# Uniform Orderings for Generalized Coloring Numbers * 

Jan van den Heuvel ${ }^{\dagger}$ and H.A. Kierstead ${ }^{\ddagger}$

December 18, 2019


#### Abstract

The generalized coloring numbers $\operatorname{col}_{r}(G)$ (also denoted by $\operatorname{scol}_{r}(G)$ ) and $\operatorname{wcol}_{r}(G)$ of a graph $G$ were introduced by Kierstead and Yang as a generalization of the usual coloring number, and have found important theoretical and algorithmic applications. For each distance $r$, these numbers are determined by an "optimal" ordering of the vertices of $G$. We study the question of whether it is possible to find a single "uniform" ordering that is "good" for all distances $r$.

We show that the answer to this question is essentially "yes". Our results give new characterizations of graph classes with bounded expansion and nowhere dense graph classes.


Keywords: generalized coloring numbers, vertex orderings, bounded expansion graph classes, nowhere dense graph classes

## 1 Introduction and Main Results

### 1.1 Coloring Numbers

All graphs $G=(V, E)$ in this paper are finite, simple and undirected. We use $|G|$ for $|V|$. By an ordering $\sigma$ of a graph we mean a total ordering of its vertex set, i.e. for every $x, y \in V$, $x \neq y$, we have exactly one of $x<_{\sigma} y$ or $y<_{\sigma} x$. The set of all orderings of $G$ is denoted $\Pi(G)$ (or just $\Pi$, if the graph is clear from the context).

For a graph $G, \sigma \in \Pi$ and $x \in V$, let $\operatorname{col}(G, \sigma, x)$ be one more than the number of neighbors $y \in N_{G}(x)$ with $y<_{\sigma} x$. The coloring number of $G$, denoted $\operatorname{col}(G)$, is defined by

$$
\operatorname{col}(G)=\min _{\sigma \in \Pi} \max _{x \in V} \operatorname{col}(G, \sigma, x)
$$

[^0]In recent terminology, the coloring number of a graph is one more than its degeneracy; under an older definition of degeneracy they were the same. Greedily coloring the vertices of $G$ in an ordering that witnesses its coloring number, shows that

$$
\chi(G) \leq \operatorname{ch}(G) \leq \operatorname{col}(G)
$$

where $\chi(G)$ and $\operatorname{ch}(G)$ denote the chromatic and list chromatic number of $G$, respectively.
An alternative way to define $\operatorname{col}(G, \sigma, x)$ is as the number of vertices $y \leq_{\sigma} x$ that have distance at most 1 from $x$. (Since $x$ has distance 0 from itself, we count $x$ in this definition as well, avoiding having to add "one more than" as in our first definition.) In this paper we are interested in generalized coloring numbers, where we consider vertices $y \leq_{\sigma} x$ that are at some further distance $r$ from $x$. These numbers were first introduced in [15], after similar notions were explored by various authors [1, 11, 13, 14, 27] in the cases $r=2,4$.

Since there are several choices we can impose on the position of the internal vertices of a path from $x$ to $y$ with respect to an ordering $\sigma$, we define two variants. Let $r \in \mathbb{N} \cup\{\infty\}$. For a graph $G$, ordering $\sigma \in \Pi$ and $x \in V$, we say that a vertex $y$ is weakly $r$-reachable from $x$ with respect to $\sigma$ if $y \leq_{\sigma} x$ and there is an $x, y$-path $P$ with length $|E(P)| \leq r$ such that all vertices $p \in V(P)$ satisfy $p \geq_{\sigma} y ; y$ is strongly $r$-reachable from $x$ with respect to $\sigma$ if we have the stronger condition that all $p \in V(P) \backslash\{y\}$ satisfy $p \geq_{\sigma} x$. Let $W_{r}[G, \sigma, x]$ be the set of vertices that are weakly $r$-reachable from $x$ with respect to $\sigma$ and $S_{r}[G, \sigma, x]$ be the set of vertices that are strongly $r$-reachable from $x$ with respect to $\sigma$. Note that $x$ itself is included in both $W_{r}[G, \sigma, x]$ and $S_{r}[G, \sigma, x]$.

The weak $r$-coloring number of $G$, denoted $\operatorname{wcol}_{r}(G)$, and the strong $r$-coloring number of $G$, denoted $\operatorname{scol}_{r}(G)$, are defined by ${ }^{1}$ :

$$
\begin{array}{ll}
\operatorname{wcol}_{r}(G, \sigma)=\max _{x \in V}\left|W_{r}[G, \sigma, x]\right| ; & \operatorname{wcol}_{r}(G)=\min _{\sigma \in \Pi} \operatorname{wcol}_{r}(G, \sigma) ; \\
\operatorname{scol}_{r}(G, \sigma)=\max _{x \in V}\left|S_{r}[G, \sigma, x]\right| ; & \operatorname{scol}_{r}(G)=\min _{\sigma \in \Pi} \operatorname{scol}_{r}(G, \sigma) .
\end{array}
$$

We obviously have $\operatorname{col}(G)=\operatorname{wcol}_{1}(G)=\operatorname{scol}_{1}(G)$.
The following easy observations hint at the usefulness of different versions of coloring numbers. If the vertices of $G$ are colored greedily so that no vertex $v$ receives the same color as any other vertex in $S_{2}[G, \sigma, v]$, then the resulting coloring is an acyclic coloring, so

$$
\operatorname{ch}_{\mathrm{a}}(G) \leq \operatorname{scol}_{2}(G),
$$

where $\mathrm{ch}_{\mathrm{a}}(G)$ denotes the list acyclic chromatic number of $G$. If the vertices of $G$ are colored greedily so that no vertex $v$ receives the same color as any vertex in $W_{2}[G, \sigma, v]$, then the resulting coloring is a star coloring, so

$$
\operatorname{ch}_{\mathrm{s}}(G) \leq \operatorname{wcol}_{2}(G),
$$

where $\mathrm{ch}_{\mathrm{s}}(G)$ denotes the star chromatic number of $G$.

[^1]As noticed already in [15], the two types of generalized coloring numbers are related by the inequalities

$$
\begin{equation*}
\operatorname{scol}_{r}(G) \leq \operatorname{wcol}_{r}(G) \leq\left(\operatorname{scol}_{r}(G)\right)^{r} . \tag{1}
\end{equation*}
$$

Thus if one of the generalized coloring numbers is bounded for a class of graphs (for some $r$ ), then so is the other one.

An interesting aspect of generalized coloring numbers is that they can also be seen as gradations between the coloring number $\operatorname{col}(G)$ and two important graph invariants, namely the tree-width $\operatorname{tw}(G)$ and the tree-depth $\operatorname{td}(G)$. (The latter is the minimum height of a depth-first search tree for a supergraph of $G$ [19].) More explicitly, we have the following proposition.

## Proposition 1.1.

Every graph $G$ satisfies:
(a) $\operatorname{col}(G)=\operatorname{scol}_{1}(G) \leq \operatorname{scol}_{2}(G) \leq \ldots \leq \operatorname{scol}_{\infty}(G)=\operatorname{tw}(G)+1 ;$
(b)

$$
\operatorname{col}(G)=\operatorname{wcol}_{1}(G) \leq \operatorname{wcol}_{2}(G) \leq \ldots \leq \operatorname{wcol}_{\infty}(G)=\operatorname{td}(G)
$$

The equality $\operatorname{scol}_{\infty}(G)=\operatorname{tw}(G)+1$ was first proved in [5, Section 6]. The equality $\mathrm{wcol}_{\infty}(G)=$ $\operatorname{td}(G)$ is proved in [22, Lemma 6.5].

Generalized coloring numbers have been instrumental in the study of sparse graph classes. Nešetřil and Ossona de Mendez introduced the notion of graph classes with bounded expansion [20] and the more general notion of nowhere dense graph classes [21]. These concepts generalize those of graph classes with bounded tree-width, minor-closed classes, bounded degree classes, etc. See the book of Nešetřil and Ossona de Mendez [22] for a wealth of information about the properties of these graph classes.

One of the key properties of this classification is that it is remarkably robust. Not only can results for particular classes that have bounded expansion (or are nowhere dense) often be generalized to all classes with that property, but these generalizations often yield new characterizations. For example, classes with bounded generalized coloring numbers were studied in [15] because they had bounded generalized game coloring numbers (see Section 3 for definitions). Later, Zhu [28] proved bounds on the generalized coloring numbers that gives the following characterizations of bounded expansion and nowhere dense classes in terms of those numbers. We will use these characterizations as definitions.

## Definition 1.2.

(a) A graph class $\mathcal{G}$ has bounded expansion if and only if there exists a function $c: \mathbb{N} \rightarrow \mathbb{N}$ such that $\operatorname{scol}_{r}(G) \leq c(r)$ for all $r$ and all $G \in \mathcal{G}$.
(b) A graph class $\mathcal{G}$ is nowhere dense if and only if there exists a function $n_{0}: \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{N}$ such that for every $\epsilon>0, r \in \mathbb{N}$ and $G \in \mathcal{G}$ we have that $\operatorname{scol}_{r}(H) \leq|H|^{\epsilon}$ for all subgraphs $H$ of $G$ with $|H| \geq n_{0}(\epsilon, r)$.

Note that by the inequalities in (1) we equally well could have defined bounded expansion and nowhere dense in terms of the weak coloring numbers.

Here is a different example demonstrating the surprising power of this classification of sparse graph classes. Streib and Trotter [24] proved that every poset whose cover graph
is planar, has dimension bounded by a function of its height. Then Joret et al. 9 used generalized coloring numbers to prove that every monotone graph class $\mathcal{G}$ is nowhere dense if and only if for every integer $h \geq 1$ and real number $\epsilon>0$, every $n$-element poset of height at most $h$ whose cover graph is in $\mathcal{G}$ has dimension $O\left(n^{\epsilon}\right)$.

Generalized coloring numbers are an important tool in the context of algorithmic sparse graphs theory; see again [22]. More recently they have played a key role in algorithmic results on model-checking for first-order logic on bounded expansion and nowhere dense graph classes [4, 7, 10].

### 1.2 The Guiding Question

An obvious question concerning generalized coloring numbers is whether an ordering that is "good" for one distance $r$ is also "good" for a different distance $r$ '. In fact, this need not be the case: in Example 2.1 we will show that for all $r, r^{\prime} \in \mathbb{N}$ with $r \neq r^{\prime}$, there exists a graph $G$ such that for all $\sigma \in \Pi(G)$ either $\operatorname{scol}_{r}(G)<\operatorname{scol}_{r}(G, \sigma)$ or $\operatorname{scol}_{r^{\prime}}(G)<\operatorname{scol}_{r^{\prime}}(G, \sigma)$.

The existence of examples as above also has consequences for the many algorithms that for a graph class $\mathcal{G}$ with bounded expansion and some $r$, use explicitly an ordering $\sigma$ which shows that $\operatorname{scol}_{r}(G) \leq c(r)$. It looks as if for every $r$ a different ordering is needed.

Given a function $c: \mathbb{N} \rightarrow \mathbb{N}$, let $\mathcal{G}_{c}$ be the graph class defined by: $G \in \mathcal{G}_{c}$ if and only if $\operatorname{scol}_{r}(G) \leq c(r)$ for all $r \in \mathbb{N}$. Then the class $\mathcal{G}_{c}$ has bounded expansion, and every class with bounded expansion is contained in $\mathcal{S}_{c^{\prime}}$ for some $c^{\prime}$.

In this paper we investigate the following problem that was raised by Dvořák [23]. Kreutzer et al. [18, Section 6] state that it is "tempting to conjecture" that the answer to this problem is yes.

## Problem 1.3.

Is it true that for all functions $c: \mathbb{N} \rightarrow \mathbb{N}$, there exists a function $c^{*}: \mathbb{N} \rightarrow \mathbb{N}$, such that for every graph $G \in \mathcal{G}_{c}$, there exists an ordering $\sigma^{*} \in \Pi(G)$ such that $\operatorname{scol}_{r}\left(G, \sigma^{*}\right) \leq c^{*}(r)$ for all $r \in \mathbb{N}$ ?

The main reason this issue was raised by several people was that for all known bounds on the generalized coloring numbers on graph classes such as (topological) minor closed classes, a single ordering of all graphs in the class gave those bounds for all distances $r$; see e.g. [8, 18].

### 1.3 Results

Our main result provides a positive answer for Problem 1.3.

## Theorem 1.4.

For any graph $G$, there exists an ordering $\sigma^{*}$ of $G$ such that for all $r \in \mathbb{N}$ we have

$$
\operatorname{scol}_{r}\left(G, \sigma^{*}\right) \leq\left(2^{r}+1\right) \cdot\left(\operatorname{scol}_{2 r}(G)\right)^{4 r}
$$

In the terminology of Problem 1.3, this means we can set $c^{*}(r)=\left(2^{r}+1\right) \cdot(c(2 r))^{4 r}$ for all $r$.
We immediately obtain the following new characterizations of graph classes with bounded expansion and nowhere dense graph classes.

## Corollary 1.5.

A graph class $\mathcal{G}$ has bounded expansion if and only if there exists a function $c^{*}: \mathbb{N} \rightarrow \mathbb{N}$, such that for every graph $G \in \mathcal{G}$ there exists an ordering $\sigma^{*}(G)$ of $G$ such that $\operatorname{scol}_{r}\left(G, \sigma^{*}(G)\right) \leq$ $c^{*}(r)$ for all $r$.

## Corollary 1.6.

A graph class $\mathcal{G}$ is nowhere dense if and only if there exists a function $n_{0}^{*}: \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{N}$ such that for every subgraph $H$ of a graph $G \in \mathcal{G}$, there exists an ordering $\sigma^{*}(H)$ of $H$ such that for all $\epsilon>0$ and $r \in \mathbb{N}$, if $|H| \geq n_{0}^{*}(\epsilon, r)$, then $\operatorname{scol}_{r}\left(H, \sigma^{*}(H)\right) \leq|H|^{\epsilon}$.

By the definition of the strong coloring number it follows that if $G$ is a graph with some ordering $\sigma^{*}(G)$, then for every subgraph $H$ of $G$, if we take $\sigma^{*}(H)$ the ordering of $H$ induced by $\sigma^{*}(G)$, we have $\operatorname{scol}_{r}\left(H, \sigma^{*}(H)\right) \leq \operatorname{scol}_{r}\left(G, \sigma^{*}(G)\right)$ for all $r$. This means that in Corollary 1.5 once we have an ordering $\sigma^{*}(G)$ for some graph $G \in \mathcal{G}$, for every $H \in \mathcal{G}$ that is a subgraph of $G$ we can take the ordering $\sigma^{*}(H)$ of $H$ induced by $\sigma^{*}(G)$. In view of this it is natural to ask whether a similar statement is possible for the condition in Corollary 1.6 for nowhere dense classes of graphs. In Subsection 2.2 we will show that this is in fact not possible.

Theorem 1.4 above follows from a technical, more general, result that deals with different graphs on the same vertex set; see Section 4. Another consequence of this more general result is the following theorem, which may be of independent interest.

## Theorem 1.7.

Let $G_{1}, \ldots, G_{k}$ be a collection of graphs, all on the same vertex set $V$, and let $r_{1}, \ldots, r_{k} \in \mathbb{N}$. Then there exists a ordering $\sigma^{*}$ of the common vertex set $V$ such that for all $i=1, \ldots, k$,

$$
\operatorname{scol}_{r_{i}}\left(G_{i}, \sigma^{*}\right) \leq(k+1)\left(\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)\right)^{2} \leq(k+1)\left(\operatorname{scol}_{2 r_{i}}\left(G_{i}\right)\right)^{4 r_{i}} .
$$

The proof of the general result, which also can be found in Section 4 , has at its basis arguments developed in [15, 16].

The remainder of this paper is organized as follows. In the next subsection we give essential terminology and notation. The two classes of examples referred to earlier can be found in Section 2. In Section 3) we describe the essential concepts and the result from [15] that provided the inspiration for our proof of the main theorem. In Section 4 we state and prove our main technical result, and give the proofs of its corollaries. In the next section we discuss some algorithmic aspects of our results. We discuss some open questions in the final section.

### 1.4 Terminology and Notation

Most of our graph theory terminology and notation is standard and can be found in text books such as [2].

If $P=v_{1} v_{2} \ldots v_{n}$ is a path, then we call $v_{1}$ and $v_{n}$ the ends of $P$. The subpath of $P$ that has ends $a$ and $b$ is denoted by $a P b$. Finally, $\stackrel{\perp}{P}$ is $P$ minus its ends. The length of a path is the number of edges in it. (So one fewer than the number of vertices.)

For two vertices $x$ and $y$ in the same component of a graph $G=(V, E)$, the distance $\operatorname{dist}_{G}(x, y)$ between $x$ and $y$ is the length of a shortest $x, y$-path in $G$. For $v \in V, N_{G}(v)$ denotes the set of vertices in $G$ adjacent to $v ; N_{G}[v]=N_{G}(v) \cup\{v\}$. For a subset $X \subseteq V$, $G[X]$ denotes the subgraph of $G$ induced on the vertex set $X$.

For a positive integer $k$, we write $[k]=\{1,2, \ldots, k\}$.
If $\sigma$ is an ordering of some set $X$ and $S, T$ are non-empty subsets of $X$, then by $S<_{\sigma} T$ we mean that $s<_{\sigma} t$ for all $s \in S, t \in T$. We abbreviate $\{s\}<_{\sigma} T$ to $s<_{\sigma} T$. The element in $S$ that is minimum with respect to $\sigma$ is denoted by $\sigma-\min (S)$. The ordering $\sigma_{S}$ on $S$ induced by $\sigma$ is the ordering given by: $s_{1}<_{\sigma_{S}} s_{2}$ if and only if $s_{1}<_{\sigma} s_{2}$, for all $s_{1}, s_{2} \in S$.

## 2 Examples

### 2.1 Graphs with No "Good" Ordering

The following examples show that in answering Problem 1.3 we cannot take $c^{*}=c$.

## Example 2.1.

Let $\varphi$ be the largest solution to $x^{2}=x+1$ (the golden ratio $\varphi=\frac{1}{2}(1+\sqrt{5}) \approx 1.62$ ). For all $r, r^{\prime} \in \mathbb{N}$ with $r<r^{\prime}$, there exists a graph $G$ such that for all $\sigma \in \Pi(G)$, either

$$
\operatorname{scol}_{r}(G, \sigma)>.08\left(\operatorname{scol}_{r}(G)\right)^{\varphi} \quad \text { or } \quad \operatorname{scol}_{r^{\prime}}(G, \sigma)>.08\left(\operatorname{scol}_{r^{\prime}}(G)\right)^{\varphi}
$$

Proof. Fix $t, n \in \mathbb{N}$ with $4 \leq t \leq n$. Let $Z=\left\{z_{i}^{h} \mid i \in[n], h \in[t]\right\}$ be a set of vertices, and partition $Z$ into $n$ sets $Z_{i}=\left\{z_{i}^{h} \mid h \in[t]\right\}$ of size $t$. We construct $G$ by connecting each ordered pair $\left(Z_{i}, Z_{j}\right), i \neq j$, with isomorphic graphs $H_{i, j}$ so that $G=\bigcup\left\{H_{i, j} \mid i, j \in[n], i \neq j\right\}$ and the $H_{i, j}$ are pairwise disjoint except for their ends in $Z$. In particular, the sets $Z_{i}$ and $Z_{j}$ are connected by both $H_{i, j}$ and $H_{j, i}$.

For all $h \in[t]$ and $i, j \in[n], i \neq j$, add a vertex $x_{i, j}$ and choose independent paths $P_{i, j}^{h}=z_{i}^{h} \ldots x_{i, j}$ of length $r$ and $Q_{i, j}^{h}=x_{i, j} \ldots z_{j}^{h}$ of length $r^{\prime}-r$. Let

$$
H_{i, j}=\bigcup_{h \in[t]} P_{i, j}^{h} \cup \bigcup_{h \in[t]} Q_{i, j}^{h} .
$$

See Figure 1 for a sketch. Set $Y_{i, j}=V\left(H_{i, j}\right) \backslash\left(Z_{i} \cup Z_{j} \cup\left\{x_{i, j}\right\}\right)$, so $H_{i, j}\left[Y_{i, j}\right]=\bigcup_{h \in[t]} P_{i, j}^{h} \cup$ $\bigcup_{h \in[t]} \stackrel{\circ}{i, j}_{h}$. Finally, set $X=\left\{x_{i, j} \mid i, j \in[n], i \neq j\right\}, X_{i}=\left\{x_{i, j}, x_{j, i} \mid j \in[n]-i\right\}$ and $Y=\bigcup_{\substack{i, j \in[n] \\ i \neq j}} Y_{i, j}$. Note that $V(G)=X \cup Y \cup Z$.

Observe the following facts:
(E1) $\operatorname{dist}_{G}\left(z_{i}^{h}, z_{j}^{h^{\prime}}\right)=r^{\prime}$, for all $h, h^{\prime} \in[t]$ and $i, j \in[n], i \neq j$;
(E2) every $Z$-path meets $X$, and every $Z_{i}, Z_{j}$-path with length $r^{\prime}$ meets one of $x_{i, j}, x_{j, i}$;
(E3) $\operatorname{dist}_{G}\left(x_{i, j}, x\right)>r^{\prime}$, for all $x \in X \backslash\left(X_{i} \cup X_{j}\right)$.
The result follows from the next three claims by an easy calculation.
Claim 1. Let $\sigma \in \Pi(G)$ satisfy $Z<_{\sigma} X<_{\sigma} Y$. Then $\operatorname{scol}_{r}(G) \leq \operatorname{scol}_{r}(G, \sigma) \leq 2 t+1$.


Figure 1: A connecting graph $H_{i, j}$ for $t=4, r=3$ and $r^{\prime}=7$.

Proof. Consider any vertex $v \in X \cup Y \cup Z$, and suppose $w \in S_{r}[G, \sigma, v]$ is witnessed by the path $R$.

If $v \in Z_{i} \subseteq Z$, then $w \leq_{\sigma} v<_{\sigma} X \cup Y$, so $w \in Z$. By (E1) we have $w \in Z_{i}$, so $\left|S_{r}[G, \sigma, v]\right| \leq\left|Z_{i}\right|=t$.

If $v=x_{i, j} \in X$, then $Z<_{\sigma} v<_{\sigma} Y$. Thus $V(\stackrel{\circ}{R}) \subseteq V\left(H_{i, j}\right) \backslash\left(Z_{i} \cup Z_{j}\right)$, and $w \in$ $Z_{i} \cup Z_{j} \cup\left\{x_{i, j}\right\}$, so $\left|S_{r}[G, \sigma, v]\right| \leq 2 t+1$.

If $v \in Y$, then $R \subseteq R^{\prime}$ for some $R^{\prime} \in\left\{P_{i, j}^{h}, Q_{i, j}^{h} \mid h \in[t]\right\}$. Thus

$$
\begin{equation*}
S_{r}[G, \sigma, v] \subseteq S_{r^{\prime}}[G, \sigma, v] \subseteq\left\{v, v_{1}, v_{2}\right\} \tag{2}
\end{equation*}
$$

where $v_{1}, v_{2} \leq_{\sigma} v$ and $v_{1}, v_{2} \in V\left(R^{\prime}\right)$. The vertices $v_{1}, v_{2}$ exist since the ends of $R^{\prime}$ come before $v$ with respect to $\sigma$. Thus $\left|S_{r}[G, \sigma, v]\right| \leq 3$.

So in all cases we have $\left|S_{r}[G, \sigma, v]\right| \leq 2 t+1$, hence $\operatorname{scol}_{r}(G, \sigma) \leq 2 t+1$.
Claim 2. Let $\sigma \in \Pi(G)$ so that $X<_{\sigma} Z<_{\sigma} Y$. Then $\operatorname{scol}_{r^{\prime}}(G) \leq \operatorname{scol}_{r^{\prime}}(G, \sigma) \leq 4 n-6$.
Proof. Consider any vertex $v \in X \cup Y \cup Z$, and suppose $w \in S_{r^{\prime}}[G, \sigma, v]$ is witnessed by the path $R$.

If $v=x_{i, j} \in X$, then $w \leq_{\sigma} v<_{\sigma} Y \cup Z$, so $w \in X$. By (E3) we have $w \in X_{i} \cup X_{j}$, so

$$
\left|S_{r^{\prime}}[G, \sigma, v]\right| \leq\left|X_{i} \cup X_{j}\right|=\left|X_{i}\right|+\left|X_{j}\right|-\left|X_{i} \cap X_{j}\right|=2(2 n-2)-2=4 n-6
$$

If $v \in Z_{i} \subseteq Z$, then $X<_{\sigma} v<_{\sigma} Y$, so $w \in X$ if $R$ meets $X$. By (E2), $R$ meets $X_{i}$ if $R$ meets $Z \backslash\{v\}$. Thus $w \in X_{i} \cup\{v\}$. This gives $\left|S_{r^{\prime}}[G, \sigma, v]\right| \leq\left|X_{i}\right|+1=2 n-1 \leq 4 n-6$.

If $v \in Y$, then $\left|S_{r^{\prime}}[G, \sigma, v]\right| \leq 3$, by (2).
Thus in all cases we have $\left|S_{r^{\prime}}[G, \sigma, v]\right| \leq 4 n-6$, hence $\operatorname{scol}_{r^{\prime}}(G, \sigma) \leq 4 n-6$.
Claim 3. For any $\sigma \in \Pi(G)$, either $\operatorname{scol}_{r}(G, \sigma) \geq .246 n$ or $\operatorname{scol}_{r^{\prime}}(G, \sigma) \geq .754 n t$.
Proof. Let $z_{i}^{h}$ be the $\sigma$-largest vertex of $Z, J=\left\{j \in[n]-i \mid z_{i}^{h} \leq_{\sigma} V\left(P_{i, j}^{h}\right)\right\}$ and $\bar{J}=[n] \backslash J$. For all $j \in \bar{J} \backslash\{i\}$ there exists a vertex $u_{j} \in V\left(P_{i, j}^{h}\right)$ with $u_{j}<_{\sigma} z_{i}^{h}$; choose $u_{j}$ as close (along the path $P_{i, j}^{h}$ ) to $z_{i}^{h}$ as possible. Then $\left\{u_{j} \mid j \in \bar{J}\right\} \cup\left\{z_{i}^{h}\right\} \subseteq S_{r}\left[G, \sigma, z_{i}^{h}\right]$. Thus $\operatorname{scol}_{r}(G, \sigma) \geq|\bar{J}|+1$, and so we are done if $|\bar{J}| \geq .246 n-1$.

Otherwise $|J| \geq(n-1)-(.246 n-1)=.754 n$. For all $j \in J$ and $h^{\prime} \in[t]$, let $v_{i, j}^{h^{\prime}}$ be the vertex of $Q_{i, j}^{h^{\prime}}$ with $v_{i, j}^{h^{\prime}}<_{\sigma} z_{i}^{h}$ that is closest to $x_{i, j}$ (along the path $Q_{i, j}^{h^{\prime}}$ ); it exists because $z_{j}^{h^{\prime}}<_{\sigma} z_{i}^{h}<x_{i, j}$ by the choice of $z_{i}^{h}$ and the definition of $J$. Then we have $\left\{v_{j}^{h^{\prime}} \mid j \in J\right.$, $\left.h^{\prime} \in[t]\right\} \subseteq S_{r^{\prime}}\left[G, \sigma, z_{i}^{h}\right]$. Thus $\operatorname{scol}_{r^{\prime}}(G, \sigma) \geq t|J| \geq .754 n t$.

Now choose $t, n$ such that $t>1000$ and $t^{\varphi} \leq n<t^{\varphi}+1$. Let $C=.08$. Then we have (using $\varphi<1.6181$ ):

$$
\begin{equation*}
.246>C 2.001^{\varphi} \quad \text { and } \quad .754>C 4^{\varphi} . \tag{3}
\end{equation*}
$$

Consider any $\sigma \in \Pi(G)$. By Claim $3, \operatorname{scol}_{r}(G, \sigma) \geq .246 n$ or $\operatorname{scol}_{r^{\prime}}(G, \sigma) \geq .754 n t$. In the first case, Claim 1 yields (using $n \geq t^{\varphi}$, (3) and $t>1000$ ):

$$
\operatorname{scol}_{r}(G, \sigma) \geq .246 n \geq .246 t^{\varphi}>C 2.001^{\varphi} t^{\varphi}=C(2.001 t)^{\varphi}>C(2 t+1)^{\varphi} \geq C\left(\operatorname{scol}_{r}(G)\right)^{\varphi}
$$

In the second case, Claim 2 yields (using (3), $n \geq t^{\varphi}>n-1$ and $\varphi^{2}=\varphi+1$ ):

$$
\operatorname{scol}_{r^{\prime}}(G, \sigma) \geq .754 n t>C 4^{\varphi} n t \geq C 4^{\varphi} t^{\varphi+1}=C\left(4 t^{\varphi}\right)^{\varphi}>C(4(n-1))^{\varphi}>C\left(\operatorname{scol}_{r^{\prime}}(G)\right)^{\varphi}
$$

### 2.2 Nowhere Dense Classes and Orderings

In the discussion after Corollary 1.6 we raised the possibility of strengthening the corollary to the following. "A graph class $\mathcal{G}$ is nowhere dense if and only if there exists a function $n_{0}^{*}: \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{N}$ such that for every graph $G \in \mathcal{G}$ there exists an ordering $\sigma^{*}(G)$ of $G$ such that for every subgraph $H$ of $G$, the ordering $\sigma^{*}(H)$ of $H$ induced by $\sigma^{*}(G)$ has the property that for all $\epsilon>0$ and $r \in \mathbb{N}$ such that $|H| \geq n_{0}^{*}(\epsilon, r)$ we have $\operatorname{scol}_{r}\left(H, \sigma^{*}(H)\right) \leq|H|^{\epsilon}$." In this subsection we show that such a strengthening is not possible, even for monotone nowhere dense classes. (A class is monotone if it closed under taking subgraphs.)

## Example 2.2.

There exists a monotone graph class $\mathcal{G}$ that is nowhere dense and with the following property. There does not exist a function $n_{0}^{*}: \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{N}$ such that for every graph $G \in \mathcal{G}$ there exists an ordering $\sigma^{*}(G)$ of $G$ such that for every subgraph $H$ of $G$, the ordering $\sigma^{*}(H)$ of $H$ induced by $\sigma^{*}(G)$ has the property that for all $\epsilon>0$ and $r \in \mathbb{N}$ such that $|H| \geq n_{0}^{*}(\epsilon, r)$ we have $\operatorname{scol}_{r}\left(H, \sigma^{*}(H)\right) \leq|H|^{\epsilon}$.

Proof. Let $\mathcal{G}$ be the class of graphs whose maximum degree is at most their girth. (The girth of a graph is the length of the smallest cycle in it.) Note that this class is obviously monotone. It is shown in [22, pages 105-106] that this class is nowhere dense (but not with bounded expansion!). One other well-known fact we use is that this class contains graphs with arbitrarily large minimum degree.

Now suppose for a contradiction that there exists a function $n_{0}^{*}: \mathbb{R} \times \mathbb{N} \rightarrow \mathbb{N}$ satisfying the properties in the statement above. Take $0<\epsilon<1$ and $r \in \mathbb{N}$, and choose an integer $d$ such that $d \geq n_{0}^{*}(\epsilon, r)$. Let $G$ be a graph in $\mathcal{G}$ with minimum degree at least $d$. By supposition there is an ordering $\sigma^{*}(G)$ of $G$ satisfying the properties in the statement.

Now let $v$ be the vertex that is last in the ordering $\sigma^{*}(G)$, and set $H=G\left[N_{G}[v]\right]$. Then $H$ has at least $d+1>n_{0}^{*}(\epsilon, r)$ vertices. In the ordering $\sigma^{*}(H)$ of $H$ induced by $\left.\sigma^{*} G\right)$, the vertex $v$ is still the last one, which gives $\operatorname{scol}_{r}\left(H, \sigma^{*}(H)\right)=\left|N_{G}[v]\right| \geq d+1$. Since $|H|^{\epsilon}<d+1$ for $\epsilon<1$, we cannot have $\operatorname{scol}_{r}\left(H, \sigma^{*}(H)\right) \leq|H|^{\epsilon}$.

## 3 Inspiration for the Proof of the Main Theorem

The inspiration for the proof of Theorem 1.4 comes from the theory of generalized game coloring numbers, which were introduced in [15]. In this section we define these numbers, and use a basic result about them to give a very easy proof of a simplified version of Theorem 1.4. The full proof follows in Section 4

The $r$-ordering game is played on a graph $G$ by two players, Alice and Bob. The game lasts for $n=|G|$ turns. The players take turns choosing unchosen vertices with Alice playing first until there are no unchosen vertices left. This creates an ordering $\sigma \in \Pi(G)$ of $G$, where $v_{i}$ is the vertex chosen at the $i$-th turn and $v_{1}<_{\sigma} v_{2}<_{\sigma} \cdots<_{\sigma} v_{n}$. The score of the game is $\operatorname{scol}_{r}(G, \sigma)$. Alice's goal is to minimize the score while Bob's goal is to maximize the score. The game r-coloring number of $G$, denoted $\operatorname{gcol}_{r}(G)$, is the least $s$ such that Alice can always achieve a score of at most $s$, regardless of how Bob plays.

The next result bounds the generalized game coloring numbers for any graph class with bounded expansion.

Theorem 3.1 (Kierstead \& Yang [15]).
All graphs $G$ satisfy $\operatorname{gcol}_{r}(G) \leq 3\left(\operatorname{wcol}_{2 r}(G)\right)^{2} \leq 3\left(\operatorname{scol}_{2 r}(G)\right)^{4 r}$ for all $r$.
Now we are ready to prove the result that inspired our general approach.

## Theorem 3.2.

For any graph $G$ and $r, r^{\prime} \in \mathbb{N}$, there exists an ordering $\sigma^{*} \in \Pi(G)$ such that

$$
\operatorname{scol}_{r}\left(G, \sigma^{*}\right) \leq 3\left(\operatorname{scol}_{2 r}(G)\right)^{4 r} \quad \text { and } \quad \operatorname{scol}_{r^{\prime}}\left(G, \sigma^{*}\right) \leq 3\left(\operatorname{scol}_{2 r^{\prime}}(G)\right)^{4 r^{\prime}}+1
$$

Proof. We will create the ordering by having two players A and B play the ordering game. Player A plays by following Alice's optimal strategy in the $r$-ordering game on $G$ and interprets Player B's moves as Bob's moves in this game. Player B ignores Alice's first move, and from then on plays by following Alice's optimal strategy in the the $r^{\prime}$-ordering game on the remaining graph and interprets player A's moves as Bob's moves in this game.

By Theorem 3.1, the resulting ordering $\sigma^{*}$ has the desired properties, where we need to be aware that Player B had to ignore the first chosen vertex, which may lead to one more reachable vertex.

## 4 The Main Theorem

In this section we prove our main results, which are all corollaries of the following technical theorem.

## Theorem 4.1.

Let $G_{1}, \ldots, G_{k}$ be a collection of graphs, all on the same vertex set $V$, and $a_{1}, \ldots, a_{k}$ and $r_{1}, \ldots, r_{k}$ be positive integers. Set $A=a_{1}+\cdots+a_{k}$. Then there exists an ordering $\sigma^{*}$ of the common vertex set $V$ such that for all $i=1, \ldots, k$ we have

$$
\operatorname{scol}_{r_{i}}\left(G_{i}, \sigma^{*}\right) \leq \frac{A}{a_{i}}\left(\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)\right)^{2}+\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)
$$

Proof. In what follows, for a graph $G$, ordering $\sigma \in \Pi(G), r \in \mathbb{N}$ and $x \in V(G)$ we use $S_{r}(G, \sigma, x)$ and $W_{r}(G, \sigma, x)$ to denote $S_{r}[G, \sigma, x] \backslash\{x\}$ and $W_{r}[G, \sigma, x] \backslash\{x\}$, respectively. We also set

$$
\begin{array}{lll}
V_{\sigma}^{l}(x)=\left\{y \in V \mid y<_{\sigma} x\right\}, & V_{\sigma}^{l}[x]=V_{\sigma}^{l}(x) \cup\{x\} ; & \text { and } \\
V_{\sigma}^{r}(x)=\left\{y \in V \mid y>_{\sigma} x\right\}, & V_{\sigma}^{r}[x]=V_{\sigma}^{r}(x) \cup\{x\} .
\end{array}
$$

For all $i$, choose an ordering $\sigma_{i}$ of $V$ such that $\operatorname{wcol}_{2 r_{i}}\left(G_{i}, \sigma_{i}\right)=\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)$. Define the graph $H_{i}$ with vertex set $V$ by setting $E\left(H_{i}\right)=\left\{u v \mid u \in W_{r_{i}}\left(G_{i}, \sigma_{i}, v\right)\right\}$.

Claim 4. For all $i$ we have $\operatorname{scol}_{2}\left(H_{i}, \sigma_{i}\right) \leq \operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)$.
Proof. If $w \in S_{2}\left(H_{i}, \sigma_{i}, v\right)$, then $w<_{\sigma_{i}} v$, and either $w v \in E\left(H_{i}\right)$ or there is a $u>_{\sigma_{i}} v$ with $v u, u w \in E\left(H_{i}\right)$. In the first case we have $w \in W_{r_{i}}\left(G_{i}, \sigma_{i}, v\right) \subseteq W_{2 r_{i}}\left(G_{i}, \sigma_{i}, v\right)$. In the second case there are paths $P=v \ldots u$ and $Q=u \ldots w$ in $G_{i}$ of length at most $r_{i}$ with $v \leq_{\sigma_{i}} V(P \cup Q) \backslash\{w\}$. This again gives $w \in W_{2 r_{i}}\left(G_{i}, \sigma_{i}, v\right)$.

We construct $\sigma^{*}$ one vertex at the time, by collecting one by one vertices from $V$. Each time a vertex is collected it is deleted from the set $U$ of uncollected vertices and put at the end of the initial segment of $\sigma^{*}$ already constructed. We maintain a vector $\boldsymbol{m}_{v}:[k] \rightarrow\{0,1, \ldots\}$ for each vertex $v$. When $\boldsymbol{m}_{v}=\mathbf{0}$, we collect $v$.

We start without any collected vertex, so $U=V$, and for all $v \in V$ and $i \in[k]$ we set $\boldsymbol{m}_{v}(i)=a_{i}$. We now run the following algorithm.

```
pick any \(v \in U\);
while \(U \neq \emptyset\) do
    pick any \(i \in[k]\) with \(\boldsymbol{m}_{v}(i) \neq 0 ; \quad\) \{such \(i\) exists, since at this point always \(\left.v \in U\right\}\)
    \(\boldsymbol{m}_{v}(i) \leftarrow \boldsymbol{m}_{v}(i)-1 ;\)
    if \(\boldsymbol{m}_{v}=\mathbf{0}\) then
        collect \(v\)
    end if;
    if \(N_{H_{i}}[v] \cap U \neq \varnothing\) then
        \(v \leftarrow \sigma_{i}-\min \left(N_{H_{i}}[v] \cap U\right)\)
    else if \(U \neq \varnothing\) then
        pick any \(v \in U\)
    end if;
```

Claim 5. At any time in the algorithm and for all $i \in[k]$, every uncollected vertex $w$ satisfies: the number of collected vertices in $N_{H_{i}}(w) \cap V_{\sigma_{i}}^{r}(w)$ is at most $\frac{A}{a_{i}} \mathrm{wcol}_{2 r_{i}}\left(G_{i}\right)$. In other words,

$$
\begin{equation*}
\left|N_{H_{i}}(w) \cap V_{\sigma_{i}}^{r}(w) \cap V_{\sigma^{*}}^{l}(w)\right| \leq \frac{A}{a_{i}} \operatorname{wcol}_{2 r_{i}}\left(G_{i}\right) \tag{4}
\end{equation*}
$$

Proof. We say that a vertex is processed when it plays the role of $v$ at Line 3 of the algorithm. Observe that each vertex is processed on exactly $A$ rounds-on $\boldsymbol{m}_{v}(i)=a_{i}$ rounds with each index $i \in[k]$ —before it is collected at Line 6 , and then it is never processed again.

Suppose $w$ is uncollected at Line 2 of some round of the algorithm. Let $s$ be the number of collected vertices $v$ in $N_{H_{i}}(w) \cap V_{\sigma_{i}}^{r}(w)$. On each round that such a vertex $v$ was processed with index $i$, the if-clause at Line 8 was witnessed by $w$. As $w$ is uncollected at Line 2 , there were at most $A$ rounds on which $w$ was chosen at Line 1 or Line 9 to be processed next. (If equality holds, then $w$ is the last vertex chosen at Line 9 of the previous round.) On all other such rounds, a vertex $w^{\prime} \in N_{H_{i}}[v] \cap U$ with $w^{\prime}<_{\sigma_{i}} w$ was picked to be processed next. Clearly, $w^{\prime} \in S_{2}\left(H_{i}, \sigma_{i}, w\right)$. Moreover, as $w^{\prime} \in U$, it is chosen on at most $A$ rounds.

So all in all we get that $s \cdot a_{i} \leq A+A \cdot\left|S_{2}\left(H_{i}, \sigma_{i}, w\right)\right|=A \cdot\left|S_{2}\left[H_{i}, \sigma_{i}, w\right]\right|$. Using Claim 4 this gives

$$
s \leq \frac{A}{a_{i}} \operatorname{scol}_{2}\left[H_{i}, \sigma_{i}, w\right] \leq \frac{A}{a_{i}} \operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)
$$

as claimed.
Let $\sigma^{*}$ be the ordering obtained by the algorithm. Take $i \in[k]$. We will bound $\left|S_{r_{i}}\left[G_{i}, \sigma^{*}, w\right]\right|$ for each $w \in V$. First notice that $S_{r_{i}}\left[G_{i}, \sigma^{*}, w\right]$ is determined at the moment $w$ is collected (since then the sets $V_{\sigma^{*}}^{l}[w]$ and $V_{\sigma^{*}}^{r}[w]$ are known).

For all $u \in S_{r_{i}}\left(G_{i}, \sigma^{*}, w\right)$, pick a path $P_{u}=u \ldots w$ in $G_{i}$ of length at most $r_{i}$ with $V\left(\stackrel{\circ}{P}_{u}\right) \subseteq V_{\sigma^{*}}^{r}(w)$. Let $p_{u}=\sigma_{i}-\min \left(V\left(P_{u}\right)\right)$. Then

$$
\begin{equation*}
\text { (a) } u<_{\sigma^{*}} w \quad \text { and } \quad \text { (b) } \quad p_{u} \leq_{\sigma_{i}} u \tag{5}
\end{equation*}
$$

Partition $S_{r_{i}}\left(G_{i}, \sigma^{*}, w\right)$ by:

$$
\begin{aligned}
& X_{1}=\left\{u \in S_{r_{i}}\left(G_{i}, \sigma^{*}, w\right) \mid p_{u}=u\right\} \\
& X_{2}=\left\{u \in S_{r_{i}}\left(G_{i}, \sigma^{*}, w\right) \mid p_{u}=w\right\} \quad \text { and } \\
& X_{3}=\left\{u \in S_{r_{i}}\left(G_{i}, \sigma^{*}, w\right) \mid p_{u}<_{\sigma_{i}}\{u, w\}\right\}
\end{aligned}
$$

If $u \in X_{1}$, then $P_{u}$ witnesses that $u \in W_{r_{i}}\left(G_{i}, \sigma_{i}, w\right)$. By the choice of $\sigma_{i}$ this gives $\left|X_{1}\right| \leq \operatorname{wcol}_{r_{i}}\left(G_{i}\right)-1 \leq \operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)-1$.

Next consider a vertex $u \in X_{2}$. Then $w=p_{u} \leq_{\sigma_{i}} V\left(P_{u}\right)$, and hence $w \in W_{r_{i}}\left[G_{i}, \sigma_{i}, u\right]$. By definition, $u w \in E\left(H_{i}\right)$. On the other hand, $u<_{\sigma^{*}} w$ by (4a). Thus we have $X_{2} \subseteq$ $N_{H_{i}}(w) \cap V_{\sigma_{i}}^{r}(w) \cap V_{\sigma^{*}}^{l}(w)$. By (4) this means $\left|X_{2}\right| \leq \frac{A}{a_{i}} \operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)$.

Finally, consider a vertex $u \in X_{3}$. Then $p_{u} \in W_{r_{i}}\left(G_{i}, \sigma_{i}, u\right)$ and $p_{u} \in W_{r_{i}}\left(G_{i}, \sigma_{i}, w\right)$. By definition, $p_{u} u \in E\left(H_{i}\right)$. By (4a), $u<_{\sigma^{*}} w$, and by (4b), $p_{u}<_{\sigma_{i}} u$. Combining this all gives $u \in N_{H_{i}}\left(p_{u}\right) \cap V_{\sigma_{i}}^{r}\left(p_{u}\right) \cap V_{\sigma^{*}}^{l}\left(p_{u}\right)$. It follows that

$$
X_{3} \subseteq \bigcup_{p \in W_{r_{i}}\left(G_{i}, \sigma_{i}, w\right)} N_{H_{i}}(p) \cap V_{\sigma_{i}}^{r}(p) \cap V_{\sigma^{*}}^{l}(p) .
$$

And so (4) leads to

$$
\left|X_{3}\right| \leq\left(\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)-1\right) \cdot \frac{A}{a_{i}} \operatorname{wcol}_{2 r_{i}}\left(G_{i}\right) .
$$

Adding it all together we get

$$
\begin{aligned}
\mid S_{r_{i}}\left[G_{i}, \sigma^{*}, w\right] & \left|=1+\left|X_{1}\right|+\left|X_{2}\right|+\left|X_{3}\right|\right. \\
& \leq 1+\left(\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)-1\right)+\frac{A}{a_{i}} \operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)+\left(\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)-1\right) \cdot \frac{A}{a_{i}} \operatorname{wcol}_{2 r_{i}}\left(G_{i}\right) \\
& =\frac{A}{a_{i}}\left(\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)\right)^{2}+\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right) .
\end{aligned}
$$

Since $\operatorname{scol}_{r_{i}}\left(G_{i}, \sigma^{*}\right)=\max _{w \in V}\left|S_{r_{i}}\left[G_{i}, \sigma^{*}, w\right]\right|$, the theorem follows.
We are now ready to prove the results stated in Subsection 1.3 . We start with the easiest proof.

Proof of Theorem 1.7. Let $G_{1}, \ldots, G_{k}$ and $r_{1}, \ldots, r_{k} \in \mathbb{N}$ as in the statement of the theorem. Using Theorem 4.1 with all $a_{i}=1$, and hence $A=k$, we get that there exists an ordering $\sigma^{*}$ of $V$ such that for all $i$ we have

$$
\left.\operatorname{scol}_{r_{i}}\left(G_{i}, \sigma^{*}\right) \leq k \cdot\left(\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)\right)\right)^{2}+\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right) \leq(k+1)\left(\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)\right)^{2} .
$$

Proof of Theorem 1.4. Set $n=|G|$. It is easy to check that the result holds if $n \leq 3$, so assume $n \geq 4$ and let $k=\left\lfloor\log _{2}(n-2)\right\rfloor$.

If $i \geq k+1$, then we have $i>\log _{2}(n-2)$, hence $2^{i}+1>n-1$. This means that $\operatorname{scol}_{i}\left(G, \sigma^{*}\right) \leq\left(2^{i}+1\right) \cdot\left(\operatorname{wcol}_{2 i}(G)\right)^{2}$ trivially holds for any ordering $\sigma^{*}$.

For $i=1, \ldots, k$, set $G_{i}=G, r_{i}=i$ and $a_{i}=2^{k-i}$. Then $A=a_{1}+\cdots+a_{k}=2^{k}-1$. Using Theorem 4.1, we find that there exists an ordering $\sigma^{*}$ of $G$ such that for all $i=1, \ldots, k$ we have

$$
\begin{aligned}
\operatorname{scol}_{i}\left(G, \sigma^{*}\right) & \leq \frac{\left(2^{k}-1\right) \cdot\left(\operatorname{wcol}_{2 i}(G)\right)^{2}}{2^{k-i}}+\operatorname{wcol}_{2 i}(G) \\
& \leq 2^{i} \cdot\left(\operatorname{wcol}_{2 i}(G)\right)^{2}+\operatorname{wcol}_{2 i}(G) \leq\left(2^{i}+1\right) \cdot\left(\operatorname{wcol}_{2 i}(G)\right)^{2}
\end{aligned}
$$

By (1) this proves the bound on $\operatorname{scol}_{i}\left(G, \sigma^{*}\right)$ for $i \leq k$, and completes the proof.
We finish with a more general version of Theorem 1.4.

## Corollary 4.2.

For any graph $G$ and $\epsilon>0$, there exits an ordering $\sigma^{*}$ of $G$ such that for all $r \in \mathbb{N}$ we have

$$
\operatorname{scol}_{r}\left(G, \sigma^{*}\right) \leq\left(\frac{(1+\epsilon)^{r+1}}{\epsilon^{2}}+1\right) \cdot\left(\operatorname{scol}_{2 r}(G)\right)^{4 r}
$$

Proof. We follow the proof of Corollary 1.4 above. First choose the positive integer $k$ such that

$$
\left(\frac{(1+\epsilon)^{(k+1)+1}}{\epsilon^{2}}+1\right) \geq|G| .
$$

This means that the bound on $\operatorname{scol}_{i}\left(G, \sigma^{*}\right)$ trivially holds for $r \geq k+1$, for any ordering $\sigma^{*}$.
Now for $i=1, \ldots, k$, set $G_{i}=G, r_{i}=i$ and $a_{i}=\left\lceil(1+\epsilon)^{k+1-i}-1\right\rceil$. Then we can estimate

$$
A=a_{1}+\cdots+a_{k} \leq \sum_{i=1}^{k}(1+\epsilon)^{k+1-i}=\frac{(1+\epsilon)^{k+1}-(1+\epsilon)}{\epsilon}<\frac{(1+\epsilon)^{k+1}}{\epsilon} .
$$

For all $i=1, \ldots, k$ we get

$$
a_{i}=\left\lceil(1+\epsilon)^{k+1-i}-1\right\rceil \geq(1+\epsilon)^{k+1-i}-1>\epsilon \cdot(1+\epsilon)^{k-i}
$$

Using Theorem 4.1 again, there exists an ordering $\sigma^{*}$ of $G$ such that for all $i=1, \ldots, k$ we have

$$
\begin{aligned}
\operatorname{scol}_{i}\left(G, \sigma^{*}\right) & \leq \frac{(1+\epsilon)^{k+1} \cdot\left(\operatorname{wcol}_{2 i}(G)\right)^{2}}{\epsilon^{2} \cdot(1+\epsilon)^{k-i}}+\operatorname{wcol}_{2 i}(G) \\
& \leq \frac{(1+\epsilon)^{i+1}}{\epsilon^{2}} \cdot\left(\operatorname{wcol}_{2 i}(G)\right)^{2}+\operatorname{wcol}_{2 i}(G) \leq\left(\frac{(1+\epsilon)^{i+1}}{\epsilon^{2}}+1\right) \cdot\left(\operatorname{wcol}_{2 i}(G)\right)^{2}
\end{aligned}
$$

By (1) this proves the bound on $\operatorname{scol}_{i}\left(G, \sigma^{*}\right)$ for $i \leq k$, and completes the proof.

## 5 Algorithmic Aspects

Our main results, Theorems 1.4 and 4.1 , guarantee the existence of a specific ordering of the vertices of a graph. But the results do not indicate if such an ordering can be found efficiently. The proof of Theorem 4.1 is in fact algorithmic. If for every $i=1, \ldots, k$ we have an ordering $\sigma_{i}$ of the vertex set such that $\operatorname{wcol}_{2 r_{i}}\left(G_{i}, \sigma_{i}\right)=\operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)$, then the proof gives an algorithm that finds an ordering $\sigma^{*}$ in $O(A \cdot|V|)$ steps. (We start with a vector $\boldsymbol{m}$ with $\boldsymbol{m}_{v}(i)=a_{i}$ for each vertex $v$, and in each iteration of the while loop one coordinate $\boldsymbol{m}_{v}$ gets reduced by one.)

So the question about the existence of an efficient algorithm to find a uniform ordering depends on the existence of an efficient algorithm to find optimal orderings for the generalized coloring numbers. It is very unlikely that this is possible, though. Grohe et al. [5, 6] proved that computing wcol $_{r}(G)$ is NP-complete for all fixed $r \geq 3$. Note that calculating the coloring number $\operatorname{col}(G)$ can be done in polynomial time; it is an interesting open problem to determine the computational complexity status of finding $\operatorname{wcol}_{2}(G)$.

Nevertheless, it is possible to find orderings that approximate the generalized coloring numbers, using ideas developed in Dvořák [3]. We need a new concept. Let $r \in \mathbb{N}$. For a graph $G$, ordering $\sigma \in \Pi$ and $x \in V$, let $b_{r}[G, \sigma, x]$ be the maximum number of paths of length at most $r$ that have $x$ as one end, whose other end $y$ satisfies $y \leq_{\sigma} x$, and that are vertex-disjoint apart from $x$. Clearly, we can assume that the internal vertices of the paths appear after $x$ in the ordering. The $r$-admissibility of $G$, denoted $\operatorname{adm}_{r}(G)$, is defined as $2^{2}$

$$
\operatorname{adm}_{r}(G, \sigma)=\max _{x \in V} b_{r}[G, \sigma, x] ; \quad \operatorname{adm}_{r}(G)=\min _{\sigma \in \Pi} \operatorname{adm}_{r}(G, \sigma) .
$$

It is obvious that once again $\operatorname{adm}_{1}(G)$ is just the coloring number $\operatorname{col}(G)$; while we also have $\operatorname{adm}_{r}(G) \leq \operatorname{scol}_{r}(G) \leq \operatorname{wcol}_{r}(G)$. On the other hand, Dvořák [3, Lemma 6] gives the existence of a function $F: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that $\operatorname{wcol}_{r}(G) \leq F\left(r, \operatorname{adm}_{r}(G)\right)$ for all $r \in \mathbb{N}$ and graphs $G$.

Dvořák [3] also gives a simple algorithm that, given $r \in \mathbb{N}$ and a graph $G$, in $O\left(r^{3} \cdot|G|\right)$ steps finds an ordering $\sigma$ of $G$ such that $\operatorname{adm}_{r}(G, \sigma) \leq r \cdot \operatorname{adm}_{r}(G)$.

Combining all this with the proof of Theorem 4.1 gives the following algorithmic version of that theorem.

## Theorem 5.1.

There exists a function $\phi: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ and an algorithm $\mathcal{A}$ such that the following holds. Let $G_{1}, \ldots, G_{k}$ be a collection of graphs, all on the same vertex set $V$, and $a_{1}, \ldots, a_{k}$ and $r_{1}, \ldots, r_{k}$ be positive integers. Set $A=a_{1}+\cdots+a_{k}$. Then algorithm $\mathcal{A}$ gives an ordering $\sigma^{*}$ of the common vertex set $V$ such that for all $i=1, \ldots, k$ we have

$$
\operatorname{scol}_{r_{i}}\left(G_{i}, \sigma^{*}\right) \leq \frac{A}{a_{i}} \cdot \phi\left(r_{i}, \operatorname{wcol}_{2 r_{i}}\left(G_{i}\right)\right) .
$$

The number of steps algorithm $\mathcal{A}$ requires is polynomial in $A$ and $|G|$.
The proof of Theorem 1.4 shows that we can use the theorem above with $A \leq n$ to get an algorithmic version of that theorem.

## Theorem 5.2.

There exists a function $\phi^{\prime}: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ and an algorithm $\mathcal{A}^{\prime}$ such that the following holds. For any graph $G$, algorithm $\mathcal{A}^{\prime}$ gives an ordering $\sigma^{*}$ of $G$ such that $\operatorname{scol}_{r}\left(G, \sigma^{*}\right) \leq \phi^{\prime}\left(r, \operatorname{scol}_{2 r}(G)\right)$ for all $r \in \mathbb{N}$. The number of steps algorithm $\mathcal{A}^{\prime}$ requires is polynomial in $|G|$.

Finally, we formulate an algorithmic version of Corollary 1.5.

## Corollary 5.3.

There exists an algorithm $\mathcal{A}^{*}$ such that the following holds. A graph class $\mathcal{G}$ has bounded expansion if and only if there exists a function $c^{*}: \mathbb{N} \rightarrow \mathbb{N}$, such that for every graph $G \in \mathcal{G}$, algorithm $\mathcal{A}^{*}$ gives an ordering $\sigma^{*}$ of $G$ such that $\operatorname{scol}_{r}\left(G, \sigma^{*}\right) \leq c^{*}(r)$ for all $r \in \mathbb{N}$. The number of steps algorithm $\mathcal{A}^{*}$ requires is polynomial in $|G|$.

[^2]
## 6 Discussion

The original motivation in [15] for defining generalized coloring numbers was to study various game theoretic questions, including generalized game coloring numbers and their applications to other games. It was a major surprise that generalized coloring numbers could provide characterizations of sparse classes; indeed even generalized game coloring numbers provide these characterizations. Just as ordinary coloring numbers have proved useful in sparsity theory, one might expect that game coloring numbers should find applications. Prior to this paper, and aside from the characterization just mentioned, we know only one other application to a non-game problem. In [12], the game strong 2-coloring number is used to provide improved bounds for Bollobás-Eldridge-type questions on packing. In this paper, while we used game coloring techniques, we did not apply any theorems from that area. We limited the competitive aspects of the theory by enforcing a prioritization for the goals of multiple players (graphs) using the vector $\boldsymbol{m}$. This draws on ideas from the Harmonious Strategy in [16]. We expect that those ideas can be used in other (non-game) settings as well. Other applications of the Harmonious Strategy include [17, [25, 26]; [17] and [26] address non-game problems.

After solving Problem 1.3, it is natural to ask how good our answer is. In other words: For $c: \mathbb{N} \rightarrow \mathbb{N}$, what is the smallest function $c^{*}: \mathbb{N} \rightarrow \mathbb{N}$ such that for all $G \in \mathcal{G}_{c}$ there is an ordering $\sigma^{*} \in \Pi(G)$ such that all $r \in \mathbb{N}$ satisfy $\operatorname{scol}_{r}\left(G, \sigma^{*}\right) \leq c^{*}(r)$ ? Recall that $\varphi=\frac{1}{2}(1+\sqrt{5}) \approx 1.62$. Example 2.1 and Theorem 1.4 show that

$$
.08 c(r)^{\varphi} \leq c^{*}(r) \leq\left(2^{r}+1\right) \cdot c(2 r)^{4 r}
$$

The lower bound is polynomial in $c(r)$, while the upper bound is exponential in $c(2 r)$. We don't have enough evidence to make a justified guess on the right order of $c^{*}$ in terms of $c$.

The main result in [15], Theorem 3.1 in this paper, gives an upper bound of $\operatorname{gcol}_{r}(G)$ in terms of $\operatorname{scol}_{2 r}(G)$. It is shown in [15] that $\operatorname{gcol}_{r}(G)$ cannot be bounded in terms of $\operatorname{scol}_{2 r-1}(G)$. Hence it is tempting to conjecture that $c^{*}(r)$ cannot be upper bounded in terms of $c(2 r-1)$, but we have been unable to find examples of graphs that confirm this.

## Acknowledgment

The authors thank Patrice Ossona de Mendez for suggesting the examples in Subsection 2.2, and two anonymous referees for careful reading.

## References

[1] G. Chen and R.H. Schelp. Graphs with linearly bounded Ramsey numbers. J. Combin. Theory Ser. B, 57(1):138-149, 1993. doi:10.1006/jctb.1993.1012.
[2] R. Diestel. Graph Theory. Springer, Berlin, 5th edition, 2018. doi:10.1007/978-3-662-53622-3.
[3] Z. Dvořák. Constant-factor approximation of the domination number in sparse graphs. European J. Combin., 34(5):833-840, 2013. doi:10.1016/j.ejc.2012.12.004
[4] Z. Dvořák, D. Král', and R. Thomas. Testing first-order properties for subclasses of sparse graphs. J. ACM, 60(5):Art. 36, $24 \mathrm{pp}, 2013$. doi:10.1145/2499483.
[5] M. Grohe, S. Kreutzer, R. Rabinovich, S. Siebertz, and K. Stavropoulos. Colouring and covering nowhere dense graphs. In Graph-theoretic Concepts in Computer Science, volume 9224 of Lecture Notes in Comput. Sci., pages 325-338. Springer, Berlin, 2016. doi:10.1007/978-3-662-53174-7_23.
[6] M. Grohe, S. Kreutzer, R. Rabinovich, S. Siebertz, and K. Stavropoulos. Colouring and covering nowhere dense graphs. SIAM J. Discrete Math., 32(4):2467-2481, 2018. doi:10.1137/18M1168753.
[7] M. Grohe, S. Kreutzer, and S. Siebertz. Deciding first-order properties of nowhere dense graphs. J. ACM, $64(3):$ Art. 17, $32 \mathrm{pp}, 2017$. doi:10.1145/3051095.
[8] J. van den Heuvel, P. Ossona de Mendez, D. Quiroz, R. Rabinovich, and S. Siebertz. On the generalised colouring numbers of graphs that exclude a fixed minor. European J. Combin., 66:129-144, 2017. doi:10.1016/j.ejc.2017.06.019.
[9] G. Joret, P. Micek, P. Ossona de Mendez, and V. Wiechert. Nowhere dense graph classes and dimension. Combinatorica, 39(5):1055-1079, 2019. doi:10.1007/s00493-019-3892-8.
[10] W. Kazana and L. Segoufin. Enumeration of monadic second-order queries on trees. ACM Trans. Comput. Log., 14(4):Art. 25, 12 pp, 2013. doi:10.1145/2528928.
[11] H.A. Kierstead. A simple competitive graph coloring algorithm. J. Combin. Theory Ser. B, 78(1):57-68, 2000. doi:10.1006/jctb.1999.1927.
[12] H.A. Kierstead and A.V. Kostochka. Efficient graph packing via game colouring. Combin. Probab. Comput., 18(5):765-774, 2009. doi:10.1017/S0963548309009973.
[13] H.A. Kierstead and W.T. Trotter. Planar graph coloring with an uncooperative partner. J. Graph Theory, 18(6):569-584, 1994. doi:10.1002/jgt.3190180605.
[14] H.A. Kierstead and W.T. Trotter. Competitive colorings of oriented graphs. Electron. J. Combin., 8(2):Research Paper 12, 15 pp, 2001. URL: www.combinatorics.org/Volume_8/Abstracts/v8i2r12.html.
[15] H.A. Kierstead and D. Yang. Orderings on graphs and game coloring number. Order, 20(3):255-264, 2003. doi:10.1023/B:ORDE.0000026489.93166.cb,
[16] H.A. Kierstead and D. Yang. Very asymmetric marking games. Order, 22(2):93-107, 2005. doi:10.1007/s11083-005-9012-y.
[17] H.A. Kierstead, D. Yang, and J. Yi. On coloring numbers of graph powers. Discrete Math., 2019. In Press, Corrected Proof. doi:10.1016/j.disc.2019.111712.
[18] S. Kreutzer, M. Pilipczuk, R. Rabinovich, and S. Siebertz. The generalised colouring numbers on classes of bounded expansion. In 41st International Symposium on Mathematical Foundations of Computer Science (MFCS 2016), volume 58 of LIPIcs. Leibniz Int. Proc. Inform., pages 85:1-85:13. Schloss Dagstuhl Leibniz-Zent. Inform., 2016. doi:10.4230/LIPIcs.MFCS.2016.85,
[19] J. Nešetřil and P. Ossona de Mendez. Tree-depth, subgraph coloring and homomorphism bounds. European J. Combin., 27(6):1022-1041, 2006. doi:10.1016/j.ejc.2005.01.010
[20] J. Nešetřil and P. Ossona de Mendez. Grad and classes with bounded expansion I. Decompositions. European J. Combin., 29(3):760-776, 2008. doi:10.1016/j.ejc.2006.07.013
[21] J. Nešetřil and P. Ossona de Mendez. On nowhere dense graphs. European J. Combin., 32(4):600-617, 2011. doi:10.1016/j.ejc.2011.01.006.
[22] J. Nešetřil and P. Ossona de Mendez. Sparsity - Graphs, Structures, and Algorithms. Springer, Heidelberg, 2012. doi:10.1007/978-3-642-27875-4.
[23] Problems presented during the Workshop on Algorithms, Logic and Structure, Warwick, UK. 12-14 December 2016, $2016 . \quad$ URL: www.warwick.ac.uk/fac/sci/maths/research/events/2016-17/nonsymposium/als/.
[24] N. Streib and W.T. Trotter. Dimension and height for posets with planar cover graphs. European J. Combin., 35:474-489, 2014. doi:10.1016/j.ejc.2013.06.017.
[25] D. Yang and X. Zhu. Game colouring directed graphs. Electron. J. Combin., 17(1):Research Paper 11, 19 pp , $2010 . \quad$ URL: Www.combinatorics.org/Volume_17/Abstracts/v17i1r11.html.
[26] D. Yang and X. Zhu. Strong chromatic index of sparse graphs. J. Graph Theory, 83(4): 334-339, 2016. doi:10.1002/jgt.21999.
[27] X. Zhu. Refined activation strategy for the marking game. J. Combin. Theory Ser. B, 98(1):1-18, 2008. doi:10.1016/j.jctb.2007.04.004.
[28] X. Zhu. Colouring graphs with bounded generalized colouring number. Discrete Math., 309(18):5562-5568, 2009. doi:10.1016/j.disc.2008.03.024.


[^0]:    ${ }^{*}$ The research for this paper was started during a visit of HAK to the London School of Economics, and continued during a return visit of JvdH to Arizona State University. The authors would like to thank both universities for their hospitality and support.
    ${ }^{\dagger}$ Department of Mathematics, London School of Economics and Political Science, London WC2A 2AE, UK.
    ${ }^{\ddagger}$ School of Mathematical and Statistical Sciences, Arizona State University, Tempe, AZ 85287, USA.
    Email: j.van-den-heuvel@lse.ac.uk, kierstead@asu.edu.

[^1]:    ${ }^{1}$ In [15] strong coloring numbers were just called coloring numbers, and weak coloring numbers were introduced for the purpose of studying (strong) coloring numbers. As weak coloring numbers have their own merit, it now seems better to distinguish between them by using the terms strong and weak.

[^2]:    ${ }^{2}$ The definition of $\operatorname{adm}_{r}(G)$ in [3] does not include the vertex $x$ in the set $b_{r}[G, \sigma, x]$; we include it here for consistency with the now standard convention for generalized coloring numbers.

