

# Properly colored short cycles in edge-colored graphs

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## Abstract

Properly colored cycles in edge-colored graphs are closely related to directed cycles in oriented graphs. As an analogy of the well-known Caccetta-Häggkvist Conjecture, we study the existence of properly colored cycles of bounded length in an edge-colored graph. We first prove that for all integers  $s$  and  $t$  with  $t \geq s \geq 2$ , every edge-colored graph  $G$  with no properly colored  $K_{s,t}$  contains a spanning subgraph  $H$  which admits an orientation  $D$  such that every directed cycle in  $D$  is a properly colored cycle in  $G$ . Using this result, we show that for  $r \geq 4$ , if the Caccetta-Häggkvist Conjecture holds, then every edge-colored graph of order  $n$  with minimum color degree at least  $n/r + 2\sqrt{n} + 1$  contains a properly colored cycle of length at most  $r$ . In addition, we also obtain an asymptotically tight total color degree condition which ensures a properly colored (or rainbow)  $K_{s,t}$ .

**Keywords:** Color degree; Properly colored  $K_{s,t}$ ; Caccetta-Häggkvist Conjecture

## 1 Introduction

In this paper, all graphs considered are simple graphs. All the terminology and notation used but not defined can be found in [4]. An *edge-colored graph* is a graph with each edge assigned a color. Given an edge-colored graph  $G$ , we say  $G$  is a *properly colored* graph if any two adjacent edges receive different colors, and  $G$  is a *rainbow* graph if all the edges receive pairwise different colors. For every vertex  $v \in V(G)$ , the *color degree* of  $v$ , denoted by

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$d_G^c(v)$ , is the number of distinct colors appearing on the incident edges of  $v$ . The *minimum color degree* of  $G$ , denoted by  $\delta^c(G)$ , is the minimum  $d_G^c(v)$  over all vertices  $v \in V(G)$ . By  $\Delta^{mon}(v)$ , we denote the maximum number of incident edges of  $v$  with the same color, and  $\Delta^{mon}(G)$  is the maximum  $\Delta^{mon}(v)$  over all vertices  $v \in V(G)$ .

The study on the existence of properly colored cycles in edge-colored graphs has a long history. Grossman and Häggkvist [9] provided a sufficient condition on the existence of properly colored cycles in edge-colored graphs with two colors. Later, Yeo [29] extended the result to edge-colored graphs with any number of colors. During the past decades, establishing sufficient conditions forcing properly colored (rainbow) cycles of certain lengths has received considerable attention [1, 3, 8, 13, 14, 16, 18]. In many classical problems the host graph  $G$  is complete. For instance, Bollobás and Erdős [3] conjectured that every edge-colored  $K_n$  with  $\Delta^{mon}(K_n) \leq \lfloor n/2 \rfloor - 1$  contains a properly colored Hamilton cycle and this conjecture was asymptotically resolved by Lo [19]. Later, Lo [17] considered the existence of properly colored Hamilton cycles under color degree conditions. Using the absorbing technique and stability method, Lo [18] recently proved that for sufficiently large  $n$ , every edge-colored graph  $G$  on  $n$  vertices with  $\delta^c(G) \geq 2n/3$  contains a properly colored Hamilton cycle.

The study of properly colored cycles is closely related to directed cycles in oriented graphs. On the one hand, oriented graphs are often used as auxiliary tools to find properly colored cycles. More details can be found in [7, 13, 14, 17]. On the other hand, finding directed cycles can be formulated as a special case of finding properly colored cycles. To see this, we have the following construction which was first introduced by Li [13]. Let  $D$  be an orientation of a simple graph  $G$  with  $V(G) = \{v_1, v_2, \dots, v_n\}$ . For every vertex  $v \in V(G)$ , we write  $d_D^+(v)$  and  $d_D^-(v)$  for its outdegree and indegree in  $D$ , respectively. Define an edge coloring  $\tau$  of  $G$  by coloring the edge  $v_i v_j$  with  $j$  for all arcs  $(v_i, v_j)$  in  $D$ . The resulting edge-colored graph, denoted by  $(D, \tau)$ , is called the *signature* of  $D$ . Then the following three properties hold.

- (1) For every vertex  $v \in V(G)$ ,  $d_{(D, \tau)}^c(v) = d_D^+(v)$  if  $d_D^-(v) = 0$ , otherwise  $d_{(D, \tau)}^c(v) = d_D^+(v) + 1$ .
- (2) A cycle in  $G$  is a directed cycle in  $D$  if and only if it is a properly colored cycle in  $(D, \tau)$ .
- (3)  $(D, \tau)$  contains no properly colored  $K_{s,t}$  for all integers  $s \geq 2$  and  $t \geq 3$ .

Properties (1) and (2) can be easily observed. To see (3), any properly colored  $K_{s,t}$  in  $(D, \tau)$  corresponds to an oriented  $K_{s,t}$  in which each vertex has at most one in-neighbor. It follows that  $st \leq s + t$ , which is impossible when  $s \geq 2, t \geq 3$ .

A fundamental question in digraph theory is to establish outdegree conditions ensuring that a digraph contains certain structures. For all positive integers  $n$  and  $r$ , let  $f(n, r)$  be the least integer such that every digraph  $D$  of order  $n$  with  $\delta^+(D) \geq f(n, r)$  contains a directed cycle of length at most  $r$ . Recall the following well-known Caccetta-Häggkvist Conjecture [5].

**Conjecture 1** ([5]). *For all positive integers  $n, r$  with  $n \geq r$ ,  $f(n, r) = \lceil n/r \rceil$ .*

Conjecture 1 is trivial when  $r \leq 2$ . The case  $r = 3$  remains open and there are numerous partial results, see [10, 11, 22, 24, 25]. The best known bound is provided by Hladký, Král' and Norin [11] stating that  $f(n, 3) \leq 0.3465n$ . For more results, we refer the reader to a survey of Sullivan [26].

Analogously, we can consider the following problem in edge-colored graphs.

**Problem 2.** *For all positive integers  $n, r$  with  $n \geq r \geq 3$ , what is the least integer  $f_c(n, r)$  such that every edge-colored graph  $G$  on  $n$  vertices with  $\delta^c(G) \geq f_c(n, r)$  contains a properly colored cycle of length at most  $r$ ?*

By the signatures of oriented graphs, it clearly holds that  $f_c(n, r) \geq f(n, r) + 1$  for  $r \geq 3$ . Moreover, in the case  $r = 3$ ,  $f_c(n, 3) > n/2$ . So there is a fundamental difference between  $f_c(n, 3)$  and  $f(n, 3)$ . In [13], Li proved that  $f_c(n, 3) = \lceil (n+1)/2 \rceil$ . Here we determine an asymptotically tight upper bound for  $f_c(n, r)$  when  $r \geq 4$ .

**Theorem 3.** *For all integers  $r \geq 4$ ,  $f_c(n, r) \leq f(n, r) + 2\sqrt{n} + 1$ .*

**Remark 1.** In Section 5, one can see that the term  $2\sqrt{n}$  in Theorem 3 can not be replaced by any (sufficiently large) absolute constant when  $r = \Theta(n)$ .

Very recently, Seymour and Spirkl [23] considered a bipartite version of Caccetta-Häggkvist Conjecture and proposed the following conjecture in which  $g(n, r)$  denotes the least integer such that every bipartite digraph  $D$  with  $n$  vertices in each part and  $\delta^+(D) \geq g(n, r)$  contains a directed cycle of length at most  $2r$ .

**Conjecture 4** ([23]). *For all positive integers  $n, r$  with  $n \geq r$ ,  $g(n, r) = \lfloor n/(r+1) \rfloor + 1$ .*

In the same paper [23], the authors proved Conjecture 4 when  $r = 1, 2, 3, 4, 6$  or  $r \geq 224539$ . Similarly, let  $g_c(n, r)$  be the least integer such that every edge-colored bipartite graph  $G$  with  $n$  vertices in each part and  $\delta^c(G) \geq g_c(n, r)$  contains a properly colored cycle of length at most  $2r$ . We also obtain the following theorem.

**Theorem 5.** *For all integers  $r \geq 2$ ,  $g(n, r) + 1 \leq g_c(n, r) \leq g(n, r) + 2\sqrt{n} + 1$ .*

We now provide a unified approach for Theorems 3 and 5. Recall the definition of  $(D, \tau)$  and its properties, we attempt to reduce the problem on properly colored cycles to the problem on directed cycles in oriented graphs. From this point, a natural but ambitious question is what condition would guarantee an orientation  $D$  of an edge-colored graph  $G$  that preserves properties (1) and (2). Inspired by property (3), we answer this question in a weaker sense as follows.

**Theorem 6.** *For all positive integers  $n, s$  and  $t$  with  $2 \leq s \leq t < n$ , every edge-colored graph  $G$  of order  $n$  with no properly colored  $K_{s,t}$  contains a spanning subgraph  $H$  of  $G$  which admits an orientation  $D$  satisfying the following.*

- (i) *For each  $v \in V(G)$ ,  $d_D^+(v) > d_G^c(v) - \left(\frac{(t-1)}{(s-1)!}\right)^{1/s} sn^{1-1/s} - s$ .*
- (ii) *Every directed cycle in  $D$  is a properly colored cycle in  $H$ .*

**Remark 2.** From the proof of Theorem 6, we can also derive that for every edge-colored bipartite graph  $G = (V_1, V_2, E)$  with  $|V_i| = n_i$  for each  $i \in \{1, 2\}$ , the above statement still holds when we replace (i) with the following assertion.

$$\text{For each } v \in V_i \text{ with } i \in \{1, 2\}, d_D^+(v) > d_G^c(v) - \left(\frac{(t-1)}{(s-1)!}\right)^{1/s} sn_{3-i}^{1-1/s} - s.$$

Using Theorem 6, we now provide a short proof of Theorem 3. Since the proof of Theorem 5 is very similar to that of Theorem 3, we omit it here.

**Proof of Theorem 3.** Let  $G$  be an edge-colored graph with  $\delta^c(G) \geq f(n, r) + 2\sqrt{n} + 1$ . We may assume that  $G$  contains no properly colored  $K_{2,2}$ . Applying Theorem 6 with  $s = t = 2$ , we obtain a subgraph  $H$  and an orientation  $D$  with  $\delta^+(D) \geq f(n, r)$  such that any directed cycle in  $D$  corresponds to a properly colored cycle in  $H$ . Therefore Theorem 3 easily follows from the definition of  $f(n, r)$ .  $\square$

The rest of the paper is organized as follows. In order to prove Theorem 6, we provide a crucial lemma (Lemma 7) in which we partially describe the typical structure of an edge-colored graph with no properly colored  $K_{s,t}$  for any given integers  $s, t$  with  $t \geq s \geq 2$ . The proofs of Lemma 7 and Theorem 6 are presented in Section 3. In Section 4, we give more applications of Theorem 6 and Lemma 7. Particularly, for all integers  $s, t$  with  $t \geq s \geq 2$ , we obtain asymptotically tight color degree conditions forcing a properly colored (or rainbow)  $K_{s,t}$ . Finally, some comments and open problems are proposed in Section 5.

## 2 Basic Notation

Given an edge-colored graph  $G$ , for every edge  $uv \in E(G)$ , let  $c_G(uv)$  be the color of  $uv$ . For each vertex  $v \in V(G)$ , denote by  $C_G(v)$  the set of colors appearing on the incident edges of  $v$ . For any two disjoint vertex sets  $U$  and  $V$ , let  $G[U, V]$  be the subgraph induced by all the edges between  $U$  and  $V$ . Let  $C_G(U, V)$  be the set of all colors appearing on the edges between  $U$  and  $V$ . If  $G$  is an edge-colored bipartite graph with two parts  $V_1, V_2$  and all the incident edges of each  $v \in V_1$  have pairwise distinct colors, then we say  $G$  is  $V_1$ -proper. We say that  $G$  is *pseudo  $V_1$ -canonical* if all edges incident to each  $v \in V_1$  have the same color. If the graph  $G$  is clear from the context, then the subscripts are usually omitted.

The *dual graph*  $\widehat{G} = (V_1, V_2; \widehat{E})$  of  $G$  is defined as follows. (i) Let  $V_1 = \{v^{(1)} \mid v \in V(G)\}$  and  $V_2 = \{v^{(2)} \mid v \in V(G)\}$ , respectively; (ii) For all edges  $uv \in E(G)$ , add two edges  $u^{(1)}v^{(2)}, v^{(1)}u^{(2)}$  in  $\widehat{G}$  and assign  $u^{(1)}v^{(2)}, v^{(1)}u^{(2)}$  the same color  $c_G(uv)$ . It is easy to see that  $d^c(v) = d^c(v^{(1)}) = d^c(v^{(2)})$  for all vertices  $v \in V(G)$ . In particular, we observe the following simple but subtle fact.

**Fact 1.** *For all positive integers  $s$  and  $t$ , the dual graph  $\widehat{G}$  contains a properly colored (or rainbow)  $K_{s,t}$  if and only if  $G$  contains a properly colored (or rainbow)  $K_{s,t}$ .*

Given an oriented graph  $D$  and a positive integer  $k$ , the  $k$ -blow-up of  $D$  is an oriented graph obtained from  $D$  by replacing each vertex  $v_i \in V(D)$  with an independent vertex set  $V_i$  of size  $k$  and adding all possible arcs from  $V_i$  to  $V_j$  for every arc  $(v_i, v_j)$  in  $D$ .

## 3 Proof of Theorem 6

For all integers  $t \geq s \geq 2$ , let  $\sigma_{s,t} = s \left( \frac{t-1}{(s-1)!} \right)^{1/s}$ . We start our proof with the following lemma.

**Lemma 7.** *Let  $2 \leq s \leq t$  be positive integers and  $G = (V_1, V_2; E)$  be a bipartite graph with  $|V_1| = n_1$  and  $|V_2| = n_2$ . If  $G$  contains no properly colored  $K_{s,t}$ , then  $G$  contains a spanning subgraph  $H$  with  $d_H^c(u) \geq d_G^c(u) - \sigma_{s,t} n_2^{1-1/s}$  for each  $u \in V_1$  and  $d_H^c(v) \leq s - 1$  for each  $v \in V_2$ . In particular, if  $s = 2$ , then  $H$  is a pseudo  $V_2$ -canonical graph.*

**Proof.** At the beginning, we extract a  $V_1$ -proper subgraph  $G_0$  from  $G$  such that  $d_{G_0}(v) = d_G^c(v)$  for each  $v \in V_1$ . Then in the rest of the proof, we only consider  $G_0$ .

Given a subset  $A \subseteq V_1$ , a vertex  $v \in V_2$  is *saturated* by  $A$  if  $|C(v, A)| \geq s - 1$ , and let  $S(A) = \{v \in V_2 \mid v \text{ is saturated by } A\}$ . Let  $U_l = \{v_1, v_2, \dots, v_l\} \subseteq V_1$  be a maximal set such that for each  $i \in [l]$ ,  $v_i$  has at least  $x$  neighbors  $v$  with  $v \notin S(U_{i-1})$  and  $c(vv_i) \notin C(v, U_{i-1})$ ,

where  $U_0 = \emptyset$ ,  $U_{i-1} = \{v_1, v_2, \dots, v_{i-1}\}$  for  $2 \leq i \leq l$  and  $x$  is an integer to be determined later. In addition, for each  $v \in S(U_l)$ , let  $U_v = U_i$  if  $v \in S(U_i) \setminus S(U_{i-1})$  for some  $i \in [l]$  and  $U_v = U_l$  if  $v \in V_2 \setminus S(U_l)$ . Therefore  $|C(v, U_v)| = s-1$  for any  $v \in S(U_l)$  and  $|C(v, U_v)| \leq s-2$  for any  $v \in V_2 \setminus S(U_l)$ . Let  $H$  be the subgraph of  $G_0$  obtained as follows. For each  $u \in V_1$ , we delete all incident edges  $uv$  with  $c(uv) \notin C(v, U_v)$ . Next, we show that  $H$  is the desired subgraph.

For all  $v \in V_2$ , we know that  $d_H^c(v) = |C(v, U_v)| \leq s-1$ . Hence, it remains to prove that  $d_H^c(u) \geq d_G^c(u) - \sigma_{s,t} n_2^{1-1/s}$  for every  $u \in V_1$ . Now we bound the number of incident edges of each  $u \in V_1$  which are deleted as described above.

**Claim 7.1.** *For each  $u \in V_1$ , we deleted at most  $(t-1)\binom{l}{s-1} + x$  incident edges.*

*Proof.* By the maximality of  $U_l$ , we know that each  $u \in V_1 \setminus U_l$  has less than  $x$  neighbors  $v$  outside  $S(U_l)$  with  $c(uv) \notin C(v, U_v)$ . So it suffices to prove that each  $u \in V_1$  has at most  $(t-1)\binom{l}{s-1}$  neighbors  $v$  inside  $S(U_l)$  with  $c(uv) \notin C(v, U_v)$ . In this case, for such a neighbor  $v$ , there exists a subset  $T \subset U_v$  of size  $s-1$  such that  $T \cup \{u, v\}$  induces a rainbow star centered at  $v$ . Since  $G$  contains no properly colored  $K_{s,t}$ ,  $T \cup \{u\}$  has at most  $t-1$  such common neighbors as  $v$ . Therefore, by double counting, each  $u$  has at most  $(t-1)\binom{|U_l|}{s-1}$  neighbors  $v$  inside  $S(U_l)$  with  $c(uv) \notin C(v, U_v)$ .  $\square$

Next, we obtain an upper bound on  $l$ .

**Claim 7.2.**  $l \leq \frac{(s-1)n_2}{x}$ .

*Proof.* We construct a bipartite digraph  $D$  between  $U_l$  and  $V_2$  as follows. For each  $v_i \in U_l$  and  $v \in V_2$ , we add an arc from  $v_i$  to  $v$  if and only if  $v \in N(v_i) \setminus S(U_{i-1})$  and  $c(vv_i) \notin C(v, U_{i-1})$ . By the definition of  $U_l$ , we know that  $d^+(v_i) \geq x$  for each  $v_i \in U_l$ . Since  $|C(v, U_l)| \leq s-2$  for each  $v \in V_2 \setminus S(U_l)$  and  $|C(v, U_i)| = s-1$  for each  $v \in S(U_i) \setminus S(U_{i-1})$  ( $i \in [l]$ ), we obtain that  $d^-(v) = s-1$  for  $v \in S(U_l)$  and  $d^-(v) \leq s-2$  for  $v \in V_2 \setminus S(U_l)$ . Therefore,

$$lx \leq |E(D)| \leq (s-1)|S(U_l)| + (s-2)(n_2 - |S(U_l)|),$$

which implies that  $l \leq \frac{(s-1)n_2}{x}$ .  $\square$

By the two claims above, it holds that for each  $u \in V_1$ ,

$$\begin{aligned} d_H^c(u) &\geq d_G^c(u) - (t-1) \binom{\frac{(s-1)n_2}{x}}{s-1} - x \\ &\geq d_G^c(u) - \frac{t-1}{(s-1)!} \left( \frac{(s-1)n_2}{x} \right)^{s-1} - x. \end{aligned}$$

Let  $x = (s-1) \left( \frac{t-1}{(s-1)!} \right)^{1/s} n_2^{1-1/s}$ , and it follows that  $d_H^c(u) \geq d_G^c(u) - \left( \frac{t-1}{(s-1)!} \right)^{1/s} s n_2^{1-1/s}$  for each  $u \in V_1$ . This completes the proof of Lemma 7.  $\square$

With Lemma 7, we are ready to prove Theorem 6.

**Proof of Theorem 6.** Let  $G$  be an edge-colored graph of order  $n$  with no properly colored  $K_{s,t}$ . By Fact 1, the dual graph  $\widehat{G} = (V_1, V_2; \widehat{E})$  also contains no properly colored  $K_{s,t}$ . Therefore, by Lemma 7, there exists a  $V_1$ -proper subgraph  $H_0 \subseteq \widehat{G}$  such that  $d_{H_0}^c(u) \geq d_G^c(u) - \sigma_{s,t} n^{1-1/s}$  for each  $u \in V_1$  and  $d_{H_0}^c(v) \leq s-1$  for each  $v \in V_2$ . Let  $H'$  be the subgraph of  $H_0$  obtained by deleting all edges  $e$  incident to  $v^{(1)}$  with  $c(e) \in C_{H_0}(v^{(2)})$  for each  $v \in V(G)$  in order. Recall that for each edge  $uv \in E(G)$ ,  $c(u^{(1)}v^{(2)}) = c(v^{(1)}u^{(2)})$  in  $\widehat{G}$ . Hence, at most one of  $u^{(1)}v^{(2)}, v^{(1)}u^{(2)}$  is included in the graph  $H'$ . Since  $d_{H_0}^c(v) \leq s-1$  for each  $v \in V_2$ , we have  $d_{H'}^c(u) \geq d_G^c(u) - \sigma_{s,t} n^{1-1/s} - (s-1)$  for each  $u \in V_1$ . Let  $H$  be a spanning subgraph of  $G$  with  $E(H) = \{uv \in E(G) \mid u^{(1)}v^{(2)} \in E(H')\}$ .

Now we show that the orientation  $D$ , defined on  $H$  by orienting  $uv \in E(H)$  from  $u$  to  $v$  if  $u^{(1)}v^{(2)} \in E(H')$ , is as desired. First of all, one can observe that any two consecutive edges of a directed path in  $D$  have different colors. Otherwise there are three vertices  $u, v, w \in V(H)$  such that both  $u^{(1)}v^{(2)}$  and  $v^{(1)}w^{(2)}$  belong to  $E(H')$  while  $c(u^{(1)}v^{(2)}) = c(v^{(1)}w^{(2)}) \in C_{H_0}(v^{(2)})$ . It follows that  $v^{(1)}w^{(2)}$  is deleted from  $H_0$ , which is a contradiction. Therefore property (ii) holds. For each vertex  $v \in V(H)$ , we know that  $d_D^+(v) = d_{H'}^c(v^{(1)}) \geq d_G^c(v) - \sigma_{s,t} n^{1-1/s} - (s-1)$  and  $d_D^-(v) = d_{H'}^c(v^{(2)})$ , so property (i) holds.  $\square$

## 4 More Applications of Theorem 6 and Lemma 7

### 4.1 Vertex-disjoint Cycles

For any positive integer  $k$ , let  $f(k)$  be the smallest integer so that every digraph of minimum out-degree at least  $f(k)$  contains  $k$  vertex disjoint directed cycles. The well-known Bermond-Thomassen Conjecture [2] states that  $f(k) = 2k-1$  for all  $k \geq 1$ , and it is true when  $k \leq 3$  [21, 27]. Motivated by Bermond-Thomassen Conjecture, Lichiardopol [15] proposed the following conjecture regarding vertex disjoint directed cycles of different lengths.

**Conjecture 8.** [15] *For every integer  $k \geq 2$ , there is an integer  $g(k)$  such that any digraph with minimum out-degree at least  $g(k)$  contains  $k$  vertex disjoint cycles of different lengths.*

Using Theorem 6, we obtain the following result on vertex disjoint properly colored cycles. Denote by  $C_4(G)$  a maximum set of vertex disjoint properly colored  $C_4$ 's in the edge-colored graph  $G$ .

**Theorem 9.** *For all positive integers  $n, k$ , let  $G$  be an edge-colored graph on  $n$  vertices. Then the following hold.*

- (i) *If  $\delta^c(G) \geq f(k - |C_4(G)|) + 4|C_4(G)| + 2\sqrt{n} + 1$ , then  $G$  contains  $k$  vertex disjoint properly colored cycles. In particular, if the Bermond-Thomassen Conjecture is true, then  $\delta^c(G) \geq 4k + 2\sqrt{n} + 1$  suffices.*
- (ii) *If Conjecture 8 is true, then  $\delta^c(G) \geq g(k) + (k+1)(k + n^{1-1/(k+1)})$  suffices to ensure  $k$  vertex disjoint properly colored cycles of different lengths.*

*Proof.* Let  $G'$  be the resulting subgraph of  $G$  by deleting all  $C_4$ 's in  $C_4(G)$ . So  $G'$  contains no properly colored  $C_4$ 's. By Theorem 6, there exists a subgraph  $H \subset G'$  and an orientation  $D$  of  $H$  such that  $d_D^+(v) \geq f(k - |C_4(G)|)$  and any directed cycle in  $D$  corresponds to a properly colored cycle in  $G'$ . Therefore, there are  $k - |C_4(G)|$  vertex disjoint properly colored cycles in  $G'$ , which together with  $C_4$ 's in  $C_4(G)$  form  $k$  vertex disjoint properly colored cycles in  $G$ . This completes the proof of (i).

We proceed the proof of (ii) by finding a maximal collection  $\mathcal{F}$  of vertex disjoint subsets  $A_1, A_2, \dots, A_l$  in  $G$  such that each  $G[A_i]$  contains a properly colored  $C_{2i+2}$  for  $i \in [l]$ . We may assume that  $|A_i| = 2i + 2$  for all  $i \in [l]$  and assume  $l < k$ , otherwise  $k$  vertex disjoint properly colored cycles of different lengths would be found. By the maximality of  $\mathcal{F}$ , we know that  $G'' = G - \bigcup_{i=1}^l A_i$  contains no properly colored  $K_{l+2, l+2}$  and  $\delta^c(G'') \geq \delta^c(G) - \sum_{i=1}^l |A_i|$ . Applying Theorem 6 with  $s = t = l + 2$ , we obtain a subgraph  $H \subset G''$  and an orientation  $D$  of  $H$  such that  $\delta^+(D) \geq \delta^c(G'') - \left(\frac{1}{l!}\right)^{1/(l+2)} (l+2)n^{1-1/(l+2)} - l - 1 > g(k)$ . By the definition of  $g(k)$ , there are  $k$  vertex disjoint directed cycles of different lengths in  $D$ , which are actually  $k$  vertex disjoint properly colored cycles of different lengths in  $G$ .  $\square$

## 4.2 Properly Colored Complete Bipartite Graphs

Recall that the signature of any oriented graph contains no properly colored  $K_{s,t}$  for all integers  $s \geq 2$  and  $t \geq 3$ . Let  $T$  be a transitive tournament. Then there is no properly colored  $K_{s,t}$  in the signature  $(T, \tau)$  and  $\sum_{v \in V(T, \tau)} d^c(v) = n(n+1)/2 - 1$ . In this part, using Theorem 6, we give the following asymptotically tight total color degree condition forcing a properly colored  $K_{s,t}$ .

**Theorem 10.** *For all positive integers  $n, s$  and  $t$  with  $n \geq t \geq s \geq 2$ , every edge-colored graph  $G$  on  $n$  vertices with  $\sum_{v \in V(G)} d^c(v) > n^2/2 + \sigma_{s,t}n^{2-1/s} + sn$  contains a properly colored  $K_{s,t}$ .*

Indeed, Theorem 10 is easily derived from the following theorem by Fact 1.



**Theorem 11.** *Let  $G = (V_1, V_2; E)$  be an edge-colored bipartite graph with  $|V_1| = n_1, |V_2| = n_2$ . For all positive integers  $t \geq s \geq 2$ , if  $\sum_{v \in V(G)} d^c(v) > n_1 n_2 + \sigma_{s,t}(n_1 n_2^{1-1/s} + n_2 n_1^{1-1/s}) + s(n_1 + n_2)$ , then  $G$  contains a properly colored  $K_{s,t}$ .*

*Proof.* Suppose that  $G$  contains no properly colored  $K_{s,t}$ . By Remark 2, there exists a subgraph  $H \subset G$  and an orientation  $D$  of  $H$  such that  $d_D^+(u) > d_G^c(u) - \sigma_{s,t} n_{3-i}^{1-1/s} - s$  for each  $u \in V_i$  ( $i = 1, 2$ ). Therefore,  $|E(D)| = \sum_{v \in V(G)} d_D^+(v) > n_1 n_2$ , which is a contradiction.  $\square$

Note that the signatures of a transitive tournament and a bipartite tournament with all arcs from one part to the other part imply the asymptotical sharpness of color degree conditions in Theorems 10 and 11. Indeed, if  $t \geq 3$  and  $s \geq 2$ , then the signature of every tournament (or bipartite tournament) shows that the bound in Theorem 10 (or Theorem 11) is asymptotically tight. Hence, it is interesting to know the exact estimate on the low order term  $n^{2-1/s}$ . Here we provide a lower bound as follows.

**Theorem 12.** *For all integers  $s, t$  with  $st > 2(s + t)$ , there exists a constant  $\gamma = \gamma(s, t)$  such that for every sufficiently large integer  $n$ , there exists an edge-colored complete graph  $K_n^c$  with no properly colored  $K_{s,t}$  and  $\delta^c(K_n^c) > n/2 + \gamma n^{1-\frac{s+t}{st-s-t}}$ .*

*Proof.* Let  $T$  be a tournament with vertex set  $\{v_1, v_2, \dots, v_n\}$  and  $\min\{\delta^+(T), \delta^-(T)\} \geq \lfloor \frac{n-1}{2} \rfloor$ . Clearly,  $(T, \tau)$  does not contain any copy of a properly colored  $K_{s,t}$  and  $\delta^c(T, \tau) \geq n/2$ . Next we show that we can slightly improve the color degree of each  $v \in (T, \tau)$  while no properly colored  $K_{s,t}$  arises. In the following, we always choose  $n$  to be sufficiently large whenever it is needed.

Let  $G = G(n, p)$  be the random graph on vertex set  $\{v_1, v_2, \dots, v_n\}$  with  $p$  to be determined later. For any  $S \in \binom{V(G)}{s+t}$ , denote by  $\mathcal{F}_S(s, t)$  the set consisting of all the distinct subgraphs of  $G(n, p)$  on  $S$  and with at least  $st - s - t$  edges, then we have

$$\mathbb{P}(|\mathcal{F}_S(s, t)| \geq 1) \leq \binom{\binom{s+t}{2}}{st-s-t} p^{st-s-t}.$$

Thus, letting  $X = \sum_{S \in \binom{V(G)}{s+t}} |\mathcal{F}_S(s, t)|$  and using the union bound, we obtain

$$\mathbb{P}(X \geq 1) \leq \binom{n}{s+t} \binom{\binom{s+t}{2}}{st-s-t} p^{st-s-t}.$$

In particular, there exists a constant  $\gamma = \gamma(s, t)$  such that  $\mathbb{P}(X \geq 1) < 1/2$  holds for  $p = 8\gamma n^{-\frac{s+t}{st-s-t}}$ .

For all  $i \in [n]$ , denote by  $A_i$  the set of in-neighbors of  $v_i$  in  $T$ . Let  $B$  be the event that  $|N_G(v_j) \cap A_i| \leq \gamma n^{1-\frac{s+t}{st-s-t}}$  for some  $j \in [n]$ . Since  $Y_i = |N_G(v_i) \cap A_i|$  has the binomial

distribution  $\mathbf{B}(|A_i|, p)$  for each  $i \in [n]$ , by Chernoff's bound [12], we have

$$\begin{aligned} \mathbb{P}(|N_G(v_i) \cap A_i| \leq \gamma n^{1-\frac{s+t}{st-s-t}}) &< \mathbb{P}(|N_G(v_i) \cap A_i| \leq \frac{3\gamma}{2} n^{1-\frac{s+t}{st-s-t}}) \\ &< \mathbb{P}(Y_i \leq \frac{\mathbb{E}[Y_i]}{2}) \leq e^{-\frac{\mathbb{E}[Y_i]}{s}}. \end{aligned}$$

Thus,

$$\mathbb{P}(B) < \sum_{i \in [n]} \mathbb{P}(Y_i \leq \frac{\mathbb{E}[Y_i]}{2}) < n e^{-\frac{\gamma}{3} n^{1-\frac{s+t}{st-s-t}}} < \frac{1}{2}.$$

By the union bound, it follows that with positive probability,  $B$  does not occur and  $X = 0$ . So there exists a subgraph  $G$  of  $(T, \tau)$  such that  $|N_G(v_i) \cap A_i| > \gamma n^{1-\frac{s+t}{st-s-t}}$  for all  $i \in [n]$  and every subgraph of  $G$  on  $s+t$  vertices has less than  $st-s-t$  edges.

Now we construct a new coloring of  $(T, \tau)$  by recoloring each  $uv \in E(G)$  differently with a new color not appearing in  $\tau$  and denote the resulting edge-colored complete graph by  $K_n^c$ . So it holds that  $\delta^c(K_n^c) > n/2 + \gamma n^{1-\frac{s+t}{st-s-t}}$ . Now we show that  $K_n^c$  contains no properly colored  $K_{s,t}$ . Recall that every properly colored subgraph in  $(T, \tau)$  corresponds to an oriented graph in which each vertex has at most one in-neighbor. Hence, any properly colored subgraph of  $(T, \tau)$  on  $s+t$  vertices has at most  $s+t$  edges. Since no subgraph of  $G$  on  $s+t$  vertices has at least  $st-s-t$  edges, we know that no properly colored  $K_{s,t}$  exists in  $K_n^c$  when  $st > s+t$ .  $\square$

By the following proposition, one can extend Theorems 10 and 11 to rainbow versions.

**Proposition 13.** For all positive integers  $s, t$ , every properly colored  $K_{s,t+s(t-1)(s-1)}$  on a bipartition  $(A, B)$  contains a rainbow  $K_{s,t}$  on a bipartition  $(A, B')$  for some  $B' \subseteq B$ .

*Proof.* We proceed the proof by induction on  $t$ . We may assume  $s \geq 2$  and  $t \geq 2$ . Hence, every properly colored  $K_{s,t+s(t-1)(s-1)}$  with a bipartition  $(A, B)$  contains a rainbow  $K_{s,t-1}$  with a bipartition  $(A, B^*)$ , where  $|A| = s, |B^*| = t-1$ . For each  $u \in A$ , define  $F_u \subset B \setminus B^*$  by declaring  $v \in F_u$  if only  $c(uv) \in C(A, B^*)$ . Since  $|F_u| \leq (s-1)(t-1)$ , we have  $|B| > |\bigcup_{u \in A} F_u| + |B^*|$ , i.e., there exists a vertex  $u' \in B \setminus B^*$  such that  $A \cup B^* \cup \{u'\}$  induces a rainbow  $K_{s,t}$ .  $\square$

### 4.3 Rainbow $C_4$

From Theorem 5 and the correctness of Conjecture 4 when  $r = 2$  [23], one can immediately obtain that every edge-colored bipartite graph  $G$  with  $n$  vertices in each part and  $\delta^c(G) > n/3 + 2\sqrt{n} + 1$  contains a properly colored  $C_4$ . It is easy to see that the minimum color degree condition is asymptotically tight by considering the signature of the  $n/3$ -blow-up of a directed  $C_6$ .

Compared to properly colored cycles, finding rainbow cycles seems much more difficult (see [8, 17]). For rainbow  $C_4$ , the first result comes from Li [13], which asserts that every bipartite graph  $G$  with  $n$  vertices in each part and  $\delta^c(G) > 3n/5 + 1$  contains a rainbow  $C_4$ . By Proposition 13 and Theorem 11, the minimum color degree condition can be easily improved to  $\delta^c(G) > n/2 + 2\sqrt{3n} + 2$ . In this part, we resolve this problem asymptotically by proving the following stronger result using Lemma 7.

**Theorem 14.** *Let  $G = (V_1, V_2; E)$  be an edge-colored bipartite graph with  $|V_i| = n_i$ ,  $\delta_i^c = \min_{v \in V_i} d^c(v)$  for  $i \in \{1, 2\}$ . If  $\delta_1^c \delta_2^c > n_1 n_2 / 9 + 8n_1 \sqrt{n_2} + 8n_2 \sqrt{n_1}$ , then  $G$  contains a rainbow  $C_4$ . Therefore, if  $\delta_1^c > n_2/3 + 24\sqrt{n_2}$  and  $\delta_2^c > n_1/3 + 24\sqrt{n_1}$ , then  $G$  contains a rainbow  $C_4$ . Moreover, this is asymptotically best possible by considering the signature of the  $n/3$ -blow-up of a directed  $C_6$ .*

*Proof.* Let  $G, n_1, n_2, \delta_1^c, \delta_2^c$  be given as in Theorem 14, and suppose  $G$  contains no rainbow  $C_4$ 's. We may assume that  $G$  is edge-critical, that is, every edge deletion would lead to a decrease in  $d^c(v)$  for some vertex  $v \in V(G)$ . Therefore, every monochromatic subgraph of  $G$  is a union of vertex-disjoint stars. Let  $\Delta_1 = \max_{u \in V_1} \Delta^{\text{mon}}(u)$  and  $\Delta_2 = \max_{u \in V_2} \Delta^{\text{mon}}(u)$ . Choose a vertex  $v_0 \in V_1$  with  $\Delta^{\text{mon}}(v_0) = \Delta_1$ . Let  $V_2^m, V_2^c, V_2'$  be a partition of  $V_2$  such that

- $V_2^m \subseteq N(v_0)$ ,  $|V_2^m| = \Delta_1$  and  $G[v_0, V_2^m]$  is a monochromatic star;
- $V_2^c \subseteq N(v_0) \setminus V_2^m$ ,  $|V_2^c| = \delta_1^c - 1$  and  $G[v_0, V_2^c]$  is a properly colored star;
- $V_2' = V_2 \setminus (V_2^c \cup V_2^m)$ .

Let  $c_1$  be the color on the edges in  $G[v_0, V_2^m]$ . For every vertex  $u \in V_2^c$ , let  $N^c(u)$  be a maximal subset of  $N(u)$  such that  $G[u, N^c(u)]$  is a properly colored star with no colors  $c_1$  and  $c(uv_0)$ . Note that  $|N^c(u)| \geq \delta_2^c - 2$  for every vertex  $u \in V_2^c$ . Let  $U = \bigcup_{u \in V_2^c} N^c(u)$ .

**Claim 14.1.** *For every  $v \in U$ ,  $|C(v, V_2^m)| \leq 2$ .*

*Proof.* Suppose that  $v \in N^c(w)$  for some vertex  $w \in V_2^c$ , it is easy to see that  $c(wv), c(v_0w)$  and  $c_1$  are pairwise distinct. The case when  $|V_2^m| \leq 2$  is trivial, so it suffices to consider the case  $|V_2^m| > 2$ . Since  $G$  is edge-critical, we have that  $c_1 \notin C(v, V_2^m)$ . Therefore, if  $|C(v, V_2^m)| > 2$ , then we can find a color  $c' \in C(v, V_2^m)$  and a corresponding vertex  $v' \in V_2^m$  such that  $c(vv') = c'$  and  $c(wv), c(vv'), c(v'v_0), c(v_0w)$  are pairwise distinct. So we find a rainbow  $C_4$ , which is a contradiction.  $\square$

Let  $G' \subseteq G$  be a maximum pseudo  $V_2'$ -canonical subgraph between  $U$  and  $V_2'$ . Next we give a lower bound on the number of edges in  $G'$ .

**Claim 14.2.**  $|E(G')| \geq \delta_1^c \delta_2^c - 4n_1 \sqrt{n_2} - 4n_2 \sqrt{n_1}$ .

*Proof.* Let  $G_1 \subseteq G$  be a maximal  $V_2^c$ -proper subgraph between  $V_2^c$  and  $U$ . Clearly,  $|E(G_1)| \geq (\delta_1^c - 1)(\delta_2^c - 2)$ . By Proposition 13,  $G$  also contains no properly colored  $K_{2,4}$ . Applying Lemma 7 to  $G_1$  with  $s = 2$  and  $t = 4$ , there is a pseudo  $U$ -canonical subgraph  $H_1 \subseteq G_1$  of size at least  $|E(G_1)| - 4|V_2^c|\sqrt{|U|}$ . Since  $|C(v, V_2^m)| \leq 2$  for each vertex  $v \in U$ , there is a  $U$ -proper subgraph  $G_2$  of  $G$  between  $U$  and  $V_2'$  such that

$$|E(G_2)| \geq \delta_1^c|U| - 2|U| - (|U||V_2^c| - |E(H_1)| + |U|).$$

Applying Lemma 7 to  $G_2$  with  $s = 2$  and  $t = 4$ , we obtain a pseudo  $V_2'$ -canonical subgraph  $H_2$  such that  $|E(H_2)| \geq |E(G_2)| - 4|U|\sqrt{|V_2'|}$ . Hence,

$$\begin{aligned} |E(G')| &\geq |E(H_2)| \geq |E(G_2)| - 4|U|\sqrt{|V_2'|} \\ &\geq \delta_1^c|U| - 2|U| - (|U||V_2^c| - |E(H_1)| + |U|) - 4|U|\sqrt{|V_2'|} \\ &\geq |E(H_1)| - 2|U| - 4|U|\sqrt{|V_2'|} \\ &\geq \delta_1^c\delta_2^c - 4n_1\sqrt{n_2} - 4n_2\sqrt{n_1}. \end{aligned}$$

□

Since  $G'$  is pseudo  $V_2'$ -canonical, we have  $|V_2'|\Delta_2 \geq |E(G')| \geq \delta_1^c\delta_2^c - 4n_1\sqrt{n_2} - 4n_2\sqrt{n_1}$ , which implies  $\Delta_2 \geq n_1/9$ . Hence,

$$\begin{aligned} n_2 &= |V_2'| + |V_2^c| + |V_2^m| \\ &\geq \frac{\delta_1^c\delta_2^c - 4n_1\sqrt{n_2} - 4n_2\sqrt{n_1}}{\Delta_2} + \Delta_1 + \delta_1^c - 1 \\ &\geq \frac{\delta_1^c\delta_2^c}{\Delta_2} + \Delta_1 + \delta_1^c - 36\sqrt{n_2} - \frac{36n_2}{\sqrt{n_1}}. \end{aligned}$$

By symmetry, we also have  $n_1 \geq \frac{\delta_1^c\delta_2^c}{\Delta_1} + \Delta_2 + \delta_2^c - 36\sqrt{n_1} - \frac{36n_1}{\sqrt{n_2}}$ . Therefore,

$$n_1n_2 \geq 3\delta_1^c\delta_2^c + f(\delta_1^c\delta_2^c, \Delta_1\Delta_2) + \delta_2^cf(\delta_1^c, \Delta_1) + \delta_1^cf(\delta_2^c, \Delta_2) - 72(n_1\sqrt{n_2} + n_2\sqrt{n_1}),$$

where  $f(a, b) = b + a^2/b$ . Since  $f(a, b) \geq 2a$  for any positive numbers  $a, b$ , it follows that

$$n_1n_2 \geq 9\delta_1^c\delta_2^c - 72(n_1\sqrt{n_2} + n_2\sqrt{n_1}) > n_1n_2,$$

which is a contradiction. This completes the proof of Theorem 14. □

By Fact 1, the following result can be easily derived from Theorem 14.

**Corollary 15.** *Let  $G$  be an edge-colored graph on  $n$  vertices. Then the following hold.*

(i) *If  $\delta^c(G) > n/3 + 2\sqrt{n} + 1$ , then  $G$  contains a properly colored  $C_4$ .*

(ii) If  $\delta^c(G) > n/3 + 24\sqrt{n}$ , then  $G$  contains a rainbow  $C_4$ .

If the host graph  $G$  is an edge-colored triangle-free graph on  $n$  vertices, then  $\delta^c(G) > n/3 + 1$  forces a rainbow  $C_4$  by a result of Čada et al. [6]. Here, we also asymptotically resolve this problem using Lemma 7.

**Theorem 16.** *Let  $G$  be an edge-colored triangle-free graph on  $n$  vertices. If  $\delta^c(G) > n/5 + 3\sqrt{n}$ , then  $G$  contains a rainbow  $C_4$ , and this is asymptotically best possible by considering the signature of the  $n/5$ -blow-up of a directed  $C_5$ .*

*Proof.* Let  $G$  be an edge-colored triangle-free graph with no rainbow  $C_4$  and  $\delta^c(G) = \delta^c > n/5 + 3\sqrt{n}$ . Then by triangle-freeness, we have that  $n/5 + 3\sqrt{n} < n/2$ , and thus  $n > 100$ . We also assume that  $G$  is edge-critical, and proceed in our proof with the following claim.

**Claim 16.1.** *There exists an edge  $uv$  in  $G$  such that  $\Delta^{\text{mon}}(u) + \Delta^{\text{mon}}(v) \geq 2\delta^c - 8\sqrt{n}$ .*

*Proof.* Let  $G'$  be a maximum  $V_1$ -proper subgraph of the dual graph  $\widehat{G} = (V_1, V_2; E')$  of  $G$ . By Proposition 13,  $G'$  contains no properly colored  $K_{2,4}$ . Applying Lemma 7 to  $G'$  with  $s = 2$  and  $t = 4$ , we obtain a pseudo  $V_2$ -canonical graph  $H \subseteq \widehat{G}$  such that for each vertex  $v^{(1)} \in V_1$ ,  $d_H^c(v^{(1)}) \geq \delta^c - 4\sqrt{n}$ . Therefore,

$$\begin{aligned} \sum_{v^{(1)} u^{(2)} \in E(H)} (d_H(v^{(2)}) + d_H(u^{(2)})) &= \sum_{v^{(1)} \in V_1} (d_H(v^{(1)}) + d_H(v^{(2)})) d_H(v^{(2)}) \\ &\geq (\delta^c - 4\sqrt{n})|E(H)| + |E(H)|^2/n \\ &\geq 2(\delta^c - 4\sqrt{n})|E(H)|, \end{aligned}$$

where the second inequality is derived from the Cauchy-Schwartz inequality that

$$\sum_{v^{(1)} \in V_1} (d_H(v^{(2)}))^2 \geq \left( \sum_{v^{(1)} \in V_1} d_H(v^{(2)}) \right)^2 / n = |E(H)|^2/n.$$

By the pigeonhole principle, there exists an edge  $uv \in E(G)$  such that  $\Delta^{\text{mon}}(u) + \Delta^{\text{mon}}(v) \geq 2\delta^c - 8\sqrt{n}$ .  $\square$

Let  $uv \in E(G)$  be an edge with  $\Delta^{\text{mon}}(u) + \Delta^{\text{mon}}(v) \geq 2\delta^c - 8\sqrt{n}$ , and choose  $A_1 \subseteq N(u)$  and  $A_2 \subseteq N(v)$  such that  $G[u, A_1]$  and  $G[v, A_2]$  are monochromatic stars and  $|A_1| + |A_2| \geq 2\delta^c - 8\sqrt{n}$ . Let  $c_1$  and  $c_2$  be the colors on the edges in  $G[u, A_1]$  and  $G[v, A_2]$  respectively. Since  $G$  is edge-critical, we may assume that  $c(uv) \neq c_2$ . Let  $B_1$  be a maximal subset of  $N(u) \setminus A_1$  such that  $G[u, B_1]$  is a properly colored star with no colors  $c(uv)$  and  $c_2$ . Let  $B_2$  be a subset of  $N(v) \setminus A_2$  such that  $G[v, B_2]$  is a properly colored star of size  $\delta^c - 3$ . Then by triangle-freeness,  $A_1, A_2, B_1, B_2$  are pairwise disjoint and  $|B_1| \geq |B_2|$ .

**Claim 16.2.** *For every vertex  $w \in B_1$ ,  $|C(w, A_2)| \leq 2$ .*

*Proof.* Suppose there exists a vertex  $w \in B_1$  such that there are at least three colors appearing between  $w$  and  $A_2$ , then we can greedily find a vertex  $r \in A_2$  such that  $c(wr) \notin \{c_2, c(uv), c(uw)\}$ . Hence,  $uvrwu$  is a rainbow  $C_4$ , which is a contradiction.  $\square$

For each vertex  $w \in B_1$ , let  $S_w$  be the maximal subset of  $N(w) \cap B_2$  such that  $G[w, S_w]$  is a properly colored star with no colors  $c_1$  and  $c(uw)$ .

**Claim 16.3.** *For every  $w \in B_1$ , we have  $|S_w| > 4\sqrt{n}$ .*

*Proof.* If some  $w \in B_1$  satisfies that  $|S_w| \leq 4\sqrt{n}$ , then by Claim 16.2 and triangle-freeness,  $w$  has at least  $\delta^c - 4\sqrt{n} - 4$  neighbors outside  $A_1 \cup A_2 \cup B_1 \cup B_2$ . Hence,  $n \geq |A_1| + |A_2| + |B_1| + |B_2| + \delta^c - 4\sqrt{n} - 4 \geq 5\delta^c - 15\sqrt{n}$  (the second inequality follows from  $n > 100$ ), i.e.,  $\delta^c \leq n/5 + 3\sqrt{n}$ , which is a contradiction.  $\square$

Let  $B'_2 = \bigcup_{w \in B_1} S_w$ . Then we have the following claim similar to Claim 16.2, and a simple proof is included for completeness.

**Claim 16.4.** *For each vertex  $w' \in B'_2$ ,  $|C(w', A_1)| \leq 2$ .*

*Proof.* Suppose that there is a vertex  $w' \in B'_2$  such that there are at least three colors between  $w'$  and  $A_1$ . By the definition of  $B'_2$ ,  $w' \in S_w$  for some vertex  $w \in B_1$ . In this case, we can greedily find a neighbor of  $w'$  in  $A_1$ , say  $r$ , such that  $c(rw') \notin \{c(ww'), c(wu), c_1\}$ , which implies that  $uw w' r u$  is a rainbow  $C_4$ , which is a contradiction.  $\square$

Let  $G'$  be a maximum  $B_1$ -proper subgraph of  $G[B_1, B'_2]$ . Applying Lemma 7 to  $G'$  with  $s = 2$  and  $t = 4$ , we obtain a pseudo  $B'_2$ -canonical subgraph  $H' \subseteq G'$  such that  $d_{H'}(w) \geq |S_w| - 4\sqrt{n} \geq 1$  for each vertex  $w \in B_1$ . Since  $B'_2 \subseteq B_2$  and

$$\sum_{xy \in E(H'), x \in B_1, y \in B'_2} \left( \frac{1}{d_{H'}(x)} - \frac{1}{d_{H'}(y)} \right) = |B_1| - |B'_2| \geq |B_1| - |B_2| \geq 0,$$

there is an edge  $u'v' \in E(H')$  with  $u' \in B_1$ ,  $v' \in B'_2$  and  $d_{H'}(u') \leq d_{H'}(v')$ . Hence, we can find a set  $S \subseteq N_G(u') \setminus (A_1 \cup A_2 \cup B_1 \cup B_2)$  of size at least  $\delta^c - d_{H'}(v') - 4\sqrt{n} - 4$  and a set  $T \subseteq N_G(v') \setminus (A_1 \cup A_2 \cup B_1 \cup B_2)$  of size at least  $d_{H'}(v') - 1$ . Since  $G$  is triangle-free,  $S$  and  $T$  are disjoint. As  $n > 100$ , it follows that

$$\begin{aligned} n &\geq |A_1| + |A_2| + |B_1| + |B_2| + |S| + |T| \\ &\geq 2\delta^c - 8\sqrt{n} + 2\delta^c - 6 + \delta^c - 4\sqrt{n} - 5 \\ &\geq 5\delta^c - 15\sqrt{n}, \end{aligned}$$

i.e.,  $\delta^c \leq n/5 + 3\sqrt{n}$ , which is a contradiction.  $\square$

## 5 Concluding Remarks

In this paper we mainly study color degree conditions forcing properly colored cycles of length at most  $r$  or properly colored complete bipartite graphs. As a crucial tool, Theorem 6 reveals a close relationship between edge-colored graphs and oriented graphs. Using Theorem 6, we have reduced some problems on properly colored cycles to the problems on directed cycles in oriented graphs.

It is worth noting that  $f(n, r) \leq n/(r - 73)$  for all  $r > 73$  by a result of Shen [24]. Based on this result, we claim that the term  $2\sqrt{n}$  in Theorem 3 can not be replaced by any absolute constant when  $n$  is sufficiently large and  $r = cn$  for some fixed constant  $c$ . Indeed, suppose in the case  $r = cn$ , the term  $2\sqrt{n}$  in Theorem 3 can be replaced by an absolute constant, say  $C$ . Then by Theorem 3,  $f^c(n, cn) \leq C + f(n, cn) \leq C + 2/c$ , which contradicts a conclusion of Li and Wang [28] which asserts that for every positive integer  $l$ , there exists an edge-colored graph  $G$  with  $\delta^c(G) \geq l$  and no properly colored cycle. Recently, Fujita, Li and Zhang [8] obtained a tight bound.

**Theorem 17.** [8] *For all positive integers  $n, d$  with  $d! \sum_{i=1}^d 1/i! > n$ , every edge-colored graph  $G$  on  $n$  vertices with  $\delta^c(G) \geq d$  contains a properly colored cycle.*

So an interesting problem is to know the exact estimate on the term  $2\sqrt{n}$  in Theorem 3. Hence, the first open case is  $r = 4$ .

Theorem 10 establishes the color degree condition forcing a properly colored  $K_{s,t}$ . It would be interesting to know whether there exists a construction indicating that the order of magnitude  $n^{2-1/s}$  in Theorem 10 is tight up to the constant  $\sigma_{s,t}$ .

In [13], Li proved that every edge-colored graph  $G$  on  $n$  vertices with  $\delta^c(G) \geq (n+1)/2$  contains a rainbow triangle. As an extension, one can consider the minimum color degree conditions for larger rainbow cliques. In addition, it would be also interesting to determine the minimum color degree condition forcing a rainbow cycle of length at most  $r$  for  $r \geq 4$ .

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