Block decomposition and statistics arising from permutation tableaux

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Abstract

Permutation statistics $w\overline{m}$ and rlm are both arising from permutation tableaux. $w\overline{m}$ was introduced by Chen and Zhou, which was proved equally distributed with the number of unrestricted rows of a permutation tableau. While rlm is shown by Nadeau equally distributed with the number of 1's in the first row of a permutation tableau.

In this paper, we investigate the joint distribution of $w\overline{m}$ and rlm. Statistic (rlm, $w\overline{m}$, rlmin, des, (<u>321</u>)) is shown equally distributed with (rlm, rlmin, $w\overline{m}$, des, (<u>321</u>)) on S_n . Then the generating function of (rlm, $w\overline{m}$) follows. An involution is constructed to explain the symmetric property of the generating function. Also, we study the triple statistic ($w\overline{m}$, rlm, asc), which is shown to be equally distributed with (rlmax -1, rlmin, asc) as studied by Josuat-Vergès. The main method we adopt throughout the paper is constructing bijections based on a block decomposition of permutations.

Keywords: bijection, involution, permutation tableaux, block decomposition, pattern

1 Introduction

In this paper, we mainly investigate two permutation statistics $w\overline{m}$ and rlm which are arising from permutation tableaux.

Permutation tableaux were introduced by Steingrímsson and Williams [19]. They are related to the enumeration of totally positive Grassmannian cells [15, 17, 18, 20] and a statistical physics model called Partially Asymmetric Exclusion Process (PASEP) [5, 8, 9, 10, 11, 13]. Several papers on the combinatorics of permutation tableaux have also been published, see [2, 3, 4, 6, 7, 12, 16].

A *permutation tableau* is a Ferrers diagram with possibly empty rows together with a 0, 1-filling of the cells satisfying the following conditions:

1. each column has at least one 1,

2. there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row.

The *length* of a permutation tableau is defined to be the number of rows plus the number of columns. Let $\mathcal{PT}(n)$ denote the set of permutations of length n.

Several statistics over permutation tableaux are defined, among which urr and topone are two interesting ones. A 0 in a permutation tableau is *row-restricted* if there is a 1 above in the same column. A row is said to be *unrestricted* if it contains no row-restricted 0. For $T \in \mathcal{PT}(n)$, as given by Corteel and Kim [6], let urr(T) be the number of unrestricted rows of T and let topone(T) be the number of 1's in the first row of T. Using recurrence relations, Corteel and Nadeau [7] obtained an explicit formula for the generating function of permutation tableaux of length n with respect to the statistics urr and topone. Corteel and Kim [6] rewrote this formula as follows

$$\sum_{T \in \mathcal{PT}(n)} x^{\operatorname{urr}(T) - 1} y^{\operatorname{topone}(T)} = (x + y)_{n - 1}, \tag{1.1}$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ for $n \ge 1$ with $(x)_0 = 1$. Moreover, they gave two beautiful bijective proofs of (1.1).

Permutation tableaux are in bijections with permutations. Let $[n] = \{1, 2, ..., n\}$ and S_n be the set of permutations on [n]. Steingrímsson and Williams [19] gave a Zig-Zag map Φ from $\mathcal{PT}(n)$ to S_n . Given $T \in \mathcal{PT}(n)$, we label T as follows. First, label the steps in the south-east border with $1, 2, \dots, n$ from north-east to south-west. Then, label a row (resp. column) with i if the row contains the south (resp. west) step with label i. A zigzag path on a permutation tableau is a path entering from the left of a row or the top of a column, going to the east or to the south changing the direction alternatively whenever it meets a 1 until exiting the tableau. Let $\pi = \pi_1 \pi_2 \cdots \pi_n = \Phi(T)$, where $\pi_i = j$ if the zigzag path corresponding to label i exits T from a row or a column labeled by j. As an example, $\Phi(\pi) = 8, 6, 1, 5, 3, 4, 9, 2, 7, 11, 10$ for π given in the left of Figure 1.1.

Corteel and Nadeau [7] found another two bijections between permutation tableaux and permutations, one of which we denote by Γ is given depend on the alternative representation of permutation tableaux. As given in Corteel and Kim [6], the alternative representation of a permutation tableau T is the diagram obtained from T by replacing the topmost 1's by \uparrow 's and the rightmost restricted 0's by \leftarrow 's and removing the remaining 0's and 1's, see



Figure 1.1: A permutation tableau (left) and its corresponding alternative representation (right).

Figure 1.1 as an example. Given $T \in \mathcal{PT}(n)$, $\pi = \pi_1 \pi_2 \cdots \pi_n = \Gamma(T)$ can be obtained as follows

1. write down the labels of the unrestricted rows of T in increasing order,

2. for each column *i* from left to right, if row *j* contains a \uparrow in column *i* and $i_1, i_2, \dots, i_r (i_1 < \dots < i_r)$ contains a \leftarrow in column *i*, then add i_1, i_2, \dots, i_r, i in increasing order before *j* in π .

As an example, $\Gamma(\pi) = 9, 4, 6, 5, 2, 8, 3, 1, 7, 11, 10$ for π given in the right of Figure 1.1.

Statistics \overline{wm} and rlm on permutations are closely related to statistics urr and topone by Φ and Γ . We present the definitions of \overline{wm} and rlm first. Given $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, the index *i* is said to be a *weak excedance* of π if $\pi_i \geq i$. Otherwise, it is called a *non-weak excedance*. An index *i* is called a *mid-point* of π if there exist j < i and k > i such that $\pi_j > \pi_i > \pi_k$. Otherwise, *i* is called a *non-mid-point*. Let $\overline{wm}(\pi) = |WM(\pi)|$ and

 $W\overline{M}(\pi) = \{\pi_i | i \text{ is a weak excedance and a non-mid-point of } \pi \}.$

For a word $w = w_1 w_2 \cdots w_n$ of distinct integers, w_i is called a *RL-maximum* of w if $w_i > w_j$ for all j > i. While w_i is called a *LR-maximum*, if $w_i > w_j$ for all j < i. The *RL*-minimum and *LR*-minimum can be defined similarly. Let Rlmax(w), Lrmax(w), Rlmin(w) and Lrmin(w) be the set of *RL*maxima, *LR*-maxima, *RL*-minima and *LR*-minima of w, respectively. Set rlmax(w), lrmax(w), rlmin(w) and lrmin(w) to be the corresponding numerical statistics. Let $\text{Rlm}(\pi)$ be the set of the RL-maxima of the subword of π which is to the left of the 1 in π . Write $\text{rlm}(\pi) = |\text{Rlm}(\pi)|$. As an example, for $\pi = 6, 5, 1, 10, 4, 3, 8, 9, 2, 11, 7, 12$, we have $W\overline{M}(\pi) = \{6, 10, 11, 12\}$, $w\overline{m}(\pi) = 4$, $\text{Rlm}(\pi) = \{5, 6\}$ and $\text{rlm}(\pi) = 2$.

Corteel and Nadeau [7], Nadeau [16], Chen and Zhou [3] proved the first, the second and the third item in the following proposition, respectively.

Proposition 1.1. For $T \in \mathcal{PT}(n)$, let $\Gamma(T) = \pi$ and $\Phi(T) = \sigma$, then

- 1. $\operatorname{urr}(T) = \operatorname{rlmin}(\pi);$
- 2. topone(T) = $\operatorname{rlm}(\pi)$;
- 3. $\operatorname{urr}(T) = w\overline{m}(\sigma)$.

It can be checked that $\operatorname{topone}(T) \neq \operatorname{rlm}(\Phi(T))$. So it is interesting to investigate the distributions of $(\operatorname{rlm}, \operatorname{w\overline{m}})$. On the other hand, from the perspective of permutations, we see that the definition of rlm is closely related to rlmax, while $\operatorname{w\overline{m}}$ is indeed the statistic lrmax (proved in Lemma 2.1). rlmax and lrmax are interesting Stirling statistics over permutations and have been widely studied. So this is another motivation of our work. More definitions and notations needed in this paper are listed as follows.

Given $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, its reverse $\pi^r \in S_n$ is given by $\pi^r(i) = \pi(n+1-i)$. Its complement π^c is given by $\pi^c(i) = n+1-\pi(i)$. Let π^{-1} denote the inverse of π , where $\pi^{-1}(j) = i$ if and only if $\pi(i) = j$. For convenient, we also write $c(\pi) = \pi^c$ and $i(\pi) = \pi^{-1}$. Assume that $W = \{i_1, i_2, \ldots, i_n\}$ with $i_1 < i_2 < \cdots < i_n$. Given a permutation w of W, we define st $(w) = (\sigma, W)$, where $\sigma \in S_n$ and σ is order-isomorphic to w. Conversely, set $st^{-1}(\sigma, W) = w$. As an example, $st(31574) = (21453, \{1, 3, 4, 5, 7\})$ and $st^{-1}(21453, \{1, 3, 4, 5, 7\}) = 31574$.

A descent (ascent) of π is a position $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$ ($\pi_i < \pi_{i+1}$). The descent set and the ascent set of π are given by $\text{Des}(\pi) = \{i: \pi_i > \pi_{i+1}\}$ and $\text{Asc}(\pi) = \{i: \pi_i < \pi_{i+1}\}$. Let $\text{des}(\pi) = |\text{Des}(\pi)|$ and $\text{asc}(\pi) = |\text{Asc}(\pi)|$ be the descent number and ascent number of π , respectively. Set $\text{ides}(\pi) = \text{des}(\pi^{-1})$.

An occurrence of a classical *pattern* p in a permutation σ is a subsequence of σ that is order-isomorphic to p. For instance, 41253 has two occurrences of the pattern 3142 in its subsequences 4153 and 4253. σ is said to *avoid* p if there exists no occurrence of p in σ . The *vincular pattern* is a generalization of the classical pattern. Adjacent letters that are underlined must stay adjacent when they are placed back to the original permutation. As an example, 41253 now contains only one occurrence of the vincular pattern <u>3142</u> in its subsequence 4153, but not in 4253. See [14] for more details about vincular patterns. Given a vincular pattern τ and a permutation π , we denote by $(\tau)\pi$ the number of occurrences of the pattern τ in π . We write $S_n(\tau)$ as the set of permutations of length n that avoid τ .

An inversion sequence of length n is a word $s = s_1 s_2 \cdots s_n$ with $0 \le s_i \le i - 1$ for $1 \le i \le n$. Let I_n be the set of all inversion sequences of length n.

Assume that

Zero(s) =
$$\{i : 1 \le i \le n, s_i = 0\},$$

Max(s) = $\{i : 1 \le i \le n, s_i = i - 1\},$
Dist(s) = $\{2 \le i \le n : s_i \ne 0 \text{ and } s_i \ne s_j \text{ for all } j > i\},$

and $\operatorname{zero}(s)$, $\max(s)$ and $\operatorname{dist}(s)$ is the numerical statistics, respectively.

In this paper, we find that (rlm, wm) and (rlm, rlmin) are equally distributed on $S_n(321)$, as well as on S_n . Particularly, we have the following theorems.

Theorem 1.2. Statistic (rlm, rlmin, wm, des, ides) are equally distributed with statistic (rlm, wm, rlmin, des, ides) over $S_n(321)$.

Theorem 1.3. Statistic (rlm, rlmin, \overline{wm} , des, (<u>321</u>))) are equally distributed with statistic (rlm, \overline{wm} , rlmin, des, (<u>321</u>))) on S_n , and hence we have

$$\sum_{\pi \in S_n} x^{\mathrm{w}\overline{\mathbf{m}}(\pi) - 1} y^{\mathrm{rlm}(\pi)} = (x + y)_{n-1}.$$
 (1.2)

Notice that x and y are symmetric in (1.2). We have the following theorem, which we will reprove by an involution over S_n .

Theorem 1.4. Statistics (rlm, $\overline{wm} - 1$) and ($\overline{wm} - 1$, rlm) are equally distributed on S_n .

Josuat-Vergès [13] showed that

$$Z_N = \sum_{\pi \in S_{N+1}} \alpha^{-\operatorname{rlmax}(\pi)+1} \beta^{-\operatorname{rlmin}(\pi)+1} y^{\operatorname{asc}(\pi)-1} q^{(\underline{31}2)\pi},$$

where Z_N is the partition function of a partially asymmetric exclusion process (PASEP) on a finite number of sites with open and directed boundary conditions, see Theorem 1.3.1 in [13]. Inspired by this, we obtain the following equidistribution by constructing bijections.

Theorem 1.5. Statistics (rlm, \overline{wm} , asc) and (rlm, rlmin, asc) are equally distributed with (rlmax -1, rlmin, asc) on S_n .

The paper is organized as follows. In Section 2, we present bijective proofs of Theorem 1.2 and 1.3. In Section 3, we construct an involution which implies Theorem 1.4. In section 4, we bijectively prove Theorem 1.5 by using inversion sequences.

2 Bijective proofs of Theorem 1.2 and 1.3

In this section, we first deduce that statistic $w\overline{m}$ is indeed statistic lrmax. Then, two bijections based on a block decomposition of permutations are given which imply Theorem 1.2 and 1.3, respectively.

Lemma 2.1. Given $\pi \in S_n$, $\pi_i \in W\overline{M}(\pi)$ if and only if π_i is a LR-maximum of π .

Proof. Suppose that $\pi_i \in W\overline{M}(\pi)$, we claim that π_i is a LR-maximum of π . Assume to the contrary that there exists j < i such that $\pi_j > \pi_i$. Since π_i is a weak excedance of π , it is easily checked that there exists k > i such that $\pi_k < \pi_i$. Hence, $\pi_j \pi_i \pi_k$ is a 321-pattern of π . It follows that π_i is a mid-point of π , which contradicts with the fact that $\pi_i \in W\overline{M}(\pi)$. The claim is verified.

Conversely, assume that π_i is a LR-maximum of π . Since it is larger that all the elements to its left, then $\pi_i \geq i$ and π_i is a non-mid-point. It follows that $\pi_i \in W\overline{M}(\pi)$. This completes the proof.

Now, we proceed to prove Theorem 1.2. As pointed out by Burstein [2], 321-avoiding permutations have the property given in Lemma 2.2. Based on this, Corollary 2.3 follows obviously.

Lemma 2.2. π is 321-avoiding if and only if each element of π is either a LR-maximum or an RL-minimum, i.e. if and only if π is identity or a union of two nondecreasing subsequences.

Corollary 2.3. Given $\pi \in S_n(321)$, if $\pi_1 = 1$, then $\operatorname{rlm}(\pi) = 0$. Otherwise, $\operatorname{rlm}(\pi) = 1$.

The following lemma can be easily checked. And then, we are prepared to give a proof of Theorem 1.2.

Lemma 2.4. Assume that $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n(321)$ with $\pi_1 \neq 1$ and $\pi_n \neq n$. *n.* Let $\sigma = \pi^{rc}$, then $\operatorname{rlm}(\pi) = \operatorname{rlm}(\sigma) = 1$ and

 $(\text{rlmin}, \text{lrmax}, \text{des}, \text{ides})\pi = (\text{lrmax}, \text{rlmin}, \text{des}, \text{ides})\sigma.$

Proof of Theorem 1.2. It suffices to construct a bijection χ over $S_n(321)$, which maps (rlm, rlmin, lrmax, des, ides) to (rlm, lrmax, rlmin, des, ides).

Assume that $\pi = \pi_1 \pi_2 \cdots \pi_n$ is 321-avoiding. If $\pi_1 = 1$, we set $l = \max\{j \mid \pi_1 = 1, \cdots, \pi_j = j\}$, otherwise, l = 0. If $\pi_n = n$, we set $r = \min\{j \mid \pi_j = j, \cdots, \pi_n = n\}$, otherwise, r = n + 1.

Let $p = p_1 \cdots p_n = \chi(\pi)$ be the permutation $1 \ 2 \cdots l \ h \ r \ r + 1 \cdots n$, where $h = \operatorname{st}((n+1-\pi_{r-1})(n+1-\pi_{r-2})\cdots(n+1-\pi_{l+1}), \{l+1, \cdots, r-1\}).$

It should be noted that $12 \cdots l$ is assumed to be empty if l = 0, while $rr + 1 \cdots n$ is assumed to be empty if r = n + 1. Clearly, we have $p \in S_n(321)$. Based on Lemma 2.4, it can be easily verified that

 $(\text{rlmin}, \text{lrmax}, \text{des}, \text{ides})(\pi) = (\text{lrmax}, \text{rlmin}, \text{des}, \text{ides})(p).$

By Corollary 2.3, we have $\operatorname{rlm}(p) = \operatorname{rlm}(\pi) = 0$ if l = 0, while $\operatorname{rlm}(p) = \operatorname{rlm}(\pi) = 1$ if l > 0. This completes the proof.

Example 2.5. Let $\pi = 123468759$, then l = 4, r = 9 and $h = \text{st}(5324, \{5, 6, 7, 8\}) = 8657$. Thus we have $p = \chi(\pi) = 123486579$. It is easy to check that $\text{rlmin}(\pi) = \text{lrmax}(p) = 6$, $\text{lrmax}(\pi) = \text{rlmin}(p) = 7$, $\text{des}(\pi) = \text{des}(p) = 2$ and $\text{ides}(\pi) = \text{ides}(p) = 2$.

In the remaining part of this section, we are dedicated to proving Theorem 1.3. Given a permutation π , put a bar after each RL-minimum, and then put a bar before each LR-maximum if there is no bar before it. Thus we obtain a block decomposition of π . Write the block decomposition of π as $B_1B_2\cdots B_k$, we define

- $N(\pi) = \{B_i | B_i \text{ contains neither LR-maximum nor RL-minimum}\},\$
- $T(\pi) = \{B_i | B_i \text{ contains both LR-maximum and RL-minimum}\},\$
- $A(\pi) = \{B_i | B_i \text{ contains a LR-maximum and no RL-minimum}\},\$

 $I(\pi) = \{B_i | B_i \text{ contains an RL-minimum and no LR-maximum}\}.$

For convenience, we call a block in $N(\pi)$ a N-block. The T-block, A-block and I-block are defined similarly. Propositions of the block decomposition below can be easily verified.

Proposition 2.6. For any $\pi \in S_n$, write $\pi = B_1 B_2 \cdots B_k$, we have

- 1. $|T(\pi)| \ge 1$.
- 2. $N(\pi) \cup T(\pi) \cup A(\pi) \cup I(\pi) = \bigcup_{1 \le i \le k} \{B_i\}.$
- 3. If $B_i \in N(\pi)$, then there exist integers j < i and h > i such that $B_j \in T(\pi)$, $B_h \in T(\pi)$, $\{B_{j+1}, \cdots, B_{i-1}\} \subset I(\pi)$ and $\{B_{i+1}, \cdots, B_{h-1}\} \subset A(\pi)$.
- 4. Let $T(\pi) \cup A(\pi) = \{B_{x_1}, \cdots, B_{x_h}\}, \text{ where } x_1 < \cdots < x_h, \text{ then }$

$$\max(B_{x_1}) < \cdots < \max(B_{x_h})$$

5. Let
$$T(\pi) \cup I(\pi) = \{B_{x_1}, \cdots, B_{x_h}\}, \text{ where } x_1 < \cdots < x_h, \text{ then}$$
$$\min(B_{x_1}) < \cdots < \min(B_{x_h}).$$

In the following, we define two operations on permutations. Given $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, assume that $\min(\pi) = \pi_i$ and $\max(\pi) = \pi_j$, let

$$L(\pi) = \pi_{i+1} \cdots \pi_n \pi_1 \cdots \pi_i,$$

$$R(\pi) = \pi_j \cdots \pi_n \pi_1 \cdots \pi_{j-1}.$$

Proposition 2.7. For $\pi = \pi_1 \cdots \pi_n \in S_n$, we have

- 1. $R \circ L(\pi) = \pi$ if and only if $\max(\pi) = \pi_1$.
- 2. $L \circ R(\pi) = \pi$ if and only if $\min(\pi) = \pi_n$.
- 3. If $\max(\pi) = \pi_1$, then $(\underline{321})(\pi) = (\underline{321})(L(\pi))$.
- 4. If $\min(\pi) = \pi_n$, then $(\underline{321})(\pi) = (\underline{321})(R(\pi))$.

Now we are ready to present the map φ over S_n such that for any $\pi \in S_n$

 $(\operatorname{rlm}, \operatorname{rlmin}, \operatorname{w\overline{m}}, \operatorname{des}, (\underline{321}))\pi = (\operatorname{rlm}, \operatorname{w\overline{m}}, \operatorname{rlmin}, \operatorname{des}, (\underline{321}))\varphi(\pi).$

Let $\pi = B_1 B_2 \cdots B_k \in S_n$ and assume that

$$N(\pi) = \{B_{N_1}, \cdots, B_{N_h}\}, \ T(\pi) = \{B_{T_1}, \cdots, B_{T_l}\}, A(\pi) = \{B_{A_1}, \cdots, B_{A_p}\}, \ I(\pi) = \{B_{I_1}, \cdots, B_{I_q}\}.$$
(2.1)

If $N(\pi) = \emptyset$, then we may view h = 0. It is similar for $A(\pi)$ and $I(\pi)$. We can obtain $\sigma = \varphi(\pi)$ through the following there steps:

- **Step 1** Write down the blocks in $N(\pi)$ and $T(\pi)$, which keeps the relative order in π , we obtain σ' ;
- Step 2 Insert $R(B_{I_1}), \dots, R(B_{I_q})$ to σ' by letting the maximal letter (i.e. the first letter) of $R(B_{I_1}), \dots, R(B_{I_q}), B_{T_1}, \dots, B_{T_l}$ increase. Between two T-blocks, $R(B_{I_c})(1 \le c \le q)$ is always to the right of a N-block, if there is any. Then we obtain σ'' ;
- Step 3 Insert $L(B_{A_1}), \dots, L(B_{A_p})$ to σ'' by letting the minimal letter (i.e. the last letter) of $L(B_{A_1}), \dots, L(B_{A_p}), B_{T_1}, \dots, B_{T_l}$ increase. Between two T-blocks, $L(B_{A_d})(1 \le d \le p)$ is always to the left of a N-block and $R(B_{A_c})(1 \le c \le q)$, if there is any. Then we obtain σ .

Example 2.8. Let $\pi = 10, 2, 6, 11, 1, 8, 13, 3, 5, 9, 4, 12, 7$, then the block decomposition of π is

 $\begin{vmatrix} 10 & 2 & 6 & | 11 & 1 & | & 8 & | 13 & 3 & | & 5 & 9 & 4 & | & 12 & 7 \end{vmatrix}$

and $N(\pi) = \{8\}, T(\pi) = \{11\ 1, 13\ 3\}, A(\pi) = \{10\ 2\ 6\}, I(\pi) = \{5\ 9\ 4, 12\ 7\}.$ By the three steps given above, we have

$$\begin{aligned} \sigma' &= 11\ 1,8,13\ 3, \quad \sigma'' = 9\ 4\ 5,11\ 1,8,12\ 7,13\ 3 \\ \sigma &= 9\ 4\ 5,11\ 1,6\ 10\ 2,8,12\ 7,13\ 3 \end{aligned}$$

Proposition 2.9. Let $\sigma = \varphi(\pi)$, we have

(1) $T(\sigma) = \{B_{T_1}, \cdots, B_{T_l}\};$ (2) $A(\sigma) = \{R(B_{I_1}), \cdots, R(B_{I_q})\};$ (3) $I(\sigma) = \{L(B_{A_1}), \cdots, L(B_{A_p})\};$ (4) $N(\sigma) = \{B_{N_1}, \cdots, B_{N_h}\}.$

Proof. Firstly, we wish to show that the first letter of $B_{T_j}(\text{i.e.max}\{B_{T_j}\})$, where $1 \leq j \leq l$, is a LR-maximum of σ , while the last letter of B_{T_j} (i.e.min $\{B_{T_j}\}$) is an RL-minimum of σ . By definition of step 1 in the description of φ , we easily check that the first letter of B_{T_j} is a LR-maximum of σ' . Since the maximal letter $R(B_{I_1}), \dots, R(B_{I_q}), B_{T_1}, \dots, B_{T_l}$ increase in step 2, we have the first letter of B_{T_j} is a LR-maximum of σ'' . Assume that $L(B_{A_i})$ is to the left of B_{T_j} in σ , then we have min $\{B_{A_i}\} < \min\{B_{T_j}\}$. It follows that B_{A_i} is to the left of B_{T_j} in π , which means that max $\{B_{A_i}\} < \max\{B_{T_j}\}$. Above all, the first letter of B_{T_j} is a LR-maximum of σ .

Now we proceed to show that the last letter of B_{T_j} is an RL-minimum of σ . Clearly, min $\{B_{T_j}\}$ is an RL-minimum of σ' . the maximal letter (i.e. the first letter) of $R(B_{I_1}), \dots, R(B_{I_q}), B_{T_1}, \dots, B_{T_l}$ increase.

Assume that $R(B_{I_i})$ is to the right of B_{T_j} in σ , then $\max\{B_{T_j}\} < \max\{B_{I_i}\}$. It means that B_{I_i} is to the right of B_{T_j} in π . Hence, $\min\{B_{I_i}\} > \min\{B_{T_j}\}$. Assume that $L(B_{A_i})$ is to the right of B_{T_j} in σ , then by the definition of step 3, it is easily seen that $\min\{B_{A_i}\} > \min\{B_{T_j}\}$. Hence, the last letter of B_{T_j} is an RL-minimum of σ , as desired.

Secondly, we need to show that the first letter of $R(B_{I_j})$ (i.e.max $\{B_{I_j}\}$), where $1 \leq j \leq q$, is a LR-maximum of σ and it contains no RL-minimum of σ . Clearly, the first letter of $R(B_{I_j})$ is a LR-maximum of σ'' . Assume that $L(B_{A_i})$ is to the left of $R(B_{I_j})$ in σ , we wish to prove that max $\{B_{A_i}\} <$ max $\{B_{I_j}\}$. Let B_{T_x} be the nearest T-block that is to the right of B_{A_i} in π , then we have $\max\{B_{A_i}\} < \max\{B_{T_x}\}$ and $\min\{B_{A_i}\} > \min\{B_{T_x}\}$. By step 3 in the description of φ , B_{T_x} is to the left of $L(B_{A_i})$, and hence to the left of $R(B_{I_j})$ in σ . It follows that $\max\{B_{T_x}\} < \max\{B_{I_j}\}$. Thus, $\max\{B_{A_i}\} < \max\{B_{I_j}\}$. Hence, the first letter of $R(B_{I_j})$ is a LR-maximum of σ .

Let B_{T_y} be the nearest T-block that is to the left of B_{I_j} in π , then we have $\max\{B_{I_j}\} < \max\{B_{T_y}\}$ and $\min\{B_{I_j}\} > \min\{B_{T_y}\}$. By step 2, B_{T_y} is to the right of $R(B_{I_j})$ in σ . It follows from the fact $\min\{B_{I_j}\} > \min\{B_{T_y}\}$ that $R(B_{I_j})$ contains no RL-minimum of σ , as desired.

Thirdly, we wish to show that the last letter of $L(B_{A_j})(1 \leq j \leq p)$ is an RL-minimum of σ and it contains no LR-maximum of σ . By description of step 3 in φ , the last letter of $L(B_{A_j})$ (i.e.min $\{B_{A_j}\}$) is smaller than all letters of T-blocks and N-blocks which are to the right of it. Now we assume that $R(B_{I_i})$ is to the right of $L(B_{A_j})$ in σ , if there is any, we aim to show that min $\{B_{I_i}\} > \min\{B_{A_j}\}$. Let B_{T_x} be the nearest T-block that is to the left of B_{I_i} in π . Thus, we have max $\{B_{I_i}\} < \max\{B_{T_x}\}$ and min $\{B_{I_i}\} >$ min $\{B_{T_x}\}$. It follows from description of step 2 that B_{T_x} is to the right of $R(B_{I_i})$ in σ , and hence to the right of $L(B_{A_j})$. Thus, by step 3, we see that min $\{B_{A_j}\} < \min\{B_{T_x}\}$. Hence, min $\{B_{I_i}\} > \min\{B_{A_j}\}$ and the last letter of $L(B_{A_j})(1 \leq j \leq p)$ is an RL-minimum of σ .

Let B_{T_y} be the nearest *T*-block that is to the right of B_{A_j} in π , then $\max\{B_{A_j}\} < \max\{B_{T_y}\}$ and $\min\{B_{A_j}\} > \min\{B_{T_y}\}$. By step 3, B_{T_y} is to the left of $L(B_{A_j})$ in σ . Then, B_{A_j} contains no LR-maximum follows from the fact that $\max\{B_{A_j}\} < \max\{B_{T_y}\}$, as desired.

Notice that $B_{N_j}(1 \le j \le h)$ contains no RL-minimum nor LR-maximum of σ . By all the analysis above, we may obtain a block decomposition of σ and propositions (1) - (4) follows. This completes the proof.

Proof of Theorem 1.3. Let $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$ with a block decomposition given in (2.1) and $\sigma = \varphi(\pi)$. It suffices to show that φ is an involution over S_n such that

 $(\operatorname{rlm}, \operatorname{rlmin}, \operatorname{lrmax}, \operatorname{des}, (\underline{321}))(\pi) = (\operatorname{rlm}, \operatorname{lrmax}, \operatorname{rlmin}, \operatorname{des}, (\underline{321}))(\sigma).$ (2.2)

Firstly, we prove that φ is an involution, i.e. $\varphi(\sigma) = \pi$. Assume that $p = \varphi(\sigma)$, by applying Proposition 2.9 twice, we have

$$N(p) = \{B_{N_1}, \cdots, B_{N_h}\},\$$

$$T(p) = \{B_{T_1}, \cdots, B_{T_l}\},\$$

$$A(p) = \{R \circ L(B_{A_1}), \cdots, R \circ L(B_{A_p})\},\$$

$$I(p) = \{L \circ R(B_{I_1}), \cdots, L \circ R(B_{I_q})\}.\$$

Notice that $\max\{B_{A_c}\}$ is the first letter of B_{A_c} for $1 \le c \le p$, while $\min\{B_{I_d}\}$ is the last letter of B_{I_d} for $1 \le d \le q$. Then from items 1, 2 in Proposition 2.7 we deduce that

$$N(p) = \{B_{N_1}, \cdots, B_{N_h}\}, \ T(p) = \{B_{T_1}, \cdots, B_{T_l}\}, A(p) = \{B_{A_1}, \cdots, B_{A_p}\}, \ I(p) = \{B_{I_1}, \cdots, B_{I_q}\}.$$

Comparing with the block decomposition of π given in (2.1), we see that $p = \pi$. Hence $\varphi^2(\pi) = \pi$ and φ is an involution over S_n .

Now, we proceed to prove (2.2). Viewing (2.1) and Proposition 2.9, we have $\operatorname{lrmax}(\pi) = \operatorname{rlmin}(\sigma) = p$ and $\operatorname{rlmin}(\pi) = \operatorname{lrmax}(\sigma) = q$. If $\pi_1 = 1$, then it is easy to check that $\sigma_1 = 1$. Hence, we have $\operatorname{rlm}(\pi) = \operatorname{rlm}(\sigma) = 0$. Otherwise, suppose that $\operatorname{Rlm}(\pi) = \{\pi_{r_1}, \pi_{r_2}, \cdots, \pi_{r_s}\}$, where $r_1 < r_2 < \cdots < r_s$. Then, π_{r_1} is the nearest LR-maximum of π that is to the left of 1. It follows that $\pi_{r_1} \cdots 1$ is a T-block of π . Hence, $\pi_{r_1} \cdots 1$ remains a T-block of σ by Proposition 2.9. Thus, $\operatorname{Rlm}(\sigma) = \{\pi_{r_1}, \pi_{r_2}, \cdots, \pi_{r_s}\}$ and we obtain that $\operatorname{rlm}(\pi) = \operatorname{rlm}(\sigma) = s$. We claim that all descents of a permutation are always contained in blocks. Assume that i is a descent of π with $\pi_i > \pi_{i+1}$, then π_i is not an RL-minimum and π_{i+1} is not a LR-maximum. Hence there is no bar neither after π_i nor before π_{i+1} . The claim is verified. It follows directly that $\operatorname{des}(\pi) = \operatorname{des}(\sigma)$. Combining with items 3, 4 in Proposition 2.7, we have $(\underline{321})(\pi) = (\underline{321})(\sigma)$. This completes the proof.

It should be mentioned that φ does not keep the statistic ides. We check the following conjecture by computer for $n \leq 9$.

Conjecture 2.10. Statistic (rlm, rlmin, lrmax, des, ides, $(\underline{321})$) are equally distributed with Statistic (rlm, lrmax, rlmin, des, ides, $(\underline{321})$) over S_n .

3 A bijective proof of Theorem 1.4

In this section, we present an involution over S_n to give a combinatorial interpretation of Theorem 1.4. In view of Lemma 2.1, it is enough to prove the following theorem.

Theorem 3.1. There exists an involution ϕ on S_n such that

$$rlm(\pi) = lrmax(\phi(\pi)) - 1,$$
(3.1)

$$\operatorname{lrmax}(\pi) - 1 = \operatorname{rlm}(\phi(\pi)). \tag{3.2}$$

In the following, we shall give such an involution. We first consider some

special cases. Define

$$S_n^1 = \{ \pi = \pi_1 \cdots \pi_n | \pi_n = 1 \},\$$

$$S_n^n = \{ \pi = \pi_1 \cdots \pi_n | \pi_n = n \}.$$

Lemma 3.2. There is a bijection ρ from S_n^n to S_n^1 , such that

$$\operatorname{rlm}(\pi) = \operatorname{lrmax}(\rho(\pi)) - 1, \tag{3.3}$$

$$\operatorname{lrmax}(\pi) - 1 = \operatorname{rlm}(\rho(\pi)). \tag{3.4}$$

Proof. Given a permutation $\pi \in S_n^n$, assume that $\pi_k = 1$ and $\pi = w \ln n$, where w and u can be empty. Let $\pi_i = \max(w)$ and j be the least element such that k < j < n and $\pi_j > \pi_i$, if there exist. Then, assume that a = $\pi_1 \cdots \pi_{i-1}, b = \pi_i \cdots \pi_{k-1}, c = \pi_{k+1} \cdots \pi_{j-1}$ and $d = \pi_j \cdots \pi_{n-1}$ Thus, we decompose π into six blocks, namely, $\pi = ab1cdn$. It should be noted that each of the blocks a, b, c, d can be empty. Define $\rho(\pi)$ to be $\pi' = b^r cnd^r a^r 1$ Clearly, $\pi' \in S_n^1$. It follows that ρ is a map from S_n^n to S_n^1 .

To prove that ρ is a bijection, we give the inverse map of ρ . Given a permutation $\tau \in S_n^1$, let $\tau = pnq1$. Both of p and q can be empty. If there exists, assume that τ_l is the largest element of p. Let τ_s is the rightmost element of q that is larger than τ_l , if there exists. Suppose that $\tau_t = n$ where t < n. We decompose τ into six blocks by setting $\tau = efngh1$, where $e = \tau_1 \cdots \tau_l$, $f = \tau_{l+1} \cdots \tau_{t-1}$, $g = \tau_{t+1} \cdots \tau_s$ and $h = \tau_{s+1} \cdots \tau_{n-1}$. Define $\chi(\tau)$ to be the permutation τ' where $\tau' = h^r e^r 1 f g^r n$. It can be easily checked that χ is the inverse map of ρ . Hence, ρ is a bijection.

Next, we proceed to prove relations (3.3) and (3.4). It is not hard to check that the following relations.

 $\operatorname{rlm}(\pi) = \operatorname{the} \operatorname{number} \operatorname{of} \operatorname{LR-maxima} \operatorname{of} b^r,$ $\operatorname{lrmax}(\pi) - 1 = \operatorname{the} \operatorname{number} \operatorname{of} \operatorname{LR-maxima} \operatorname{of} a\pi_i d,$ $\operatorname{rlm}(\pi') = \operatorname{the} \operatorname{number} \operatorname{of} \operatorname{LR-maxima} \operatorname{of} adn,$ $\operatorname{lrmax}(\pi') - 1 = \operatorname{the} \operatorname{number} \operatorname{of} \operatorname{LR-maxima} \operatorname{of} b^r.$

Notice that the number of LR-maxima of $a\pi_i d$ equals to the number of LR-maxima of adn. Hence relations (3.3) and (3.4) follows, as desired.

Based on Lemma 3.2, we are now ready to give the involution ϕ on S_n . *Proof of Theorem 3.1.* Firstly, we give the description of ϕ . For a permutation $\pi \in S_n$, there are two cases to consider.

Case 1: 1 is to the left of *n*. Assume that $\pi = unv$ and $(e, S) = \operatorname{st}(un)$. Then ϕ is defined by letting $\phi(\pi) = \operatorname{st}^{-1}(\rho(e), S)v$.

Case 2: 1 is to the right of *n*. Assume that $\pi = p1q$ and (o, T) = st(p1). Then ϕ is defined by letting $\phi(\pi) = st^{-1}(\rho^{-1}(o), T)q$.

From the construction of ϕ , it is easily seen that ϕ is an involution on S_n . In the following, we proceed to prove relations (3.1) and (3.2).

By Lemma 3.2, $\operatorname{lrmax}(e) - 1 = \operatorname{rlm}(\rho(e))$ and $\operatorname{rlm}(e) = \operatorname{lrmax}(\rho(e)) - 1$. By order-isomorphic, we deduce that $\operatorname{lrmax}(un) - 1 = \operatorname{rlm}(st^{-1}(\rho(e), S))$ and $\operatorname{rlm}(un) = \operatorname{lrmax}(st^{-1}(\rho(e), S)) - 1$. Notice that in case 1, there is no element z in subword v such that $z \in \operatorname{Rlm}(\pi)$ nor $z \in \operatorname{Lrmax}(\pi)$. Thus, $\operatorname{lrmax}(\pi) - 1 = \operatorname{rlm}(\phi(\pi))$ and $\operatorname{rlm}(\pi) = \operatorname{lrmax}(\phi(\pi)) - 1$ hold for case 1. The fact that (3.1) and (3.2) hold for case 2 can be proved similarly and we omit it here. We complete the proof.

We end this section by giving examples of bijections ρ and ϕ .

Example 3.3. Let $\pi = 372514869$, then

$$a = 3, b = 725, c = 4, d = 86.$$

Hence, $\rho(\pi) = 527496831$. Let $\sigma = 38251496107$, then

 $st(3825149610) = (372514869, \{1, 2, 3, 4, 5, 6, 8, 9, 10\})$

and hence $\phi(\sigma) = 52841069317$.

4 A bijective proof of Theorem 1.5

In this section, we first prove Lemma 4.1 by giving an involution γ over the set of inversion sequences of length n. This allows us to construct a bijection α on S_n implying Lemma 4.2. Based on Lemma 4.2, another bijection β over S_n is given, which proves Theorem 1.5.

Lemma 4.1. Statistics (dist, zero, max, rlmin) and (dist, zero, rlmin, max) are equally distributed over I_n . Particularly, there is an involution γ over I_n such that for each $e \in I_n$ we have

$$(\text{dist}, \text{zero}, \max, \text{rlmin})e = (\text{dist}, \text{zero}, \text{rlmin}, \max)\gamma(e).$$
 (4.1)

Lemma 4.2. Statistics (asc, rlmax, lrmax, rlmin) and (asc, rlmax, rlmin, lrmax) are equally distributed over S_n . Particularly, there is an involution α over S_n such that for each $\pi \in S_n$ we have

$$(\operatorname{asc}, \operatorname{rlmax}, \operatorname{lrmax}, \operatorname{rlmin})\pi = (\operatorname{asc}, \operatorname{rlmax}, \operatorname{rlmin}, \operatorname{lrmax})\alpha(\pi).$$
 (4.2)

To prove Lemma 4.1, we construct γ over I_n by induction. Let $\gamma(0) = 0$. For $e = e_1 e_2 \cdots e_{n-1} e_n \in I_n$, assume that $r' = \gamma(e_1 e_2 \cdots e_{n-1})$. Then, $r = \gamma(e)$ is obtained by inserting e_n to the $e_n + 1$ -th position of r'.

Example 4.3. Let e = 00113213, then $\gamma(e)$ can be obtained as follows

 $0 \rightarrow 00 \rightarrow 010 \rightarrow 0110 \rightarrow 01130 \rightarrow 012130 \rightarrow 0112130 \rightarrow 01132130.$

And $\gamma^2(e) = \gamma(01132130)$ can be obtained as follows

 $0 \rightarrow 01 \rightarrow 011 \rightarrow 0113 \rightarrow 01213 \rightarrow 011213 \rightarrow 0113213 \rightarrow 00113213.$

Clearly, γ is well-defined and we can easily verify the following propositions.

Proposition 4.4. Let $e = e_1 e_2 \cdots e_n \in I_n$ and $r = r_1 r_2 \cdots r_n = \gamma(e)$. Then

- (1) $e_n + 1$ is the largest element in Max(r).
- (2) Assume that j is the largest element in Max(e), then

$$r = \gamma(e_1 \cdots e_{j-1} e_{j+1} \cdots e_n) e_j.$$

Proof of Lemma 4.1. It suffices to show that γ is an involution over I_n and satisfies (4.1).

We proceed to prove that γ is an involution by induction. When n = 1, $\gamma^2(0) = 0$. Suppose that $\gamma^2(t) = t$ for each $t \in I_{n-1}$ with $n \ge 2$. We claim that $\gamma^2(e) = e$ for each $e \in I_n$. By Proposition 4.4, we have $e_n + 1$ is the largest in $Max(\gamma(e_1e_2\cdots e_n))$ and hence e_n is the last element of $\gamma^2(e_1e_2\cdots e_n)$. Combining the construction of γ and (2) in Proposition 4.4, we deduce that

$$\gamma^2(e_1e_2\cdots e_n) = \gamma^2(e_1e_2\cdots e_{n-1})e_n$$
$$= e_1e_2\cdots e_{n-1}e_n.$$

The claim is verified. Hence, γ is an involution.

Now, we shall prove relation (4.1). It is easy to check that $(\text{dist}, \text{zero})e = (\text{dist}, \text{zero})\gamma(e)$. It is left to show that

$$(\max, \operatorname{rlmin})e = (\operatorname{rlmin}, \max)\gamma(e)$$
 (4.3)

Obviously, it holds for n = 1. Suppose that (4.3) holds for n - 1, where $n \ge 2$, we claim that it also holds for n. There are two cases to consider. If $e_n = n - 1$, then $\gamma(e_1 e_2 \cdots e_n) = \gamma(e_1 e_2 \cdots e_{n-1})(n-1)$. Thus,

$$\max(e_1 e_2 \cdots e_n) = \max(e_1 e_2 \cdots e_{n-1}) + 1, \qquad (4.4)$$

$$\operatorname{rlmin}(\gamma(e_1e_2\cdots e_n)) = \operatorname{rlmin}(\gamma(e_1e_2\cdots e_{n-1})) + 1.$$
(4.5)

Combining (4.4) (4.5) and the hypothesis that $\max(e_1e_2\cdots e_{n-1}) = \operatorname{rlmin}(\gamma(e_1e_2\cdots e_{n-1}))$, we deduce that $\max(e) = \operatorname{rlmin}(\gamma(e))$. Then $\operatorname{rlmin}(e) = \max(\gamma(e))$ follows from the fact that γ is an involution.

If $e_n < n-1$, by Proposition 4.4, e_n+1 is the largest element of $Max(\gamma(e))$. It follows that e_n is not an *RL*-minimum of $\gamma(e)$. Thus, we have

$$\max(e_1 e_2 \cdots e_n) = \max(e_1 e_2 \cdots e_{n-1}), \qquad (4.6)$$

$$\operatorname{rlmin}(\gamma(e_1e_2\cdots e_n)) = \operatorname{rlmin}(\gamma(e_1e_2\cdots e_{n-1})).$$
(4.7)

Similarly, in view of (4.6) (4.7) and the hypothesis, we have $(\max, \operatorname{rlmin})e = (\operatorname{rlmin}, \max)\gamma(e)$ in this case. This completes the proof.

To prove Lemma 4.2, we need the permutation code b, namely, a bijection between permutations and inversion sequences, given by Baril and Vajnovszki [1]. We give a brief review of the code b first.

An interval [m, n] with m < n is the set $\{x \in \mathbb{N} : m \le x \le n\}$, where $\mathbb{N} = \{0, 1, \dots\}$. A labeled interval is a pair (I, l), where I is an interval and l is an integer. Given $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$ and an integer i with $0 \le i < n$, let the *i*-th slice of π , $U_i(\pi)$, to be a sequence of labelled intervals constructed recursively by the following process. Set $U_0(\pi) = ([0, n], 0)$. For $i \ge 1$, assume that $U_{i-1}(\pi) = (I_1, l_1), (I_2, l_2), \cdots, (I_k, l_k)$ is the (i-1)-th slide of π and v is the index such that $\pi_i \in I_v$, then $U_i(\pi)$ is constructed as follows.

- If $\min(I_v) < \pi_i = \max(I_v)$, then $U_i(\pi)$ equals $(I_1, l_1), \cdots, (I_{v-1}, l_{v-1}), (J, l_{v+1}), (I_{v+1}, l_{v+2}), \cdots, (I_{k-1}, l_k), (I_k, l_k + 1),$ where $J = [\min(I_v), \pi_i - 1].$
- If $\min(I_v) < \pi_i < \max(I_v)$, then $U_i(\pi)$ equals $(I_1, l_1), \cdots, (I_{v-1}, l_{v-1})(H, l_v), (J, l_{v+1}), (I_{v+1}, l_{v+2}), \cdots, (I_{k-1}, l_k), (I_k, l_k+1),$ where $H = [\pi_i + 1, \max(I_v)]$ and $J = [\min(I_v), \pi_i - 1].$
- If $\min(I_v) = \pi_i < \max(I_v)$, then $U_i(\pi)$ equals $(I_1, l_1), \cdots, (I_{v-1}, l_{v-1})(H, l_v), (I_{v+1}, l_{v+1}), \cdots, (I_{k-1}, l_{k-1}), (I_k, l_k + 1),$ where $H = [\pi_i + 1, \max(I_v)].$
- If $\min(I_v) = \pi_i = \max(I_v)$, then $U_i(\pi)$ equals

$$(I_1, l_1), \cdots, (I_{v-1}, l_{v-1}), (I_{v+1}, l_{v+1}), \cdots, (I_{k-1}, l_{k-1}), (I_k, l_k+1).$$

Let $b(\pi) = b_1 b_2 \cdots b_n \in I_n$, where $b_i = l_v$ such that (I_v, l_v) is a labelled interval in the (i-1)-th slice of π with $\pi_i \in I_v$.

Example 4.5. For $\pi = 24135$ and $\sigma = 14352$, we have $b(\pi) = 00210$ and $b(\sigma) = 00102$ with

 $\begin{array}{ll} U_0(\pi) = ([0,5],0), & U_0(\sigma) = ([0,5],0), \\ U_1(\pi) = ([3,5],0)([0,1],1), & U_1(\sigma) = ([2,5],0)([0,0],1), \\ U_2(\pi) = ([5,5],0)([3,3],1)([0,1],2), & U_2(\sigma) = ([5,5],0)([2,3],1)([0,0],2), \\ U_3(\pi) = ([5,5],0)([3,3],1)([0,0],3), & U_3(\sigma) = ([5,5],0)([2,2],2)([0,0],3), \\ U_4(\pi) = ([5,5],0)([0,0],4), & U_4(\sigma) = ([2,2],2)([0,0],4). \end{array}$

Baril and Vajnovszki also proved a set-valued equidistribution as follows.

Lemma 4.6. For any $\pi \in S_n$,

(Des, Ides, Lrmax, Lrmin, Rlmax) $\pi = (Asc, Dist, Zero, Max, Rlmin) b(\pi),$

and so statistics (Des, Ides, Lrmax, Lrmin, Rlmax) on S_n has the same distribution as (Asc, Dist, Zero, Max, Rlmin) on I_n .

Let $\alpha = c \circ i \circ b^{-1} \circ \gamma \circ b \circ i \circ c$, then it is easy to check that α is an involution on S_n . Now, we are ready to give the proof of Lemma 4.2.

Proof of Lemma 4.2. Given $\pi \in S_n$, it is enough to show that

 $(\operatorname{asc}, \operatorname{rlmax}, \operatorname{lrmax}, \operatorname{rlmin}) \pi = (\operatorname{asc}, \operatorname{rlmax}, \operatorname{rlmin}, \operatorname{lrmax}) \alpha(\pi)$ (4.8)

Notice that $\operatorname{asc}(\pi) = \operatorname{des}(\operatorname{c}(\pi))$ and $\operatorname{des}(\pi) = \operatorname{ides}(\operatorname{i}(\pi))$. Combining with Lemma 4.1 and Lemma 4.6, we see that $\operatorname{asc}(\pi) = \operatorname{asc}(\alpha(\pi))$.

Furthermore, the following properties are easy to check.

- 1) π_i is an RL-maximum of π if and only if i is an RL-maximum of π^{-1} .
- 2) π_i is a LR-minimum of π if and only if *i* is a LR-minimum of π^{-1} .
- 3) π_i is a LR-maximum of π if and only if *i* is a RL-minimum of π^{-1} .
- 4) π_i is an RL-minimum of π if and only if i is an LR-maximum of π^{-1} .

It follows that

 $(\text{rlmax}, \text{lrmin}, \text{lrmax}, \text{rlmin})\pi = (\text{rlmax}, \text{lrmin}, \text{rlmin}, \text{lrmax})\pi^{-1}.$ (4.9)

Also, we have

 $(\text{rlmax}, \text{lrmin}, \text{lrmax}, \text{rlmin})\pi = (\text{rlmin}, \text{lrmax}, \text{lrmin}, \text{rlmax})\pi^c.$ (4.10)

Based on equations (4.9), (4.10), Lemma 4.1 and Lemma 4.6, we deduce that

$$(\text{rlmax}, \text{lrmax}, \text{rlmin}) \pi = (\text{rlmax}, \text{rlmin}, \text{lrmax}) \alpha(\pi),$$

as desired. This completes the proof.

For a set X, let n - X be the set obtained by n minus each element in X. We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. In view of Theorem 1.3, (rlm, $\overline{\text{wm}}$, asc) is equally distributed with (rlm, rlmin, asc) on S_n . It is enough to construct a bijection β over S_n such that

$$(\operatorname{rlm}, \operatorname{w\overline{m}}, \operatorname{asc})\pi = (\operatorname{rlmax} -1, \operatorname{rlmin}, \operatorname{asc})\beta(\pi)$$
 (4.11)

for each $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$.

Assume that $\pi = x1y$, where x and y can be empty. Let $st(x) = (\overline{x}, X)$ and $st(y) = (\overline{y}, Y)$. Then set $\beta(\pi) = \sigma$, where $\sigma = y'nx'$, $x' = st^{-1}(\alpha(\overline{x}), n + 1 - X)$ and $y' = st^{-1}(\overline{y}^{rc}, n + 1 - Y)$.

To show that β is a bijection, it suffices to construct its inverse. Given $\sigma = wnv \in S_n$, where w and v can be empty. Let $st(w) = (\overline{w}, W)$ and $st(v) = (\overline{v}, V)$. Then set $\delta(\sigma) = \pi$, where $\pi = v'1w', v' = st^{-1}(\alpha(\overline{v}), n+1-V)$ and $w' = st^{-1}(\overline{w}^{rc}, n+1-W)$. Notice that α is an involution, δ is the inverse of β . Hence, β is a bijection.

In the following, we proceed to prove (4.11). Notice that $\operatorname{rlm}(\pi) = \operatorname{rlmax}(x)$ and $\operatorname{rlmax}(\sigma) = \operatorname{rlmax}(x') + 1$. By (4.8), we have $\operatorname{rlmax}(x) = \operatorname{rlmax}(x')$. It follows that $\operatorname{rlm}(\pi) = \operatorname{rlmax}(\sigma) - 1$.

To prove $\overline{\mathrm{wm}}(\pi) = \mathrm{rlmin}(\sigma)$, it is enough to show that $\mathrm{lrmax}(\pi) = \mathrm{rlmin}(\sigma)$ in view of Lemma 2.1. Let $\mathrm{lrmax}_{>s}(u)$ be the number of LR-maxima of the word u which are larger than s, and $\mathrm{rlmin}_{<s}(u)$ be the number of RL-minima of the word u which are smaller than s. We consider the following two cases.

- x is empty. Thus $\pi = 1y$. It follows that $\operatorname{lrmax}(\pi) = 1 + \operatorname{lrmax}(y)$ and $\operatorname{rlmin}(\sigma) = 1 + \operatorname{rlmin}(y')$. Clearly, $\operatorname{lrmax}(y) = \operatorname{rlmin}(y')$. Hence, we have $\operatorname{lrmax}(\pi) = \operatorname{rlmin}(\sigma)$.
- x is not empty. By the block decomposition, we have $\operatorname{lrmax}(\pi) = \operatorname{lrmax}(x) + \operatorname{lrmax}_{>max(X)}(y)$ and $\operatorname{rlmin}(\sigma) = \operatorname{rlmin}(x') + \operatorname{rlmin}_{<min(n+1-X)}(y')$. Since $\operatorname{lrmax}(x) = \operatorname{rlmin}(x')$ and $\operatorname{lrmax}_{>max(X)}(y) = \operatorname{rlmin}_{<min(n+1-X)}(y')$, then $\operatorname{lrmax}(\pi) = \operatorname{rlmin}(\sigma)$ follows.

Finally, we notice that $\operatorname{asc}(\pi) = \operatorname{asc}(x) + 1 + \operatorname{asc}(y)$ and $\operatorname{asc}(\sigma) = \operatorname{asc}(x') + 1 + \operatorname{asc}(y')$. Since $\operatorname{asc}(x) = \operatorname{asc}(x')$ and $\operatorname{asc}(y) = \operatorname{asc}(y')$, we deduce that $\operatorname{asc}(\pi) = \operatorname{asc}(\sigma)$. This completes the proof.

Example 4.7. Let $\pi = 593721684$, then n = 9, x = 59372 and y = 684. $(\bar{x}, X) = \text{st}^{-1}(x) = (35241, \{2, 3, 5, 7, 9\})$ and $(\bar{y}, Y) = \text{st}^{-1}(y) = (231, \{4, 6, 8\})$. $\alpha(\bar{x}) = 51342$ can be obtained as follows

 $35241 \xrightarrow{c} 31425 \xrightarrow{i} 24135 \xrightarrow{b} 00210 \xrightarrow{\gamma} 00102 \xrightarrow{b^{-1}} 14352 \xrightarrow{i} 15324 \xrightarrow{c} 51342.$

Then, $x' = \operatorname{st}^{-1}(51342, \{1, 3, 5, 7, 8\}) = 81573$, $y' = \operatorname{st}^{-1}(312, \{2, 4, 6\}) = 624$ and $\sigma = \beta(\pi) = 624981573$. It is easy to check that $\operatorname{rlm}(\pi) = \operatorname{rlmax}(\sigma) - 1 = 3$, $\operatorname{wm}(\pi) = \operatorname{rlmin}(\sigma) = 2$ and $\operatorname{asc}(\pi) = \operatorname{asc}(\sigma) = 4$.

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