Since $a^{-1}=a^g=a^{b^2}=a^{i^2}$, we have $i^2=-1$ in $\mathbb{Z}_p$, and since $p=3$ or 5, we have $i=\pm 2$ and $p=5$. Hence H is isomorphic to the Frobenius group F_{20} of order 20 and so $H \notin \mathcal{BC}$ by Lemma 3.3, a contradiction.

Let $P_2 = \langle b,g\rangle \cong \mathbb{Z}_2^2$. Considering the action of P_2 on P, we have $H = P \rtimes P_2 \cong \langle a,b,g \mid a^p=b^2=g^2=[b,g]=1, a^b=a, a^g=a^{-1}\rangle \cong D_{4p}$, and by Proposition 1.2, $H \notin \mathcal{BC}$, a contradiction.

It follows that $P_2 \cong D_8$, Q_8, or $Q_8 \times \mathbb{Z}_2$, as claimed.

Claim 2: $P \trianglelefteq H$.

Let N be a minimal normal subgroup of H. Since $P \cong \mathbb{Z}_p$, we have $N=P$ or $N=\mathbb{Z}_2^\ell$ for some $\ell \geq 1$. If $N=P$ then $P \trianglelefteq H$, as claimed. If $N = \mathbb{Z}_2^\ell$, then $(P \rtimes \langle g\rangle)N > P \rtimes \langle g\rangle$ because $P \rtimes \langle g\rangle$ ($\cong D_6$ or D_{10}) has non-normal Sylow 2-subgroups. By the minimality of H, we have $H = N(P \rtimes \langle g\rangle)$. Since $P_2 \ntrianglelefteq H$, we have $N < P_2$ and hence $NP < H$. Clearly, $NP \ncong D_6$ or D_{10}, and the minimality of H implies that NP is abelian. It follows $NP = N \times P$ and $H = (N \times P) \rtimes \langle g\rangle$. Thus, $P \trianglelefteq H$, as claimed.

By Claim 2, $PK \leq H$ for any subgroup $K \leq H$, and by Claim 1, $P_2 \cong D_8$, Q_8, or $Q_8 \times \mathbb{Z}_2$. If $P_2 \cong Q_8 \times \mathbb{Z}_2$ then H has a proper subgroup isomorphic to $P \rtimes Q_8$, and the minimality of H implies that either $P \rtimes Q_8$ is abelian, or $P \rtimes Q_8 \cong D_6$ or D_{10}, both of which are impossible. If $P_2 \cong Q_8$, then $P \rtimes \langle a\rangle$ is a proper subgroup of H for any element a of order 4 in P_2, and by the minimality of H, $P \rtimes \langle a\rangle$ is abelian because $P \rtimes \langle a\rangle \ncong D_6$ or D_{10}. This implies that $[P, P_2]=1$ as $P_2 \cong Q_8$, and hence $H = P \times P_2$, which is impossible because $P \rtimes \langle g\rangle \cong D_6$ or D_{10}. If $P_2 \cong D_8$, let $P_2 = \langle a, g \mid a^4=g^2=1, a^g=a^{-1}\rangle$ and

$P = \langle b \rangle$, and a similar argument as above implies $[P, \langle a \rangle] = 1$. Since $P \rtimes \langle g \rangle \cong D_6$ or D_{10}, we have $b^g = b^{-1}$. It follows that

$$H \cong \mathbb{Z}_p \rtimes D_8 = \langle a, b, g \mid b^p = a^4 = g^2 = 1, a^g = a^{-1}, b^a = b, b^g = b^{-1} \rangle \cong D_{8p}.$$

By Proposition 1.2, $H \notin \mathcal{BC}$, a contradiction.

□

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4: By Lemmas 4.1 and 4.2, D_8, Q_8, $Q_8 \times \mathbb{Z}_2$, D_6 and D_{10} belong to $\mathcal{BC}$. To prove the necessity, let H be a non-abelian group with $H \in \mathcal{BC}$.

By Proposition 2.3 (i), H is solvable, and by Proposition 2.3 (iii), $2 \mid |H|$. Let L and K be a Sylow 2-subgroup and a Hall $2'$-subgroup of H, respectively. By Proposition 2.3, $K \in \mathcal{BC}$ and $L \in \mathcal{BC}$. Furthermore, $H = KL$, and by Proposition 2.3 (iii), K is abelian. Let $p_1, \ldots, p_k$ be all distinct odd prime divisors of $|H|$, and let P_i be a Sylow p_i-subgroup of H contained in K for $1 \leq i \leq k$. Then $K = P_1 \times P_2 \times \cdots \times P_k$. If $k = 0$ then $H = L$ is a 2-group, and by Lemma 4.1, $H \cong D_8$, Q_8 or $Q_8 \times \mathbb{Z}_2$. If $k = 1$ then H is a $\{2, p_1\}$-group, and by Lemma 4.2, then $H \cong D_6$ or D_{10}. In what follows, we assume $k \geq 2$.

If each Hall $\{2, p_i\}$-subgroup of H is abelian for each $1 \leq i \leq k$, then L is abelian and $H = K \times L$, forcing that H is abelian, a contradiction. Hence H has a non-abelian Hall $\{2, p_\ell\}$-subgroup for some prime p_ℓ, say M. It follows from Proposition 2.3 that $M \in \mathcal{BC}$ and from Lemma 4.2 that $M \cong D_6$ or D_{10}, yielding that $L \cong \mathbb{Z}_2$. Hence $K \trianglelefteq H$ and $H = K \rtimes P_2 = (P_1 \times \cdots \times P_k) \rtimes \mathbb{Z}_2$. It follows that $P_i \trianglelefteq H$ for each $1 \leq i \leq k$, and hence $P_i L \leq H$. Furthermore, we may assume $M = P_\ell L$. Again by Lemma 4.2, for each $1 \leq i \leq k$ we have either $P_i L = P_i \times L$ (abelian), or $P_i L \cong D_6$ or D_{10}.

Suppose $P_j L = P_j \times L$ for some $1 \leq j \leq k$. Recall that $M = L P_\ell$ is a Hall $\{2, p_\ell\}$-subgroup of H, and $M \cong D_6$ or D_{10}. Clearly, $p_\ell \neq p_j$, and $M P_j = L P_\ell P_j = M \times P_j$. Then H contains a subgroup isomorphic to $D_6 \times \mathbb{Z}_{p_j}$ or $D_{10} \times \mathbb{Z}_{p_j}$, which is impossible by Lemma 3.1. Note that if $p_i \neq 3, 5$, then $P_i L = P_i \times L$ because $P_i L \ncong D_6$ or D_{10}. This implies that $k = 2$ as $k \geq 2$, and $\{p_1, p_2\} = \{3, 5\}$. Furthermore, $\{P_1 L, P_2 L\} = \{D_6, D_{10}\}$ and hence $H = P_1 P_2 L \cong D_{30}$ because a group of order 15 must be cyclic, which is impossible by Proposition 1.2. This completes the proof. □

References

[1] I. Antončič, A. Hujdurović and K. Kutnar, *A classification of pentavalent arc-transitive bicirculants*, J. Algebraic Combin. **41** (2015), 643–668.

[2] A. Araluze, I. Kovács, K. Kutnar, L. Martínez and D. Marušič, *Partial sum quadruples and bi-abelian digraphs*, J. Combin. Theory Ser. A **119** (2012), 1811–1831.

[3] M. Aschbacher, *Finite group theory*, Cambridge University Press, Cambridge, 1986.

[4] L. Babai, *On a conjecture of M. E. Watkins on graphical regular representations of finite groups*, Comp. Math. **37** (1978), 291–296.

[5] W. Bosma, C. Cannon and C. Playoust, *The* MAGMA *algebra system I: The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[6] M. Conder, I. Estélyi and T. Pisanski, *Vertex-transitive Haar graphs that are not Cayley graphs*, In: Conder, M., Deza, A., Weiss, A. (eds.) Discrete Geometry and Symmetry. GSC 2015. Springer Proceedings in Mathematics & Statistics, Vol **234**, pp. 61–70. Springer, Cham, 2018.

[7] T. Dobson and P. Spiga, *Cayley numbers with arbitrarily many distinct prime factors*, J. Combin. Theory B. **122** (2017), 301–310.

[8] J.-L. Du, Y.-Q. Feng and P. Spiga, *A Classification of the m-graphical regular representation of finite groups*, preprint arXiv:1901.07133 [math.CO].

[9] I. Estélyi and T. Pisanski, *Which Haar graphs are Cayley graphs*, Electronic J. Combin. **23** (2016), #P3.10.

[10] Y.-Q. Feng, I. Kovács and D.-W. Yang, *On groups all of whose Haar graphs are Cayley graphs*, J. Algebraic Combin. (2019). https://doi.org/10.1007/s10801-019-00894-7

[11] C.D. Godsil, *GRR's for non-solvable groups*, in: Algebraic methods in graph theory, Vol. I, II Szeged (1978), pp. 221–239, Colloq. Math. Soc. János Bolyai, Amsterdam-New York 1981.

[12] Y.-Q. Feng, K. Kutnar, D. Marušič and D.-W. Yang, *On cubic symmetric non-Cayley graphs with solvable automorphism groups*, preprint arXiv:1607.02618 [math.CO]

[13] M. Hall and J.K. Senior, *The Groups of Order* 2^n $(n \leq 6)$, Macmillan, New York, 1964.

[14] M. Hladnik, *Schur norms of bicirculant matrices*, Lin. Alg. Appl. **286** (1999), 261–272.

[15] M. Hladnik, D. Marušič and T. Pisanski, *Cyclic Haar graphs*, Discrete Math. **244** (2002), 137–153.

[16] A. Hujdurović, K. Kutnar and D. Marušič, *On normality of n-Cayley graphs*, Appl. Math. Comput. **332** (2018), 469-476.

[17] W. Imrich, *Graphs with transitive abelian automorphism group*, in: Combinatorial theory and its applications, Balatonfüred, 1969 (P. ErdHos et. al eds.), pp. 651–656, Coll. Math. Soc. János Bolyai 4, North-Holland 1969.

[18] W. Imrich, *Graphical regular representations of groups of odd order*, in Combinatorics (A. Hajnal and V. T. Sós eds.), pp. 611–621, Coll. Math. Soc. János Bolyai 18, North-Holland 1976.

[19] W. Imrich and M. E. Watkins, *On graphical regular representation of cyclic extension of groups*, Pacific J. Math. **55** (1974), 461–477.

[20] H. Koike and I. Kovács, *Isomorphic tetravalent cyclic Haar graphs*, Ars Math. Contemp. **7** (2014), 215–235.

[21] C.H. Li and Á. Seress, *On vertex-transitive non-Cayley graphs of square-free order*, Des. Codes Cryptogr. **34** (2005), 265–281.

[22] Z.P. Lu, C.Q. Wang and M.Y. Xu, *Semisymmetric cubic graphs constructed from bi-Cayley graphs of A_n*, Ars Combin. **80** (2006), 177–187.

[23] Z.P. Lu, C.Q. Wang and M.Y. Xu, *On semisymmetric cubic graphs of order $6p^2$*, Sci. China Ser. A **47** (2004), 1–17.

[24] D. Marušič, *Cayley properties of vertex symmetric graphs*, Ars Combin. **16** (1983), 297–302.

[25] L.A. Nowitz and M.E. Watkins, *On graphical regular representations of non-abelian groups I*, Canad. J. Math. **24** (1972), 993–1008.

[26] L.A. Nowitz and M.E. Watkins, *On graphical regular representations of non-abelian groups II*, Canad. J. Math. **24** (1972), 1009–1018.

[27] M.E. Watkins, *On the action of non-abelian groups on graphs*, J. Combin. Theory **11** (1971), 95–104.

[28] J.-X. Zhou and Y.-Q. Feng, *The automorphisms of bi-Cayley graphs*, J. Combin. Theory Ser. B **116** (2016), 504–532.

[29] J.-X. Zhou and Y.-Q. Feng, *Cubic bi-Cayley graphs over abelian groups*, European J. Combin. **36** (2014), 679–693.

[30] J.-X. Zhou, *Tetravlent vertex-transitive graphs of order $4p$*, J. Graph Theory **71** (2012), 402–415.

[31] J.-X. Zhou, *Every finite group has a normal bi-Cayley graph*, Ars Math. Contemp. **14** (2018), 177-186.