

TILINGS IN GRAPHONS

JAN HLADKÝ, PING HU, AND DIANA PIGUET

ABSTRACT. We introduce a counterpart to the notion of tilings, that is vertex-disjoint copies of a fixed graph F , to the setting of graphons. The case $F = K_2$ gives the notion of matchings in graphons. We give a transference statement that allows us to switch between the finite and limit notion, and derive several favorable properties, including the LP-duality counterpart to the classical relation between the fractional vertex covers and fractional matchings/tilings, and discuss connections with property testing.

As an application of our theory, we determine the asymptotically almost sure F -tiling number of inhomogeneous random graphs $G(n, W)$. As another application, in an accompanying paper [Hladký, Hu, Piguet: Komlós's tiling theorem via graphon covers, J. Graph Theory, 2019] we give a proof of a strengthening of a theorem of Komlós [Komlós: Tiling Turán Theorems, Combinatorica, 2000].

1. TOWARDS LIMITS OF TILINGS¹

The emergence of graph limit theories has brought numerous novel views on classical problems in graph theory. More precisely, the problems in which these theories helped concern comparing subgraph densities. This is a very broad area lying in the heart of extremal graph theory. In this explanatory section, we focus only on the applications of dense graphs limits, firstly because these have been richer, and secondly because this is the direction which we pursue in this paper. Two closely related theories have emerged. Razborov's flag algebras method [21] represents an abstract approach to graph limits. The method gives universal methods for calculations with subgraph densities (most notably the semidefinite method and differential calculus), and has led to the complete solutions of, or at least to a breakthrough progress on, several prominent problems in the area. Of these breakthroughs let us mention a result of Razborov [22] who determined an optimal function $f : [0, 1] \rightarrow [0, 1]$ such that if an n -vertex graph G contains at least $\alpha \binom{n}{2}$ edges then

$$(1.1) \quad G \text{ contains at least } (f(\alpha) + o(1)) \binom{n}{3} \text{ triangles,}$$

thus resolving an old question of Lovász and Simonovits. On the other hand, the theory developed by Borgs, Chayes, Lovász, Szegedy, Sós and Vesztergombi [5, 18] provides explicit limit objects, the so-called graphons. This theory has been successfully applied in various parts of

Hladký: The research leading to these results has received funding from the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme (FP7/2007-2013) under REA grant agreement number 628974.

Hu: This work has received funding from the Leverhulme Trust 2014 Philip Leverhulme Prize of Daniel Kral and from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 648509).

Piguet was supported by the Czech Science Foundation, grant number GJ16-07822Y.

¹An extended abstract describing also the main results of this paper appeared in the proceedings of Eurocomb 2017, [9].

graph theory (and in particular provided insights into the properties of Szemerédi regularity partitions) and random graphs. In extremal graph theory, the theory of graphons has been used to prove a certain “local” version of Sidorenko’s conjecture, [16].

As said, the graph limit theories have been very powerful in relating subgraph densities. Some other concepts, like the one of the minimum degree have been translated to the setting of graph limits and have been explored. Extremal graph theory, however, is a much richer field, and which other statements or features can be formulated in the language of graph limits is interesting in its own right. In the present paper we develop a theory of tiling in graphons. A *tiling* \mathcal{T} by a finite graph F in another graph G (“ F -tiling in G ”, in short) is a collection of vertex-disjoint copies of F (not necessarily induced) in G . This concept is used also under the name of F -matching, as we get the usual notion of matchings, when $F = K_2$. The *size* of the tiling \mathcal{T} is simply the cardinality $|\mathcal{T}|$. We write $\text{til}(F, G)$ for the size of a maximum F -tiling in G . For example, a “tiling counterpart” to (1.1) would entail finding an optimal function $g : [0, 1] \rightarrow [0, 1]$ such that any n -vertex graph G containing at least $\alpha \binom{n}{2}$ edges satisfies

$$G \text{ contains a tiling of } K_3 \text{ of size at least } (g(\alpha) + o(1)) \frac{n}{3}.$$

Such a function g was indeed determined in [1]. Another example is the basic result of Erdős and Gallai [11], where they determined the size of a matching guaranteed in a graph of a given density.

There is a fractional relaxation of the notion of tilings. In that notion, we put $[0, 1]$ -weights on copies of F in G , and require that the total weight at each vertex in G is at most 1. However, for us it is more convenient to choose a slightly different notion in which we replace copies by homomorphic copies. More precisely, we assume that the graph F is on the vertex set $[k]$. We denote by $\mathcal{F}_F(G)$ all the copies of F in the graph G ,

$$(1.2) \quad \mathcal{F}_F(G) = \left\{ (u_1, u_2, \dots, u_k) \in V(G)^k : u_i u_j \in E(G) \text{ for each pair } ij \in E(F) \right\}.$$

More precisely, the members of $\mathcal{F}_F(G)$ correspond to ordered vertex-sets of G that represent homomorphisms of F into G . With a slight abuse of notation we shall call members of $\mathcal{F}_F(G)$ *copies of F in G* . Given a copy F' of F in G , the *vertex set* $V(F')$ is defined in an obvious way. Note that $|V(F')| \leq k$, but equality need not hold.

A *fractional F -tiling* in a finite graph G is a weight function $\mathfrak{t} : \mathcal{F}_F(G) \rightarrow [0, 1]$ that satisfies that for each $v \in V(G)$,

$$\sum_{F' \in \mathcal{F}_F(G), v \in V(F')} \mathfrak{t}(F') \leq 1.$$

The *size* $\|\mathfrak{t}\|$ of \mathfrak{t} is the total weight of $\mathcal{F}_F(G)$, i.e., $\|\mathfrak{t}\| = \sum_{F' \in \mathcal{F}_F(G)} \mathfrak{t}(F')$. A standard compactness argument shows, that there exists an F -tiling of maximum size. We denote this maximum size by $\text{ftil}(F, G)$ and call it the *fractional F -tiling number*. Each F -tiling $\mathcal{T} \subset \mathcal{F}_F(G)$ can be represented as a fractional F -tiling by simply putting weight 1 on the copies of \mathcal{T} and 0 on the copies of $\mathcal{F}_F(G) \setminus \mathcal{T}$. Thus we have $\text{til}(F, G) \leq \text{ftil}(F, G)$.

Remark 1.1. *To illustrate the notion, and in particular to emphasize the difference between copies and homomorphic copies, let us compute $\text{ftil}(F, G)$ when F is a five-cycle and G is a triangle (so, G is the smaller of the graphs!). Clearly, $\text{ftil}(C_5, K_3) \leq \frac{3}{5}$, which is just a particular instance of the general bound $\text{ftil}(F, G) \leq \frac{v(G)}{v(F)}$. On the other hand, $1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 1, 4 \mapsto 2, 5 \mapsto 3$ is a homomorphism of C_5 to K_3 . Now we can consider 2 further variants of this homomorphism obtained by cyclic shifts of K_3 , and putting weight $\frac{1}{5}$ on each of these 3 homomorphic copies. This shows that $\text{ftil}(C_5, K_3) \geq \frac{3}{5}$.*

The linear programming (LP) duality provides a very useful way of expressing $\text{ftil}(F, G)$. To this end recall that a function $c : V(G) \rightarrow [0, 1]$ is a *fractional F -cover* of G if $\sum_{v \in F'} c(v) \geq 1$ for each copy $F' \in \mathcal{F}_F(G)$ (so, in the summation, we view F' as a multiset). The *size* of the fractional cover c is the total weight of $V(G)$, and is denoted by $\|c\|$. Again, a compactness argument shows that there exists a fractional F -cover of G of minimum size, which is denoted by $\text{fcov}(F, G)$ and called the *fractional F -cover number*. Then the Duality Theorem asserts that $\text{fcov}(F, G) = \text{ftil}(F, G)$. Let us note that showing the “ \geq ” direction is easy and the difficulty lies in proving the “ \leq ” direction.

In this paper we define the notions of (fractional) F -tilings and fractional F -covers for graphons (Definition 3.1 and Definition 3.12). As we show, when the graphon in question is taken to be a representation of a finite graph then there is a correspondence between the classical finite notion and the new graphon notion (see Propositions 3.2 and 3.13). We relate these graphon notions to the limits of the corresponding finite parameters (see Theorem 3.4, Proposition 3.8, Corollary 5.2) and treat their continuity properties on the graphon space (see Theorem 3.7, Theorem 3.7, and Theorem 3.14). We derive the LP-duality between the fractional F -tiling number and fractional F -cover number for graphons (Theorem 3.16).

This paper is organized as follows. In Section 2 we introduce common notation and provide preliminaries regarding functional analysis, graph limits and the regularity lemma. In Section 3 we introduce all the main concepts in our theory and state the main results. In Section 4 we give proofs of these results. In Section 5 we determine the F -tiling number of inhomogeneous random graphs. In an accompanying paper [13] we give another application of the theory by proving a strengthened version of a theorem of Komlós [15] regarding tilings in finite graphs.

Let us also mention that Doležal and Hladký [8] study in more detail the case $F = K_2$, that is, the case of matchings in graphons.

2. NOTATION AND PRELIMINARIES

Suppose that X is an arbitrary set. Given $\mathbf{x} = (x_1, \dots, x_k) \in X^k$ and two numbers $1 \leq i < j \leq k$, we write $\pi_{ij}(\mathbf{x})$ for the projection of \mathbf{x} , $\pi_{ij}(\mathbf{x}) = (x_i, x_j)$. We extend this to projecting sets, i.e., given $Y \subset X^k$, we write $\pi_{ij}(Y) = \{\pi_{ij}(\mathbf{x}) : \mathbf{x} \in Y\}$. We will use this notion only with regard to the inverse map. That is, if $A \subset X^2$ then $\pi_{ij}^{-1}(A) = \{\mathbf{x} \in X^k : (x_i, x_j) \in A\}$.

Given a function f and a number a we define its *support* $\text{supp } f = \{x : f(x) \neq 0\}$ and its variant $\text{supp}_a f = \{x : f(x) \geq a\}$.

Our notation follows [17]. Throughout the paper we shall assume that Ω is an atomless Borel probability space equipped with a measure ν (defined on an implicit σ -algebra). The product measure on Ω^k is denoted by ν^k . Recall that a set is *null* if it has zero measure.

2.1. Banach–Alaoglu Theorem. As Ω is a Borel probability space, it is in particular a separable measure space. It is well-known that then the Banach space $\mathcal{L}^1(\Omega)$ is separable (see e.g. [6, Theorem 13.8]). The dual of $\mathcal{L}^1(\Omega)$ is $\mathcal{L}^\infty(\Omega)$.

Let us now recall the defining property of the weak* topology on $\mathcal{L}^\infty(\Omega)$: a sequence $f_1, f_2, \dots \in \mathcal{L}^\infty(\Omega)$ converges weak* to a function $f \in \mathcal{L}^\infty(\Omega)$ if for each $g \in \mathcal{L}^1(\Omega)$ we have that $\int_{x \in \Omega} f_n(x)g(x) \, d\nu \rightarrow \int_{x \in \Omega} f(x)g(x) \, d\nu$. Weak* convergence is denoted by $f_n \xrightarrow{w^*} f$.

These initial preparations can be used to verify that in the current context the assumptions of the sequential Banach–Alaoglu Theorem (as stated for example in [23, Theorem 1.9.14]) are fulfilled. Thus, let us state the theorem in the setting of $\mathcal{L}^\infty(\Omega)$.

Theorem 2.1. *If Ω is a Borel probability space then each sequence of functions of $\mathcal{L}^\infty(\Omega)$ -norm at most 1 contains a weak* convergent subsequence.*

2.2. Graphons. Our graphons will be defined on Ω^2 , i.e., symmetric measurable functions $W : \Omega^2 \rightarrow [0, 1]$. For a finite graph G on vertex set $\{v_1, \dots, v_n\}$ we can partition Ω into n sets $\Omega_1, \dots, \Omega_n$ of measure $\frac{1}{n}$ each and obtain a *representation* of G . This graphon, always denoted by W_G , is defined to be one or zero on each square $\Omega_i \times \Omega_j$ depending on whether the pair $v_i v_j$ does or does not form an edge, respectively. Note that W_G is not uniquely defined; it depends on the partition $\Omega_1, \dots, \Omega_n$. Given a bounded symmetric measurable function $W : \Omega^2 \rightarrow \mathbb{R}$, define its *cut norm* $\|W\|_\square$ to be $\sup_{S, T \subseteq \Omega} \left| \int_{(x, y) \in S \times T} W(x, y) \, d\nu^2 \right|$, where the supremum is taken over all measurable subsets S and T . Let W be a graphon and let $\varphi : \Omega \rightarrow \Omega$ be a measure preserving map. Define W^φ by $W^\varphi(x, y) = W(\varphi(x), \varphi(y))$. Given two graphons U and W , we define the *cut distance* of them by $\delta_\square(U, W) = \inf_\varphi \|U - W^\varphi\|_\square$, where the infimum is taken over all measure preserving map $\varphi : \Omega \rightarrow \Omega$. We denote $U \leq W$ ν^2 -almost everywhere by $U \leq W$.

Suppose that we are given an arbitrary graphon $W : \Omega^2 \rightarrow [0, 1]$ and a graph F whose vertex set is $[k]$. We write $W^{\otimes F} : \Omega^k \rightarrow [0, 1]$ for a function defined by $W^{\otimes F}(x_1, \dots, x_k) = \prod_{1 \leq i < j \leq k, ij \in E(F)} W(x_i, x_j)$, which we call the *density* of F in W . Let $\mathcal{F}_F(W) = \text{supp } W^{\otimes F}$. Note that if we take W to be a graphon representation a finite graph G then there is a natural correspondence between $\mathcal{F}_F(W)$ and $\mathcal{F}_F(G)$ defined in (1.2).

We shall need the following easy statement about graphons, which is essentially given in [18, Lemma 4.1].

Lemma 2.2. *Suppose that U and W are two graphons such that $\|W - U\|_\square < \delta$, and let F be a finite graph with k vertices. Then for arbitrary measurable sets $P_1, \dots, P_k \subset \Omega$ we have*

$$\left| \int_{(x_1, \dots, x_k) \in \prod_i P_i} U^{\otimes F}(x_1, \dots, x_k) \, d\nu^k - \int_{(x_1, \dots, x_k) \in \prod_i P_i} W^{\otimes F}(x_1, \dots, x_k) \, d\nu^k \right| \leq \binom{k}{2} \delta.$$

□

2.3. Removal lemma. Let us state a graphon version of the Removal lemma.

Lemma 2.3. *Suppose that F is a graph on a vertex set $[k]$. Then for every $\epsilon > 0$, there exists a constant $\delta > 0$ such that whenever $W : \Omega^2 \rightarrow [0, 1]$ is a graphon with $\int_{(x_1, \dots, x_k) \in \Omega^k} W^{\otimes F} \, d\nu^k < \delta$, then W can be made F -free by decreasing it in a suitable way such that it changes by at most ϵ in the $\mathcal{L}^1(\Omega^2)$ -distance.*

2.4. Partite versions of graphons. In this section we introduce an auxiliary notion of partite graphons which we shall use in Section 4.5. If A and B are (not necessarily disjoint) sets then we write $A \amalg B$ for their disjoint union, i.e., for the set $(\{1\} \times A) \cup (\{2\} \times B)$. If A and B are measurable subsets of a measure space (Ω, ν) then we can view A as a measure space of total measure $\nu(A)$ and B as a measure space of total measure $\nu(B)$. Then, $A \amalg B$ can be viewed as a measure space of total measure $\nu(A) + \nu(B)$.

Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon and A_1, \dots, A_k are (not necessarily disjoint) measurable subsets of Ω . Then we can construct a (A_1, \dots, A_k) -partite version of W , which is a function $U : (A_1 \amalg \dots \amalg A_k)^2 \rightarrow [0, 1]$ defined by the formula

$$U_{\upharpoonright_{(\{i\} \times A_i) \times (\{j\} \times A_j)}} = W_{\upharpoonright_{A_i \times A_j}}$$

for each $i, j \in [k]$ distinct, and by setting $U_{\upharpoonright_{(\{i\} \times A_i) \times (\{i\} \times A_i)}}$ to 0 for each $i \in [k]$. U is almost a graphon; the only reason why it need not be is that the measure on $A_1 \amalg \dots \amalg A_k$ defined above, and henceforth denoted by $\nu_{A_1 \amalg \dots \amalg A_k}$, need not have total measure 1. So, we shall use graphon terminology even for partite versions.

If $X \subset (A_1 \amalg \dots \amalg A_k)^2$ then the *folded version* of X is simply the set of all $(x, y) \in \Omega$ such that there exist some $i, j \in [k]$ such that $((i, x), (i, y)) \in X$. Note that

$$(2.1) \quad \begin{array}{l} \text{the } \Omega^2\text{-measure of a folded set is at most} \\ \text{the } (A_1 \amalg \dots \amalg A_k)^2\text{-measure of the original set} \end{array}$$

When k is fixed, the total measure of $\nu_{A_1 \amalg \dots \amalg A_k}$ is upper-bounded (by k), and thus it is easy to translate Lemma 2.3 as follows.

Lemma 2.4. *Suppose that F is a graph on a vertex set $[k]$. Then for every $\epsilon > 0$ there exists a constant $\delta > 0$ such that the following holds. If $W : \Omega^2 \rightarrow [0, 1]$ is a graphon and A_1, \dots, A_k are measurable subsets of Ω with the property that for the (A_1, \dots, A_k) -version of W , denoted by U , we have $\int_{(x_1, \dots, x_k) \in (A_1 \amalg \dots \amalg A_k)^k} U^{\otimes F} d\nu_{A_1 \amalg \dots \amalg A_k}^k < \delta$ then U can be made F -free by decreasing it in a suitable way such that it changes by at most ϵ in the $\mathcal{L}^1((A_1 \amalg \dots \amalg A_k)^2)$ -distance. \square*

Let us make one easy and well-known comment about the Removal lemma (this comment applies equally to Lemma 2.3 and 2.4). When making a graphon F -free with a small change in the \mathcal{L}^1 -distance it only makes sense to nullify the graphon at each point where its value is changed. In particular, when positive values of the graphon are uniformly lower bounded, the nullification occurs on a set of small measure. This is stated in the lemma below.

Lemma 2.5. *Suppose that F is a graph on a vertex set $[k]$. Then for every $\alpha > 0$ and $d > 0$ there exists a constant $\beta > 0$ such that the following holds. If $W : \Omega^2 \rightarrow \{0\} \cup [d, 1]$ is a graphon and A_1, \dots, A_k are measurable subsets of Ω with the property that for the (A_1, \dots, A_k) -version of W , denoted by U , we have $\int_{(x_1, \dots, x_k) \in (A_1 \amalg \dots \amalg A_k)^k} U^{\otimes F} d\nu_{A_1 \amalg \dots \amalg A_k}^k < \beta$ then U can be made F -free by nullifying it on a suitable set of $\nu_{A_1 \amalg \dots \amalg A_k}^2$ -measure at most α . \square*

2.5. Regularity lemma. Here, we introduce Szemerédi's Regularity Lemma in a form that is convenient for our later purposes. Recall that the *density* of a bipartite graph with colour classes A and B is defined as $d(A, B) = \frac{e(A, B)}{|A||B|}$. We say that (A, B) forms an ϵ -regular pair if $|d(A, B) - d(X, Y)| < \epsilon$ for each $X \subset A$ and $Y \subset B$ with $|X| \geq \epsilon|A|$ and $|Y| \geq \epsilon|B|$. Suppose that G is a graph on a vertex set V . Let $\mathcal{V} = \{V_0, V_1, \dots, V_\ell\}$ be a partition of V . We say that a subgraph H of G is an (ϵ, d) -regularization of G if

- $V(H) = V$,
- $|V_0| < \epsilon n$,
- for each $1 \leq i < j \leq \ell$, we have that $|V_i| = |V_j|$,
- for each $1 \leq i < j \leq \ell$, we have that the density of the pair (V_i, V_j) in the graph H is either 0 or at least d , and the pair is ϵ -regular in H ,

- there are no edges in H incident to any vertex of V_0 ,
- there are no edges in the graphs $H[V_1], H[V_2], \dots, H[V_\ell]$, and
- $e(H) \geq e(G) - 2d|V|^2$.

In the setting above, the sets V_1, \dots, V_ℓ are called *clusters*, and the number ℓ is called the *complexity* of the regularization. The Szemerédi Regularity Lemma then can be stated as follows.

Lemma 2.6. *For each $d, \epsilon > 0$ there exists a number L such that any graph admits an (ϵ, d) -regularization of complexity at most L .*

Suppose that G is a graph and H is a (ϵ, d) -regularization of G for a partition $\mathcal{V} = \{V_0, V_1, \dots, V_\ell\}$. We can then create the *cluster graph* R corresponding to the pair (H, \mathcal{V}) . This is an edge-weighted graph on the vertex set $[\ell]$ in which an edge ij is present if the bipartite graph $H[V_i, V_j]$ has positive density. The weight on the edge ij in R is then the density of $H[V_i, V_j]$. A standard and well-known fact states that if H is an (ϵ, d) -regularization of a graph G , and R is the corresponding cluster graph, then for the representations W_G and W_R we have that

$$(2.2) \quad \text{dist}_\square(W_G, W_R) \leq 4d + 2\epsilon .$$

The so-called Slicing Lemma tells us that a subpair (X, Y) of an ϵ -regular pair (A, B) is (ϵ/x) -regular, if $|X| \geq x|A|$ and $|Y| \geq x|B|$. This fact, however, is useless when we take x smaller than ϵ . The following lemma shows that we can still save the situation if we take X and Y at random. The lemma is a weak version of the the main result of [12]. There, the statement ([12, Theorem 3.6]) is given even in the setting of so-called sparse regular pairs which is a generalization of the regularity concept which we do not need in the present paper.

Lemma 2.7. *Suppose that $\beta, \epsilon_1 > 0$ are given. Then there exist numbers $\epsilon_0 > 0$ and $C > 0$ such that for every ϵ_0 -regular pair (A, B) of an arbitrary density $d = d(A, B)$ and for any $a, b > C/d$, the number of pairs (A', B') , $A' \subset A$, $B' \subset B$, $|A'| = a$, $|B'| = b$ which induce ϵ_1 -regular pairs of density at least $d - \epsilon_1$ is at least*

$$\exp(-\beta^a - \beta^b) \binom{|A|}{a} \binom{|B|}{b} .$$

Last, we need a weak version of the Blow-up Lemma. In our version, we do not require perfect tiling.

Lemma 2.8. *Suppose that we are given a graph F on a vertex set $[k]$ and numbers $\gamma, d > 0$. Then there exist numbers $\epsilon > 0$ and $m_0 \in \mathbb{N}$ such that the following holds. If B is a graph whose vertices are partitioned into sets V_1, \dots, V_k with $|V_1| = \dots = |V_k| > m_0$ and which has the property that for each $ij \in E(F)$ the pair (V_i, V_j) is ϵ -regular with density at least d , then $\text{til}(F, B) \geq (1 - \gamma)|V_1|$.*

2.6. Subgraphons converging to a subgraphon. In this section, we provide a proof of the following simple lemma (which seems to be new).

Lemma 2.9. *Suppose that $(W_n)_n$ is a sequence of graphons on Ω converging to a graphon $W : \Omega^2 \rightarrow [0, 1]$ in the cut-norm. Let $U \leq W$ be an arbitrary graphon. Then there exists a sequence $(U_n)_n$ with $U_n \leq W_n$ which converges to U in the cut-norm.*

Proof. Let $\epsilon > 0$ be arbitrary. There exist k and a partition of Ω into sets of measure $1/k$ each, $\Omega = \Omega_1 \sqcup \dots \sqcup \Omega_k$, such that there exist step-functions with steps $\Omega_i \times \Omega_j$ which approximate U and W up to an error at most ϵ^2 in the $\mathcal{L}^1(\Omega^2)$ -norm (each one separately). Such partition and approximation exist since squares generate the sigma-algebra on Ω^2 . It is also well known

that the said step-functions can be taken to be constant $U^{ij} := k^2 \int_{(x,y) \in \Omega_i \times \Omega_j} U(x,y) \, dv^2$ and $W^{ij} := k^2 \int_{(x,y) \in \Omega_i \times \Omega_j} W(x,y) \, dv^2$ on each square $\Omega_i \times \Omega_j$. A routine calculation gives that for all but at most $2\epsilon k^2$ pairs (i,j) we have

$$(2.3) \quad \int_{(x,y) \in \Omega_i \times \Omega_j} |U(x,y) - U^{ij}| \, dv^2 \leq \epsilon/k^2 \quad \text{and} \quad \int_{(x,y) \in \Omega_i \times \Omega_j} |W(x,y) - W^{ij}| \, dv^2 \leq \epsilon/k^2.$$

Let $\mathcal{P} \subset [k]^2$ be the set of the pairs that fail (2.3).

Since $U^{ij} \leq W^{ij}$ for each i and j , when we define U_n by

$$U_n(x,y) = W_n(x,y) \cdot \frac{U^{ij}}{W^{ij}} \text{ for } (x,y) \in \Omega_i \times \Omega_j,$$

we indeed get a graphon with $U_n \leq W_n$. Let us now bound $\|U - U_n\|_{\square}$. Let $A, B \subset \Omega$ be arbitrary. We write $R^{ij} = (A \times B) \cap (\Omega_i \times \Omega_j)$. Then by the triangle inequality we have

$$(2.4) \quad \left| \int_{(x,y) \in A \times B} U(x,y) \, dv^2 - \int_{(x,y) \in A \times B} U_n \, dv^2 \right| \leq \sum_{i,j} \left| \int_{(x,y) \in R^{ij}} U(x,y) \, dv^2 - \int_{(x,y) \in R^{ij}} U_n(x,y) \, dv^2 \right| \\ = \sum_{i,j} \left| \int_{(x,y) \in R^{ij}} U(x,y) \, dv^2 - \int_{(x,y) \in R^{ij}} W_n(x,y) \cdot \frac{U^{ij}}{W^{ij}} \, dv^2 \right|.$$

The total contribution of the pairs $(i,j) \in \mathcal{P}$ to the right-hand side of (2.4) is at most 2ϵ . Suppose now that $(i,j) \notin \mathcal{P}$, and let us find an upper bound for the corresponding term in (2.4).

$$\begin{aligned} & \left| \int_{(x,y) \in R^{ij}} U(x,y) \, dv^2 - \int_{(x,y) \in R^{ij}} W_n(x,y) \cdot \frac{U^{ij}}{W^{ij}} \, dv^2 \right| \\ (2.3) \quad & \leq \left| \int_{(x,y) \in R^{ij}} U^{ij}(x,y) \, dv^2 - \int_{(x,y) \in R^{ij}} W_n(x,y) \cdot \frac{U^{ij}}{W^{ij}} \, dv^2 \right| + \epsilon/k^2 \\ & \leq U^{ij} \left| \int_{(x,y) \in R^{ij}} 1 \, dv^2 - \int_{(x,y) \in R^{ij}} \frac{W_n(x,y)}{W^{ij}} \, dv^2 \right| + \epsilon/k^2 \\ & \leq U^{ij} \left| \int_{(x,y) \in R^{ij}} 1 - \frac{1}{W^{ij}} \left(\|W_n - W\|_{\square} + \int_{(x,y) \in R^{ij}} W(x,y) \, dv^2 \right) \right| + \epsilon/k^2 \\ & = U^{ij} \left| \frac{1}{W^{ij}} \left(\|W_n - W\|_{\square} + \int_{(x,y) \in R^{ij}} (W(x,y) - W^{ij}) \, dv^2 \right) \right| + \epsilon/k^2 \\ (2.3) \quad & \leq U^{ij} \left(\frac{1}{W^{ij}} \|W_n - W\|_{\square} + \epsilon/k^2 \right) + \epsilon/k^2 \\ & \leq \|W_n - W\|_{\square} + 2\epsilon/k^2. \end{aligned}$$

Thus, the total contribution of the terms $(i,j) \notin \mathcal{P}$ is at most $k^2 \|W_n - W\|_{\square} + 2\epsilon$ which is less than 3ϵ for large enough n . Consequently, $\left| \int_{(x,y) \in A \times B} U \, dv^2 - \int_{(x,y) \in A \times B} U_n \, dv^2 \right| \leq 5\epsilon$.

Thus, the proof proceeds by letting ϵ go to zero and diagonalizing. \square

3. THE THEORY OF TILINGS IN GRAPHONS

3.1. A naive approach. We want to gain some understanding what an F -tiling in a graphon should be. Let us take our first motivation from the world of finite graphs. Suppose that F and G are finite graphs, F is on the vertex set $[k]$. Then each F -tiling can be viewed as a set $\mathbf{X} \subset V(G)^k$ such that

- (a) for each $T \in \mathbf{X}$ we have that $T_i T_j$ forms an edge of G for each $ij \in E(F)$, $i < j$, and
- (b) for each $v \in V(G)$ there is at most one pair $(T, i) \in \mathbf{X} \times [k]$ such that $v = T_i$.

The size of the F -tiling \mathbf{X} is obviously $|\mathbf{X}|$, which can be rewritten using the projection $\pi_1 : V(G)^k \rightarrow V(G)$ on the first coordinate as

$$(3.1) \quad |\mathbf{X}| = |\pi_1(\mathbf{X})| .$$

Let A be the adjacency matrix of G . Then Condition (a) can be rewritten as

$$(a') \text{ for each } T \in \mathbf{X} \text{ we have that } \prod_{ij \in E(F), i < j} A_{T_i T_j} > 0.$$

These conditions can be translated directly to graphons. That is, we might want to say that a set $\mathbf{Y} \subset \Omega^k$ is an F -tiling in a graphon $W : \Omega^2 \rightarrow [0, 1]$ if

- for each $1 \leq i < j \leq k$ the map $\pi_j \circ \pi_i^{-1}$ is a ν -measure preserving bijection between $\pi_i(\mathbf{Y})$ and $\pi_j(\mathbf{Y})$,
- for each $T \in \mathbf{Y}$ we have that $\prod_{ij \in E(F), i < j} W(T_i, T_j) > 0$,
- and for each $x \in \Omega$ there is at most one pair $(T, i) \in \mathbf{Y} \times [k]$ such that $x = T_i$.

Using (3.1), we would then say that $\nu(\pi_1(\mathbf{Y}))$ is the size of the F -tiling \mathbf{Y} .

There is however a substantial problem with this approach in that whether the condition $\prod_{ij \in E(F), i < j} W(T_i, T_j) > 0$ holds depends on values of W of ν^2 -measure zero. The theory of graphons cannot in principle achieve this level on sensitivity. Thus, a different approach is needed.

3.2. Introducing tilings in graphons. As we saw in the previous section the main problem with the straightforward graphon counterpart to F -tilings of finite graphs was that the limiting object was too centralized; so centralized that all the information was needed to be carried on a set of measure zero. Observe that in a setting of a finite graph G , fractional F -tilings can be more “spread out” than integer F -tilings in the sense that the weight of the latter is always supported on at most $v(G)/v(F)$ many copies of F while it can be supported on up to $\binom{v(G)}{v(F)}$ many copies in the former. Thus, the notion of fractional F -tilings seems better admissible to a limit counterpart. Indeed, as we shall see in Section 3.3, in the world of graphons there is no difference between F -tilings and fractional F -tilings. Thus, we want to approach the notion of F -tilings in graphons by looking at fractional F -tilings in finite graphs.

Suppose that F and G are finite graphs, F is on the vertex set $[k]$, the order of G is n , and let $t_G : \mathcal{F}_F(G) \rightarrow [0, 1]$ be a fractional F -tiling in G . Let $W_G : \Omega^2 \rightarrow [0, 1]$ be a graphon representation of G . Let $(\Omega_u)_{u \in V(G)}$ be the partition of Ω corresponding to this representation. We can then associate a measure μ on Ω^k corresponding to t_G by the defining formula

$$(3.2) \quad \mu(A) = \sum_{u_1 u_2 \dots u_k \in \mathcal{F}_F(G)} \frac{t_G(u_1 u_2 \dots u_k)}{n} \cdot \frac{\nu^k((\Omega_{u_1} \times \Omega_{u_2} \times \dots \times \Omega_{u_k}) \cap A)}{\nu^k(\Omega_{u_1} \times \Omega_{u_2} \times \dots \times \Omega_{u_k})} ,$$

for each measurable $A \subset \Omega^k$. In words, we introduce a linear renormalization on t_G . Then the measure μ spreads the total of $\frac{t_G(u_1 u_2 \dots u_k)}{n}$ uniformly over each rectangle $\Omega_{u_1} \times \Omega_{u_2} \times \dots \times \Omega_{u_k}$. An example is given in Figure 3.1. The following properties of μ are obvious:

- (1) μ is absolutely continuous with respect to ν^k ,
- (2) μ is supported on a subset of the set $\mathcal{F}_F(W_G)$,
- (3) for each $X \subset \Omega$ we have $\sum_{\ell=1}^k \mu\left(\prod_{i=1}^{\ell-1} \Omega \times X \times \prod_{i=\ell+1}^k \Omega\right) \leq \nu(X)$.

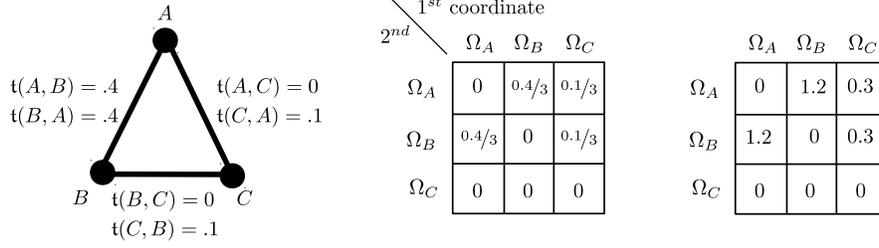


FIGURE 3.1. The left-hand picture shows a sample graph G and a fractional K_2 -tiling in it. The middle picture shows the total masses of the nine rectangles with respect to the measure μ defined by (3.2). The right-hand picture then shows the point-wise values of the Radon–Nikodym derivative $\frac{d\mu}{dv^2}$.

Property (1) allows us to take the Radon–Nikodym derivative $\frac{d\mu}{dv^k}$ of μ . We define functions that arise in this way as F -tilings.²

Definition 3.1. Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon, and that F is a graph on the vertex set $[k]$. An $\mathcal{L}^1(\Omega^k)$ -function $t : \Omega^k \rightarrow [0, +\infty)$ is called an F -tiling in W if

$$(3.3) \quad \text{supp } t \subset \mathcal{F}_F(W),$$

and we have for each $x \in \Omega$ that

$$(3.4) \quad \sum_{\ell=1}^k \int_{(x_1, \dots, x_{\ell-1}, x_{\ell+1}, \dots, x_k) \in \Omega^{k-1}} t(x_1, \dots, x_{\ell-1}, x, x_{\ell+1}, \dots, x_k) dv^{k-1} \leq 1.$$

The *size* of an F -tiling t is $\|t\| = \int_{(x_1, \dots, x_k) \in \Omega^k} t(x_1, \dots, x_k) dv^k$. The F -tiling number of W , denoted by $\text{til}(F, W)$, is the supremum of sizes over all F -tilings in W .

A couple of remarks is in place. First, if F and F' are copies of the same graph on the vertex set $[k]$ with the vertex-labels permuted we can get an F' -tiling from an F -tiling by permuting the corresponding coordinates. Second, observe that the definition does not depend on the values of W , only on $\text{supp } W$. Third, we have $\text{til}(F, W) \leq \frac{1}{k}$. This is a counterpart to the finite statement $\text{til}(F, G) \leq \frac{v(G)}{k}$. Fourth, the supremum in the definition of $\text{til}(F, W)$ need not be attained. An example is given in Proposition 3.3. Fifth, when W_G is a representation of a finite graph G , we have $\text{ftil}(F, G) = v(G)\text{til}(F, W_G)$ (we emphasize that the relevant graph parameter is $\text{ftil}(F, G)$, not $\text{til}(F, G)$). While this would easily follow from tools we develop later (specifically, from Proposition 3.13 and Theorem 3.16), here we give a self-contained proof.

Proposition 3.2. Suppose that F and G are finite graphs and that W_G is a graphon representation of G . Let n be the number of vertices of G . Then we have $\text{ftil}(F, G) = n \cdot \text{til}(F, W_G)$.

Proof. Let us assume that the vertex set of G is $V(G) = [n]$. Let us consider the partition $\Omega = \Omega_1 \sqcup \dots \sqcup \Omega_n$ of the space that hosts W_G into sets representing the individual vertices of G .

²As said, later we shall see that there is no distinction between integral and fractional tilings for graphons. Thus, even though the motivation for the current notion comes from *fractional* tilings in finite graphs, we call the resulting graphon concept simply *tilings*.

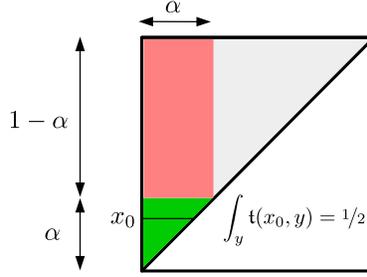


FIGURE 3.2. For the graphon defined by (3.5), any symmetric K_2 -tiling t of weight $\|t\| = 1/2$ satisfies that t is zero almost everywhere on the red rectangle $[0, \alpha] \times [0, 1 - \alpha]$.

Suppose that $t_G : \mathcal{F}_F(G) \rightarrow [0, 1]$ is an arbitrary fractional F -tiling in G . Define $t : \Omega^k \rightarrow [0, +\infty)$ to be constant $t_G(i_1, i_2, \dots, i_k) / n$ on each set $\Omega_{i_1} \times \Omega_{i_2} \times \dots \times \Omega_{i_k}$. It is straightforward to check that t is an F -tiling in W of size $\|t\| / n$. We conclude that $\text{ftil}(F, G) \leq n \cdot \text{til}(F, W_G)$.

Suppose that $t : \Omega^k \rightarrow [0, +\infty)$ is an arbitrary F -tiling in W . Define $t_G : \mathcal{F}_F(G) \rightarrow [0, 1]$ to be constant $n \cdot \int_{(x_1, \dots, x_k) \in \Omega_{i_1} \times \Omega_{i_2} \times \dots \times \Omega_{i_k}} t(x_1, \dots, x_k) \, d\nu^k$ for each $(i_1, i_2, \dots, i_k) \in \mathcal{F}_F(G)$. It is straightforward to check that t_G is an F -tiling in W of size $n\|t\|$. We conclude that $\text{ftil}(F, G) \geq n \cdot \text{til}(F, W_G)$. \square

Proposition 3.3. Consider the graphon $W : [0, 1]^2 \rightarrow [0, 1]$ defined as

$$(3.5) \quad W(x, y) = \begin{cases} 0 & \text{if } x + y > 1/2 \\ 1 & \text{if } x + y \leq 1/2 \end{cases}.$$

Then $\text{til}(K_2, W) = 1/2$, but there exists no K_2 -tiling of size $1/2$.

Proof. For an arbitrary $C \geq 1$ we can take a function $t_C : [0, 1]^2 \rightarrow [0, +\infty)$ defined by

$$t_C(x, y) = \begin{cases} 0 & \text{if } x + y > 1/2 \text{ or } x + y < 1/2 - 1/C \\ C & \text{if } x + y \in [1/2 - 1/C, 1/2] \end{cases}.$$

It is easy to see that t_C is a K_2 -tiling and that $\|t_C\| \geq 1/2 - 1/C$. Thus $\text{til}(F, W) \geq 1/2$. Yet, there is no K_2 -tiling of size $1/2$. To see this, assume for a contradiction that t is such a K_2 -tiling. By replacing $t(x, y)$ by $\frac{1}{2}(t(x, y) + t(y, x))$, we can assume that $t(x, y)$ is symmetric. To get the contradiction, it is enough to show that t is constant zero almost everywhere on each rectangle $[0, \alpha] \times [0, 1 - \alpha]$ (see Figure 3.2). Firstly, observe that because t is symmetric, (3.4) tells us that $\int_{y \in [0, 1]} t(x_0, y) \, dy \leq 1/2$ for every $x_0 \in [0, 1]$. Since $1/2 = \|t\| = \int_{x_0 \in [0, 1]} \left(\int_{y \in [0, 1]} t(x_0, y) \, dy \right) \, dx_0$, we conclude that $\int_{y \in [0, 1]} t(x_0, y) \, dy = 1/2$ for almost every $x_0 \in [0, 1]$. Applying this for $x_0 \geq 1 - \alpha$, we get that the integral $\int_{x \in [1 - \alpha, 1]} \left(\int_{y \in [0, x]} t(x, y) \, dy \right) \, dx$ (over the green triangle in Figure 3.2) is exactly $\alpha/2$. The value $\int_{x \in [0, 1]} \left(\int_{y \in [0, \alpha]} t(x, y) \, dy \right) \, dx$ is at most $\alpha/2$ by (3.4). We conclude that t is zero almost everywhere on $[0, \alpha] \times [0, 1 - \alpha]$. \square

3.3. Graphon tilings versus fractional graphon tilings. Let us now explain that there should be no difference between F -tilings and fractional F -tilings for graphons. In the world of finite graphs, the former is a proper subset of the latter. So, let us show that in the graphon world

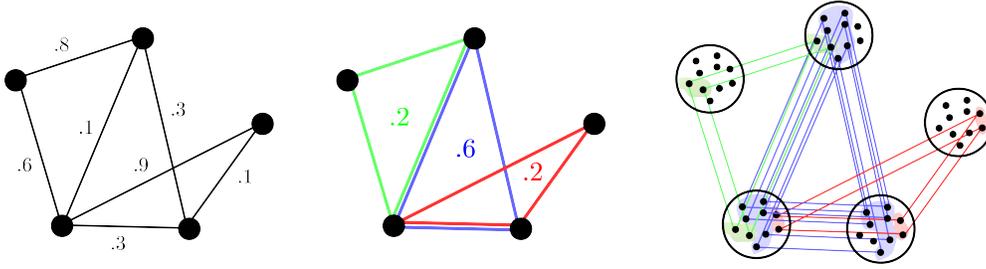


FIGURE 3.3. The left-hand picture shows the graph R_n with the edge-weights. The middle picture shows a fractional triangle-tiling t in R_n (the different copies of a triangle are depicted with different colors). On the right-hand picture, we split the clusters of G_n according to t , thus obtaining subpairs of the original regular pairs. Using Lemma 2.8, we get an F -tiling in G_n of size proportional to the size of the fractional F -tiling in R_n . (This example is oversimplified: If the triangle-tiling is “spread out”, meaning that $t(H)$ is of order ϵ_n or less for many copies H of the triangle, then the regularity of the pairs is not inherited to the subpairs and Lemma 2.8 cannot be applied. As we shall see, this issue can be resolved.)

we have the opposite inclusion as well. That is, we want to show that a fractional F -tiling t of certain size in W yields an F -tiling of approximately the same size (after rescaling) in finite graphs G_n of a sequence that converges to W . To this end, fix a sequence $(R_n)_n$ of finer and finer Szemerédi regularizations of the graphs G_n . That is, R_n is a cluster graph whose edges carry weights in the interval $[0, 1]$ and which comes from regularizing G_n with an error ϵ_n , with $\lim_n \epsilon_n = 0$. Since the sequence $(R_n)_n$ converges to W in the cut-distance we can take t and turn into a fractional F -tiling t_n in R_n of size $\approx \|t\|v(R_n)$ (for n large). Of course, the fact that we can transfer a fractional F -tiling on a graphon to a fractional F -tiling on a graph in a sequence converging to this graphon needs a formal statement, which we give in Theorem 3.4. In the last step, we recall that Lemma 2.8 provides a standard tool for finding an F -tiling of size $\approx \|t\|n$ in G_n based on an fractional F -tiling of size $\approx \|t\|v(R_n)$ in the cluster graph R_n . A description by picture how this step is done is given in Figure 3.3.

Note that the above also explains our remark in Section 3.2 that the notion of tilings depends only on the support of the graphon, and not on its values themselves. Indeed, Lemma 2.8 was applied to subpairs of regular pairs. And Lemma 2.8 works for regular pairs of an arbitrary positive density.³

3.4. Tilings and convergence. Suppose F is a finite graph and that we have a sequence (G_n) of graphs, $v(G_n) = n$, that converges to a graphon $W : \Omega^2 \rightarrow [0, 1]$ in the cut-distance. Observe that the numbers $\text{til}(F, G_n)$ grow (up to) linearly in n . Hence, we could hope that $\frac{\text{til}(F, G_n)}{n} \rightarrow \text{til}(F, W)$ holds. Unfortunately, this is not always true. Indeed, consider $F = K_2$, and the graphs G_n are defined as a perfect matching on n vertices for n even, and as an edgeless

³Lower density will require a finer error parameter of regularity. But in the limit case, we have “infinitely fine regular pairs”.

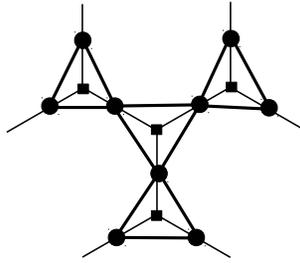


FIGURE 3.4. The transformation described in Remark 3.6. A part of the random (bipartite) graph H_n shown with thin lines and box vertices. The corresponding line graph G_n shown with thick lines and circle vertices.

n -vertex graph for n odd. Then $(G_n)_n$ converges to the zero graphon but $\frac{\text{til}(F, G_n)}{n}$ oscillates between 0 and $\frac{1}{2}$. So, while we see that the quantity $\text{til}(F, \cdot)$ cannot be continuous on the space of graphons, here we establish lower semicontinuity for graphs converging to graphon.

Theorem 3.4. *Suppose that F is a finite graph and let (G_n) be a sequence of graphs of growing orders converging to a graphon $W : \Omega^2 \rightarrow [0, 1]$ in the cut-distance. Then we have that $\liminf_n \frac{\text{til}(F, G_n)}{v(G_n)} \geq \text{til}(F, W)$.*

The proof of Theorem 3.4 is given in Section 4.4.

Remark 3.5. *Lower semicontinuity is the more applicable half of continuity for the purposes of extremal graph theory. Indeed, in extremal graph theory one typically wants to bound the F -tiling number of graphs from a graph class of interest from below (as opposed to bounding from above). Thanks to Theorem 3.4 this can be achieved (in the asymptotic sense) by finding a lower bound $\text{til}(F, W)$ for each limiting graphon W . This proof scheme is used in [13].*

Remark 3.6. *Let us compare the situation with the sparse case. That is, suppose that we have a Benjamini–Schramm convergent sequence $(G_n)_n$ of bounded degree graphs, and we are concerned with the sequence $(\text{til}(F, G_n)/v(G_n))_n$ for some fixed graph F . A theorem of Nguyen and Onak [20] (reproved later by Elek and Lippner [10] and by Bordenave, Lelarge, and Salez [4]) tells us that indeed the normalized matching ratios (i.e., the most important case $F = K_2$) converge. In contrast, Endre Csóka has communicated to us that the sequence $(\text{til}(K_3, G_n)/v(G_n))_n$ needs not to converge. The construction he uses to this end is as follows. For n even, consider a random cubic graph H_n of order $2n$, for n odd consider a random bipartite cubic graph H_n of order $2n$. It is well-known that the resulting sequence is Benjamini–Schramm convergent almost surely, and that the limit is the rooted infinite 3-regular tree. Let G_n be the line graphs of H_n . This is shown in Figure 3.4. There is a one-to-one correspondence between independent sets in H_n 's and triangle tilings in G_n 's. Thus, when H_n is bipartite (n even), we have $\text{til}(K_3, G_n) = v(G_n)/3$. On the other hand, a result of Bollobás about independence number of random cubic graphs [3] translates as $\text{til}(K_3, G_n) \leq 0.92 \cdot v(G_n)/3$ asymptotically almost surely for n odd.*

A similar construction shows that the parameter $\text{til}(F, G)/v(G)$ is discontinuous in the Benjamini–Schramm topology for each 2-connected graph F . In that construction, random cubic graphs and random bipartite cubic graphs are replaced by random $v(F)$ -regular graphs and random bipartite $v(F)$ -regular graphs, respectively. It would be interesting to fully characterize the graphs F for which the quantity $\text{til}(F, \cdot)/v(\cdot)$ is continuous. The case when F is a path or a star seems particularly interesting.

In Theorem 3.7 below we state a version of Theorem 3.4 for a sequence (W_n) of graphons converging to a graphon W . Then we have $\liminf_n \text{til}(F, W_n) \geq \text{til}(F, W)$. Note that this is not a strengthening of Theorem 3.4 since in general we do not have (even approximately) that $\frac{\text{til}(F, G)}{v(G)} = \text{til}(F, U_G)$ for a graphon representation U_G of the graph G .

Theorem 3.7. *Suppose that (W_n) is a sequence of graphons $W_n : \Omega^2 \rightarrow [0, 1]$ converging to a graphon $W : \Omega^2 \rightarrow [0, 1]$ in the cut-distance and let F be an arbitrary graph. Then we have that $\liminf_n \text{til}(F, W_n) \geq \text{til}(F, W)$.*

The proof of Theorem 3.7 is given in Section 4.3.

3.5. Robust tiling number. While — as we have explained — the function $\text{til}(F, \cdot)$ is not upper-semicontinuous, it is “upper-semicontinuous in the cut-distance after small \mathcal{L}^1 -perturbations”. This is stated below.

Proposition 3.8. *Suppose that F is an arbitrary graph and that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon. Then for an arbitrary $\eta > 0$ there exists a number $\delta > 0$ such that each graphon U with $\|W - U\|_{\square} < \delta$ can be decreased in the $\mathcal{L}^1(\Omega^2)$ -distance by at most η (in a suitable way) so that we obtain a graphon U^* for which $\text{til}(F, U^*) \leq \text{til}(F, W) + \eta$.*

The proof of Proposition 3.8 is given in Section 4.5. Let us state a version of Proposition 3.8 which deals with the situation when the the graphon W is approximated by a finite graph and not a graphon.

Proposition 3.9. *Suppose that F is an arbitrary graph and that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon. Then for an arbitrary $\eta > 0$ there exists a number $\delta > 0$ such that in each finite graph G with $\text{dist}_{\square}(W, G) < \delta$ we can erase at most $\eta v(G)^2$ edges (in a suitable way) so that for the resulting graph G^* we have $\text{til}(F, G^*) \leq n \cdot (\text{til}(F, W) + \eta)$.*

Proposition 3.9 is not stronger than Proposition 3.8 since it deals with graphs only. On the other hand, Proposition 3.9 does not follow directly from Proposition 3.8 since it contains the extra assertion that each edge is either entirely deleted, or kept entirely untouched. We do not include a proof of Proposition 3.9 as it is analogous to that of Proposition 3.8.

Propositions 3.8 motivates the following definition. Given a finite graph F , a number $\epsilon > 0$ and a graphon $W : \Omega^2 \rightarrow [0, 1]$ we define

$$(3.6) \quad \text{til}_{\epsilon}(F, W) = \inf_{W^-} \text{til}(F, W^-) ,$$

where W^- ranges over all graphons with $W^- \leq W$ and with $\|W - W^-\|_1 \leq \epsilon$. Similarly, for an n -vertex graph G , we define

$$\text{til}_{\epsilon}(F, G) = \inf_{G^-} \text{til}(F, G^-) ,$$

where G^- ranges over all subgraphs of G with at most ϵn^2 edges deleted from G . These “robust versions of the tiling number” are continuous.

Theorem 3.10. *Suppose that F is a finite graph and $\epsilon > 0$. Then the quantity $\text{til}_{\epsilon}(F, \cdot)$ is continuous on the space of graphons equipped with the cut-norm.*

Theorem 3.11. *Suppose that F is a finite graph and $\epsilon > 0$. Suppose that $(G_n)_n$ is a sequence of graphs of growing orders that converges in the cut-distance to a graphon W . Then the sequence $(\text{til}_{\epsilon}(F, G_n)/v(G_n))_n$ converges to $\text{til}_{\epsilon}(F, W)$.*

We shall give a proof of Theorem 3.10 in Section 4.6. We omit a proof of Theorem 3.11 but it follows easily by combining the proofs of Theorem 3.10 and Theorem 3.4.

Let us give an interpretation of Theorem 3.11 in property testing in the so-called dense model. Formally, let \mathcal{G} be the class of all isomorphism classes of finite graphs. We say that a function (called often a *parameter*) $f : \mathcal{G} \rightarrow \mathbb{R}$ is *testable* if for each $\epsilon > 0$ there exists a number $r = r(\epsilon)$ and a function (called often a *tester*) $g : \mathcal{G} \rightarrow \mathbb{R}$ such that for each $G \in \mathcal{G}$, we have

$$\mathbf{P} [|f(G) - g(H)| > \epsilon] < \epsilon,$$

where $H = G[A]$ is the subgraph of G induced by selecting a set A of r vertices at random. A very convenient and concise characterization of testable parameters was provided in [5] in the language of graph limits: *A graph parameter f is testable if and only if it is continuous in the cut-distance.*⁴ Also, in the positive case, we can take the tester to be $g = f$.

Thus, Theorem 3.11 tells us that $\text{til}_\epsilon(F, \cdot)$ is testable. For example, suppose that we have a large computer network G and we want to estimate the size of the largest possible matching in it. But a small number of links between the computers may become broken, and we do not know in advance which links these will be, so we actually want to estimate $\text{til}_\epsilon(K_2, G)$. Then a good estimate to this quantity can be provided just by computing $\text{til}_\epsilon(K_2, H)$ for a large, randomly selected induced subgraph H of G .

3.6. Fractional covers and LP duality. We define fractional covers in graphons in a complete analogy to the finite notion.

Definition 3.12. Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon, and F is a graph on the vertex set $[k]$. A measurable function $c : \Omega \rightarrow [0, 1]$ is called a *fractional F -cover* in W if

$$\nu^k \left(\mathcal{F}_F(W) \cap \left\{ (x_1, x_2, \dots, x_k) \in \Omega^k : \sum_{i=1}^k c(x_i) < 1 \right\} \right) = 0.$$

The *size* of c , denoted by $\|c\|$, is defined by $\|c\| = \int_{x \in \Omega} c(x) \, d\nu$. The *fractional F -cover number* $\text{fcov}(F, W)$ of W is the infimum of the sizes of fractional F -covers in W .

As with tilings, let us note that the notion of fractional covers does not depend on the values of the graphon but only on its support.

The following proposition gives us a simple but useful relation between the fractional F -cover number between a graph and its graphon representation.

Proposition 3.13. *Suppose that F and G are finite graphs and that W_G is a graphon representation of G . Let n be the number of vertices of G . Then we have $\text{fcov}(F, G) = n \cdot \text{fcov}(F, W_G)$.*

Proof. Suppose that $\Omega_1 \sqcup \Omega_2 \sqcup \dots \sqcup \Omega_n$ is the partition of the underlying probability space Ω given by the graphon representation W . Any fractional F -cover of G can be represented as a function with steps $\Omega_1, \Omega_2, \dots, \Omega_n$. This step-function is a fractional F -cover of W_G . This shows that $\text{fcov}(F, G) \geq n \cdot \text{fcov}(F, W_G)$. On the other hand, let $c : \Omega \rightarrow [0, 1]$ be an arbitrary fractional F -cover of W_G . Let us modify c on each set Ω_i by replacing it by its essential infimum on that set. Since W_G is constant on the rectangles $\Omega_i \times \Omega_j$ we get that the modified step-function c' is still a fractional F -cover of W_G , obviously with $\|c'\| \leq \|c\|$. But c' can be viewed as a fractional F -cover of G . This shows that $\text{fcov}(F, G) \leq n \cdot \text{fcov}(F, W_G)$. \square

⁴See also [19] for more advanced connections between property testing and graph limits.

The sequence of graphons representing a growing number of isolated copies of F converges to the constant zero graphon. This shows that the fractional F -cover number is not continuous. We can, however, establish lower semicontinuity using the following theorem.

Theorem 3.14. *Suppose that F is a graph on the vertex set $[k]$. Let (W_n) be a sequence of graphons $W_n : \Omega^2 \rightarrow [0, 1]$ converging to a graphon W in the cut-norm. Suppose that c_n is a fractional F -cover of W_n . Then an arbitrary weak* accumulation point c of (c_n) is a fractional F -cover of W .*

We give the proof of Theorem 3.14 in Section 4.1.

Recall that if a sequence of functions (f_n) weak* converges to a function f , we have $\lim \int f_n = \int f$. Also, recall that by Theorem 2.1 the set of all functions of \mathcal{L}^∞ -norm at most 1 is sequentially compact with respect to the weak* topology. Thus, we get the following.

Corollary 3.15. *Suppose that F is a finite graph. Suppose that (W_n) is a sequence of graphons converging to W . Then $\liminf_n \text{fcov}(F, W_n) \geq \text{fcov}(F, W)$.*

Corollary 3.15 has two important consequences. Firstly, it tells us that $\text{fcov}(F, \cdot)$ is lower-semicontinuous with respect to the cut-distance. Secondly, if we take $W_1 = W_2 = \dots = W$, and c_n a sequence of fractional F -covers whose sizes tend to $\text{fcov}(F, W)$, we get a fractional F -tiling which attains the value $\text{fcov}(F, W)$. Thus, unlike with $\text{til}(F, W)$,

(3.7) the value of $\text{fcov}(F, W)$ is attained by some fractional F -cover.

We are now ready to state the LP duality theorem for tilings in graphons.

Theorem 3.16. *Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon and F is an arbitrary finite graph. Then we have $\text{til}(F, W) = \text{fcov}(F, W)$.*

We give the proof of Theorem 3.16 in Section 4.2.

Theorem 3.16 is a very convenient tool for extremal graph theory. More specifically, suppose that we want to prove that for a fixed graph F and for a family of graphs \mathcal{G} we have that $\text{til}(F, G) \geq (\gamma + o(1))v(G)$ for each $G \in \mathcal{G}$. (Here, $o(1)$ tends to 0 as $v(G)$ goes to infinity.) By combining Theorem 3.4 and Theorem 3.16 it suffices to show that no graphon arising as a limit of graphs from \mathcal{G} has a fractional F -cover of size less than γ . We shall see one particular application of this scheme in Section 5 and another one is used in the accompanying paper [13] on Komlós's Theorem.

4. PROOFS

Figure 4.1 shows the dependencies and the key steps in the proofs of our main results.

4.1. Proof of Theorem 3.14. Without loss of generality let us assume that $c_n \xrightarrow{w^*} c$. In order to show that c is a fractional F -cover of W , we need to show that for any measurable subsets $A_1, \dots, A_k \subset \Omega$ for which there are numbers $\alpha_1, \dots, \alpha_k$, $\sum \alpha_i < 1$ with $c|_{A_i} < \alpha_i$, we have $\int_{(x_1, \dots, x_k) \in \prod A_i} W^{\otimes F}(x_1, \dots, x_k) \, d\nu^k = 0$. Let $a = 1 - \sum \alpha_i$. Suppose for contradiction that $\int_{(x_1, \dots, x_k) \in \prod A_i} W^{\otimes F}(x_1, \dots, x_k) \, d\nu^k > 0$. Let $\zeta > 0$ be such that the set

$$\mathbf{X} = \left\{ \mathbf{x} \in \prod_i A_i : W^{\otimes F}(\mathbf{x}) > \zeta \right\}$$

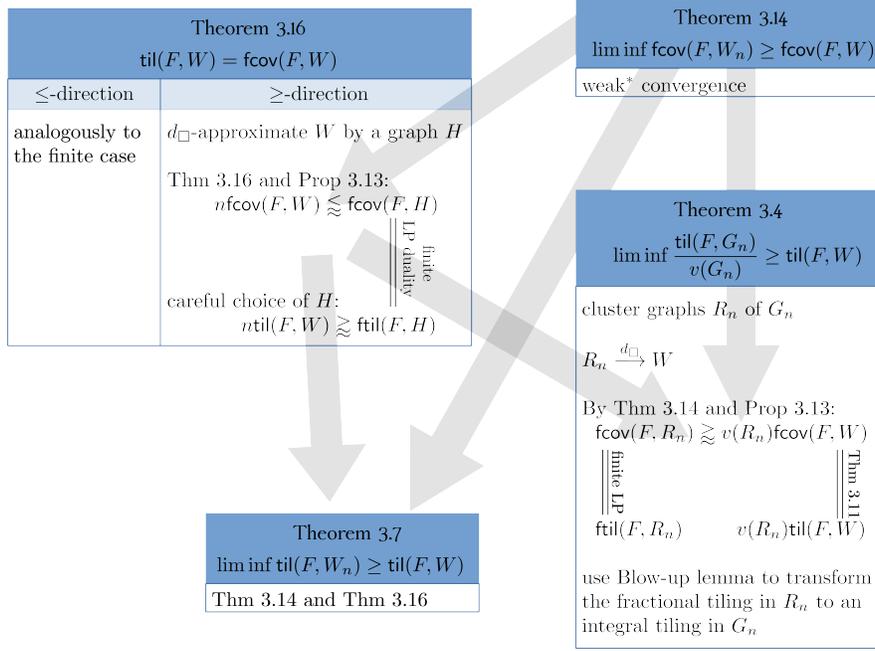


FIGURE 4.1. Main proof steps and dependencies of main results from Section 3.

has positive ν^k -measure. Since k -dimensional boxes generate the sigma-algebra on Ω^k , for each $d > 0$ we can find sets $B_1 \subset A_1, \dots, B_k \subset A_k$ of positive measure such that

$$(4.1) \quad \nu^k \left(\mathbf{X} \cap \prod_i B_i \right) \geq (1-d) \prod_i \nu(B_i).$$

Let us fix such sets for $d := (a/12k)^k$.

Let us take N_1 such that for each $n > N_1$ we have $\|W - W_n\|_\square < (a/13k)^{k+2} \cdot \prod_i \nu(B_i) \cdot \xi$. Lemma 2.2 tells us that for each collection D_1, \dots, D_k of measurable subsets of Ω we have

$$(4.2) \quad \left| \int_{(x_1, \dots, x_k) \in \prod D_i} W_n^{\otimes F}(x_1, \dots, x_k) d\nu^k - \int_{(x_1, \dots, x_k) \in \prod D_i} W^{\otimes F}(x_1, \dots, x_k) d\nu^k \right| < (a/13k)^k \cdot \prod_i \nu(B_i) \cdot \xi.$$

Since $\mathbf{c}_n \xrightarrow{w^*} \mathbf{c}$, there exists a number N_2 such that for each $n > N_2$ we have

$$(4.3) \quad \int_{x \in B_i} \mathbf{c}_n(x) d\nu \leq \int_{x \in B_i} \mathbf{c}(x) + \frac{a}{3k} d\nu \leq \nu(B_i) \left(\alpha_i + \frac{a}{3k} \right) \quad \text{for each } i = 1, \dots, k.$$

Now, fix $n > \max(N_1, N_2)$. For $i = 1, \dots, k$, define $D_i = B_i \cap \mathbf{c}_n^{-1}([0, \alpha_i + 2a/3k])$. Then we have

$$\begin{aligned} \int_{x \in B_i} \mathbf{c}_n(x) d\nu &= \int_{x \in D_i} \mathbf{c}_n(x) d\nu + \int_{x \in B_i \setminus D_i} \mathbf{c}_n(x) d\nu \\ &\geq \nu(D_i) \cdot 0 + \nu(B_i \setminus D_i) \cdot (\alpha_i + 2a/3k) \\ &= \nu(B_i) \left(\alpha_i + 2a/3k - \frac{\nu(D_i)}{\nu(B_i)} \cdot (\alpha_i + 2a/3k) \right). \end{aligned}$$

Plugging this into (4.3) we get

$$\nu(B_i) \left(\alpha_i + 2a/3k - 2 \frac{\nu(D_i)}{\nu(B_i)} \right) \leq \nu(B_i) (\alpha_i + a/3k) ,$$

and consequently,

$$(4.4) \quad \nu(D_i) \geq \frac{a}{6k} \cdot \nu(B_i) .$$

Observe also that for each $\mathbf{x} \in \prod D_i$ we have $\sum_i c_n(\mathbf{x}_i) \leq \sum_i (\alpha_i + 2a/3k) = 1 - a/3 < 1$. Since c_n is a fractional F -cover of W_n , we conclude that

$$(4.5) \quad \int_{(x_1, \dots, x_k) \in \prod D_i} W_n^{\otimes F}(x_1, \dots, x_k) d\nu^k = 0 .$$

On the other hand,

$$\begin{aligned} \int_{(x_1, \dots, x_k) \in \prod D_i} W_n^{\otimes F}(x_1, \dots, x_k) d\nu^k &\geq \int_{(x_1, \dots, x_k) \in \mathbf{X} \cap \prod D_i} \xi d\nu^k \\ &\stackrel{\text{by (4.1)}}{\geq} \left(\prod_i \nu(D_i) - d \prod_i \nu(B_i) \right) \xi \\ &\stackrel{\text{by (4.4)}}{\geq} (a/12k)^k \prod_i \nu(B_i) \cdot \xi . \end{aligned}$$

This, together with (4.5) contradicts (4.2).

4.2. Proof of Theorem 3.16. We split the proof into the easy “ \leq ”-part and the difficult “ \geq ”-part. The former is given in Proposition 4.1 and its proof is a straightforward modification of the finite version. The latter is given in Proposition 4.2. In the proof of Proposition 4.2 we first approximate a graphon by a finite graphon on which we use finite LP-duality as a black-box.

We remark that this approach seems ad-hoc and it is reasonable to ask for a more conceptual proof. Indeed, as was pointed out to us by an anonymous referee, there is fairly rich literature on “algebraic infinite-dimensional duality” (see [2] for a survey). This theory would certainly not be able to give the results we need. The referee also pointed out to us “topological infinite-dimensional duality” developed in [7] which they suggested might provide a blackbox proof of Theorem 3.16. We were, however, unable to apply this theory in our setting.

Proposition 4.1. *Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon defined on Ω and F is a graph on the vertex set $[k]$. Suppose that $c : \Omega \rightarrow [0, 1]$ is a fractional F -cover of W and that $t : \Omega^k \rightarrow [0, 1]$ is an F -tiling in W . Then $\|t\| \leq \|c\|$.*

Proof. We have

$$\begin{aligned} \|t\| &= \\ \stackrel{\text{by (3.3)}}{=} & \int_{(x_1, \dots, x_k) \in \mathcal{F}_F(W)} t(x_1, \dots, x_k) d\nu^k \\ \stackrel{\text{Definition 3.12}}{\leq} & \int_{(x_1, \dots, x_k) \in \mathcal{F}_F(W)} t(x_1, \dots, x_k) \left(\sum_{i=1}^k c(x_i) \right) d\nu^k \\ &= \int_{x \in \Omega} \left(c(x) \sum_{i=1}^k \int_{\substack{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k) \in \Omega^{k-1} \\ (x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_k) \in \mathcal{F}_F(W)}} t(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_k) \right) d\nu \\ \stackrel{\text{by (3.4)}}{\leq} & \int_{x \in \Omega} c(x) d\nu , \end{aligned}$$

as was needed. \square

Proposition 4.2. *Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon defined on Ω and F is a graph on the vertex set $[k]$. Then for an arbitrary $\epsilon > 0$ there exists an F -tiling $\mathfrak{t} : \Omega^k \rightarrow [0, 1]$ in W with $\|\mathfrak{t}\| \geq \text{fcov}(F, W) - \epsilon$.*

Proof. For each $n = 1, 2, \dots$, we find a suitable $k_n \in \mathbb{N}$, a partition $\Omega = \Omega_1^{(n)} \sqcup \Omega_2^{(n)} \sqcup \dots \sqcup \Omega_{k_n}^{(n)}$ into k_n sets of measure $\frac{1}{k_n}$ each, and a graphon $W_n : \Omega^2 \rightarrow [0, 1]$ such that W_n is constant on each rectangle $\Omega_i^{(n)} \times \Omega_j^{(n)}$ ($i, j \in [k_n]$), and $\|W - W_n\|_1 < 1/n$. This is possible as step functions with square steps are dense in $\mathcal{L}^1(\Omega^2)$. We shall now modify the graphons W_n in two steps.

First, let $W'_n : \Omega^2 \rightarrow [0, 1]$ be defined as

$$W'_n(x, y) = \begin{cases} W_n(x, y) & \text{if } W_n(x, y) \geq \sqrt{1/n} \\ 0 & \text{otherwise} \end{cases}.$$

This way, we have that

$$(4.6) \quad \|W_n - W'_n\|_1 < \sqrt{1/n}.$$

For a fixed n , we say that a rectangle $\Omega_i^{(n)} \times \Omega_j^{(n)}$ is *shoddy* if the measure of the set

$$(\text{supp } W'_n \setminus \text{supp } W) \cap (\Omega_i^{(n)} \times \Omega_j^{(n)})$$

is at least $\sqrt[4]{1/n} \cdot \frac{1}{k_n^2}$. Note that at each point of $\text{supp } W'_n \setminus \text{supp } W$ the difference between W_n and W is at least $\sqrt{1/n}$. This gives that $\nu^2(\text{supp } W'_n \setminus \text{supp } W) \leq \frac{\|W_n - W\|_1}{\sqrt{1/n}} \leq \sqrt{1/n}$, which in turn implies that at most $\sqrt[4]{1/n} \cdot k_n^2$ rectangles $\Omega_i^{(n)} \times \Omega_j^{(n)}$ are shoddy. Let us define

$$W''_n(x, y) = \begin{cases} 0 & (x, y) \text{ lies in a shoddy rectangle} \\ W'_n(x, y) & \text{otherwise} \end{cases}.$$

Observe that shoddy rectangles are symmetric, so W''_n is indeed a graphon. We changed W'_n on a set of measure at most $\sqrt[4]{1/n}$ which gives that

$$(4.7) \quad \|W'_n - W''_n\|_1 \leq \sqrt[4]{1/n}.$$

To summarize, we ended up with a graphon W''_n which is a step-function on the rectangles $\Omega_i^{(n)} \times \Omega_j^{(n)}$ and for which we have

$$\|W''_n - W\|_1 \leq \|W_n - W\|_1 + \|W'_n - W_n\|_1 + \|W''_n - W'_n\|_1 \stackrel{(4.6)(4.7)}{<} \frac{1}{n} + \sqrt{1/n} + \sqrt[4]{1/n} \xrightarrow{n \rightarrow \infty} 0.$$

Since the $\mathcal{L}^1(\Omega^2)$ -norm is stronger than the cut-norm, we have that the graphons (W''_n) converge to W . In particular, Theorem 3.14 tells us that $\liminf_n \text{fcov}(F, W''_n) \geq \text{fcov}(F, W)$. Let N_1 be such that for each $n \geq N_1$ we have $\text{fcov}(F, W''_n) \geq \text{fcov}(F, W) - \epsilon/2$. Let $N_2 = \lceil k/\epsilon \rceil^8$. Now, fix an arbitrary number $N \geq \max(N_1, N_2)$, and consider the graphon W''_N . This graphon represents a finite graph on the vertex set $V = \{1, \dots, k_N\}$. We call this graph H . We forget the edge-weights, that is we put an edge ij in H whenever W''_N is positive on $\Omega_i^{(N)} \times \Omega_j^{(N)}$. By Proposition 3.13, $\text{fcov}(F, H) = k_N \cdot \text{fcov}(F, W''_N)$.

The LP-duality of finite graphs tells us that we can find a fractional F -tiling $\mathfrak{t}_H : V^k \rightarrow [0, 1]$ on H with $\sum \mathfrak{t}_H(v_1, \dots, v_k) = \text{fcov}(F, H)$. It is easy to transform \mathfrak{t}_H to a fractional F -tiling

$t : \Omega^k \rightarrow [0, \infty)$ on W''_N by defining t to be the constant $k_N \cdot t_H(v_1, \dots, v_k)$ on the entire rectangle $\prod_i \Omega_{v_i}^{(N)}$ (for each $v_1, v_2, \dots, v_k \in [k_N]$). Then t has weight $\frac{1}{k_N} \sum t_H(v_1, \dots, v_k) = \text{fcov}(F, W''_N)$. The function t is not necessarily a fractional F -tiling on W , as condition (3.3) may be violated due to values at points $P = \text{supp} \left((W''_N)^{\otimes F} \right) \setminus \text{supp} (W^{\otimes F})$. We have

$$(4.8) \quad P \subset \bigcup_{i < j, ij \in E(F)} \pi_{ij}^{-1} (\text{supp } W''_n \setminus \text{supp } W) .$$

Let t' be zero on P , and equal to t elsewhere. Then t' is a fractional F -tiling on W .

Since $\text{supp } W''_n \setminus \text{supp } W$ is disjoint from points of shoddy rectangles, we have that for each choice of v_1, \dots, v_k , the product set $\prod_{i=1}^k \Omega_{v_i}^{(N)}$ satisfies

$$\begin{aligned} v^k \left(\prod_{i=1}^k \Omega_{v_i}^{(N)} \cap P \right) &\stackrel{(4.8)}{\leq} \sum_{\ell < j, ij \in E(F)} v^k \left(\prod_{i=1}^k \Omega_{v_i}^{(N)} \cap \pi_{\ell j}^{-1} (\text{supp } W''_N \setminus \text{supp } W) \right) \\ &\leq v^k \left(\prod_{i=1}^k \Omega_{v_i}^{(N)} \right) \binom{k}{2} \sqrt[4]{1/N} . \end{aligned}$$

Using that t is constant on $\prod_{i=1}^k \Omega_{v_i}^{(N)}$, we get that

$$\int_{(x_1, \dots, x_k) \in \prod_{i=1}^k \Omega_{v_i}^{(N)}} t'(x_1, \dots, x_k) dv^k \geq \left(1 - \binom{k}{2} \sqrt[4]{1/N} \right) \int_{(x_1, \dots, x_k) \in \prod_{i=1}^k \Omega_{v_i}} t(x_1, \dots, x_k) dv^k .$$

Summing over all product sets $\prod_{i=1}^k \Omega_{v_i}^{(N)}$ we get that t' is a fractional F -tiling on W with

$$\begin{aligned} \int_{(x_1, \dots, x_k) \in \Omega^k} t'(x_1, \dots, x_k) dv^k &\geq \left(1 - \binom{k}{2} \sqrt[4]{1/N} \right) \int_{(x_1, \dots, x_k) \in \Omega^k} t(x_1, \dots, x_k) dv^k \\ &\geq \int_{(x_1, \dots, x_k) \in \Omega^k} t(x_1, \dots, x_k) dv^k - \frac{\epsilon}{2} \\ &= \text{fcov}(F, W''_N) - \frac{\epsilon}{2} \geq \text{fcov}(F, W) - \epsilon , \end{aligned}$$

as was needed. \square

4.3. Proof of Theorem 3.7. By Theorem 3.16 it suffices to prove that $\liminf_n \text{fcov}(F, W_n) \geq \text{fcov}(F, W)$. This, however, is the subject of Corollary 3.15.

4.4. Proof of Theorem 3.4. For simplicity, let us assume that $v(G_n) = n$. Suppose that $\epsilon > 0$ is arbitrary. We want to show that for n sufficiently large, $\text{til}(F, G_n) \geq (\text{til}(F, W) - \epsilon)n = (\text{fcov}(F, W) - \epsilon)n$ (where the last equality uses Theorem 3.16). Let $\delta > 0$ be such that whenever U is a graphon of cut-distance at most δ from W we have that

$$(4.9) \quad \text{fcov}(F, U) \geq \text{fcov}(F, W) - \epsilon/3 .$$

Such a number δ exists by Corollary 3.15. Let $d = \delta/10$. Let ϵ_1 be given by Lemma 2.8 for the error parameter $\gamma = \epsilon/10$ and density $d/2$. Let ϵ_0 be given by Lemma 2.7 for the input parameter $\beta = 0.1$ and error parameter ϵ_1 . Set $\epsilon_2 = \min(\epsilon_0, \delta/10)$. Lemma 2.6 with error parameter ϵ_2 gives us a bound L on the complexity of partitions.

Suppose n is large enough, so that we have

$$(4.10) \quad \text{dist}_{\square}(W_{G_n}, W) < \delta/2 .$$

Consider a cluster graph R_n corresponding to a (ϵ_2, d) -regularization of the graph G_n . Let $\mathcal{V} = (V_1, \dots, V_\ell)$ be the corresponding partition of $V(G_n)$ into clusters, with $\ell \leq L$. Let $m = |V_1|$. By (2.2) we have $\text{dist}_\square(W_{G_n}, W_{R_n}) < \delta/2$. Combining with (4.10), triangle inequality gives $\text{dist}_\square(W, W_{R_n}) < \delta$. In particular, (4.9) can be applied to W_{R_n} . Proposition 3.13 then gives that $\text{fcov}(F, R_n) = \ell \cdot \text{fcov}(F, W_{R_n}) \geq \ell (\text{fcov}(F, W) - \epsilon/3)$. We ignore the edge weights in R_n . We apply the (standard) LP duality on R_n . We get a fractional F -tiling $\mathfrak{t}_{R_n} : [\ell]^k \rightarrow [0, 1]$. We now delete from \mathfrak{t}_{R_n} very small weights. Formally, for a k -tuple $\mathbf{v} \in [\ell]^k$ we define $\mathfrak{t}'_{R_n}(\mathbf{v})$ to be 0 if $\mathfrak{t}_{R_n}(\mathbf{v}) < \epsilon/(10\ell^{k-1})$ and to be $\mathfrak{t}_{R_n}(\mathbf{v})$ otherwise. Observe that $\|\mathfrak{t}'_{R_n}\| \geq \|\mathfrak{t}_{R_n}\| - \epsilon^\ell/10$. For each copy \tilde{F} of the graph F inside R_n which is in the support of \mathfrak{t}'_{R_n} , consider an arbitrary integer $m_{\tilde{F}}$ in the range between $[(1 - \epsilon/10) \cdot \mathfrak{t}'_{R_n}(\tilde{F}) \cdot m, \mathfrak{t}'_{R_n}(\tilde{F}) \cdot m]$. Since we can assume that n is sufficiently large, for the gap between the endpoints of this range we have

$$\frac{\epsilon}{10} \cdot \mathfrak{t}'_{R_n}(\tilde{F}) \cdot m \geq \frac{\epsilon}{10} \cdot \frac{\epsilon}{10\ell^{k-1}} \cdot \frac{(1 - \epsilon_2)n}{\ell} \geq 1.$$

In particular, we can indeed find an integer $m_{\tilde{F}}$ in the specified range.

Now, for each cluster V_i , we construct a partition of V_i into sets $V_{i,0} \sqcup \sqcup V_{i,\tilde{F}}$ where \tilde{F} runs over all copies of F in R_n that touch the vertex i . In this partition, the size of each set $V_{i,\tilde{F}}$ is $m_{\tilde{F}}$, and $V_{i,0}$ is the remainder. Since \mathfrak{t}'_{R_n} is an F -tiling $\sum_{\tilde{F}:\tilde{F}\ni i} m_{\tilde{F}} \leq \sum_{\tilde{F}:\tilde{F}\ni i} \mathfrak{t}'_{R_n}(\tilde{F}) \cdot m \leq m$. That means that at least one such partition exists. Among all such partitions $V_{i,0} \sqcup \sqcup V_{i,\tilde{F}}$, consider a random one.

Suppose that this way we randomly partitioned all the clusters V_1, \dots, V_ℓ . For a given copy \tilde{F} of F in R_n in the support of \mathfrak{t}'_{R_n} , we set up a random variable $X_{\tilde{F}}$ as follows. If for each edge $ij \in \tilde{F}$ the pair $(V_{i,\tilde{F}}, V_{j,\tilde{F}})$ is ϵ_1 -regular, we set $X_{\tilde{F}} = (1 - \epsilon/10)m_{\tilde{F}}$. If there exists an edge $ij \in \tilde{F}$ for which the pair $(V_{i,\tilde{F}}, V_{j,\tilde{F}})$ fails to be ϵ_1 -regular, we set $X_{\tilde{F}} = 0$. Lemma 2.8 gives that one can always find an F -tiling of size $X_{\tilde{F}}$ in $G_n \left[\bigcup_{i \in V(\tilde{F})} V_{i,\tilde{F}} \right]$.

Lemma 2.7 implies that $\mathbf{E}[X_{\tilde{F}}] \geq \exp(-2\epsilon(F)0.1^{m_{\tilde{F}}}) (1 - \epsilon/10)m_{\tilde{F}} \geq (1 - \epsilon/5)m_{\tilde{F}}$. By linearity of expectation,

$$\begin{aligned} \mathbf{E} \left[\sum_{\tilde{F}} X_{\tilde{F}} \right] &\geq (1 - \epsilon/5) \sum_{\tilde{F}} (1 - \epsilon/10) \cdot \mathfrak{t}'_{R_n}(\tilde{F}) \cdot m \geq (1 - \epsilon/3) \|\mathfrak{t}'_{R_n}\| m \\ &\geq (1 - \epsilon/3) \|\mathfrak{t}_{R_n}\| m - \epsilon n/10. \end{aligned}$$

So, let us fix one collection of partitions such that $\sum_{\tilde{F}} X_{\tilde{F}} \geq (1 - \epsilon/3) \|\mathfrak{t}_{R_n}\| m - \epsilon n/10$. Then by the above there is an F -tiling in G_n of size $\sum_{\tilde{F}} X_{\tilde{F}}$. Using the fact that $\text{fcov}(F, R_n) \geq \ell (\text{fcov}(F, W) - \epsilon/3)$, we obtain that $\text{til}(F, G_n) \geq (\text{fcov}(F, W) - \epsilon)n$, as needed.

4.5. Proof of Proposition 3.8. Write $k = v(F)$. Given $\eta > 0$, let the number β be given by the Removal lemma 2.5 for the graph F and the parameters $\alpha = (\eta/2)^{k+2}$ and $d = \eta/2$. Set $\delta = \beta(\eta/2)^{4k^2}/k^2$.

Suppose now that U is δ -close to W in the cut-norm. Let us fix an optimal fractional F -cover \mathfrak{c} of W (recall Theorem 3.16 and (3.7)). Let \hat{U} and \mathfrak{c}' be defined by

$$\begin{aligned} \hat{U}(x, y) &= \begin{cases} U(x, y) & \text{if } U(x, y) > \eta/2 \\ 0 & \text{if } U(x, y) \leq \eta/2 \end{cases} \\ \mathfrak{c}'(x) &= \max(\mathfrak{c}(x) + \eta, 1). \end{aligned}$$

Obviously, $\|U - \widehat{U}\|_1 \leq \eta/2$ and $\|c'\| \leq \|c\| + \eta$. Let $\ell = \lceil \eta^{-1} \rceil$. Let us consider the sets A_i , $i = 1, \dots, \ell$ defined by $A_i = c^{-1}[0, i/\ell)$. Let $X \subset \Omega^k$ be the set of points (x_1, \dots, x_k) where $\widehat{U}^{\otimes F}(x_1, \dots, x_k) > 0$ but $\sum_i c'(x_i) < 1$. Let $\mathcal{I} \subset \mathbb{N}^k$ be the set of all k -tuples of integers whose sum is ℓ . We have $|\mathcal{I}| \leq \left(\frac{2}{\eta}\right)^k$. Observe that

$$(4.11) \quad X \subset \bigcup_{\mathbf{i} \in \mathcal{I}} A_{i_1} \times A_{i_2} \times \dots \times A_{i_k} .$$

The next claim is crucial.

Claim. For each k -tuple $\mathbf{i} \in \mathcal{I}$, we have for $\mathbf{A} = A_{i_1} \times A_{i_2} \times \dots \times A_{i_k}$ that $\nu^k(\mathbf{A} \cap X) \leq \beta$.

Proof. Suppose for a contradiction that $\nu^k(\mathbf{A} \cap X) > \beta$. Therefore, we have

$$(4.12) \quad \int_{(x_1, \dots, x_k) \in \mathbf{A}} U^{\otimes F}(x_1, \dots, x_k) d\nu^k \geq \int_{(x_1, \dots, x_k) \in \mathbf{A}} \widehat{U}^{\otimes F}(x_1, \dots, x_k) d\nu^k \geq (\eta/2)^{e(F)} \cdot \beta > k^2 \delta .$$

On the other hand, for each k -tuple $\mathbf{x} \in \mathbf{A}$ we have $\sum_j c(x_j) < 1$. Since c is a fractional F -cover of W , we conclude that $W^F(\mathbf{x}) = 0$. Therefore,

$$(4.13) \quad \int_{(x_1, \dots, x_k) \in \mathbf{A}} W^{\otimes F}(x_1, \dots, x_k) d\nu^k = 0 .$$

Combining 4.12 and 4.13 with Lemma 2.2 we get that $\|U - W\|_{\square} > \delta$, which is a contradiction. \square

We shall use tools introduced in Section 2.4. For each $\mathbf{i} \in \mathcal{I}$, let $U_{\mathbf{i}}$ be the $(A_{i_1}, \dots, A_{i_k})$ -partite version of \widehat{U} defined on an auxiliary measure space $\Omega_{\mathbf{i}} = A_{i_1} \amalg \dots \amalg A_{i_k}$ with a measure $\nu_{\Omega_{\mathbf{i}}}$. The above Claim tells us that $\int_{(x_1, \dots, x_k) \in (\Omega_{\mathbf{i}})^k} \widehat{U}_{\mathbf{i}}^{\otimes F} d(\nu_{\Omega_{\mathbf{i}}}) \leq \beta$. The Removal Lemma (Lemma 2.5) tells us that there exists a set $S_{\mathbf{i}} \subset \Omega_{\mathbf{i}}^2$ of $\Omega_{\mathbf{i}}^2$ -measure at most α such that nullifying $\widehat{U}_{\mathbf{i}}$ on $S_{\mathbf{i}}$ yields an F -free graphon. Let $B_{\mathbf{i}} \subset \Omega^2$ be the folded version of $S_{\mathbf{i}}$. We have that $\nu^2(B_{\mathbf{i}}) \leq \alpha$ by (2.1). Let us now nullify \widehat{U} on $B = \bigcup_{\mathbf{i} \in \mathcal{I}} B_{\mathbf{i}}$. The set B has ν^2 -measure at most $|\mathcal{I}| \cdot \alpha \leq \frac{2}{\eta^k} \cdot \alpha \leq \eta/2$, and thus the resulting graphon U^* satisfies $\|U - U^*\|_1 \leq \eta$. The nullification together with (4.11) tells us that c' is a fractional F -cover of U^* . This finishes the proof.

4.6. Proof of Theorem 3.10. Suppose that a sequence of graphons $(W_n)_n$ converges to a graphon W in the cut-norm. We shall prove the statement in two steps:

$$(4.14) \quad \limsup_n \text{til}_{\epsilon}(F, W_n) \geq \text{til}_{\epsilon}(F, W) , \text{ and}$$

$$(4.15) \quad \limsup_n \text{til}_{\epsilon}(F, W_n) \leq \text{til}_{\epsilon}(F, W) .$$

(Note that by passing to a subsequence, we could turn the limit superior into a limit.)

First, let us prove (4.14). For each n , suppose that $U_n \leq W_n$ is an arbitrary graphon of $\mathcal{L}^1(\Omega^2)$ -norm at most ϵ . Since the space of graphons is sequentially compact, let us consider the limit U of a suitable subsequence $(U_{n_i})_i$. Since $U_{n_i} \leq W_{n_i}$ for each i , we also have $U \leq W$. Furthermore, since the $\mathcal{L}^1(\Omega^2)$ -norm is continuous with respect to the cut-distance, we have

that the $\mathcal{L}^1(\Omega^2)$ -norm of U is at most ϵ . In particular, U appears in the infimum in (3.6) for the graphon W . We then have

$$\limsup_n \text{til}(F, W_n - U_n) \geq \liminf_i \text{til}(F, W_{n_i} - U_{n_i}) \stackrel{T3.7}{\geq} \text{til}(F, W - U) \geq \text{til}_\epsilon(F, W),$$

as was needed for (4.14).

For (4.15) we shall need that the function $\text{til}_\delta(F, W)$, considered as a function in $\delta \in (0, 1)$ is left-continuous.

Claim. Suppose that $\delta \in (0, 1)$. Then in the setting above, for each $\eta > 0$ and each graphon $U \leq W$ with $\|U\|_1 = \delta$ there exists a graphon $U' \leq U$ with $\|U'\|_1 < \delta$ such that $\text{til}(F, W - U) \leq \text{til}(F, W - U') + \eta$.

Proof of Claim. Clearly, there exists a set $A \subset \Omega$ of measure at most η such that $U|_{A \times \Omega}$ is positive on a set of positive measure. Let us nullify U on $(A \times \Omega) \cup (\Omega \times A)$. For the resulting graphon U' we have $\|U'\|_1 < \delta$.

Suppose now that \mathfrak{t} is an arbitrary F -tiling in $W - U'$. By nullifying \mathfrak{t} on those $v(F)$ -tuples whose at least one coordinate lies in A , we obtain an F -tiling \mathfrak{t}^* in $W - U$. By (3.4), we have $\|\mathfrak{t}\| \leq \|\mathfrak{t}^*\| + \eta$. We conclude that $\text{til}(F, W - U) \leq \text{til}(F, W - U') + \eta$. \square

By the above claim, it suffices to prove the following weaker form of (4.15):

$$(4.16) \quad \limsup_n \text{til}_\epsilon(F, W_n) \leq \text{til}_{\epsilon-1/\ell}(F, W) + 1/\ell, \text{ for each } \ell \in \mathbb{N}.$$

So, suppose that $\ell \in \mathbb{N}$ is arbitrary. Let $U \leq W$ be an arbitrary graphon of $\mathcal{L}^1(\Omega^2)$ -norm at most $\epsilon - 1/\ell$. Let $\mathfrak{c} : \Omega \rightarrow [0, 1]$ be an arbitrary fractional F -cover of $W - U$ of size $\text{til}(F, W - U)$. Such a cover exists by Theorem 3.16. Let us consider the sets $A_i, i = 1, \dots, \ell - 1$ defined by $A_i = \mathfrak{c}^{-1}[(i-1)/\ell, i/\ell]$ and $A_\ell = \mathfrak{c}^{-1}[(\ell-1)/\ell, 1]$. We therefore have another fractional F -cover $\mathfrak{c}^* = \sum_i \frac{i}{\ell} \mathbf{1}_{A_i}$ of size at most $\|\mathfrak{c}\| + 1/\ell$.

For each n , let us take a graphon $U_n \leq W_n$ so that the sequence $(U_n)_n$ converges to U in the cut-norm. Such a sequence $(U_n)_n$ exists by Lemma 2.9. Since the $\mathcal{L}^1(\Omega^2)$ -norm of U is at most $\epsilon - 1/\ell$ and since the $\mathcal{L}^1(\Omega^2)$ topology is stronger than the cut-norm topology, we can additionally assume that the $\mathcal{L}^1(\Omega^2)$ -norm of each U_n is at most $\epsilon - 1/\ell$. By the same argument as in the proof of Proposition 3.8, for large enough n , the density of copies of F in $W_n - U_n$ not covered by the \mathfrak{c}^* is $o_n(1)$. Using the Removal lemma, for large enough n , we can find graphons $U'_n \leq W_n - U_n$ so that U'_n has $\mathcal{L}^1(\Omega^2)$ -norm at most $1/2\ell$ and that $W_n - (U_n + U'_n)$ has zero density of copies of F in not covered by the \mathfrak{c}^* . Since $U_n + U'_n$ has $\mathcal{L}^1(\Omega^2)$ -norm at most ϵ , we get that

$$\text{til}_\epsilon(F, W_n) \leq \|\mathfrak{c}^*\| \leq \|\mathfrak{c}\| + 1/\ell \leq \text{til}(F, W - U) + 1/\ell,$$

as was needed for (4.16).

5. TILINGS IN INHOMOGENEOUS RANDOM GRAPHS

In this section we give a simple application of our theory. It concerns the random graph model $\mathbb{G}(n, W)$ which was introduced by Lovász and Szegedy in [18]. Let us briefly recall the model. If $W : \Omega^2 \rightarrow [0, 1]$ is a graphon, then to sample a graph from the distribution $\mathbb{G}(n, W)$, $G \sim \mathbb{G}(n, W)$, we take deterministically $V(G) = [n]$. Further, we sample points $x_1, \dots, x_n \in \Omega$ independently at random according to the law ν . To define the edges of G , we include each

pair ij as an edge in G with probability $W(x_i, x_j)$, independently of the other choices. When W is constant p , $\mathbb{G}(n, W)$ is the usual Erdős–Rényi random graph $\mathbb{G}(n, p)$. See [17, Chapter 10] for more properties of the model $\mathbb{G}(n, W)$.

We prove that the ratios $\frac{\text{til}(F, \mathbb{G}(n, W))}{n}$ and $\frac{\text{ftil}(F, \mathbb{G}(n, W))}{n}$ converge to $\text{til}(F, W)$ asymptotically almost surely. The proof of this statement is a short application of Theorem 3.16.

Theorem 5.1. *Suppose that $W : \Omega^2 \rightarrow [0, 1]$ is a graphon. Then the values $\frac{\text{til}(F, \mathbb{G}(n, W))}{n}$ and $\frac{\text{ftil}(F, \mathbb{G}(n, W))}{n}$ converge in probability to the constant $\text{til}(F, W)$.*

Proof. It is well-known (see e.g. [17, Lemma 10.16]) that the sequence of graphs $(\mathbb{G}(n, W))_n$ converges to W in the cut-distance almost surely. Thus by Theorem 3.4, $\frac{\text{til}(F, \mathbb{G}(n, W))}{n}$ is asymptotically almost surely at least $\text{til}(F, W) - o(1)$. The analogous statement for $\frac{\text{ftil}(F, \mathbb{G}(n, W))}{n}$ holds, as for any graphs F and G we have $\text{til}(F, G) \leq \text{ftil}(F, G)$.

Now, we pick an arbitrary $\ell \in \mathbb{N}$ and we show that asymptotically almost surely, $\frac{\text{ftil}(F, \mathbb{G}(n, W))}{n}$ is at most $\text{til}(F, W) + 3/\ell$. Let us apply Theorem 3.16 and fix a fractional F -cover $\mathfrak{c} : \Omega \rightarrow [0, 1]$ of size $\text{til}(F, W)$. Let us round \mathfrak{c} up to the closest multiple of $1/2\ell$. This way, the size of the modified fractional F -cover \mathfrak{c}' increased by at most $1/2\ell$. For $i = 0, \dots, 2\ell$, define Ω_i to be the preimage of $i/2\ell$ under \mathfrak{c}' . Since \mathfrak{c}' is a fractional F -cover, we have that for each k -tuple i_1, i_2, \dots, i_k with $\sum_j i_j < 2\ell$ that

$$\int_{(x_1, \dots, x_k) \in \prod_{\ell=1}^k \Omega_{i_\ell}} W^{\otimes F}(x_1, \dots, x_k) \, d\nu^k = 0.$$

Since the integrand is non-negative, we get $W^{\otimes F}(\mathbf{x}) = 0$ for almost every k -tuple $\mathbf{x} = (x_1, \dots, x_k)$ as above. Then, for such a tuple \mathbf{x} , there exist two indices $p_{\mathbf{x}}, q_{\mathbf{x}}, 1 \leq p_{\mathbf{x}} < q_{\mathbf{x}} \leq k, p_{\mathbf{x}}q_{\mathbf{x}} \in E(F)$ such that

$$(5.1) \quad W(x_{p_{\mathbf{x}}}, x_{q_{\mathbf{x}}}) = 0.$$

Let us now sample the random graph $G \sim \mathbb{G}(n, W)$. Let $y_1, y_2, \dots, y_n \in \Omega$ be the random points that represent the n vertices $1, 2, \dots, n$ of $\mathbb{G}(n, W)$. By the Law of Large Numbers, asymptotically almost surely we have for each $j = 0, 1, \dots, 2\ell$,

$$(5.2) \quad |\{y_1, \dots, y_n\} \cap \Omega_j| \leq (\nu(\Omega_j) + 1/\ell^2) n.$$

Define a function $\mathfrak{b} : V(G) \rightarrow [0, 1]$ by mapping a vertex i to $\mathfrak{c}'(y_i)$. We then have

$$\begin{aligned} \|\mathfrak{b}\| &= \sum_{i=1}^n \mathfrak{c}'(y_i) = \sum_{j=0}^{2\ell} |\{y_1, \dots, y_n\} \cap \Omega_j| \cdot \frac{j}{2\ell} \\ &\stackrel{(5.2)}{\leq} \sum_{j=0}^{2\ell} (\nu(\Omega_j) + 1/\ell^2) n \cdot \frac{j}{2\ell} \leq (\|\mathfrak{c}'\| + 2/\ell)n \leq (\|\mathfrak{c}\| + 3/\ell)n. \end{aligned}$$

We claim that \mathfrak{b} is a fractional F -cover of G with probability 1. Indeed, let $m_1, m_2, \dots, m_k \in V(G)$ be arbitrary with

$$(5.3) \quad \sum_{j=1}^k \mathfrak{b}(m_j) < 1.$$

It is our task to show that there exist $pq \in E(F)$ such that $m_p m_q \notin E(G)$. To this end, we observe that (5.3) translates as $y_{m_1} \in \Omega_{i_1}, y_{m_2} \in \Omega_{i_2}, \dots, y_{m_k} \in \Omega_{i_k}$ for some k -tuple i_1, i_2, \dots, i_k

with $\sum_j i_j < 2\ell$. Thus, (5.1) applies for some numbers $p, q \in [k]$. But then indeed the edge $m_p m_q$ was included with probability $W(y_{m_p}, y_{m_q}) = 0$ in $\mathbb{G}(n, W)$, as was needed.

Since ℓ was arbitrary, we obtain that $\frac{\text{ftil}(F, \mathbb{G}(n, W))}{n}$ is asymptotically almost surely at most $\text{til}(F, W) + o(1)$. The analogous statement for $\frac{\text{til}(F, \mathbb{G}(n, W))}{n}$ follows from the fact that for any graphs F and G we have $\text{til}(F, G) \geq \text{ftil}(F, G)$. \square

It is plausible that Theorem 5.1 can be extended even to sparse inhomogeneous random graphs $\mathbb{G}(n, p_n W)$, where $(p_n)_n$ is a sequence of positive reals tending to zero.

As an immediate corollary of Theorem 5.1 and Theorem 3.4 we obtain the following corollary.

Corollary 5.2. *Suppose that F is an arbitrary graph and W is an arbitrary graphon. Then*

$$\liminf \frac{\text{til}(F, G_n)}{v(G_n)} = \text{til}(F, W),$$

where the limit inferior ranges over all graph sequences (G_n) that converge to W .

6. A CONCLUDING REMARK: ANALYTIC APPROACH TO LINEAR PROGRAMMING IN THE SETTING OF GRAPHONS

Our main results are proven by first approximating a graphon by a step-function and then applying discrete tools, such as linear programming. Arguments that would work directly in the setting of functional analysis would certainly be much shorter and more elegant. As was pointed out by the referee, some such tools are already available in [2] and [7], but we were unable to make use of them in the current context. In a recent paper [14], Hladký and Rocha investigate concepts of the chromatic number and the clique number of graphons. Unlike in finite graphs, the LP duality fails for the fractional versions of these parameters.

7. ACKNOWLEDGMENTS

JH would like thank Dan Král and András Máthé for useful discussions that preceded this project. He would also like to thank Martin Doležal for the discussions they have had regarding functional analysis. We thank two anonymous referees for their comments.

Part of this paper was written while JH was participating in the program *Measured group theory* at The Erwin Schrödinger International Institute for Mathematics and Physics.

The contents of this publication reflects only the authors' views and not necessarily the views of the European Commission of the European Union. This publication reflects only its authors' view; the European Research Council Executive Agency is not responsible for any use that may be made of the information it contains.

REFERENCES

- [1] P. Allen, J. Böttcher, J. Hladký, and D. Piguet. A density Corrádi-Hajnal theorem. *Canad. J. Math.*, 67(4):721–758, 2015.
- [2] E. J. Anderson and P. Nash. *Linear programming in infinite-dimensional spaces*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons, Ltd., Chichester, 1987. Theory and applications, A Wiley-Interscience Publication.
- [3] B. Bollobás. The independence ratio of regular graphs. *Proc. Amer. Math. Soc.*, 83(2):433–436, 1981.
- [4] Ch. Bordenave, M. Lelarge, and J. Salez. Matchings on infinite graphs. *Probab. Theory Related Fields*, 157(1-2):183–208, 2013.

- [5] C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, and K. Vesztegombi. Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing. *Adv. Math.*, 219(6):1801–1851, 2008.
- [6] A. Bruckner, J. Bruckner, and B. Thomson. *Real Analysis*. ClassicalRealAnalysis.com, second edition, 2008.
- [7] S. A. Clark. An infinite-dimensional LP duality theorem. *Math. Oper. Res.*, 28(2):233–245, 2003.
- [8] M. Doležal and J. Hladký. Matching polytons. *Electron. J. Combin.*, 26(4):Paper No. 4.38, 33, 2019.
- [9] M. Doležal, J. Hladký, P. Hu, and D. Piguet. First steps in combinatorial optimization on graphons: Matchings. In *European Conference on Combinatorics, Graph Theory and Applications (EuroComb 2017)*, volume 61 of *Electron. Notes Discrete Math.*, pages 359–365. Elsevier Sci. B. V., Amsterdam, 2017.
- [10] G. Elek and G. Lippner. Borel oracles. An analytical approach to constant-time algorithms. *Proc. Amer. Math. Soc.*, 138(8):2939–2947, 2010.
- [11] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.*, 10:337–356 (unbound insert), 1959.
- [12] S. Gerke, Y. Kohayakawa, V. Rödl, and A. Steger. Small subsets inherit sparse ϵ -regularity. *J. Combin. Theory Ser. B*, 97(1):34–56, 2007.
- [13] J. Hladký, P. Hu, and D. Piguet. Komlós’s tiling theorem via graphon covers. *J. Graph Theory*, 90(1):24–45, 2019.
- [14] J. Hladký and I. Rocha. Independent sets, cliques, and colorings in graphons. *European J. Combin.*, 88:103108, 18, 2020.
- [15] J. Komlós. Tiling Turán theorems. *Combinatorica*, 20(2):203–218, 2000.
- [16] L. Lovász. Subgraph densities in signed graphons and the local Simonovits-Sidorenko conjecture. *Electron. J. Combin.*, 18(1):Paper 127, 21, 2011.
- [17] L. Lovász. *Large networks and graph limits*, volume 60 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2012.
- [18] L. Lovász and B. Szegedy. Limits of dense graph sequences. *J. Combin. Theory Ser. B*, 96(6):933–957, 2006.
- [19] L. Lovász and B. Szegedy. Testing properties of graphs and functions. *Israel J. Math.*, 178:113–156, 2010.
- [20] H. N. Nguyen and K. Onak. Constant-Time Approximation Algorithms via Local Improvements. In *49th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2008, October 25-28, 2008, Philadelphia, PA, USA*, pages 327–336, 2008.
- [21] A. A. Razborov. Flag algebras. *J. Symbolic Logic*, 72(4):1239–1282, 2007.
- [22] A. A. Razborov. On the minimal density of triangles in graphs. *Combin. Probab. Comput.*, 17(4):603–618, 2008.
- [23] T. Tao. *An epsilon of room, I: real analysis*, volume 117 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2010. Pages from year three of a mathematical blog.

INSTITUTE OF MATHEMATICS, CZECH ACADEMY OF SCIENCE, ŽITNÁ 25, 110 00, PRAHA, CZECHIA. THE INSTITUTE OF MATHEMATICS OF THE CZECH ACADEMY OF SCIENCES IS SUPPORTED BY RVO:67985840.

Email address: `hladky@math.cas.cz`

SCHOOL OF MATHEMATICS, SUN YAT-SEN UNIVERSITY, GUANGZHOU, 510275, CHINA. THIS WORK HAS BEEN CARRIED OUT WHILE AT: DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UNITED KINGDOM.

Email address: `huping9@mail.sysu.edu.cn`

INSTITUTE OF COMPUTER SCIENCE, CZECH ACADEMY OF SCIENCES, POD VODÁRENSKOU VĚŽÍ 2, 182 07 PRAGUE, CZECH REPUBLIC. WITH INSTITUTIONAL SUPPORT RVO:67985807.

Email address: `piguet@cs.cas.cz`