# Block decomposition and statistics arising from permutation tableaux 

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#### Abstract

Permutation statistics $w \bar{m}$ and rlm are both arising from permutation tableaux. w $\bar{m}$ was introduced by Chen and Zhou, which was proved equally distributed with the number of unrestricted rows of a permutation tableau. While rlm is shown by Nadeau equally distributed with the number of 1's in the first row of a permutation tableau.

In this paper, we investigate the joint distribution of $w \bar{m}$ and rlm. Statistic (rlm, $\mathrm{w} \overline{\mathrm{m}}$, rlmin, des, (321)) is shown equally distributed with (rlm, rlmin, $\mathrm{w} \overline{\mathrm{m}}, \mathrm{des},(\underline{321})$ ) on $S_{n}$. Then the generating function of (rlm, $\mathrm{w} \overline{\mathrm{m}}$ ) follows. An involution is constructed to explain the symmetric property of the generating function. Also, we study the triple statistic ( $\mathrm{w} \overline{\mathrm{m}}, \mathrm{rlm}, \mathrm{asc}$ ), which is shown to be equally distributed with (rlmax -1 , rlmin, asc) as studied by Josuat-Vergès. The main method we adopt throughout the paper is constructing bijections based on a block decomposition of permutations.


Keywords: bijection, involution, permutation tableaux, block decomposition, pattern

## 1 Introduction

In this paper, we mainly investigate two permutation statistics $w \bar{m}$ and rlm which are arising from permutation tableaux.

Permutation tableaux were introduced by Steingrímsson and Williams [19]. They are related to the enumeration of totally positive Grassmannian cells $[15,17,18,20]$ and a statistical physics model called Partially Asymmetric Exclusion Process (PASEP) [5, 8, 9, 10, 11, 13]. Several papers on the combinatorics of permutation tableaux have also been published, see $[2,3,4,6,7,12,16]$.

A permutation tableau is a Ferrers diagram with possibly empty rows together with a 0 , 1-filling of the cells satisfying the following conditions:

1. each column has at least one 1 ,
2. there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row.

The length of a permutation tableau is defined to be the number of rows plus the number of columns. Let $\mathcal{P} \mathcal{T}(n)$ denote the set of permutations of length $n$.

Several statistics over permutation tableaux are defined, among which urr and topone are two interesting ones. A 0 in a permutation tableau is row-restricted if there is a 1 above in the same column. A row is said to be unrestricted if it contains no row-restricted 0 . For $T \in \mathcal{P} \mathcal{T}(n)$, as given by Corteel and $\operatorname{Kim}[6]$, let $\operatorname{urr}(T)$ be the number of unrestricted rows of $T$ and let topone $(T)$ be the number of 1's in the first row of $T$. Using recurrence relations, Corteel and Nadeau [7] obtained an explicit formula for the generating function of permutation tableaux of length $n$ with respect to the statistics urr and topone. Corteel and Kim [6] rewrote this formula as follows

$$
\begin{equation*}
\sum_{T \in \mathcal{P} \mathcal{T}(n)} x^{\operatorname{urr}(T)-1} y^{\text {topone }(T)}=(x+y)_{n-1}, \tag{1.1}
\end{equation*}
$$

where $(x)_{n}=x(x+1) \cdots(x+n-1)$ for $n \geq 1$ with $(x)_{0}=1$. Moreover, they gave two beautiful bijective proofs of (1.1).

Permutation tableaux are in bijections with permutations. Let $[n]=$ $\{1,2, \ldots, n\}$ and $S_{n}$ be the set of permutations on $[n]$. Steingrímsson and Williams [19] gave a Zig-Zag map $\Phi$ from $\mathcal{P} \mathcal{T}(n)$ to $S_{n}$. Given $T \in \mathcal{P} \mathcal{T}(n)$, we label $T$ as follows. First, label the steps in the south-east border with $1,2, \cdots, n$ from north-east to south-west. Then, label a row (resp. column) with $i$ if the row contains the south (resp. west) step with label $i$. A zigzag path on a permutation tableau is a path entering from the left of a row or the top of a column, going to the east or to the south changing the direction alternatively whenever it meets a 1 until exiting the tableau. Let $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}=\Phi(T)$, where $\pi_{i}=j$ if the zigzag path corresponding to label $i$ exits $T$ from a row or a column labeled by $j$. As an example, $\Phi(\pi)=8,6,1,5,3,4,9,2,7,11,10$ for $\pi$ given in the left of Figure 1.1.

Corteel and Nadeau [7] found another two bijections between permutation tableaux and permutations, one of which we denote by $\Gamma$ is given depend on the alternative representation of permutation tableaux. As given in Corteel and Kim [6], the alternative representation of a permutation tableau $T$ is the diagram obtained from $T$ by replacing the topmost 1's by $\uparrow$ 's and the rightmost restricted 0's by $\leftarrow$ 's and removing the remaining 0 's and 1's, see

|  | 11 9 8 6 5 3 <br> 0 1 1 0 0 1 |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| 2 | 0 | 0 | 0 | 1 | 1 | 1 |
| 4 | 0 | 0 | 0 | 0 | 1 |  |
| 7 | 0 | 1 | 1 |  |  |  |
| 10 | 1 |  |  |  |  |  |



Figure 1.1: A permutation tableau (left) and its corresponding alternative representation (right).

Figure 1.1 as an example. Given $T \in \mathcal{P} \mathcal{T}(n), \pi=\pi_{1} \pi_{2} \cdots \pi_{n}=\Gamma(T)$ can be obtained as follows

1. write down the labels of the unrestricted rows of $T$ in increasing order,
2. for each column $i$ from left to right, if row $j$ contains a $\uparrow$ in column $i$ and $i_{1}, i_{2}, \cdots, i_{r}\left(i_{1}<\cdots<i_{r}\right)$ contains a $\leftarrow$ in column $i$, then add $i_{1}, i_{2}, \cdots, i_{r}, i$ in increasing order before $j$ in $\pi$.

As an example, $\Gamma(\pi)=9,4,6,5,2,8,3,1,7,11,10$ for $\pi$ given in the right of Figure 1.1.

Statistics $w \bar{m}$ and rlm on permutations are closely related to statistics urr and topone by $\Phi$ and $\Gamma$. We present the definitions of $w \bar{m}$ and rlm first. Given $\pi=\pi_{1} \pi_{2} \cdots \pi_{n} \in S_{n}$, the index $i$ is said to be a weak excedance of $\pi$ if $\pi_{i} \geq i$. Otherwise, it is called a non-weak excedance. An index $i$ is called a mid-point of $\pi$ if there exist $j<i$ and $k>i$ such that $\pi_{j}>\pi_{i}>\pi_{k}$. Otherwise, $i$ is called a non-mid-point. Let $\mathrm{w} \overline{\mathrm{m}}(\pi)=|\mathrm{W} \overline{\mathrm{M}}(\pi)|$ and

$$
\mathrm{W} \overline{\mathrm{M}}(\pi)=\left\{\pi_{i} \mid i \text { is a weak excedance and a non-mid-point of } \pi\right\}
$$

For a word $w=w_{1} w_{2} \cdots w_{n}$ of distinct integers, $w_{i}$ is called a $R L$-maximum of $w$ if $w_{i}>w_{j}$ for all $j>i$. While $w_{i}$ is called a $L R$-maximum, if $w_{i}>w_{j}$ for all $j<i$. The $R L$-minimum and $L R$-minimum can be defined similarly. Let $\operatorname{Rlmax}(w), \operatorname{Lrmax}(w), \operatorname{Rlmin}(w)$ and $\operatorname{Lrmin}(w)$ be the set of $R L$ maxima, $L R$-maxima, $R L$-minima and $L R$-minima of $w$, respectively. Set $\operatorname{rlmax}(w), \operatorname{lrmax}(w), \operatorname{rlmin}(w)$ and $\operatorname{lrmin}(w)$ to be the corresponding numerical statistics. Let $\operatorname{Rlm}(\pi)$ be the set of the RL-maxima of the subword of $\pi$ which is to the left of the 1 in $\pi$. Write $r \operatorname{lm}(\pi)=|\mathrm{Rlm}(\pi)|$. As an example, for $\pi=6,5,1,10,4,3,8,9,2,11,7,12$, we have $\operatorname{WM}(\pi)=\{6,10,11,12\}$, $\mathrm{w} \overline{\mathrm{m}}(\pi)=4, \operatorname{Rlm}(\pi)=\{5,6\}$ and $\operatorname{rlm}(\pi)=2$.

Corteel and Nadeau [7], Nadeau [16], Chen and Zhou [3] proved the first, the second and the third item in the following proposition, respectively.

Proposition 1.1. For $T \in \mathcal{P} \mathcal{T}(n)$, let $\Gamma(T)=\pi$ and $\Phi(T)=\sigma$, then

1. $\operatorname{urr}(T)=\operatorname{rlmin}(\pi)$;
2. $\operatorname{topone}(T)=\operatorname{rlm}(\pi)$;
3. $\operatorname{urr}(T)=\mathrm{w} \overline{\mathrm{m}}(\sigma)$.

It can be checked that topone $(T) \neq \operatorname{rlm}(\Phi(T))$. So it is interesting to investigate the distributions of ( $\mathrm{rlm}, \mathrm{w} \overline{\mathrm{m}}$ ). On the other hand, from the perspective of permutations, we see that the definition of rlm is closely related to rlmax, while $w \bar{m}$ is indeed the statistic lrmax (proved in Lemma 2.1). rlmax and lrmax are interesting Stirling statistics over permutations and have been widely studied. So this is another motivation of our work. More definitions and notations needed in this paper are listed as follows.

Given $\pi=\pi_{1} \pi_{2} \cdots \pi_{n} \in S_{n}$, its reverse $\pi^{r} \in S_{n}$ is given by $\pi^{r}(i)=$ $\pi(n+1-i)$. Its complement $\pi^{c}$ is given by $\pi^{c}(i)=n+1-\pi(i)$. Let $\pi^{-1}$ denote the inverse of $\pi$, where $\pi^{-1}(j)=i$ if and only if $\pi(i)=j$. For convenient, we also write $\mathrm{c}(\pi)=\pi^{c}$ and $\mathrm{i}(\pi)=\pi^{-1}$. Assume that $W=\left\{i_{1}, i_{2}, \ldots, i_{n}\right\}$ with $i_{1}<i_{2}<\cdots<i_{n}$. Given a permutation $w$ of $W$, we define $\operatorname{st}(w)=$ $(\sigma, W)$, where $\sigma \in S_{n}$ and $\sigma$ is order-isomorphic to $w$. Conversely, set $\mathrm{st}^{-1}(\sigma, W)=w$. As an example, $\operatorname{st}(31574)=(21453,\{1,3,4,5,7\})$ and $\mathrm{st}^{-1}(21453,\{1,3,4,5,7\})=31574$.

A descent (ascent) of $\pi$ is a position $i \in[n-1]$ such that $\pi_{i}>\pi_{i+1}\left(\pi_{i}<\right.$ $\left.\pi_{i+1}\right)$. The descent set and the ascent set of $\pi$ are given by $\operatorname{Des}(\pi)=\left\{i: \pi_{i}>\right.$ $\left.\pi_{i+1}\right\}$ and $\operatorname{Asc}(\pi)=\left\{i: \pi_{i}<\pi_{i+1}\right\}$. Let $\operatorname{des}(\pi)=|\operatorname{Des}(\pi)|$ and $\operatorname{asc}(\pi)=$ $|\operatorname{Asc}(\pi)|$ be the descent number and ascent number of $\pi$, respectively. Set $\operatorname{ides}(\pi)=\operatorname{des}\left(\pi^{-1}\right)$.

An occurrence of a classical pattern $p$ in a permutation $\sigma$ is a subsequence of $\sigma$ that is order-isomorphic to $p$. For instance, 41253 has two occurrences of the pattern 3142 in its subsequences 4153 and 4253. $\sigma$ is said to avoid $p$ if there exists no occurrence of $p$ in $\sigma$. The vincular pattern is a generalization of the classical pattern. Adjacent letters that are underlined must stay adjacent when they are placed back to the original permutation. As an example, 41253 now contains only one occurrence of the vincular pattern 3142 in its subsequence 4153 , but not in 4253 . See [14] for more details about vincular patterns. Given a vincular pattern $\tau$ and a permutation $\pi$, we denote by $(\tau) \pi$ the number of occurrences of the pattern $\tau$ in $\pi$. We write $S_{n}(\tau)$ as the set of permutations of length $n$ that avoid $\tau$.

An inversion sequence of length $n$ is a word $s=s_{1} s_{2} \cdots s_{n}$ with $0 \leq s_{i} \leq$ $i-1$ for $1 \leq i \leq n$. Let $I_{n}$ be the set of all inversion sequences of length $n$.

Assume that

$$
\begin{aligned}
\operatorname{Zero}(s) & =\left\{i: 1 \leq i \leq n, s_{i}=0\right\} \\
\operatorname{Max}(s) & =\left\{i: 1 \leq i \leq n, s_{i}=i-1\right\} \\
\operatorname{Dist}(s) & =\left\{2 \leq i \leq n: s_{i} \neq 0 \text { and } s_{i} \neq s_{j} \text { for all } j>i\right\},
\end{aligned}
$$

and zero $(s), \max (s)$ and $\operatorname{dist}(s)$ is the numerical statistics, respectively.
In this paper, we find that (rlm, $w \bar{m}$ ) and (rlm, rlmin) are equally distributed on $S_{n}(321)$, as well as on $S_{n}$. Particularly, we have the following theorems.

Theorem 1.2. Statistic (rlm, rlmin, $w \bar{m}$, des, ides) are equally distributed with statistic (rlm, w $\bar{m}$, rlmin, des, ides) over $S_{n}(321)$.

Theorem 1.3. Statistic (rlm, rlmin, $\overline{\mathrm{m}}$, des, (321))) are equally distributed


$$
\begin{equation*}
\sum_{\pi \in S_{n}} x^{\mathrm{w} \overline{\mathrm{~m}}(\pi)-1} y^{\mathrm{rlm}(\pi)}=(x+y)_{n-1} \tag{1.2}
\end{equation*}
$$

Notice that $x$ and $y$ are symmetric in (1.2). We have the following theorem, which we will reprove by an involution over $S_{n}$.

Theorem 1.4. Statistics (rlm, $\mathrm{w} \overline{\mathrm{m}}-1$ ) and ( $\mathrm{w} \overline{\mathrm{m}}-1, \mathrm{rlm}$ ) are equally distributed on $S_{n}$.

Josuat-Vergès [13] showed that

$$
Z_{N}=\sum_{\pi \in S_{N+1}} \alpha^{-\mathrm{rlmax}(\pi)+1} \beta^{-\mathrm{rlmin}(\pi)+1} y^{\operatorname{asc}(\pi)-1} q^{(\underline{312) \pi}}
$$

where $Z_{N}$ is the partition function of a partially asymmetric exclusion process (PASEP) on a finite number of sites with open and directed boundary conditions, see Theorem 1.3.1 in [13]. Inspired by this, we obtain the following equidistribution by constructing bijections.

Theorem 1.5. Statistics (rlm, $\mathrm{w} \overline{\mathrm{m}}, \mathrm{asc})$ and (rlm, rlmin, asc) are equally distributed with (rlmax -1, rlmin, asc) on $S_{n}$.

The paper is organized as follows. In Section 2, we present bijective proofs of Theorem 1.2 and 1.3. In Section 3, we construct an involution which implies Theorem 1.4. In section 4, we bijectively prove Theorem 1.5 by using inversion sequences.

## 2 Bijective proofs of Theorem 1.2 and 1.3

In this section, we first deduce that statistic $w \bar{m}$ is indeed statistic lrmax. Then, two bijections based on a block decomposition of permutations are given which imply Theorem 1.2 and 1.3, respectively.

Lemma 2.1. Given $\pi \in S_{n}, \pi_{i} \in \mathrm{~W} \overline{\mathrm{M}}(\pi)$ if and only if $\pi_{i}$ is a $L R$-maximum of $\pi$.

Proof. Suppose that $\pi_{i} \in \mathrm{~W} \overline{\mathrm{M}}(\pi)$, we claim that $\pi_{i}$ is a LR-maximum of $\pi$. Assume to the contrary that there exists $j<i$ such that $\pi_{j}>\pi_{i}$. Since $\pi_{i}$ is a weak excedance of $\pi$, it is easily checked that there exists $k>i$ such that $\pi_{k}<\pi_{i}$. Hence, $\pi_{j} \pi_{i} \pi_{k}$ is a 321-pattern of $\pi$. It follows that $\pi_{i}$ is a mid-point of $\pi$, which contradicts with the fact that $\pi_{i} \in \mathrm{~W} \overline{\mathrm{M}}(\pi)$. The claim is verified.

Conversely, assume that $\pi_{i}$ is a LR-maximum of $\pi$. Since it is larger that all the elements to its left, then $\pi_{i} \geq i$ and $\pi_{i}$ is a non-mid-point. It follows that $\pi_{i} \in \mathrm{~W} \overline{\mathrm{M}}(\pi)$. This completes the proof.

Now, we proceed to prove Theorem 1.2. As pointed out by Burstein [2], 321-avoiding permutations have the property given in Lemma 2.2. Based on this, Corollary 2.3 follows obviously.

Lemma 2.2. $\pi$ is 321-avoiding if and only if each element of $\pi$ is either a LR-maximum or an RL-minimum, i.e. if and only if $\pi$ is identity or a union of two nondecreasing subsequences.

Corollary 2.3. Given $\pi \in S_{n}(321)$, if $\pi_{1}=1$, then $\operatorname{rlm}(\pi)=0$. Otherwise, $\operatorname{rlm}(\pi)=1$.

The following lemma can be easily checked. And then, we are prepared to give a proof of Theorem 1.2.

Lemma 2.4. Assume that $\pi=\pi_{1} \pi_{2} \cdots \pi_{n} \in S_{n}$ (321) with $\pi_{1} \neq 1$ and $\pi_{n} \neq$ $n$. Let $\sigma=\pi^{r c}$, then $\operatorname{rlm}(\pi)=\operatorname{rlm}(\sigma)=1$ and

$$
(\text { rlmin }, \operatorname{lrmax}, \text { des }, \operatorname{ides}) \pi=(\operatorname{lrmax}, \text { rlmin, des, ides }) \sigma .
$$

Proof of Theorem 1.2. It suffices to construct a bijection $\chi$ over $S_{n}(321)$, which maps (rlm, rlmin, lrmax, des, ides) to (rlm, lrmax, rlmin, des, ides).

Assume that $\pi=\pi_{1} \pi_{2} \cdots \pi_{n}$ is 321-avoiding. If $\pi_{1}=1$, we set $l=$ $\max \left\{j \mid \pi_{1}=1, \cdots, \pi_{j}=j\right\}$, otherwise, $l=0$. If $\pi_{n}=n$, we set $r=$ $\min \left\{j \mid \pi_{j}=j, \cdots, \pi_{n}=n\right\}$, otherwise, $r=n+1$.

Let $p=p_{1} \cdots p_{n}=\chi(\pi)$ be the permutation $12 \cdots l h r r+1 \cdots n$, where

$$
h=\operatorname{st}\left(\left(n+1-\pi_{r-1}\right)\left(n+1-\pi_{r-2}\right) \cdots\left(n+1-\pi_{l+1}\right),\{l+1, \cdots, r-1\}\right) .
$$

It should be noted that $12 \cdots l$ is assumed to be empty if $l=0$, while $r r+$ $1 \cdots n$ is assumed to be empty if $r=n+1$. Clearly, we have $p \in S_{n}(321)$. Based on Lemma 2.4, it can be easily verified that

$$
(\text { rlmin }, \operatorname{lrmax}, \text { des }, \operatorname{ides})(\pi)=(\operatorname{lrmax}, \text { rlmin }, \text { des }, \operatorname{ides})(p)
$$

By Corollary 2.3, we have $\operatorname{rlm}(p)=\operatorname{rlm}(\pi)=0$ if $l=0$, while $\operatorname{rlm}(p)=$ $\operatorname{rlm}(\pi)=1$ if $l>0$. This completes the proof.

Example 2.5. Let $\pi=123468759$, then $l=4, r=9$ and $h=\operatorname{st}(5324,\{5$, $6,7,8\})=8657$. Thus we have $p=\chi(\pi)=123486579$. It is easy to check that $\operatorname{rlmin}(\pi)=\operatorname{lrmax}(p)=6, \operatorname{lrmax}(\pi)=\operatorname{rlmin}(p)=7, \operatorname{des}(\pi)=\operatorname{des}(p)=2$ and $\operatorname{ides}(\pi)=\operatorname{ides}(p)=2$.

In the remaining part of this section, we are dedicated to proving Theorem 1.3. Given a permutation $\pi$, put a bar after each RL-minimum, and then put a bar before each LR-maximum if there is no bar before it. Thus we obtain a block decomposition of $\pi$. Write the block decomposition of $\pi$ as $B_{1} B_{2} \cdots B_{k}$, we define

$$
\begin{aligned}
\mathrm{N}(\pi) & =\left\{B_{i} \mid B_{i} \text { contains neither LR-maximum nor RL-minimum }\right\} \\
\mathrm{T}(\pi) & =\left\{B_{i} \mid B_{i} \text { contains both LR-maximum and RL-minimum }\right\} \\
\mathrm{A}(\pi) & =\left\{B_{i} \mid B_{i} \text { contains a LR-maximum and no RL-minimum }\right\} \\
\mathrm{I}(\pi) & =\left\{B_{i} \mid B_{i} \text { contains an RL-minimum and no LR-maximum }\right\}
\end{aligned}
$$

For convenience, we call a block in $N(\pi)$ a N-block. The T-block, A-block and I-block are defined similarly. Propositions of the block decomposition below can be easily verified.

Proposition 2.6. For any $\pi \in S_{n}$, write $\pi=B_{1} B_{2} \cdots B_{k}$, we have

1. $|T(\pi)| \geq 1$.
2. $N(\pi) \cup T(\pi) \cup A(\pi) \cup I(\pi)=\bigcup_{1 \leq i \leq k}\left\{B_{i}\right\}$.
3. If $B_{i} \in \mathrm{~N}(\pi)$, then there exist integers $j<i$ and $h>i$ such that $B_{j} \in$ $\mathrm{T}(\pi), B_{h} \in \mathrm{~T}(\pi),\left\{B_{j+1}, \cdots, B_{i-1}\right\} \subset \mathrm{I}(\pi)$ and $\left\{B_{i+1}, \cdots, B_{h-1}\right\} \subset$ $\mathrm{A}(\pi)$.
4. Let $\mathrm{T}(\pi) \cup \mathrm{A}(\pi)=\left\{B_{x_{1}}, \cdots, B_{x_{h}}\right\}$, where $x_{1}<\cdots<x_{h}$, then

$$
\max \left(B_{x_{1}}\right)<\cdots<\max \left(B_{x_{h}}\right) .
$$

5. Let $\mathrm{T}(\pi) \cup \mathrm{I}(\pi)=\left\{B_{x_{1}}, \cdots, B_{x_{h}}\right\}$, where $x_{1}<\cdots<x_{h}$, then

$$
\min \left(B_{x_{1}}\right)<\cdots<\min \left(B_{x_{h}}\right) .
$$

In the following, we define two operations on permutations. Given $\pi=$ $\pi_{1} \pi_{2} \cdots \pi_{n} \in S_{n}$, assume that $\min (\pi)=\pi_{i}$ and $\max (\pi)=\pi_{j}$, let

$$
\begin{aligned}
L(\pi) & =\pi_{i+1} \cdots \pi_{n} \pi_{1} \cdots \pi_{i} \\
R(\pi) & =\pi_{j} \cdots \pi_{n} \pi_{1} \cdots \pi_{j-1}
\end{aligned}
$$

Proposition 2.7. For $\pi=\pi_{1} \cdots \pi_{n} \in S_{n}$, we have

1. $R \circ L(\pi)=\pi$ if and only if $\max (\pi)=\pi_{1}$.
2. $L \circ R(\pi)=\pi$ if and only if $\min (\pi)=\pi_{n}$.
3. If $\max (\pi)=\pi_{1}$, then $(\underline{321})(\pi)=(\underline{321})(L(\pi))$.
4. If $\min (\pi)=\pi_{n}$, then $(\underline{321})(\pi)=(\underline{321})(R(\pi))$.

Now we are ready to present the map $\varphi$ over $S_{n}$ such that for any $\pi \in S_{n}$ $(\operatorname{rlm}, \operatorname{rlmin}, \mathrm{w} \overline{\mathrm{m}}, \operatorname{des},(\underline{321})) \pi=(\mathrm{rlm}, \mathrm{w} \overline{\mathrm{m}}$, rlmin $, \operatorname{des},(\underline{321})) \varphi(\pi)$.

Let $\pi=B_{1} B_{2} \cdots B_{k} \in S_{n}$ and assume that

$$
\begin{array}{ll}
N(\pi)=\left\{B_{N_{1}}, \cdots, B_{N_{h}}\right\}, & T(\pi)=\left\{B_{T_{1}}, \cdots, B_{T_{l}}\right\}, \\
A(\pi)=\left\{B_{A_{1}}, \cdots, B_{A_{p}}\right\}, & I(\pi)=\left\{B_{I_{1}}, \cdots, B_{I_{q}}\right\} . \tag{2.1}
\end{array}
$$

If $N(\pi)=\emptyset$, then we may view $h=0$. It is similar for $A(\pi)$ and $I(\pi)$. We can obtain $\sigma=\varphi(\pi)$ through the following there steps:

Step 1 Write down the blocks in $N(\pi)$ and $\mathrm{T}(\pi)$, which keeps the relative order in $\pi$, we obtain $\sigma^{\prime}$;

Step 2 Insert $R\left(B_{I_{1}}\right), \cdots, R\left(B_{I_{q}}\right)$ to $\sigma^{\prime}$ by letting the maximal letter (i.e. the first letter) of $R\left(B_{I_{1}}\right), \cdots, R\left(B_{I_{q}}\right), B_{T_{1}}, \cdots, B_{T_{l}}$ increase. Between two T-blocks, $R\left(B_{I_{c}}\right)(1 \leq c \leq q)$ is always to the right of a N-block, if there is any. Then we obtain $\sigma^{\prime \prime}$;

Step 3 Insert $L\left(B_{A_{1}}\right), \cdots, L\left(B_{A_{p}}\right)$ to $\sigma^{\prime \prime}$ by letting the minimal letter (i.e. the last letter) of $L\left(B_{A_{1}}\right), \cdots, L\left(B_{A_{p}}\right), B_{T_{1}}, \cdots, B_{T_{l}}$ increase. Between two T-blocks, $L\left(B_{A_{d}}\right)(1 \leq d \leq p)$ is always to the left of a N-block and $R\left(B_{A_{c}}\right)(1 \leq c \leq q)$, if there is any. Then we obtain $\sigma$.

Example 2.8. Let $\pi=10,2,6,11,1,8,13,3,5,9,4,12,7$, then the block decomposition of $\pi$ is

$$
\begin{array}{|lllllllllllll|}
\mid 10 & 2 & 6 & \mid 11 & 1 \mid & 8 & \mid 13 & 3 \mid & 5 & 9 & 4 \mid & 12 & 7 \mid
\end{array}
$$

 By the three steps given above, we have

$$
\begin{aligned}
\sigma^{\prime} & =111,8,133, \quad \sigma^{\prime \prime}=945,111,8,127,133 \\
\sigma & =945,111,6102,8,127,133
\end{aligned}
$$

Proposition 2.9. Let $\sigma=\varphi(\pi)$, we have
(1) $T(\sigma)=\left\{B_{T_{1}}, \cdots, B_{T_{l}}\right\}$;
(2) $A(\sigma)=\left\{R\left(B_{I_{1}}\right), \cdots, R\left(B_{I_{q}}\right)\right\}$;
(3) $I(\sigma)=\left\{L\left(B_{A_{1}}\right), \cdots, L\left(B_{A_{p}}\right)\right\}$;
(4) $N(\sigma)=\left\{B_{N_{1}}, \cdots, B_{N_{h}}\right\}$.

Proof. Firstly, we wish to show that the first letter of $B_{T_{j}}\left(\right.$ i.e. $\left.\max \left\{B_{T_{j}}\right\}\right)$, where $1 \leq j \leq l$, is a LR-maximum of $\sigma$, while the last letter of $B_{T_{j}}$ (i.e. $\min \left\{B_{T_{j}}\right\}$ ) is an RL-minimum of $\sigma$. By definition of step 1 in the description of $\varphi$, we easily check that the first letter of $B_{T_{j}}$ is a LR-maximum of $\sigma^{\prime}$. Since the maximal letter $R\left(B_{I_{1}}\right), \cdots, R\left(B_{I_{q}}\right), B_{T_{1}}, \cdots, B_{T_{l}}$ increase in step 2, we have the first letter of $B_{T_{j}}$ is a LR-maximum of $\sigma^{\prime \prime}$. Assume that $L\left(B_{A_{i}}\right)$ is to the left of $B_{T_{j}}$ in $\sigma$, then we have $\min \left\{B_{A_{i}}\right\}<\min \left\{B_{T_{j}}\right\}$. It follows that $B_{A_{i}}$ is to the left of $B_{T_{j}}$ in $\pi$, which means that $\max \left\{B_{A_{i}}\right\}<\max \left\{B_{T_{j}}\right\}$. Above all, the first letter of $B_{T_{j}}$ is a LR-maximum of $\sigma$.

Now we proceed to show that the last letter of $B_{T_{j}}$ is an RL-minimum of $\sigma$. Clearly, $\min \left\{B_{T_{j}}\right\}$ is an RL-minimum of $\sigma^{\prime}$. the maximal letter (i.e. the first letter) of $R\left(B_{I_{1}}\right), \cdots, R\left(B_{I_{q}}\right), B_{T_{1}}, \cdots, B_{T_{l}}$ increase.

Assume that $R\left(B_{I_{i}}\right)$ is to the right of $B_{T_{j}}$ in $\sigma$, then $\max \left\{B_{T_{j}}\right\}<\max \left\{B_{I_{i}}\right\}$. It means that $B_{I_{i}}$ is to the right of $B_{T_{j}}$ in $\pi$. Hence, $\min \left\{B_{I_{i}}\right\}>\min \left\{B_{T_{j}}\right\}$. Assume that $L\left(B_{A_{i}}\right)$ is to the right of $B_{T_{j}}$ in $\sigma$, then by the definition of step 3 , it is easily seen that $\min \left\{B_{A_{i}}\right\}>\min \left\{B_{T_{j}}\right\}$. Hence, the last letter of $B_{T_{j}}$ is an RL-minimum of $\sigma$, as desired.

Secondly, we need to show that the first letter of $R\left(B_{I_{j}}\right)\left(\right.$ i.e. $\left.\max \left\{B_{I_{j}}\right\}\right)$, where $1 \leq j \leq q$, is a LR-maximum of $\sigma$ and it contains no RL-minimum of $\sigma$. Clearly, the first letter of $R\left(B_{I_{j}}\right)$ is a LR-maximum of $\sigma^{\prime \prime}$. Assume that $L\left(B_{A_{i}}\right)$ is to the left of $R\left(B_{I_{j}}\right)$ in $\sigma$, we wish to prove that $\max \left\{B_{A_{i}}\right\}<$ $\max \left\{B_{I_{j}}\right\}$. Let $B_{T_{x}}$ be the nearest T-block that is to the right of $B_{A_{i}}$ in
$\pi$, then we have $\max \left\{B_{A_{i}}\right\}<\max \left\{B_{T_{x}}\right\}$ and $\min \left\{B_{A_{i}}\right\}>\min \left\{B_{T_{x}}\right\}$. By step 3 in the description of $\varphi, B_{T_{x}}$ is to the left of $L\left(B_{A_{i}}\right)$, and hence to the left of $R\left(B_{I_{j}}\right)$ in $\sigma$. It follows that $\max \left\{B_{T_{x}}\right\}<\max \left\{B_{I_{j}}\right\}$. Thus, $\max \left\{B_{A_{i}}\right\}<\max \left\{B_{I_{j}}\right\}$. Hence, the first letter of $R\left(B_{I_{j}}\right)$ is a LR-maximum of $\sigma$.

Let $B_{T_{y}}$ be the nearest T-block that is to the left of $B_{I_{j}}$ in $\pi$, then we have $\max \left\{B_{I_{j}}\right\}<\max \left\{B_{T_{y}}\right\}$ and $\min \left\{B_{I_{j}}\right\}>\min \left\{B_{T_{y}}\right\}$. By step 2, $B_{T_{y}}$ is to the right of $R\left(B_{I_{j}}\right)$ in $\sigma$. It follows from the fact $\min \left\{B_{I_{j}}\right\}>\min \left\{B_{T_{y}}\right\}$ that $R\left(B_{I_{j}}\right)$ contains no RL-minimum of $\sigma$, as desired.

Thirdly, we wish to show that the last letter of $L\left(B_{A_{j}}\right)(1 \leq j \leq p)$ is an RL-minimum of $\sigma$ and it contains no LR-maximum of $\sigma$. By description of step 3 in $\varphi$, the last letter of $L\left(B_{A_{j}}\right)$ (i.e. $\left.\min \left\{B_{A_{j}}\right\}\right)$ is smaller than all letters of $T$-blocks and $N$-blocks which are to the right of it. Now we assume that $R\left(B_{I_{i}}\right)$ is to the right of $L\left(B_{A_{j}}\right)$ in $\sigma$, if there is any, we aim to show that $\min \left\{B_{I_{i}}\right\}>\min \left\{B_{A_{j}}\right\}$. Let $B_{T_{x}}$ be the nearest $T$-block that is to the left of $B_{I_{i}}$ in $\pi$. Thus, we have $\max \left\{B_{I_{i}}\right\}<\max \left\{B_{T_{x}}\right\}$ and $\min \left\{B_{I_{i}}\right\}>$ $\min \left\{B_{T_{x}}\right\}$. It follows from description of step 2 that $B_{T_{x}}$ is to the right of $R\left(B_{I_{i}}\right)$ in $\sigma$, and hence to the right of $L\left(B_{A_{j}}\right)$. Thus, by step 3, we see that $\min \left\{B_{A_{j}}\right\}<\min \left\{B_{T_{x}}\right\}$. Hence, $\min \left\{B_{I_{i}}\right\}>\min \left\{B_{A_{j}}\right\}$ and the last letter of $L\left(B_{A_{j}}\right)(1 \leq j \leq p)$ is an RL-minimum of $\sigma$.

Let $B_{T_{y}}$ be the nearest $T$-block that is to the right of $B_{A_{j}}$ in $\pi$, then $\max \left\{B_{A_{j}}\right\}<\max \left\{B_{T_{y}}\right\}$ and $\min \left\{B_{A_{j}}\right\}>\min \left\{B_{T_{y}}\right\}$. By step 3, $B_{T_{y}}$ is to the left of $L\left(B_{A_{j}}\right)$ in $\sigma$. Then, $B_{A_{j}}$ contains no LR-maximum follows from the fact that $\max \left\{B_{A_{j}}\right\}<\max \left\{B_{T_{y}}\right\}$, as desired.

Notice that $B_{N_{j}}(1 \leq j \leq h)$ contains no RL-minimum nor LR-maximum of $\sigma$. By all the analysis above, we may obtain a block decomposition of $\sigma$ and propositions (1) - (4) follows. This completes the proof.

Proof of Theorem 1.3. Let $\pi=\pi_{1} \pi_{2} \cdots \pi_{n} \in S_{n}$ with a block decomposition given in (2.1) and $\sigma=\varphi(\pi)$. It suffices to show that $\varphi$ is an involution over $S_{n}$ such that
$($ rlm, rlmin, lrmax, des, $(\underline{321}))(\pi)=($ rlm, lrmax, rlmin, des, $(\underline{321}))(\sigma)$.
Firstly, we prove that $\varphi$ is an involution, i.e. $\varphi(\sigma)=\pi$. Assume that $p=\varphi(\sigma)$, by applying Proposition 2.9 twice, we have

$$
\begin{aligned}
N(p) & =\left\{B_{N_{1}}, \cdots, B_{N_{h}}\right\}, \\
T(p) & =\left\{B_{T_{1}}, \cdots, B_{T_{l}}\right\}, \\
A(p) & =\left\{R \circ L\left(B_{A_{1}}\right), \cdots, R \circ L\left(B_{A_{p}}\right)\right\}, \\
I(p) & =\left\{L \circ R\left(B_{I_{1}}\right), \cdots, L \circ R\left(B_{I_{q}}\right)\right\} .
\end{aligned}
$$

Notice that $\max \left\{B_{A_{c}}\right\}$ is the first letter of $B_{A_{c}}$ for $1 \leq c \leq p$, while $\min \left\{B_{I_{d}}\right\}$ is the last letter of $B_{I_{d}}$ for $1 \leq d \leq q$. Then from items 1,2 in Proposition 2.7 we deduce that

$$
\begin{aligned}
N(p) & =\left\{B_{N_{1}}, \cdots, B_{N_{h}}\right\}, T(p)=\left\{B_{T_{1}}, \cdots, B_{T_{l}}\right\}, \\
A(p) & =\left\{B_{A_{1}}, \cdots, B_{A_{p}}\right\}, \quad I(p)=\left\{B_{I_{1}}, \cdots, B_{I_{q}}\right\} .
\end{aligned}
$$

Comparing with the block decomposition of $\pi$ given in (2.1), we see that $p=\pi$. Hence $\varphi^{2}(\pi)=\pi$ and $\varphi$ is an involution over $S_{n}$.

Now, we proceed to prove (2.2). Viewing (2.1) and Proposition 2.9, we have $\operatorname{lrmax}(\pi)=\operatorname{rlmin}(\sigma)=p$ and $\operatorname{rlmin}(\pi)=\operatorname{lrmax}(\sigma)=q$. If $\pi_{1}=1$, then it is easy to check that $\sigma_{1}=1$. Hence, we have $\operatorname{rlm}(\pi)=\operatorname{rlm}(\sigma)=0$. Otherwise, suppose that $\operatorname{Rlm}(\pi)=\left\{\pi_{r_{1}}, \pi_{r_{2}}, \cdots, \pi_{r_{s}}\right\}$, where $r_{1}<r_{2}<\cdots<$ $r_{s}$. Then, $\pi_{r_{1}}$ is the nearest LR-maximum of $\pi$ that is to the left of 1 . It follows that $\pi_{r_{1}} \cdots 1$ is a T-block of $\pi$. Hence, $\pi_{r_{1}} \cdots 1$ remains a T-block of $\sigma$ by Proposition 2.9. Thus, $\operatorname{Rlm}(\sigma)=\left\{\pi_{r_{1}}, \pi_{r_{2}}, \cdots, \pi_{r_{s}}\right\}$ and we obtain that $\operatorname{rlm}(\pi)=\operatorname{rlm}(\sigma)=s$. We claim that all descents of a permutation are always contained in blocks. Assume that $i$ is a descent of $\pi$ with $\pi_{i}>\pi_{i+1}$, then $\pi_{i}$ is not an RL-minimum and $\pi_{i+1}$ is not a LR-maximum. Hence there is no bar neither after $\pi_{i}$ nor before $\pi_{i+1}$. The claim is verified. It follows directly that $\operatorname{des}(\pi)=\operatorname{des}(\sigma)$. Combining with items 3, 4 in Proposition 2.7, we have $(\underline{321})(\pi)=(\underline{321})(\sigma)$. This completes the proof.

It should be mentioned that $\varphi$ does not keep the statistic ides. We check the following conjecture by computer for $n \leq 9$.

Conjecture 2.10. Statistic (rlm, rlmin, lrmax, des, ides, (321)) are equally distributed with Statistic (rlm, lrmax, rlmin, des, ides, (321)) over $S_{n}$.

## 3 A bijective proof of Theorem 1.4

In this section, we present an involution over $S_{n}$ to give a combinatorial interpretation of Theorem 1.4. In view of Lemma 2.1, it is enough to prove the following theorem.

Theorem 3.1. There exists an involution $\phi$ on $S_{n}$ such that

$$
\begin{align*}
& \operatorname{rlm}(\pi)=\operatorname{lrmax}(\phi(\pi))-1  \tag{3.1}\\
& \operatorname{lrmax}(\pi)-1=\operatorname{rlm}(\phi(\pi)) . \tag{3.2}
\end{align*}
$$

In the following, we shall give such an involution. We first consider some
special cases. Define

$$
\begin{aligned}
& S_{n}^{1}=\left\{\pi=\pi_{1} \cdots \pi_{n} \mid \pi_{n}=1\right\}, \\
& S_{n}^{n}=\left\{\pi=\pi_{1} \cdots \pi_{n} \mid \pi_{n}=n\right\} .
\end{aligned}
$$

Lemma 3.2. There is a bijection $\rho$ from $S_{n}^{n}$ to $S_{n}^{1}$, such that

$$
\begin{align*}
& \operatorname{rlm}(\pi)=\operatorname{lrmax}(\rho(\pi))-1,  \tag{3.3}\\
& \operatorname{lrmax}(\pi)-1=\operatorname{rlm}(\rho(\pi)) . \tag{3.4}
\end{align*}
$$

Proof. Given a permutation $\pi \in S_{n}^{n}$, assume that $\pi_{k}=1$ and $\pi=w 1 u n$, where $w$ and $u$ can be empty. Let $\pi_{i}=\max (w)$ and $j$ be the least element such that $k<j<n$ and $\pi_{j}>\pi_{i}$, if there exist. Then, assume that $a=$ $\pi_{1} \cdots \pi_{i-1}, b=\pi_{i} \cdots \pi_{k-1}, c=\pi_{k+1} \cdots \pi_{j-1}$ and $d=\pi_{j} \cdots \pi_{n-1}$ Thus, we decompose $\pi$ into six blocks, namely, $\pi=a b 1 c d n$. It should be noted that each of the blocks $a, b, c, d$ can be empty. Define $\rho(\pi)$ to be $\pi^{\prime}=b^{r} c n d^{r} a^{r} 1$ Clearly, $\pi^{\prime} \in S_{n}^{1}$. It follows that $\rho$ is a map from $S_{n}^{n}$ to $S_{n}^{1}$.

To prove that $\rho$ is a bijection, we give the inverse map of $\rho$. Given a permutation $\tau \in S_{n}^{1}$, let $\tau=p n q 1$. Both of $p$ and $q$ can be empty. If there exists, assume that $\tau_{l}$ is the largest element of $p$. Let $\tau_{s}$ is the rightmost element of $q$ that is larger than $\tau_{l}$, if there exists. Suppose that $\tau_{t}=n$ where $t<n$. We decompose $\tau$ into six blocks by setting $\tau=$ efngh1, where $e=\tau_{1} \cdots \tau_{l}, f=\tau_{l+1} \cdots \tau_{t-1}, g=\tau_{t+1} \cdots \tau_{s}$ and $h=\tau_{s+1} \cdots \tau_{n-1}$. Define $\chi(\tau)$ to be the permutation $\tau^{\prime}$ where $\tau^{\prime}=h^{r} e^{r} 1 f g^{r} n$. It can be easily checked that $\chi$ is the inverse map of $\rho$. Hence, $\rho$ is a bijection.

Next, we proceed to prove relations (3.3) and (3.4). It is not hard to check that the following relations.

$$
\begin{aligned}
\operatorname{lm}(\pi) & =\text { the number of LR-maxima of } b^{r} \\
\operatorname{lrmax}(\pi)-1 & =\text { the number of LR-maxima of } a \pi_{i} d, \\
\operatorname{rlm}\left(\pi^{\prime}\right) & =\text { the number of LR-maxima of } a d n \\
\operatorname{lrmax}\left(\pi^{\prime}\right)-1 & =\text { the number of LR-maxima of } b^{r}
\end{aligned}
$$

Notice that the number of LR-maxima of $a \pi_{i} d$ equals to the number of LRmaxima of adn. Hence relations (3.3) and (3.4) follows, as desired.

Based on Lemma 3.2, we are now ready to give the involution $\phi$ on $S_{n}$.
Proof of Theorem 3.1. Firstly, we give the description of $\phi$. For a permutation $\pi \in S_{n}$, there are two cases to consider.

Case 1: 1 is to the left of $n$. Assume that $\pi=u n v$ and $(e, S)=\operatorname{st}(u n)$. Then $\phi$ is defined by letting $\phi(\pi)=s t^{-1}(\rho(e), S) v$.

Case 2: 1 is to the right of $n$. Assume that $\pi=p 1 q$ and $(o, T)=\operatorname{st}(p 1)$. Then $\phi$ is defined by letting $\phi(\pi)=s t^{-1}\left(\rho^{-1}(o), T\right) q$.

From the construction of $\phi$, it is easily seen that $\phi$ is an involution on $S_{n}$. In the following, we proceed to prove relations (3.1) and (3.2).

By Lemma 3.2, $\operatorname{lrmax}(e)-1=\operatorname{rlm}(\rho(e))$ and $\operatorname{rlm}(e)=\operatorname{lrmax}(\rho(e))-1$. By order-isomorphic, we deduce that $\operatorname{lrmax}(u n)-1=\operatorname{rlm}\left(s t^{-1}(\rho(e), S)\right)$ and $\operatorname{rlm}(u n)=\operatorname{lrmax}\left(s t^{-1}(\rho(e), S)\right)-1$. Notice that in case 1 , there is no element $z$ in subword $v$ such that $z \in \operatorname{Rlm}(\pi)$ nor $z \in \operatorname{Lrmax}(\pi)$. Thus, $\operatorname{lrmax}(\pi)-1=\operatorname{rlm}(\phi(\pi))$ and $\operatorname{rlm}(\pi)=\operatorname{lrmax}(\phi(\pi))-1$ hold for case 1 . The fact that (3.1) and (3.2) hold for case 2 can be proved similarly and we omit it here. We complete the proof.

We end this section by giving examples of bijections $\rho$ and $\phi$.
Example 3.3. Let $\pi=372514869$, then

$$
a=3, \quad b=725, \quad c=4, \quad d=86
$$

Hence, $\rho(\pi)=527496831$. Let $\sigma=38251496107$, then

$$
\operatorname{st}(3825149610)=(372514869,\{1,2,3,4,5,6,8,9,10\})
$$

and hence $\phi(\sigma)=52841069317$.

## 4 A bijective proof of Theorem 1.5

In this section, we first prove Lemma 4.1 by giving an involution $\gamma$ over the set of inversion sequences of length $n$. This allows us to construct a bijection $\alpha$ on $S_{n}$ implying Lemma 4.2. Based on Lemma 4.2, another bijection $\beta$ over $S_{n}$ is given, which proves Theorem 1.5.

Lemma 4.1. Statistics (dist, zero, max, rlmin) and (dist, zero, rlmin, max) are equally distributed over $I_{n}$. Particularly, there is an involution $\gamma$ over $I_{n}$ such that for each $e \in I_{n}$ we have

$$
\begin{equation*}
(\text { dist, zero, } \max , \text { rlmin }) e=(\text { dist, zero, rlmin, } \max ) \gamma(e) . \tag{4.1}
\end{equation*}
$$

Lemma 4.2. Statistics (asc, rlmax, lrmax, rlmin) and (asc, rlmax, rlmin, lrmax) are equally distributed over $S_{n}$. Particularly, there is an involution $\alpha$ over $S_{n}$ such that for each $\pi \in S_{n}$ we have

$$
\begin{equation*}
(\text { asc }, \text { rlmax }, \operatorname{lrmax}, \mathrm{rlmin}) \pi=(\text { asc }, \mathrm{rlmax}, \text { rlmin }, \operatorname{lrmax}) \alpha(\pi) \tag{4.2}
\end{equation*}
$$

To prove Lemma 4.1, we construct $\gamma$ over $I_{n}$ by induction. Let $\gamma(0)=0$. For $e=e_{1} e_{2} \cdots e_{n-1} e_{n} \in I_{n}$, assume that $r^{\prime}=\gamma\left(e_{1} e_{2} \cdots e_{n-1}\right)$. Then, $r=$ $\gamma(e)$ is obtained by inserting $e_{n}$ to the $e_{n}+1$-th position of $r^{\prime}$.

Example 4.3. Let $e=00113213$, then $\gamma(e)$ can be obtained as follows

$$
0 \rightarrow 00 \rightarrow 010 \rightarrow 0110 \rightarrow 01130 \rightarrow 012130 \rightarrow 0112130 \rightarrow 01132130 .
$$

And $\gamma^{2}(e)=\gamma(01132130)$ can be obtained as follows

$$
0 \rightarrow 01 \rightarrow 011 \rightarrow 0113 \rightarrow 01213 \rightarrow 011213 \rightarrow 0113213 \rightarrow 00113213 .
$$

Clearly, $\gamma$ is well-defined and we can easily verify the following propositions.

Proposition 4.4. Let $e=e_{1} e_{2} \cdots e_{n} \in I_{n}$ and $r=r_{1} r_{2} \cdots r_{n}=\gamma(e)$. Then
(1) $e_{n}+1$ is the largest element in $\operatorname{Max}(r)$.
(2) Assume that $j$ is the largest element in $\operatorname{Max}(e)$, then

$$
r=\gamma\left(e_{1} \cdots e_{j-1} e_{j+1} \cdots e_{n}\right) e_{j}
$$

Proof of Lemma 4.1. It suffices to show that $\gamma$ is an involution over $I_{n}$ and satisfies (4.1).

We proceed to prove that $\gamma$ is an involution by induction. When $n=1$, $\gamma^{2}(0)=0$. Suppose that $\gamma^{2}(t)=t$ for each $t \in I_{n-1}$ with $n \geq 2$. We claim that $\gamma^{2}(e)=e$ for each $e \in I_{n}$. By Proposition 4.4, we have $e_{n}+$ 1 is the largest in $\operatorname{Max}\left(\gamma\left(e_{1} e_{2} \cdots e_{n}\right)\right)$ and hence $e_{n}$ is the last element of $\gamma^{2}\left(e_{1} e_{2} \cdots e_{n}\right)$. Combining the construction of $\gamma$ and (2) in Proposition 4.4, we deduce that

$$
\begin{aligned}
\gamma^{2}\left(e_{1} e_{2} \cdots e_{n}\right) & =\gamma^{2}\left(e_{1} e_{2} \cdots e_{n-1}\right) e_{n} \\
& =e_{1} e_{2} \cdots e_{n-1} e_{n}
\end{aligned}
$$

The claim is verified. Hence, $\gamma$ is an involution.
Now, we shall prove relation (4.1). It is easy to check that (dist, zero) $e=$ (dist, zero) $\gamma(e)$. It is left to show that

$$
\begin{equation*}
(\max , \operatorname{rlmin}) e=(\operatorname{rlmin}, \max ) \gamma(e) \tag{4.3}
\end{equation*}
$$

Obviously, it holds for $n=1$. Suppose that (4.3) holds for $n-1$, where $n \geq 2$, we claim that it also holds for $n$. There are two cases to consider. If $e_{n}=n-1$, then $\gamma\left(e_{1} e_{2} \cdots e_{n}\right)=\gamma\left(e_{1} e_{2} \cdots e_{n-1}\right)(n-1)$. Thus,

$$
\begin{align*}
\max \left(e_{1} e_{2} \cdots e_{n}\right) & =\max \left(e_{1} e_{2} \cdots e_{n-1}\right)+1  \tag{4.4}\\
\operatorname{rlmin}\left(\gamma\left(e_{1} e_{2} \cdots e_{n}\right)\right) & =\operatorname{rlmin}\left(\gamma\left(e_{1} e_{2} \cdots e_{n-1}\right)\right)+1 \tag{4.5}
\end{align*}
$$

Combining (4.4) (4.5) and the hypothesis that max $\left(e_{1} e_{2} \cdots e_{n-1}\right)=\operatorname{rlmin}\left(\gamma\left(e_{1}\right.\right.$ $\left.e_{2} \cdots e_{n-1}\right)$ ), we deduce that $\max (e)=\operatorname{rlmin}(\gamma(e))$. Then $\operatorname{rlmin}(e)=\max (\gamma(e))$ follows from the fact that $\gamma$ is an involution.

If $e_{n}<n-1$, by Proposition $4.4, e_{n}+1$ is the largest element of $\operatorname{Max}(\gamma(e))$. It follows that $e_{n}$ is not an $R L$-minimum of $\gamma(e)$. Thus, we have

$$
\begin{align*}
\max \left(e_{1} e_{2} \cdots e_{n}\right) & =\max \left(e_{1} e_{2} \cdots e_{n-1}\right)  \tag{4.6}\\
\operatorname{rlmin}\left(\gamma\left(e_{1} e_{2} \cdots e_{n}\right)\right) & =\operatorname{rlmin}\left(\gamma\left(e_{1} e_{2} \cdots e_{n-1}\right)\right) \tag{4.7}
\end{align*}
$$

Similarly, in view of (4.6) (4.7) and the hypothesis, we have (max, rlmin) $e=$ (rlmin, max $) \gamma(e)$ in this case. This completes the proof.

To prove Lemma 4.2, we need the permutation code $b$, namely, a bijection between permutations and inversion sequences, given by Baril and Vajnovszki [1]. We give a brief review of the code $b$ first.

An interval $[m, n]$ with $m<n$ is the set $\{x \in \mathbb{N}: m \leq x \leq n\}$, where $\mathbb{N}=\{0,1, \cdots\}$. A labeled interval is a pair $(I, l)$, where $I$ is an interval and $l$ is an integer. Given $\pi=\pi_{1} \pi_{2} \cdots \pi_{n} \in S_{n}$ and an integer $i$ with $0 \leq i<n$, let the $i$-th slice of $\pi, U_{i}(\pi)$, to be a sequence of labelled intervals constructed recursively by the following process. Set $U_{0}(\pi)=([0, n], 0)$. For $i \geq 1$, assume that $U_{i-1}(\pi)=\left(I_{1}, l_{1}\right),\left(I_{2}, l_{2}\right), \cdots,\left(I_{k}, l_{k}\right)$ is the $(i-1)$-th slide of $\pi$ and $v$ is the index such that $\pi_{i} \in I_{v}$, then $U_{i}(\pi)$ is constructed as follows.

- If $\min \left(I_{v}\right)<\pi_{i}=\max \left(I_{v}\right)$, then $U_{i}(\pi)$ equals

$$
\left(I_{1}, l_{1}\right), \cdots,\left(I_{v-1}, l_{v-1}\right),\left(J, l_{v+1}\right),\left(I_{v+1}, l_{v+2}\right), \cdots,\left(I_{k-1}, l_{k}\right),\left(I_{k}, l_{k}+1\right)
$$

where $J=\left[\min \left(I_{v}\right), \pi_{i}-1\right]$.

- If $\min \left(I_{v}\right)<\pi_{i}<\max \left(I_{v}\right)$, then $U_{i}(\pi)$ equals

$$
\left(I_{1}, l_{1}\right), \cdots,\left(I_{v-1}, l_{v-1}\right)\left(H, l_{v}\right),\left(J, l_{v+1}\right),\left(I_{v+1}, l_{v+2}\right), \cdots,\left(I_{k-1}, l_{k}\right),\left(I_{k}, l_{k}+1\right)
$$

where $H=\left[\pi_{i}+1, \max \left(I_{v}\right)\right]$ and $J=\left[\min \left(I_{v}\right), \pi_{i}-1\right]$.

- If $\min \left(I_{v}\right)=\pi_{i}<\max \left(I_{v}\right)$, then $U_{i}(\pi)$ equals

$$
\left(I_{1}, l_{1}\right), \cdots,\left(I_{v-1}, l_{v-1}\right)\left(H, l_{v}\right),\left(I_{v+1}, l_{v+1}\right), \cdots,\left(I_{k-1}, l_{k-1}\right),\left(I_{k}, l_{k}+1\right),
$$

where $H=\left[\pi_{i}+1, \max \left(I_{v}\right)\right]$.

- If $\min \left(I_{v}\right)=\pi_{i}=\max \left(I_{v}\right)$, then $U_{i}(\pi)$ equals

$$
\left(I_{1}, l_{1}\right), \cdots,\left(I_{v-1}, l_{v-1}\right),\left(I_{v+1}, l_{v+1}\right), \cdots,\left(I_{k-1}, l_{k-1}\right),\left(I_{k}, l_{k}+1\right)
$$

Let $b(\pi)=b_{1} b_{2} \cdots b_{n} \in I_{n}$, where $b_{i}=l_{v}$ such that $\left(I_{v}, l_{v}\right)$ is a labelled interval in the $(i-1)$-th slice of $\pi$ with $\pi_{i} \in I_{v}$.

Example 4.5. For $\pi=24135$ and $\sigma=14352$, we have $b(\pi)=00210$ and $b(\sigma)=00102$ with

$$
\begin{array}{ll}
U_{0}(\pi)=([0,5], 0), & U_{0}(\sigma)=([0,5], 0), \\
U_{1}(\pi)=([3,5], 0)([0,1], 1), & U_{1}(\sigma)=([2,5], 0)([0,0], 1), \\
U_{2}(\pi)=([5,5], 0)([3,3], 1)([0,1], 2), & U_{2}(\sigma)=([5,5], 0)([2,3], 1)([0,0], 2), \\
U_{3}(\pi)=([5,5], 0)([3,3], 1)([0,0], 3), & U_{3}(\sigma)=([5,5], 0)([2,2], 2)([0,0], 3), \\
U_{4}(\pi)=([5,5], 0)([0,0], 4), & U_{4}(\sigma)=([2,2], 2)([0,0], 4) .
\end{array}
$$

Baril and Vajnovszki also proved a set-valued equidistribution as follows.
Lemma 4.6. For any $\pi \in S_{n}$,
(Des, Ides, Lrmax, Lrmin, Rlmax) $\pi=($ Asc, Dist, Zero, Max, Rlmin $) b(\pi)$,
and so statistics (Des, Ides, Lrmax, Lrmin, Rlmax) on $S_{n}$ has the same distribution as (Asc, Dist, Zero, Max, Rlmin) on $I_{n}$.

Let $\alpha=\mathrm{c} \circ \mathrm{i} \circ b^{-1} \circ \gamma \circ b \circ \mathrm{i} \circ \mathrm{c}$, then it is easy to check that $\alpha$ is an involution on $S_{n}$. Now, we are ready to give the proof of Lemma 4.2.

Proof of Lemma 4.2. Given $\pi \in S_{n}$, it is enough to show that

$$
\begin{equation*}
(\text { asc }, \operatorname{rlmax}, \operatorname{lrmax}, \operatorname{llmin}) \pi=(\text { asc }, \text { rlmax }, \text { rlmin }, \operatorname{lrmax}) \alpha(\pi) \tag{4.8}
\end{equation*}
$$

Notice that $\operatorname{asc}(\pi)=\operatorname{des}(c(\pi))$ and $\operatorname{des}(\pi)=\operatorname{ides}(\mathrm{i}(\pi))$. Combining with Lemma 4.1 and Lemma 4.6, we see that $\operatorname{asc}(\pi)=\operatorname{asc}(\alpha(\pi))$.

Furthermore, the following properties are easy to check.

1) $\pi_{i}$ is an RL-maximum of $\pi$ if and only if $i$ is an RL-maximum of $\pi^{-1}$.
2) $\pi_{i}$ is a LR-minimum of $\pi$ if and only if $i$ is a LR-minimum of $\pi^{-1}$.
3) $\pi_{i}$ is a LR-maximum of $\pi$ if and only if $i$ is a RL-minimum of $\pi^{-1}$.
4) $\pi_{i}$ is an RL-minimum of $\pi$ if and only if $i$ is an LR-maximum of $\pi^{-1}$.

It follows that

$$
\begin{equation*}
(\mathrm{rlmax}, \operatorname{lrmin}, \operatorname{lrmax}, \operatorname{rlmin}) \pi=(\mathrm{rlmax}, \operatorname{lrmin}, \operatorname{rlmin}, \operatorname{lrmax}) \pi^{-1} . \tag{4.9}
\end{equation*}
$$

Also, we have
$($ rlmax $, \operatorname{lrmin}, \operatorname{lrmax}$, rlmin $) \pi=($ rlmin $, \operatorname{lrmax}, \operatorname{lrmin}$, rlmax $) \pi^{c}$.

Based on equations (4.9), (4.10), Lemma 4.1 and Lemma 4.6, we deduce that

$$
(\mathrm{rlmax}, \operatorname{lrmax}, \mathrm{rlmin}) \pi=(\mathrm{rlmax}, \mathrm{rlmin}, \operatorname{lrmax}) \alpha(\pi),
$$

as desired. This completes the proof.
For a set $X$, let $n-X$ be the set obtained by $n$ minus each element in $X$. We are now ready to prove Theorem 1.5.
Proof of Theorem 1.5. In view of Theorem 1.3, (rlm, $\mathrm{w} \overline{\mathrm{m}}, \mathrm{asc}$ ) is equally distributed with (rlm, rlmin, asc) on $S_{n}$. It is enough to construct a bijection $\beta$ over $S_{n}$ such that

$$
\begin{equation*}
(\mathrm{rlm}, \mathrm{w} \overline{\mathrm{~m}}, \operatorname{asc}) \pi=(\mathrm{rlmax}-1, \mathrm{rlmin}, \operatorname{asc}) \beta(\pi) \tag{4.11}
\end{equation*}
$$

for each $\pi=\pi_{1} \pi_{2} \cdots \pi_{n} \in S_{n}$.
Assume that $\pi=x 1 y$, where $x$ and $y$ can be empty. Let $\operatorname{st}(x)=(\bar{x}, X)$ and $\operatorname{st}(y)=(\bar{y}, Y)$. Then set $\beta(\pi)=\sigma$, where $\sigma=y^{\prime} n x^{\prime}, x^{\prime}=s t^{-1}(\alpha(\bar{x}), n+$ $1-X)$ and $y^{\prime}=s t^{-1}\left(\bar{y}^{r c}, n+1-Y\right)$.

To show that $\beta$ is a bijection, it suffices to construct its inverse. Given $\sigma=w n v \in S_{n}$, where $w$ and $v$ can be empty. Let $\operatorname{st}(w)=(\bar{w}, W)$ and $s t(v)=(\bar{v}, V)$. Then set $\delta(\sigma)=\pi$, where $\pi=v^{\prime} 1 w^{\prime}, v^{\prime}=s t^{-1}(\alpha(\bar{v}), n+1-V)$ and $w^{\prime}=s t^{-1}\left(\bar{w}^{r c}, n+1-W\right)$. Notice that $\alpha$ is an involution, $\delta$ is the inverse of $\beta$. Hence, $\beta$ is a bijection.

In the following, we proceed to prove (4.11). Notice that $\operatorname{rlm}(\pi)=$ $\operatorname{rlmax}(x)$ and $\operatorname{rlmax}(\sigma)=\operatorname{rlmax}\left(x^{\prime}\right)+1$. By (4.8), we have $\operatorname{rlmax}(x)=$ $\operatorname{rlmax}\left(x^{\prime}\right)$. It follows that $\operatorname{rlm}(\pi)=\operatorname{rlmax}(\sigma)-1$.

To prove $\mathrm{w} \overline{\mathrm{m}}(\pi)=\operatorname{rlmin}(\sigma)$, it is enough to show that $\operatorname{lrmax}(\pi)=$ $\operatorname{rlmin}(\sigma)$ in view of Lemma 2.1. Let $\operatorname{lrmax}_{>s}(u)$ be the number of LR-maxima of the word $u$ which are larger than $s$, and $\operatorname{rlmin}_{<s}(u)$ be the number of RLminima of the word $u$ which are smaller than $s$. We consider the following two cases.

- x is empty. Thus $\pi=1 y$. It follows that $\operatorname{lrmax}(\pi)=1+\operatorname{lrmax}(y)$ and $\operatorname{rlmin}(\sigma)=1+\operatorname{rlmin}\left(y^{\prime}\right)$. Clearly, $\operatorname{lrmax}(y)=\operatorname{rlmin}\left(y^{\prime}\right)$. Hence, we have $\operatorname{lrmax}(\pi)=\operatorname{rlmin}(\sigma)$.
- x is not empty. By the block decomposition, we have $\operatorname{lrmax}(\pi)=$ $\operatorname{lrmax}(x)+\operatorname{lrmax}_{>\max (X)}(y)$ and $\operatorname{rlmin}(\sigma)=\operatorname{rlmin}\left(x^{\prime}\right)+\operatorname{rlmin}_{<\min (n+1-X)}\left(y^{\prime}\right)$. Since $\operatorname{lrmax}(x)=\operatorname{rlmin}\left(x^{\prime}\right)$ and $\operatorname{lrmax}_{>\max (X)}(y)=\operatorname{rlmin}_{<\min (n+1-X)}\left(y^{\prime}\right)$, then $\operatorname{lrmax}(\pi)=\operatorname{rlmin}(\sigma)$ follows.

Finally, we notice that $\operatorname{asc}(\pi)=\operatorname{asc}(x)+1+\operatorname{asc}(y)$ and $\operatorname{asc}(\sigma)=\operatorname{asc}\left(x^{\prime}\right)+$ $1+\operatorname{asc}\left(y^{\prime}\right)$. Since $\operatorname{asc}(x)=\operatorname{asc}\left(x^{\prime}\right)$ and $\operatorname{asc}(y)=\operatorname{asc}\left(y^{\prime}\right)$, we deduce that $\operatorname{asc}(\pi)=\operatorname{asc}(\sigma)$. This completes the proof.

Example 4.7. Let $\pi=593721684$, then $n=9, x=59372$ and $y=$ 684. $(\bar{x}, X)=\mathrm{st}^{-1}(x)=(35241,\{2,3,5,7,9\})$ and $(\bar{y}, Y)=\mathrm{st}^{-1}(y)=$ (231, $\{4,6,8\}) . \alpha(\bar{x})=51342$ can be obtained as follows

$$
35241 \xrightarrow{c} 31425 \xrightarrow{i} 24135 \xrightarrow{b} 00210 \xrightarrow{\gamma} 00102 \xrightarrow{b^{-1}} 14352 \xrightarrow{i} 15324 \xrightarrow{c} 51342 .
$$

Then, $x^{\prime}=\operatorname{st}^{-1}(51342,\{1,3,5,7,8\})=81573, y^{\prime}=\operatorname{st}^{-1}(312,\{2,4,6\})=624$ and $\sigma=\beta(\pi)=624981573$. It is easy to check that $\operatorname{rlm}(\pi)=\operatorname{rlmax}(\sigma)-1=$ 3 , $\mathrm{w} \overline{\mathrm{m}}(\pi)=\operatorname{rlmin}(\sigma)=2$ and $\operatorname{asc}(\pi)=\operatorname{asc}(\sigma)=4$.

## Acknowledgement

We wish to thank the referees for valuable suggestions. The author was supported by the National Natural Science Foundation of China (No. 11701420) and the Natural Science Foundation Project of Tianjin Municipal Education Committee (No. 2017KJ243, No. 2018KJ193).

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