PARTITION AND COHEN–MACAULAY EXTENDERS

JOSEPH DOOLITTLE, BENNET GOECKNER, AND ALEXANDER LAZAR

Abstract. If a pure simplicial complex is partitionable, then its h-vector has a combinatorial interpretation in terms of any partitioning of the complex. Given a non-partitionable complex Δ, we construct a complex $\Gamma \supseteq \Delta$ of the same dimension such that both Γ and the relative complex (Γ, Δ) are partitionable. This allows us to rewrite the h-vector of any pure simplicial complex as the difference of two h-vectors of partitionable complexes, giving an analogous interpretation of the h-vector of a non-partitionable complex.

By contrast, for a given complex Δ it is not always possible to find a complex Γ such that both Γ and (Γ, Δ) are Cohen–Macaulay. We characterize when this is possible, and we show that the construction of such a Γ in this case is remarkably straightforward. We end with a note on a similar notion for shellability and a connection to Simon's conjecture on extendable shellability for uniform matroids.

1. Introduction

The h-vector of a simplicial complex contains important and well-studied information about the complex and its associated Stanley–Reisner ring. If a pure complex is *partitionable*, then the entries of its h-vector are non-negative and have a combinatorial interpretation in terms of the partitioning of the face poset. In general, the h-vector can be described algebraically in terms of the Stanley–Reisner ring of Δ, but the aforementioned combinatorial interpretation for the h-vector of a partitionable complex does not apply to non-partitionable complexes.

We introduce a new object of study, which we will use to extend the combinatorial interpretation for the h-vector.

Definition 1.1. Let Δ be a pure d-dimensional simplicial complex. A pure d-dimensional complex Γ is a *partition extender* for Δ if

- $\Delta \subseteq \Gamma$.
- Γ is partitionable.
- The relative complex (Γ, Δ) is partitionable.

Theorem 1.2 (Theorem 4.1)**.** *Every pure simplicial complex has a partition extender.*

For any relative complex (Γ, Δ) with $\dim \Gamma = \dim \Delta$ we can write

$$h(\Delta) = h(\Gamma) - h(\Gamma, \Delta).$$

When Γ is a partition extender for Δ, then both of the right-hand h-vectors have combinatorial interpretations. This allows us to view the h-vector of Δ as an "error term" between the h-vector of Γ and the h-vector of (Δ, Γ). Specifically, every h-vector of a simplicial complex is the difference between the h-vector of a partitionable complex and the h-vector of a partitionable relative complex.

Our construction of a partition extender can be generalized to nonpure complexes. In the nonpure case, partitionability is a more subtle condition than in the pure case (see [11]). However, we show that our construction satisfies strong enough properties to yield a combinatorial interpretation of the h-triangle of an arbitrary nonpure complex.

We further show that if $\operatorname{depth} \Bbbk[\Delta] = \dim \Bbbk[\Delta] - 1$, then for any Cohen–Macaulay complex Γ of the same dimension that contains Δ, the relative complex (Γ, Δ) is Cohen–Macaulay. This similarly allows us to write the h-vector of any such complex as the difference between the h-vector of a Cohen–Macaulay complex and the h-vector of a relative Cohen–Macaulay complex. We also show that such a Γ does not exist if the depth of $\Bbbk[\Delta]$ is any lower.

While an equivalent notion for shellability is straightforward to define, it is unclear when shellable extenders exist. They certainly cannot exist whenever $\operatorname{depth} \Bbbk[\Delta] < \dim \Bbbk[\Delta] - 1$, since relative shellability implies relative Cohen–Macaulayness. We conclude with a connection to Simon's conjecture on shellability of uniform matroids [16, Conjecture 4.2.1].

In Section 2, we review standard definitions and background material. In Section 3, we give explicit constructions which have the required properties to make our proofs work. In Section 4, we provide our main result on partition extenders for pure complexes. In Section 5, we consider the case of nonpure partitionability. In Section 6, we prove parallel results with the Cohen–Macaulay property in place of partitionable. In Section 7, we survey the current state of the problem with the shellable property. In Section 8, we discuss possible future directions of investigation.

2. Preliminaries

A *simplicial complex* Δ is a collection of sets such that if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$. The elements of Δ are called *faces* of Δ, and maximal faces are called *facets*. If σ is a face of Δ, the *dimension* of σ is $\dim(\sigma) := |\sigma| - 1$. The *dimension* of Δ is defined to be the maximum of the dimensions of the faces of Δ. We say that Δ is *pure* if all its facets have the same dimension. Let Δ be a d-dimensional simplicial complex. The *f-vector* of Δ is the vector

$$f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), f_1(\Delta), \dots, f_d(\Delta)),$$

where $f_i(\Delta)$ is the number of i-dimensional faces of Δ. Note that $f_{-1}(\Delta) = 1$ unless Δ is the *empty complex* $\Delta = \varnothing$.

The *h-vector* of Δ is the vector $h(\Delta) = (h_0(\Delta), h_1(\Delta), \dots, h_{d+1}(\Delta))$, whose entries are defined by the relation

$$\sum_{i=0}^{d+1} f_{i-1}(\Delta)(x-1)^{d-i+1} = \sum_{i=0}^{d+1} h_i(\Delta)x^{d-i+1}. \tag{1}$$

The *face poset* $P(\Delta)$ of a simplicial complex Δ is the set of all faces of Δ, partially-ordered by inclusion. An *interval* I in a poset P, denoted $I = [\sigma, \tau]$, is the set of elements e of P such that $\sigma \le e \le \tau$. When this interval I is itself a Boolean poset (i.e., $I \cong 2^{[k]}$ for some $k \in \mathbb{Z}_{\ge 0}$), we say it is a *Boolean interval.*

Let Γ be a simplicial complex and Δ be a subcomplex of Γ. The *relative complex* (Γ, Δ) consists of the faces of Γ not contained in Δ. A relative complex is *pure* if all its maximal faces have the same dimension. If (Γ, Δ) is a relative complex, we can define $f(\Gamma, \Delta) = (f_{-1}(\Gamma, \Delta), \dots, f_d(\Gamma, \Delta))$ by $f_j(\Gamma, \Delta) = f_j(\Gamma) - f_j(\Delta)$ for all j. We can further define $h(\Gamma, \Delta)$ via (1) above.

A poset P is said to be *partitionable* if P can be written as a disjoint union of intervals $I_1 \sqcup \cdots \sqcup I_k$ such that each I_j is a Boolean interval and the maximum element of each I_j is a maximal element of P. A (relative) complex is said to be *partitionable* if its face poset is partitionable.

Proposition 2.1. [18, Page 118] *If a pure relative complex is partitionable, then* $h_i(\Gamma, \Delta)$ *is the number of Boolean intervals in any partitioning of the face poset of* (Γ, Δ) *whose minimal element is an* $(i-1)$*-dimensional face of* (Γ, Δ)*.*

We note that for any simplicial complex Γ that $(\Gamma, \varnothing) = \Gamma$, so Proposition 2.1 holds for simplicial complexes as well. There is no previously known combinatorial interpretation of the h-vectors for non-partitionable complexes.

The notation $[n]$ indicates the set of integers $\{1, 2, \dots, n\}$. We take as a convention that $[0] = \varnothing$. Throughout the rest of this paper, we assume that all simplicial complexes are collections of subsets of $[n]$.

If σ is a face of Δ, the *link* of σ in Δ is the simplicial complex

$$\mathrm{lk}_\Delta(\sigma) = \{\tau \in \Delta \mid \sigma \cup \tau \in \Delta,\ \sigma \cap \tau = \varnothing\}.$$

A simplicial complex Δ is said to be *Cohen–Macaulay* (over $\Bbbk$) if, for all faces $\sigma \in \Delta$,

$$\tilde{H}_i(\mathrm{lk}_\Delta(\sigma), \Bbbk) = \begin{cases} \Bbbk^{\beta_\sigma}, & i = \dim(\Delta) - \dim(\sigma) - 1 \\ 0, & \text{otherwise} \end{cases}$$

where $\tilde{H}_i(X, \Bbbk)$ is the i^{th} reduced homology group of X with coefficients in $\Bbbk$ and $\beta_\sigma \in \mathbb{N}$ is the top Betti number of the link. By a result of Reisner [15], this definition is equivalent to $\Bbbk[\Delta]$ being Cohen–Macaulay, i.e., that $\operatorname{depth} \Bbbk[\Delta] = \dim \Bbbk[\Delta]$. Here $\Bbbk[\Delta]$ is the *Stanley–Reisner ring* (or *face ring*) of Δ. For a complex Δ on n vertices $\Bbbk[\Delta] \coloneqq \Bbbk[x_1, \dots, x_n]/I_\Delta$ where I_Δ is the monomial ideal generated by non-faces of Δ.

Given a face $\sigma \in \Delta$, we distinguish between the face σ and the complex $\langle\sigma\rangle$ whose only facet is σ. If $\dim \sigma = d$, we call this latter object a *d-simplex.*

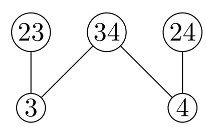
3. Intermediate Constructions

Our main goal is to write the h-vector of any pure complex as the difference of h-vectors of two partitionable (relative) complexes. We will prove that this is always possible in Section 4. In this section we introduce two intermediate constructions.

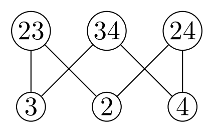
Definition 3.1. A (d, k)-*partition extender* is a pure d-dimensional simplicial complex Δ with a specified facet F and a k-dimensional face σ in F such that both $(\Delta, \langle F\rangle)$ and $(\Delta, \langle F\rangle) \cup \{\sigma\}$ are partitionable.

Remark 3.2. It is not true that the object $(\Delta, \langle F\rangle) \cup \{\sigma\}$ in Definition 3.1 is a relative complex in general, but we can still determine whether its face poset is partitionable or not.

Example 3.3. An example of a $(1, 0)$-partition extender is $\Delta = \langle 12, 23, 34, 24\rangle$ with $F = 12$ and $\sigma = 2$. The face poset of $(\Delta, \langle F\rangle)$ is pictured below. A partitioning of this poset is given by the intervals $[23, 23]$, $[3, 34]$, and $[4, 24]$.



The poset of $(\Delta, \langle F\rangle) \cup \{\sigma\}$, which has a partitioning into the intervals $[2, 23]$, $[3, 34]$, and $[4, 24]$, is shown below.



Definition 3.4. A (d, k)-*prepartition extender* is a pure d-dimensional simplicial complex Δ with a specified facet F, and a face σ in F of dimension k such that $(\Delta, \langle F\rangle) \cup \{\sigma\}$ is partitionable.

This differs from a (d, k)-partition extender in that we do not require $(\Delta, \langle F\rangle)$ to be partitionable.

Note that σ is in F, so there are no elements below it in $(\Delta, \langle F\rangle) \cup \{\sigma\}$. Therefore in any partitioning of the poset $(\Delta, \langle F\rangle) \cup \{\sigma\}$, σ must be a bottom element of some interval in the partitioning.

Proposition 3.5. *For all $-1 \le k \le d$, there exists a (d, k)-prepartition extender.*

Proof. We prove this proposition by directly constructing a (d, k)-prepartition extender for arbitrary k and d. Consider two d-simplices, D_1 and D_2 such that $D_1 \cap D_2 = \sigma$, where σ is a k-face. Label the vertices of D_1 not in σ as $\{1, \dots, d-k\}$, the vertices of D_2 not in σ as $\{d-k+1, \dots, 2d-2k\}$, and the vertices of σ as $\{2d-2k+1, \dots, 2d-k+1\}$.

Define $W_{1,j} = \{j+1, \dots, j+d-k+1\}$ for all j such that $0 \le j \le d-k-1$, and $W_{2,i} = \sigma \setminus i$ for all i in σ. Let Δ be the simplicial complex on $2d-k+1$ vertices whose facets are D_1, D_2, and all sets of the form $W_{1,j} \cup W_{2,i}$. We emphasize that when $d = k$, there are no valid choices for j, and so Δ is the complex on $d+1$ vertices whose facets are D_1 and D_2, which are in fact the same facet. We also emphasize that when $k = -1$, there are no valid choices for i, so Δ is the complex on $2d+2$ vertices whose facets are D_1 and D_2. For all other choices of d and k, we see that $|W_{1,j}| = d-k+1$ and $|W_{2,i}| = k$. Therefore, in all cases Δ is a pure simplicial complex of dimension d.

The following is a set of Boolean intervals in the face poset of Δ.

$$\begin{aligned} I =& [\sigma, D_1] \\ I' =& [\varnothing, D_2] \\ I_{i,j} =& [\{j+1\} \cup \{v \in \sigma : v < i\}, W_{1,j} \cup W_{2,i}] \text{ for } i \in \sigma \text{ and } 0 \leq j \leq d-k-1. \end{aligned}$$

We claim that every face of Δ is in exactly one of these intervals, except for the face σ which is in both I and I'.

Note that $I \cap I' = \sigma$. Furthermore, I is disjoint from each $I_{i,j}$, since every face in I contains σ, and no face of $I_{i,j}$ contains σ. Likewise, I' is disjoint from each $I_{i,j}$, since $j+1$ is a vertex of D_1 that is not contained in σ, and therefore not contained in D_2.

Consider some face τ not in I or I', that is, τ is not contained in D_2 and τ does not contain σ. Let $j+1$ be the least vertex of τ. Since τ is not in D_2, this means that $j+1$ is in $[d-k]$, and so $0 \leq j \leq d-k-1$. Since $\tau \subseteq W_{1,j'} \cup W_{2,i'}$ for some j', i', the difference between the largest index in τ not in σ and $j'+1$ is at most $d-k$. Therefore, $\tau \cap [2d-2k] \subseteq W_{1,j}$. Let i be the largest vertex of σ such that all smaller labeled vertices of σ are in τ. This implies that i is the smallest vertex of σ not in τ. Since $\tau \not\supseteq \sigma$, there is some vertex of σ not in τ, and therefore this i exists. This shows that $\tau \cap \sigma \subseteq W_{2,i}$. We conclude that τ is in the interval $I_{i,j}$.

Furthermore, we will show that τ is not in any other interval. By assumption, τ is not in I or I'.

Let $I_{i',j'}$ be an interval which contains τ. Since τ contains all vertices of σ less than i, and $W_{2,i'}$ does not contain i', then i' cannot be less than i, since that would imply that τ both does and does not contain i'. Likewise, i' cannot be greater than i, since every face in $I_{i',j'}$ contains the vertices of σ less than i', and τ does not contain i, which is one of those vertices. Therefore $i' = i$.

Furthermore, we see that j' cannot be greater than j, since otherwise $W_{1,j'}$ does not contain $j+1$, and τ does contain $j+1$. Similarly, j' cannot be less than j, because every face in $I_{i,j'}$ contains $j'+1$, but $j+1$ was the smallest vertex that τ contained. Therefore $j' = j$.

Therefore the only interval that contains τ is $I_{i,j}$.

This means that Δ is a (d,k)-prepartition extender, with D_2 as the specified facet, σ as the specified face, and the set $\{I\} \cup \bigcup_{i,j}\{I_{i,j}\}$ as a partition of $(\Delta, \langle D_2 \rangle) \cup \{\sigma\}$. $\square$

Example 3.6. We describe the facets of (d,k)-prepartition extenders given in Proposition 3.5 for $d-2 \leq k \leq d$.

A (d,d)-prepartition extender is a d-simplex.

A $(d,d-1)$-prepartition extender has the following set of facets:

$$\begin{aligned} D_1 &= \{1,3,4,\ldots,d+2\} \\ D_2 &= \{2,3,\ldots,d+2\} \\ W_{1,0} \cup W_{2,i} &= \{1,2,\ldots,\hat{i},\ldots,d+2\}, \ \ 3 \leq i \leq d+2, \end{aligned}$$

where $\{1,2,\ldots,\hat{i},\ldots,d+2\}$ is the set $\{1,2,\ldots,d+2\} \setminus \{i\}$. We therefore see that a $(d,d-1)$-prepartition extender is the boundary of the $(d+1)$-simplex on vertex set $[d+2]$.

A $(d,d-2)$-prepartition extender has the following set of facets:

$$\begin{aligned} D_1 &= \{1,2,5,6\ldots,d+3\} \\ D_2 &= \{3,4,5,\ldots,d+3\} \\ W_{1,0} \cup W_{2,i} &= \{1,2,3,5,\ldots,\hat{i},\ldots,d+3\} \ \ 5 \leq i \leq d+3 \\ W_{1,1} \cup W_{2,i} &= \{2,3,4,5,\ldots,\hat{i},\ldots,d+3\} \ \ 5 \leq i \leq d+3. \end{aligned}$$

Remark 3.7. Let Δ be a (d,k)-prepartition extender given in Proposition 3.5 with specified facet F and specified k-face $\sigma \in F$. Then, if we define $h_\ell((\Delta, \langle F \rangle) \cup \{\sigma\})$ to be the number of Boolean intervals in the partitioning of $(\Delta, \langle F \rangle) \cup \{\sigma\}$ whose bottom element has size ℓ,

$$h_\ell((\Delta, \langle F \rangle) \cup \{\sigma\}) = \begin{cases} d-k, & \ell < k+1 \\ d-k+1, & \ell = k+1 \\ 0, & \text{otherwise} \end{cases}.$$

Proof. For all $\ell < k+1$, there are exactly $(d-k)$ intervals $I_{i,j}$ in the partitioning above whose bottom elements have size ℓ. If $\ell = k+1$, there are $d-k$ intervals $I_{i,j}$ whose bottom elements have size ℓ, and the interval $I = [\sigma, D_1]$ also has a bottom element whose size is ℓ. $\square$

Proposition 3.8. *For all $-1 \leq k \leq d$, there exists a (d,k)-partition extender.*

Proof. Recall from Definition 3.1 that a (d,k)-partition extender consists of a pure d-dimensional complex Δ, along with a specified facet F and specified k-dimensional face σ in F. We construct our (d,k)-partition extender inductively, starting with $k = d$ and decreasing k. First we note that a (d,d)-prepartition extender is in fact a (d,d)-partition extender. Indeed, in any partitioning of a (d,d)-prepartition extender, one of the intervals must be $[\sigma,\sigma]$, and so removing σ and this interval gives the required partitioning of $(\Delta, \langle F \rangle)$.

Suppose that (d,h)-partition extenders exist for all $h > k$. We will construct a (d,k)-partition extender K with specified facet F, and specified k-face σ. Let K' be a (d,k)-prepartition extender with specified facet F and specified k-face σ.

First, fix a partitioning of $(K', \langle F\rangle) \cup \{\sigma\}$. Let $\tilde{F}$ be the top element in the interval containing σ in this partitioning. Let τ be an h-face of K' such that $\sigma \subsetneq \tau \subseteq \tilde{F}$. By induction, there exists a (d, h)-partition extender K_τ with specified facet F_τ and specified h-face σ_τ. Attach this (d, h)-partition extender to K' by identifying F_τ with $\tilde{F}$ and identifying σ_τ with τ. We define K to be the complex obtained from K' by attaching K_τ for each τ with $\sigma \subsetneq \tau \subseteq \tilde{F}$.

The complex K with specified facet F and specified k-face σ is a (d, k)-partition extender. To verify this, we need a partitioning of $(K, \langle F\rangle) \cup \{\sigma\}$ and a partitioning of $(K, \langle F\rangle)$. We note that K consists of a (d, k)-prepartition extender K', and many (d, h)-partition extenders K_τ, for each $k < h \leq d$.

First, $(K, \langle F\rangle) \cup \{\sigma\}$ admits a partitioning consisting of

(1) the partitioning of $(K', \langle F\rangle) \cup \{\sigma\}$ arising from its status as a prepartition extender,
(2) the partitionings of the K_τ such that τ is not included in the partitioned set.

Furthermore, $(K, \langle F\rangle)$ admits a partitioning consisting of

(1) the partitioning of $(K', \langle F\rangle) \cup \{\sigma\}$ excluding the interval $[\sigma, \tilde{F}]$,
(2) the partitionings of the K_τ such that τ is included in the partitioned set.

Since both of these partitionings exist, K is a (d, k)-partition extender, and by induction, (d, k)-partition extenders exist for all pairs (d, k) with $d \geq k$. $\square$

Previously, we had described (d, k)-prepartition extenders. Both (d, d)- and $(d, d-1)$-prepartition extenders are in fact partition extenders. To illustrate the full construction of a (d, k)-partition extender, we give a small example in which the partition extender differs from the prepartition extender.

Example 3.9. We give an example of a $(3, 1)$-partition extender K using the construction in Proposition 3.8. We start with a $(3, 1)$-prepartition extender: Following Proposition 3.5, we construct the prepartition extender

$$K' = \langle 1256, 3456, 1236, 1235, 2346, 2345\rangle$$

with specified facet $F = 3456$ and specified face $\sigma = 56$. This labeling is identical to the canonical $(3, 1)-$prepartition extender as constructed in Proposition 3.5.

We observe that K' is exactly the canonical $(3, 1)$-prepartition extender that we constructed earlier. The following is a partitioning of $(K', \langle 3456\rangle) \cup \{56\}$ given by our construction:

$$[56, 1256] \quad [1, 1236] \quad [2, 2346] \quad [15, 1235] \quad [25, 2345]. \tag{2}$$

We now must create partition extenders for each $56 \subsetneq \tau \subsetneq 1256$, i.e., we create $(3, 2)$-partition extenders for the faces 156 and 256. Recall that the other intervals in (2) are fixed and will be part of both partitionings.

For the face 156, we construct the partition extender

$$K'' = \langle 7156, 2156, 7256, 7216, 7215 \rangle$$

with specified facet $F = 2156$ and specified face $\sigma = 156$. The bijection to our canonical $(3, 2)-$prepartion extender is induced by $(7, 2, 1, 5, 6) \mapsto (1, 2, 3, 4, 5)$. The following is a partitioning of $(K'', \langle 2156 \rangle)$.

$$[7156, 7156] \quad [7, 7256] \quad [71, 216] \quad [715, 7215] \tag{3}$$

For the face 256, we construct the partition extender

$$K''' = \langle 8256, 1256, 8156, 8126, 8125 \rangle$$

with specified facet $F = 1256$ and specified face $\sigma = 256$. The bijection to our canonical $(3, 2)-$prepartion extender is induced by $(8, 1, 2, 5, 6) \mapsto (1, 2, 3, 4, 5)$. The following is a partitioning of $(K''', \langle 1256 \rangle)$:

$$[8256, 8256] \quad [8, 8156] \quad [82, 8126] \quad [825, 8125]. \tag{4}$$

The $(3, 1)$-partition extender is $K = K' \cup K'' \cup K'''$ with specified facet $F = 3456$ and specified face $\sigma = 56$. Equations (2), (3), and (4) together give a partitioning of $(K, \langle 3456 \rangle) \cup \{56\}$.

For a partitioning of $(K, \langle 3456 \rangle)$, we take the partitionings from equations (2), (3), and (4) and modify only the first interval in each line. We get the following:

[1256,1256]	[1,1236]	[2,2346]	[15,1235]	[25,2345]
[156,1567]	[7,2567]	[17,1267]	[157,1257]	
[256,2568]	[8,1568]	[28,1268]	[258,1258].	

Thus K is a $(3, 1)$-partition extender.

4. Main Theorem

Now we are prepared to prove our main result.

Theorem 4.1. *Every pure simplicial complex has a partition extender.*

Proof. Let Δ be a pure d-dimensional complex. For each k-face σ of Δ, attach a (d, k)-partition extender to Δ by identifying σ and a facet containing σ to the specified faces of the (d, k)-partition extender. Call this complex Γ. By Proposition 3.8, Γ is a pure partitionable d-dimensional complex, with the partition where each (d, k)-extender uses the σ it was attached to. Furthermore, (Γ, Δ) is partitionable, with the partition where each (d, k)-extender is partitioned without the σ it was attached to. Therefore Γ is a partition extender for Δ. $\square$

We now provide a combinatorial interpretation of the h-vector of a pure simplicial complex Δ with a partition extender Γ. We can write the f-vector of Δ as

$$f_i(\Delta) = f_i(\Gamma) - f_i(\Gamma, \Delta).$$

Since the h-vector is a bijective linear transformation of the f-vector, we transform the above equation into

$$h_i(\Delta) = h_i(\Gamma) - h_i(\Gamma, \Delta).$$

Since both Γ and (Γ, Δ) are pure and partitionable, we may use the combinatorial interpretation of these values to give a combinatorial interpretation of $h_i(\Delta)$.

Corollary 4.2. *If Δ is a pure simplicial complex, then*

$$\begin{aligned} h_i(\Delta) =& |\{\textit{intervals in a partitioning of } \Gamma \textit{ with bottom element of size } i\}| \\ &- |\{\textit{intervals in a partitioning of } (\Gamma, \Delta) \textit{ with bottom element of size } i\}| \end{aligned}$$

for any partition extender Γ of Δ.

In our construction of the partition extender Γ of Δ, there is significant overlap between the sets of intervals in the partitioning of Γ and the partitioning of (Γ, Δ). Keeping track of the heights of the intervals that differ between the partitioning of (Γ, Δ) and that of Γ yields

$$h_i(\Gamma) - h_i(\Gamma, \Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}(\Delta),$$

which is exactly the formula for $h_i(\Delta)$ in terms of the $f_j(\Delta)$. Thus our construction gives a combinatorial witness to the algebraic transformation between $h(\Delta)$ and $f(\Delta)$.

5. Nonpure Partitionability

Our construction of a partition extender can be applied to nonpure complexes in a natural way. Suppose that Δ is a nonpure complex. If σ is a face of Δ, we write

$$d_\Delta(\sigma) \coloneqq \max_{\tau \in \Delta} \{\dim(\tau) \mid \sigma \subseteq \tau\}.$$

In [5, Definition 3.1], Björner and Wachs define a two-dimensional array called the *f-triangle* $f^{\triangle}(\Delta)$ that refines the f-vector of Δ, with entries given by

$$f_{i,j}(\Delta) = |\{\sigma \in \Delta \mid d_\Delta(\sigma) = i - 1,\ \dim(\sigma) = j - 1\}|.$$

Björner and Wachs also define a refinement of the h-vector called the h-triangle $h^{\triangle}(\Delta)$ which is a two-dimensional array with entries $h_{i,j}(\Delta)$ that is obtained from $f^{\triangle}(\Delta)$ by applying the f-vector to h-vector transformation on each row of $f^{\triangle}(\Delta)$. More precisely,

$$h_{i,j}(\Delta) = \sum_{k=0}^{j} (-1)^{j-k} \binom{i-k}{j-k} f_{i,k}.$$

The f- and h-triangles of a relative complex (Γ, Δ) are defined analogously.[1]

[1]Note that if $\dim(\Gamma) \neq \dim(\Delta)$ then the f-triangles of Γ and Δ will have different dimensions.

Remark 5.1. If $\Gamma \supseteq \Delta$ with $\dim(\Gamma) = \dim(\Delta)$ and $d_\Delta(\sigma) = d_\Gamma(\sigma)$ for all $\sigma \in \Delta$, then $f^\triangle(\Gamma, \Delta) = f^\triangle(\Gamma) - f^\triangle(\Delta)$. Indeed, suppose that $\sigma \in \Gamma$ contributes to $f_{i,j}(\Gamma)$. Either $\sigma \in \Delta$, in which case by assumption it contributes to $f_{i,j}(\Delta)$, or $\sigma \in (\Gamma, \Delta)$, in which case it contributes to $f_{i,j}(\Gamma, \Delta)$. Since the f-triangle to h-triangle transformation is linear and $d_\Delta(\sigma) = d_\Gamma(\sigma)$, we also have $h^\triangle(\Gamma, \Delta) = h^\triangle(\Gamma) - h^\triangle(\Delta)$.

It is natural to assume that the entries $h_{i,j}$ of the h-triangle of a partitionable nonpure complex have an analogous interpretation to the entries of the h-vector of a pure partitionable complex. This is *false* in general. In [11, Example 1], Hachimori gives an example of a partitionable nonpure complex whose h-triangle has a negative entry.

However, Hachimori introduces several strictly stronger variants of partitionability for nonpure complexes [11]; among these is the existence of an *h-compatible* partitioning of Δ, i.e., a partitioning of the face poset of Δ where $h_{i,j}(\Delta)$ is the number of Boolean intervals in the partitioning whose bottom element is a face of size j and whose top element is a facet of size i. In [11, Theorem 2], Hachimori shows that h-compatibility is equivalent to a property he calls layer-compatibility: A partitioning

$$P(\Delta) = \bigsqcup_{F \text{ facet of } \Delta} [\Psi(F), F]$$

of the face poset of Δ is *layer-compatible* if the restriction

$$\bigsqcup_{\substack{F \text{ facet of } \Delta \\ \dim(F) \geq r}} [\Psi(F), F]$$

is a partitioning of the face poset of $\langle F \mid F \text{ facet of } \Delta,\ \dim(F) \geq r \rangle$ for all $0 \leq r \leq \dim(\Delta)$.

Remark 5.2. While [11, Theorem 2] is stated in terms of absolute complexes, the same proof works for relative complexes as well.

We can now prove a nonpure analog of Theorem 4.1.

Theorem 5.3. *Let Δ be a nonpure complex. Then there is a complex $\Gamma \supseteq \Delta$ with $\dim(\Gamma) = \dim(\Delta)$ such that Γ and (Γ, Δ) are layer-compatibly partitionable.*

Proof. Let Δ be a nonpure complex, and let Γ be the complex obtained by attaching a $(d_\Delta(\sigma), k)$-partition extender to each k-face σ of Δ for all k. Clearly, $d_\Delta(\sigma) = d_\Gamma(\sigma)$ for all $\sigma \in \Delta$, so we can write the $h^\triangle(\Delta)$ as the difference $h^\triangle(\Gamma) - h^\triangle(\Gamma, \Delta)$.

It is easy to check that the partitionings of Γ and (Γ, Δ) we construct in Proposition 3.8 are both layer-compatible. □

Since layer-compatibility implies h-compatibility, we now have a combinatorial interpretation of the h-triangle of any nonpure complex. We define an *(i, j)-interval* of Δ to be a Boolean interval of $P(\Delta)$ whose bottom element has size j and whose top element is a facet of size i.

Corollary 5.4. *For any nonpure complex* Δ*, we have*

$$\begin{aligned} h_{i,j}(\Delta) =& |\{(i,j)\text{-intervals in an } h\text{-compatible partitioning of } \Gamma\}| \\ &- |\{(i,j)\text{-intervals in an } h\text{-compatible partitioning of } (\Gamma,\Delta)\}|, \end{aligned}$$

where Γ *is the partition extender constructed in Theorem 5.3.*

6. Cohen–Macaulay Extenders

Given the existence of partition extenders of pure simplicial complexes, it seems natural to ask if extenders exist for other well-studied combinatorial properties of simplicial complexes. A relative complex (Γ, Δ) is *relative Cohen–Macaulay* if I_Γ/I_Δ is a Cohen–Macaulay $\Bbbk[\mathbf{x}]$-module. Equivalently, a relative complex is relative Cohen–Macaulay if the relative homology $\tilde{H}_i(\mathrm{lk}_\Gamma(\sigma), \mathrm{lk}_\Delta(\sigma))$ is trivial except possibly when $|\sigma| + i = d$, where d is the dimension of Γ [18, Theorem III.7.2].

Definition 6.1. Let Δ be a pure d-dimensional simplicial complex. A d-dimensional complex Γ is a *Cohen–Macaulay extender* for Δ if

- $\Delta \subseteq \Gamma$.
- Γ is Cohen–Macaulay.
- The relative complex (Γ, Δ) is relative Cohen–Macaulay.

Unlike the case for partition extenders, there is a large class of pure complexes for which Cohen–Macaulay extenders do not exist. The *depth* of a simplicial complex Δ is defined as $\operatorname{depth} \Bbbk[\Delta]$, the depth of its Stanley–Reisner ring. By applying Hochster's formula [13], it can be shown that $\operatorname{depth} \Bbbk[\Delta]$ is the largest integer h such that $\tilde{H}_i(\mathrm{lk}_\Delta(\sigma))$ is trivial whenever $|\sigma| + i + 1 < h$ for all $-1 < i < d$ and $\sigma \in \Delta$. We recall that for a d-dimensional simplicial complex Δ, $\dim \Bbbk[\Delta] = d + 1$.

Proposition 6.2. *If* Δ *is a simplicial complex such that* $\operatorname{depth} \Bbbk[\Delta] < \dim \Bbbk[\Delta] - 1$*, then* Δ *does not have a Cohen–Macaulay extender.*

Proof. Let Δ be a d-dimensional complex with $\operatorname{depth} \Bbbk[\Delta] < \dim \Bbbk[\Delta] - 1$. By definition, there is a face $\sigma \in \Delta$ and an index i such that $\tilde{H}_i(\mathrm{lk}_\Delta(\sigma))$ is nontrivial where $|\sigma| + i + 1 = \operatorname{depth} \Bbbk[\Delta] \le d - 1$; equivalently, $i + 1 \le d - |\sigma| - 1$.

Suppose Γ is a d-dimensional complex such that Γ is Cohen–Macaulay and $\Delta \subseteq \Gamma$. We can write the long exact sequence of relative homology for the pair $(\mathrm{lk}_\Gamma(\sigma), \mathrm{lk}_\Delta(\sigma))$.

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \tilde{H}_{d-|\sigma|}(\mathrm{lk}_\Delta(\sigma)) \arrow[r] & \tilde{H}_{d-|\sigma|}(\mathrm{lk}_\Gamma(\sigma)) \arrow[r] & \tilde{H}_{d-|\sigma|}((\mathrm{lk}_\Gamma(\sigma), \mathrm{lk}_\Delta(\sigma))) \arrow[dll] \\
& \tilde{H}_{d-|\sigma|-1}(\mathrm{lk}_\Delta(\sigma)) \arrow[r] & \tilde{H}_{d-|\sigma|-1}(\mathrm{lk}_\Gamma(\sigma)) \arrow[r] & \tilde{H}_{d-|\sigma|-1}((\mathrm{lk}_\Gamma(\sigma), \mathrm{lk}_\Delta(\sigma))) \arrow[dll] \\
& \tilde{H}_{d-|\sigma|-2}(\mathrm{lk}_\Delta(\sigma)) \arrow[r] & \tilde{H}_{d-|\sigma|-2}(\mathrm{lk}_\Gamma(\sigma)) \arrow[r] & \tilde{H}_{d-|\sigma|-2}((\mathrm{lk}_\Gamma(\sigma), \mathrm{lk}_\Delta(\sigma))) \arrow[dll] \\
& \tilde{H}_{d-|\sigma|-3}(\mathrm{lk}_\Delta(\sigma)) \arrow[r] & \tilde{H}_{d-|\sigma|-3}(\mathrm{lk}_\Gamma(\sigma)) \arrow[r, dashed] & {}
\end{tikzcd}$$

Since Γ is Cohen–Macaulay, we know that $\tilde{H}_j(\mathrm{lk}_\Gamma(\sigma))$ is trivial whenever $j < d-|\sigma|$. This observation lets us break up the long exact sequence into the following exact sequences for each $\ell \geq 1$:

$$0 \longrightarrow \tilde{H}_{d-|\sigma|-\ell}((\mathrm{lk}_\Gamma(\sigma), \mathrm{lk}_\Delta(\sigma))) \longrightarrow \tilde{H}_{d-|\sigma|-\ell-1}(\mathrm{lk}_\Delta(\sigma)) \longrightarrow 0$$

Each of these middle maps is an isomorphism. Since $\tilde{H}_i(\mathrm{lk}_\Delta(\sigma))$ is nontrivial, $\tilde{H}_{i+1}((\mathrm{lk}_\Gamma(\sigma), \mathrm{lk}_\Delta(\sigma)))$ is also nontrivial. Since $i+1 \leq d-|\sigma|-1$, the relative complex (Γ, Δ) is not relative Cohen–Macaulay. Therefore there is no Cohen–Macaulay extender for Δ. □

Theorem 6.3. *Let Δ be a simplicial complex. Then Δ has a Cohen–Macaulay extender if and only if* $\operatorname{depth} \Bbbk[\Delta] \geq \dim \Bbbk[\Delta] - 1$.

Proof. The case that $\operatorname{depth} \Bbbk[\Delta] < \dim \Bbbk[\Delta] - 1$ is covered by Proposition 6.2, so we assume that $\operatorname{depth} \Bbbk[\Delta] \geq \dim \Bbbk[\Delta] - 1$.

Let Δ be a d-dimensional simplicial complex with depth at least d, and let Γ be a Cohen–Macaulay d-dimensional complex that contains Δ. We begin by writing a short exact sequence of modules over $\Bbbk[x_1, \ldots, x_n]$ with I_Δ and I_Γ as the Stanley–Reisner ideals associated to Δ and Γ.

$$0 \to I_\Delta/I_\Gamma \to \Bbbk[\Gamma] \to \Bbbk[\Delta] \to 0$$

By the assumptions on Δ and Γ, we can see that $\operatorname{depth} \Bbbk[\Gamma] = \dim \Bbbk[\Gamma]$ and $\operatorname{depth} \Bbbk[\Delta] \geq \dim \Bbbk[\Delta] - 1 = \dim \Bbbk[\Gamma] - 1$. By the Depth Lemma [6, Proposition 1.2.9], we get that $\operatorname{depth}(I_\Delta/I_\Gamma) = \dim \Bbbk[\Gamma] - 1$. This is equivalent to saying that (Γ, Δ) is relative Cohen–Macaulay. Therefore Γ is a Cohen–Macaulay extender of Δ. □

Theorem 6.3 shows that if $\operatorname{depth} \Bbbk[\Delta] \geq \dim \Bbbk[\Delta] - 1$, then *any* Cohen–Macaulay complex Γ of the same dimension that contains Δ is a Cohen–Macaulay extender for Δ. If Δ is a d-dimensional complex on $n+1$ vertices, then perhaps the most natural Cohen–Macaulay extender to consider is the d-skeleton of the n-simplex $\Delta_n^{(d)}$, which is

$$\Delta_n^{(d)} = \{\sigma \subseteq [n+1] \ : \ |\sigma| \leq d+1\}.$$

In particular, we note that if a Cohen–Macaulay extender exists for a complex, then we can construct one without introducing new vertices.

Codenotti, Katthän, and Sanyal recently classified the h-vectors of relative Cohen–Macaulay complexes. In [7, Theorem 5.7], it is shown that $(h_0, \ldots, h_{d+1})$ is the h-vector of a proper Cohen–Macaulay relative complex if and only if $h_0 = 0$ and $h_i \geq 0$ for all i, answering a question of Björner in [17]. (Here "proper" means that the subcomplex in question is not the void complex.) They find more a restrictive characterization in [7, Theorem 1.3] for Cohen–Macaulay relative complexes on ground set $[n]$. Theorem 6.3 is a result in the same vein, with the further constraint that the total complex be Cohen–Macaulay.

7. Shelling extenders and Simon's conjecture

A relative complex (Γ, Δ) is *shellable* if its facets can be ordered $F_1, \dots, F_k$ such that $\langle F_{i+1} \rangle \setminus \langle F_1, \dots, F_i, \Delta \rangle$ has a unique minimal face for all $i \in [k-1]$. Such an ordering of the facets is a *shelling order*. If a pure relative complex is shellable, then it is relative Cohen–Macaulay [18, Page 118]. Therefore, in our search for a similar notion of an extender for shellability, we limit our search to complexes Δ such that $\operatorname{depth} \Bbbk[\Delta] \geq \dim \Bbbk[\Delta] - 1$.

Definition 7.1. Let Δ be a pure d-dimensional simplicial complex. A d-dimensional complex Γ is a *shelling extender* for Δ if

- $\Delta \subseteq \Gamma$.
- Γ is shellable.
- The relative complex (Γ, Δ) is shellable.

Conjecture 7.2. *If* Δ *is a simplicial complex such that* $\operatorname{depth} \Bbbk[\Delta] \geq \dim \Bbbk[\Delta] - 1$ *for all fields* $\Bbbk$*, then* Δ *has a shelling extender.*

Such shellable extenders may have application to a conjecture of Simon. We first recall that a pure complex Δ is *extendably shellable* if every partial shelling order $F_1, \dots, F_j$ can be extended to a shelling order $F_1, \dots, F_j, F_{j+1}, \dots, F_k$ of Δ.

Conjecture 7.3. [16, Conjecture 4.2.1] *If* Δ *is the* d*-skeleton of an* n*-simplex, then* Δ *is extendably shellable.*

Some partial results about extendable shellability are known. Simon's conjecture is known to be true in certain cases. For $d \leq 1$ and $d \geq n-1$, the conjecture is clearly true. The case $d = n-2$ was proved by Bigdeli, Yazdan Pour, and Zaare-Nahandi in [2] and by Dochtermann in [9] (and was strengthened by Culbertson, Dochtermann, Guralnik and Stiller in [8]).

The case $d = 2$ was shown by Björner and Eriksson in [4] as a consequence of the fact that matroid complexes of rank ≤ 3 are extendably shellable, since the d-skeleton of the n-simplex is the independence complex of the uniform matroid of rank $d+1$ over $n+1$ elements. On the other hand, in [12, Theorem 2.3.1] Hall shows that the boundary of the d-crosspolytope is not extendably shellable for $d \geq 12$. In [1], Benedetti and Bolognini found a counterexample to a strengthening of Simon's conjecture that had been posed by Bigdeli and Faridi [3], Dochtermann [9], and Nikseresht [14].

We note the connection between Conjecture 7.2 and Simon's conjecture.

Question 7.4. If a shelling extender exists for Δ, then is it possible to create a shelling extender Γ without introducing any new vertices?

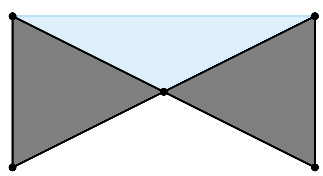
Remark 7.5. If Question 7.4 has a positive answer, then this would prove Conjecture 7.3.

Theorem 6.3 shows that the d-skeleton of the n-simplex is a Cohen–Macaulay extender for Δ whenever such an extender exists. Thus it is reasonable to ask whether this construction is possible in the case of shelling extenders. We note that the h-vector characterizations of shellable relative complexes is the same as in the Cohen–Macaulay case [7], so there is no direct numerical obstruction to this construction.

8. Questions and Future Directions

One may ask how close a given complex Δ is to being partitionable by considering the "smallest" possible partition extender Γ. Our construction produces partition extenders that are quite large, but it is often possible to find smaller extenders by hand. The bow-tie pictured below is a standard example of a non-partitionable complex, with a negative entry in the h-vector.

Example 8.1. Below, the dark complex is the bow-tie with f-vector equal to $(1, 5, 6, 2)$ and h-vector equal to $(1, 2, -1, 0)$. The entire complex pictured has f-vector $(1, 5, 7, 3)$ and h-vector $(1, 2, 0, 0)$. The lighter shaded relative complex has f-vector $(0, 0, 1, 1)$ and h-vector $(0, 0, 1, 0)$. Both the larger complex and relative complex are partitionable, and the h-vector of the bow-tie is given by the difference of the two other h-vectors.



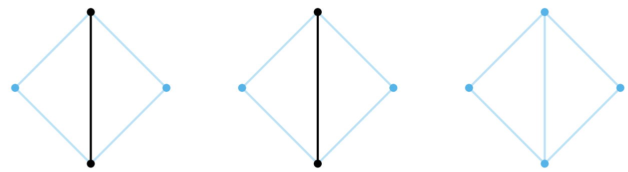
The above example of a partition extender is far smaller than those constructed in the proof of Theorem 4.1. This observation leads naturally to the following questions:

Question 8.2. Is it possible to construct a minimal partition extender with respect to the number of faces added? With respect to the size of the h-vector of the relative complex? With respect to some other measure of size?

Question 8.3. Assuming that a minimal partition extender exists, is it unique?

If, for example, Δ is a complete graph on four vertices together with two additional disjoint edges, then $h(\Delta) = (1, 6, 1)$ but Δ is not partitionable. This means that the number and sizes of the negative entries of the h-vector of a complex does not capture how many faces need to be added to create a partition extender, since there are non-partitionable complexes whose h-vectors are all positive. In fact, a result of Duval, Goeckner, Klivans, and Martin [10] shows that that there are even Cohen–Macaulay complexes (which have much stronger conditions on their h-vectors than positivity) that are non-partitionable.

Example 8.4. Here we explicitly realize our construction on a pair of edges in black, with the partition extender drawn in a lighter shade. Our construction adds 8 vertices and 13 edges, but a minimal partition extender can be created by introducing a single edge to connect the two edges in black.



Given a complex Δ, we might ask for an upper bound on how many faces must be added to create a partition extender Γ via our construction. If $g(k)$ is the number of faces in a $(d, d-k)$-partition extender, then $g(k)$ is defined by the recurrence relation

$$g(k) = k(2^{d+1} - 2^k) + \sum_{j=0}^{k-1} \binom{k}{j} g(j).$$

Since g is an increasing function, if we ignore the term -2^k, we obtain a simple one-term recurrence relation bound

$$g(k) \leq k2^{d+1} + 2^k g(k-1).$$

As long as $g(k-1) > 2^{d+1}$,

$$g(k) \leq 2(2^k) g(k-1).$$

The starting term is $g(0) = 0$, and $g(1) \leq 2^{d+1}$. Therefore, an upper bound for $g(k)$ is

$$g(k) \leq 2^{2^k - 1 + d}.$$

Thus, given a complex Δ with $f(\Delta) = (f_{-1}, f_0, \dots, f_d)$, our construction will add

$$\sum_{-1 \leq k \leq d} f_k \cdot g(d-k) \leq \sum_{-1 \leq k \leq d} f_k \cdot 2^{2^{d-k} - 1 + d}$$

total faces. This bound is not exact, but we expect it to be of the correct order of magnitude. As seen in Example 8.1, the number of faces added in a minimal partition extender can be much lower.

In Section 5 we constructed nonpure partition extenders. Along the same lines, given some condition on the depths of the pure skeletons of a nonpure complex Δ, we expect that it should be possible to construct a *sequentially Cohen–Macaulay extender* Γ, that is, a $\Gamma \supseteq \Delta$ such that $d_\Delta(\sigma) = d_\Gamma(\sigma)$ for all $\sigma \in \Delta$, and Γ and (Γ, Δ) are both sequentially Cohen–Macaulay.

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Email address: `jdoolittle@tugraz.at`

Institute of Geometry, TU Graz, Austria

Email address: `goeckner@uw.edu`

Department of Mathematics, University of Washington

Email address: `alelaz@kth.se`

Department of Mathematics, KTH Royal Institute of Technology, Sweden