Lower bounds for the chromatic number of certain Kneser-type hypergraphs

Soheil Azarpendar Amir Jafari

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Abstract

Let $n \geq 1, r \geq 2$, and $s \geq 0$ be integers and $\mathcal{P} = \{P_1, \ldots, P_l\}$ be a partition of $[n] = \{1, \ldots, n\}$ with $|P_i| \leq r$ for $i = 1, \ldots, l$. Also, let \mathcal{F} be a family of nonempty subsets of [n]. The *r*-uniform Kneser-type hypergraph KG^r($\mathcal{F}, \mathcal{P}, s$) is the hypergraph with the vertex set of all \mathcal{P} -admissible elements $F \in \mathcal{F}$, that is $|F \cap P_i| \leq 1$ for $i = 1, \ldots, l$ and the edge set of all *r*-subsets $\{F_1, \ldots, F_r\}$ of the vertex set that $|F_i \cap F_j| \leq s$ for all $1 \leq i < j \leq r$. In this article, we extend the equitable *r*-colorability defect $\operatorname{ecd}^r(\mathcal{F})$ of Abyazi Sani and Alishahi to the case when one allows intersection among the vertices of an edge. It will be denoted by $\operatorname{ecd}^r(\mathcal{F}, s)$. We then, give (under certain assumptions) lower bounds for the chromatic number of $\operatorname{KG}^r(\mathcal{F}, \mathcal{P}, s)$ and some of its variants in terms of $\operatorname{ecd}^r(\mathcal{F}, \lfloor s/2 \rfloor)$. This work generalizes many existing results in the literature of the Kneser hypergraphs. It generalizes the previous results of the current authors from the special family of all k-subsets of [n] to a general family \mathcal{F} of subsets.

1 Introduction

Let $n \geq 1$ and $r \geq 2$ be integers and $\mathcal{P} = \{P_1, \ldots, P_l\}$ be a partition of $[n] = \{1, \ldots, n\}$ with $|P_i| \leq r$ for $i = 1, \ldots, l$. Let \mathcal{F} be a family of non-empty subsets in [n]. The *r*uniform Kneser-type hypergraph KG^{*r*}(\mathcal{F}, \mathcal{P}) is the hypergraph with the vertex set of all \mathcal{P} -admissible elements $F \in \mathcal{F}$, that is $|F \cap P_i| \leq 1$ for $i = 1, \ldots, l$, and the edge set of all *r*-subsets $\{F_1, \ldots, F_r\}$ of the vertex set that are pairwise disjoint. This hypergraph first was considered by Alishahi and Hajiabolhassan in [2]. It was later considered by Aslam, Chen, Coldren, Frick, and Setiabrata in [6]. For an integer $s \geq 0$, that is assumed to have the property¹ s < |F| for all elements F of \mathcal{F} , we may relax the condition of pairwise disjointness to $|F_i \cap F_j| \leq s$ and arrive at the definition of the *r*-uniform hypergraph KG^{*r*}($\mathcal{F}, \mathcal{P}, s$). We are interested here to find lower bounds for the chromatic number $\chi(\text{KG}^r(\mathcal{F}, \mathcal{P}, s))$ of this hypergraph in terms of a generalization of the equitable *r*-colorability defect of Abyazi Sani and Alishahi [1]. This result is an extension of the previous results of the current authors in [4].

An equitable partition of a set X is a partition of it into subsets X_i for i = 1, ..., r such that $||X_i| - |X_j|| \le 1$ for all $1 \le i \le j \le r$. The equitable r-colorability defect $\operatorname{ecd}^r(\mathcal{F})$ of a family \mathcal{F} of non-empty subsets in [n] is the minimum size of a subset $X_0 \subseteq [n]$ so that there is an equitable partition

$$[n] \setminus X_0 = X_1 \cup \dots \cup X_r$$

¹Without this assumption, we will have a loop edge $\{F, \ldots, F\}$ and the chromatic number of the associated hypergraph is by convention infinity, so there is no need to give a lower bound.

with the property that there are no elements $F \in \mathcal{F}$ and $i = 1, \ldots, r$, with $F \subseteq X_i$. Abyazi Sani and Alishahi [1] proved the following generalization of the corresponding result of Kriz in [7] and [8] for the *r*-colorability defect.

Theorem 1.1. One has

$$\chi(KG^r(\mathcal{F})) \ge \left\lceil \frac{ecd^r(\mathcal{F})}{r-1} \right\rceil.$$

Here $\mathrm{KG}^r(\mathcal{F})$ is the hypergraph with no partition condition, in other words, \mathcal{P} is the partition of [n] by the singletons. Our goal here, is to extend this result to the cases $\mathrm{KG}^r(\mathcal{F},\mathcal{P})$ and $\mathrm{KG}^r(\mathcal{F},\mathcal{P},s)$.

For $s \ge 0$ and subsets A and B of [n], we write $A \subseteq_s B$ if there is a set E of size at most s, such that $A \setminus E \subseteq B$. The general r-equitable colorability defect $\operatorname{ecd}^r(\mathcal{F}, s)$ is the minimum size of a subset $X_0 \subseteq [n]$, such that there is an equitable partition

$$[n] \setminus X_0 = X_1 \cup \cdots \cup X_r$$

with the property that there are no $F \in \mathcal{F}$ and $i = 1, \ldots, r$, such that $F \subseteq_s X_i$.

Remark 1.1. Note that $ecd^{r}(\mathcal{F}, 0)$ is the original equitable *r*-colorability defect of Abyazi Sani and Alishahi. It is easy to see that for the family of all *k*-subsets of [n], denoted by $\binom{[n]}{k}$, when $n \ge r(k-1) + 1$ and $0 \le s < k$, one has

$$ecd^{r}\binom{[n]}{k}, s) = n - r(k - s - 1).$$

We have the following results.

Theorem 1.2. Under the above condition on the partition \mathcal{P} , one has

$$\chi(KG^r(\mathcal{F},\mathcal{P})) \ge \left\lceil \frac{ecd^r(\mathcal{F})}{r-1} \right\rceil$$

Theorem 1.3. Under the above condition on s, one has

$$\chi(KG^r(\mathcal{F},s)) \ge \left\lceil \frac{ecd^r(\mathcal{F},\lfloor s/2 \rfloor)}{r-1} \right\rceil.$$

Here the partition is understood to be trivial, in other words, by the singletons.

Unfortunately to give a unified theorem that deals with the case of $\mathrm{KG}^r(\mathcal{F}, \mathcal{P}, s)$ we need to either assume that the pair $(\mathcal{F}, \mathcal{P})$ satisfies an extra condition or modify the definition of the hypergraph into $\widetilde{\mathrm{KG}}^r(\mathcal{F}, \mathcal{P}, s)$ as follows.

Definition 1.1. The pair $(\mathcal{F}, \mathcal{P})$ is said to be s-good, if for any \mathcal{P} -admissible subset A for which there exists $F \in \mathcal{F}$ so that $F \subseteq_s A$, one can find a \mathcal{P} -admissible element $F' \in \mathcal{F}$ such that $F' \subseteq_s A$.

Remark 1.2. Let us show that the pair $\binom{[n]}{k}$, \mathcal{P}) is s-good, if $n \ge r(k-1)+1$, $0 \le s < k$, and $|P_i| \le r$. Note that by the assumption on n, we have at least k non-empty partition parts in \mathcal{P} . Now suppose $F \subseteq_s A$ for a \mathcal{P} -admissible subset A and a k-subset F. If $|A| \ge k$ any k-subset F' of A is \mathcal{P} -admissible. Hence we may assume, $k - s \le |A| \le k$ and therefore one can always add at most s elements to A from different partition parts with empty intersection with A, to make it into a \mathcal{P} -admissible k-subset F' with $F' \subseteq_s A'$. Without any assumptions on the partition and the family, we need to modify the definition of the hypergraph $\mathrm{KG}^r(\mathcal{F},\mathcal{P},s)$ as follows.

Definition 1.2. We let $\widetilde{KG}^r(\mathcal{F}, \mathcal{P}, s)$ be the r-uniform hypergraph with the vertex set of all elements A of \mathcal{F} such that

$$\sum_{i=1}^{l} \max\left\{ |A \cap P_i| - 1, 0 \right\} \le \lfloor s/2 \rfloor$$

and the edge set of all r-subsets $\{A_1, \ldots, A_r\}$ of vertices such that $|A_i \cap A_j| \leq s$ for all $1 \leq i < j \leq r$.

Note that when s = 0, the above condition is the same as \mathcal{P} -admissibility. Also if \mathcal{P} is the trivial partition into singletons, this condition holds for all $A \in \mathcal{F}$. We have the following two results.

Theorem 1.4. Under the above assumptions, one has

$$\chi(\widetilde{KG}^r(\mathcal{F},\mathcal{P},s)) \ge \left\lceil \frac{ecd^r(\mathcal{F},\lfloor s/2 \rfloor)}{r-1} \right\rceil.$$

Theorem 1.5. If the pair $(\mathcal{F}, \mathcal{P})$ is $\lfloor s/2 \rfloor$ -good, then

$$\chi(KG^{r}(\mathcal{F},\mathcal{P},s)) \geq \left\lceil \frac{ecd^{r}(\mathcal{F},\lfloor s/2 \rfloor)}{r-1} \right\rceil$$

Note that theorem 1.4 implies as its special cases, theorems 1.2 and 1.3.

Remark 1.3. In [5], Daneshpajouh presents the following lower bound for the chromatic number of the hypergraph $KG^r(\binom{[n]}{k}, s)$, when $0 \le s < k$ and $n \ge r(k-1) + 1$

$$\chi(KG^r(\binom{[n]}{k},s)) \ge \left\lceil \frac{n-r(k-s-1)}{r-1} \right\rceil.$$

When $n \ge r(k-1)+1$, then $ecd^r(\binom{[n]}{k}, s) = n - r(k-s-1)$ and hence, this is a stronger lower bound than the one obtained from Theorem 1.3. It is feasible that the above theorems remain true if one replaces |s/2| with s.

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2 Reduction of Theorem 1.4 and Theorem 1.5

In this section, we prove the following lemma, which reduces the proof of Theorem 1.4 and Theorem 1.5, to the case when r is a prime number. The proof is obtained by imitating a method used by Kriz in [8], who himself followed a similar method used by Alon, Frankl, and Lovász in [3].

Lemma 2.1. If Theorem 1.4 (resp. Theorem 1.5) is true for $r = r_1$ and $r = r_2$ then Theorem 1.4 (resp. Theorem 1.5) is true for $r = r_1r_2$. *Proof.* Let $s' = \lfloor s/2 \rfloor$ and $\mathcal{P} = \{P_1, \ldots, P_l\}$ be a partition of [n] with $|P_i| \leq r_1 r_2$. Also, let $\mathcal{P}' = \{P'_1, \ldots, P'_{l'}\}$ be a partition obtained from \mathcal{P} by partitioning each P_i into at most r_1 pieces of sizes less than or equal to r_2 . For $X \subseteq [n]$, define

$$\mathcal{F}(X,s) = \{A \subseteq X \mid \text{There exists } F \in \mathcal{F} \text{ such that } A \subseteq F \subseteq_s A \}.$$

We also define a new family

$$\mathcal{F}' = \{ X \subseteq [n] \mid \operatorname{ecd}^{r_1}(\mathcal{F}(X, s')) > (r_1 - 1)t \}$$

where $t = \chi(\widetilde{\mathrm{KG}}^{r_1r_2}(\mathcal{F},\mathcal{P},s))$ (resp. $t = \chi(\mathrm{KG}^{r_1r_2}(\mathcal{F},\mathcal{P},s)))$ and let c be a proper coloring of its vertices into $\{1,\ldots,t\}$. Suppose $X \in \mathcal{F}'$ is a vertex of $\mathrm{KG}^{r_2}(\mathcal{F}',\mathcal{P}')$, then for each $P_i \in \mathcal{P}$, one has $|X \cap P_i| \leq r_1$ so $\mathcal{P}|_X := \{P_1 \cap X, \ldots, P_l \cap X\}$ is a partition of X with each piece of size at most r_1 . By the hypothesis of the lemma, for such an $X, \chi(\widetilde{\mathrm{KG}}^{r_1}(\mathcal{F}(X,s'),\mathcal{P}|_X,0)) > t$. The induced coloring c_0 on the \mathcal{P} -admissible elements $A \in \mathcal{F}(X,s')$ is defined as follows. According to the definition, let $F \in \mathcal{F}$ be such that $F \subseteq_{s'} A$, then F is a vertex of $\widetilde{\mathrm{KG}}^{r_1r_2}(\mathcal{F},\mathcal{P},s)$ and define $c_0(A) = c(F)$, in the first case. In the case of Theorem 1.5, by the goodness assumption on the pair $(\mathcal{F},\mathcal{P})$, we can find a \mathcal{P} -admissible F' such that $F' \subseteq_{s'} A$ and define $c_0(A) = c(F')$.

Since c_0 is not a proper coloring, it follows that one may find vertices

$$B_1(X),\ldots,B_{r_1}(X)$$

of $\widetilde{\mathrm{KG}}^{r_1}(\mathcal{F}(X,s'),\mathcal{P}|_X,0)$ that are pairwise disjoint and have the same color. Define the coloring c' for $\mathrm{KG}^{r_2}(\mathcal{F}',\mathcal{P}')$ by $c'(X) = c_0(B_1(X))$. Note that for each $B_i(X)$, one has a vertex $F_i(X)$ of $\widetilde{\mathrm{KG}}^{r_1r_2}(\mathcal{F},\mathcal{P},s)$ (resp. of $\mathrm{KG}^{r_1r_2}(\mathcal{F},\mathcal{P},s)$) such that $F_i(X) \subseteq_{s'} B_i(X)$, with $c(F_1(X)) = \cdots = c(F_{r_1}(X))$. Then c' is a proper coloring since otherwise there exists pairwise disjoint vertices X_1, \ldots, X_{r_2} with the same color, and hence the r_1r_2 subsets $F_i(X_j)$ of \mathcal{F} have pairwise intersection of at most s elements and the same color. This contradicts the properness of the coloring c. So by the hypothesis of the lemma, $\mathrm{ecd}^{r_2}(\mathcal{F}') \leq (r_2 - 1)t$. Hence, one may find $X_0 \subseteq [n]$ of size at most $(r_2 - 1)t$ and an equitable partition

$$[n] \setminus X_0 = X_1 \cup \dots \cup X_{r_2}$$

with the property that no $X \in \mathcal{F}'$ is a subset of one of X_1, \ldots, X_{r_2} . So in particular for $1 \leq i \leq r_2, X_i \notin \mathcal{F}'$ and hence $\operatorname{ecd}^{r_1}(\mathcal{F}(X_i, s')) \leq (r_1 - 1)t$. This implies the existence of a subset $X_{i,0} \subseteq X_i$ of size at most $(r_1 - 1)t$ and an equitable partition

$$X_i \setminus X_{i,0} = X_{i,1} \cup \dots \cup X_{i,r_1}$$

such that no $A \in \mathcal{F}(X_i, s')$ is a subset of one of $X_{i,1}, \ldots, X_{i,r_1}$. We may assume that $|X_{i,0}| = (r_1 - 1)t$, since if $|X_{i,0}| < (r_1 - 1)t$, remove one element from an $X_{i,j}$ for $j = 1, \ldots, r_1$ with the largest size and add it to the $X_{i,0}$ without violating any of the conditions. By repeating this process, we may assume $|X_{i,0}| = (r_1 - 1)t$ for $i = 1, \ldots, r_2$. If now $|X_{i,j}| - |X_{i',j'}| > 1$ for some $1 \le i, i' \le r_2$ and $1 \le j, j' \le r_1$, then it follows that $|X_i| - |X_{i'}| > 1$, which is a contradiction. The reason for this, is that if we let $a = |X_{i,j}|$, the minimum size that X_i can have is $a + (r_1 - 1)(a - 1) + t(r_1 - 1)$, and the maximum size that $X_{i'}$ can have is $a - 2 + (r_1 - 1)(a - 1) + (r_1 - 1)t$.

It follows that we have an equitable partition

$$[n] \setminus X'_0 = X_{1,1} \cup \cdots \cup X_{1,r_1} \cup \cdots \cup X_{r_2,1} \cup \cdots \cup X_{r_2,r_1}$$

where

$$X'_0 = X_0 \cup X_{1,0} \dots \cup X_{r_2,0}$$

is of size at most

$$(r_2 - 1)t + r_2(r_1 - 1)t = (r_1r_2 - 1)t$$

and this partition has the property that is no $F \in \mathcal{F}$ such that $F \subseteq_{s'} X_{i,j}$ for some $i = 1, \ldots, r_2$ and $j = 1, \ldots, r_1$. Since otherwise, $A = F \cap X_{i,j} \in \mathcal{F}(X_i, s')$ and $A \subseteq X_{i,j}$, which is a contradiction. This shows that $\operatorname{ecd}^{r_1r_2}(\mathcal{F}, s')$ is less than or equal to $(r_1r_2 - 1)t$ or in other words, t is greater than or equal to $\frac{\operatorname{ecd}^{r_1r_2}(\mathcal{F}, s')}{r_1r_2 - 1}$. This proves the lemma.

3 Proof of Theorem 1.4 and Theorem 1.5

To prove Theorems 1.4 and 1.5, hence we may suppose that r = p is a prime number. We use \mathbb{Z}_p -Tucker lemma. We recall its statement from [9]. The simplicial complex $\mathbb{E}_{n-1}(\mathbb{Z}_p)$ has $\mathbb{Z}_p \times [n]$ as its vertices and all subsets $A \subseteq \mathbb{Z}_p \times [n]$ with pairwise different second components as faces. It has a free action of \mathbb{Z}_p that acts on the first component of each vertex by multiplication. We take \mathbb{Z}_p to be the multiplicative group of all *p*th roots of unity.

Lemma 3.1. (\mathbb{Z}_p -Tucker Lemma) Let n, m > 0 and $m \ge \alpha \ge 0$ be integers and p be a prime number. If λ is a map from the non-empty faces of $E_{n-1}(\mathbb{Z}_p)$ to $\mathbb{Z}_p \times [m]$ with $\lambda(A) = (\lambda_1(A), \lambda_2(A)) \in \mathbb{Z}_p \times [m]$ that satisfies the following properties,

- 1. If $\omega \in \mathbb{Z}_p$ and A is a non-empty face of $E_{n-1}(\mathbb{Z}_p)$, then $\lambda_1(\omega \cdot A) = \omega \cdot \lambda_1(A)$ and $\lambda_2(\omega \cdot A) = \lambda_2(A)$. That is λ is \mathbb{Z}_p -equivariant.
- 2. If $A_1 \subseteq A_2$ be non-empty faces of $E_{n-1}(\mathbb{Z}_p)$ and $\lambda_2(A_1) = \lambda_2(A_2) \leq \alpha$ then $\lambda_1(A_1) = \lambda_1(A_2)$.
- 3. If $A_1 \subseteq \cdots \subseteq A_p$ be non-empty faces of $E_{n-1}(\mathbb{Z}_p)$ and $\lambda_2(A_1) = \cdots = \lambda_2(A_p) > \alpha$ then $\lambda_1(A_1), \ldots, \lambda_1(A_p)$ are not pairwise distinct.

then

$$\alpha + (m - \alpha)(p - 1) \ge n.$$

Now let us present our proof for Theorem 1.4 (resp. Theorem 1.5).

Proof. Let $t = \chi(\widetilde{\mathrm{KG}}^p(\mathcal{F},\mathcal{P},s))$ (resp. $t = \chi(\mathrm{KG}^p(\mathcal{F},\mathcal{P},s))$ for the case of Theorem 1.5) and let c be a coloring of the vertices of this hypergraph with colors $\{1,\ldots,t\}$. Let $s' = \lfloor s/2 \rfloor$, $\alpha = n - \mathrm{ecd}^p(\mathcal{F},s')$ and $m = \alpha + t$. Also for simplicity choose a complete ordering on non-empty subsets of [n], that has the property that if |A| < |B| then A < B.

We define a \mathbb{Z}_p -equivariant map λ from the non-empty faces of $\mathbb{E}_{n-1}(\mathbb{Z}_p)$ to $\mathbb{Z}_p \times [m]$ that satisfies the two properties of the \mathbb{Z}_p -Tucker lemma and hence

$$\alpha + (m - \alpha)(p - 1) = n - \operatorname{ecd}^{p}(\mathcal{F}, s') + (p - 1)t \ge n$$

and hence the result follows. For a non-empty face A of $\mathbb{E}_{n-1}(\mathbb{Z}_p)$ and $i \in \mathbb{Z}_p$, let $A^i = \{1 \leq j \leq n | (i, j) \in A\}$. The definition of $\lambda(A) = (\lambda_1(A), \lambda_2(A)) \in \mathbb{Z}_p \times [m]$ is given

in two cases. Case 1: If there is an element $F \in \mathcal{F}$ with $F \subseteq_{s'} A^i$ for some $i \in \mathbb{Z}_p$ and

$$\sum_{i=1}^{l} \max\left\{ |F \cap P_j| - 1, 0 \right\} \le s'$$

(resp. F is \mathcal{P} -admissible) for all $1 \leq j \leq l$, then choose the smallest such subset with respect to the complete ordering on subsets of [n], say $F \subseteq_{s'} A^i$ and define

$$\lambda(A) = (i, c(F) + \alpha).$$

We remark that since |F| > s, one can not have more than one $i \in \mathbb{Z}_p$ that $F \subseteq_{s'} A^i$. **Case 2:** Otherwise, choose a non-empty subset $B \subseteq A$ such that for all $i \in \mathbb{Z}_p$ and $j = 1, \ldots, l, |B^i \cap P_j| \leq 1$ and $\pi_2(B)$ is maximum with respect to the chosen complete order on subsets of [n], this is clearly unique. Here $\pi_2 : \mathbb{Z}_p \times [n] \to [n]$ is the projection onto the second component. Also, assume that

$$|B^{i_1}| = \dots = |B^{i_h}| < |B^{i_{h+1}}| \le \dots \le |B^{i_p}|$$

for some $1 \le h \le p$, where h = p means that all the sizes are equal. Define

$$\lambda_2(A) = p|B^{i_1}| + p - h.$$

Note that $\lambda_2(A) \leq \alpha$. This is because by removing elements from $B^{i_{h+1}}, \ldots, B^{i_p}$ (if there are any) arbitrarily, we may assume that their sizes are $|B^{i_1}| + 1$ to arrive at an equitable partition of a set of size $\lambda_2(A)$. If $\lambda_2(A)$ is greater than $n - \operatorname{ecd}^p(\mathcal{F}, s')$, then by the definition of $\operatorname{ecd}^p(\mathcal{F}, s')$ there is an element $F \in \mathcal{F}$ with $F \subseteq_{s'} B^{i_k}$ for some $k = 1, \ldots, p$, and therefore

$$\sum_{i=1}^{l} \max\left\{ |F \cap P_j| - 1, 0 \right\} \le s'$$

(resp. F can be chosen so that it is \mathcal{P} -admissible by the s'-goodness assumption). This contradicts the fact that we are in the Case 2.

The definition of $\lambda_1(A)$ is more delicate. We define it in several sub-cases. **Case 2.1:** If h < p, find $1 \le h' < p$ such that $hh' \equiv 1 \mod p$ and define

$$\lambda_1(A) = (i_1 \dots i_h)^{h'}.$$

Case 2.2: If h = p, find the smallest $1 \le j \le l$ that $\pi_2(B) \cap P_j$ is non-empty, and take the unique subset $B' \subseteq B$ such that $\pi_2(B') = \pi_2(B) \cap P_j$. Let $\pi_1(B') = \{j_1, \ldots, j_k\}$, where π_1 is the projection onto the first component. Then we have again two sub-cases: **Case 2.2.1:** If k < p, choose $1 \le k' < p$ such that $kk' \equiv 1 \mod p$ and define:

$$\lambda_1(A) = (j_1 \dots j_k)^{k'}$$

Case 2.2.2: If k = p, define $\lambda_1(A)$ to be the first component of the element of B' with the smallest second component.

It remains to check the properties of the \mathbb{Z}_p -Tucker lemma. First, λ is \mathbb{Z}_p -equivariant in the Case 1. That is $\lambda_1(\omega \cdot A) = \omega \cdot \lambda_1(A)$ and $\lambda_2(\omega \cdot A) = \lambda_2(A)$ for any $\omega \in \mathbb{Z}_p$. This is because, if $F \subseteq A^i$ is the required subset for A in case one then $F \subseteq (\omega A)^{\omega \cdot i}$ is the required subset for $\omega \cdot A$.

If $A_1 \subseteq \cdots \subseteq A_p$ is a chain of non-empty faces of $E_{n-1}(\mathbb{Z}_p)$ with $\lambda_2(A_1) = \cdots = \lambda_2(A_p) > 0$

 α , then we are in the Case 1. Hence with have vertices F_1, \ldots, F_p of $\mathrm{KG}^p(\mathcal{F}, \mathcal{P}, s)$ with $F_i \subseteq_{s'} A_i^{\lambda_1(A_i)}$ with $c(F_1) = \cdots = c(F_p)$. If $\lambda_1(A_1), \ldots, \lambda_1(A_p)$ are pairwise distinct, then since $A_i^{\lambda_1(A_i)} \cap A_j^{\lambda_1(A_j)} = \emptyset$ for $i \neq j$ then $|F_i \cap F_j| \leq 2s' \leq s$ and $\{F_1, \ldots, F_p\}$ will be a mono-chromic edge, which contradicts properness of c. Hence the third condition of the \mathbb{Z}_p -Tucker lemma holds.

To show that λ is \mathbb{Z}_p -equivariant in Case 2, note that if $B \subseteq A$ is the required set for A, then $\omega \cdot B_1 \subseteq \omega \cdot B_2$ is the required set in for $\omega \cdot A$, hence $\lambda_2(A) = \lambda_2(\omega \cdot A)$. Also, the corresponding $\{i_1, \ldots, i_h\}$ will be $\{\omega \cdot i_1, \ldots, \omega \cdot i_h\}$. In the Case 2.1, we have

$$\lambda_1(\omega \cdot A) = ((\omega \cdot i_1) \dots (\omega \cdot i_h))^{h'} = \omega^{hh'} \cdot (i_1 \dots i_h)^{h'} = \omega \cdot \lambda_1(A).$$

In Case 2.2, we have $\omega \cdot B'$ as the corresponding set for $\omega \cdot A$. So in both Cases 2.2.1 and 2.2.2 it follows that $\lambda_1(\omega \cdot A) = \omega \cdot \lambda_1(A)$. This proves that λ is \mathbb{Z}_p -equivariant.

If $A_1 \subseteq A_2$ are non-empty faces of $\mathbb{E}_{n-1}(\mathbb{Z}_p)$ with $\lambda_2(A_1) = \lambda_2(A_2) \leq \alpha$, then we are in the second case. With maximal subsets $B_1 \subseteq A_1$ and $B_2 \subseteq A_2$. Assume that

$$|B_1^{i_1}| = \dots = |B_1^{i_h}| < |B_1^{i_{h+1}}| \le \dots \le |B_1^{i_p}|$$
$$|B_2^{j_1}| = \dots = |B_2^{j_k}| < |B^{j_{k+1}}| \le \dots \le |B_2^{j_p}|$$

for some $1 \leq h \leq p$ and $1 \leq k \leq p$. If $\lambda_2(A_1) = \lambda_2(A_2)$, then $|B_1^{i_1}| = |B_2^{j_1}|$ and h = k. Now since $B_1 \subseteq A_1 \subseteq A_2$, by the maximality of B_2 , we have $|B_1^i| \leq |B_2^i|$. Therefore $\{i_1, \ldots, i_h\} = \{j_1, \ldots, j_h\}$. So in Case 2.1 we must have $\lambda_1(A_1) = \lambda_1(A_2)$.

If we are in Case 2.2, then $|B_1^i| = |B_2^i|$ for all $i \in \mathbb{Z}_p$ and hence $|B_1| = |B_2|$. This implies that the first $1 \leq j \leq l$ that $\pi_2(B_1) \cap P_j$ is non-empty is the same as the first $1 \leq j' \leq l$ that $\pi_2(B_2) \cap P_{j'}$ is non-empty. So by the maximality and equality of $|B_1| = |B_2|$, it follows that $\pi_1(B_1') = \pi_1(B_2')$. In the Case 2.2.1 therefore $\lambda_1(A_1) = \lambda_1(A_2)$. Finally, in the Case 2.2.2 since $|P_j| \leq p$, it follows that $B_1' = B_2'$ and hence the first component of the element with the smallest second component in both of them are the same, that is $\lambda_1(A_1) = \lambda_1(A_2)$. This finishes checking the conditions and hence the proof of the theorem is complete.

4 A generalization of a theorem of Abyazi Sani and Alishahi

In this section, using Theorem 1.2, we generalize Theorem 3 of Abyazi Sani and Alishahi in [1]. For an integer vector $S = (s_1, \ldots, s_n)$ with $0 \le s_i \le r$, the notion of an S-disjoint multi-set $\{A_1, \ldots, A_r\}$ of subsets of [n] was considered by Sarkaria and Ziegler in [10], [11], and [12]. It means that for all $1 \le i \le n$, the number of $1 \le j \le r$ that $i \in A_j$ is at most s_i . This generalizes the notion of pairwise disjoint that is just $S = (1, 1, \ldots, 1)$ -disjoint. Ziegler [11] extended the *r*-colorability defect of a family \mathcal{F} of subsets of [n], $\operatorname{cd}^r(\mathcal{F})$, to the S-disjoint *r*-colorability defect $\operatorname{cd}^r_S(\mathcal{F})$. This was also extended by Abyazi Sani and Alishahi [1] to the equitable S-disjoint *r*-colorability defect $\operatorname{ecd}^r_S(\mathcal{F})$ which is defined as follows. Let $\bar{n} = \sum_{i=1}^n s_i$. Then $\operatorname{ecd}^r_S(\mathcal{F})$ is defined by

$$\bar{n} - \max\left\{\sum_{i=1}^{r} |A_i| \mid \{A_1, \dots, A_r\} \text{ equitable and } S\text{-disjoint } \forall F \in \mathcal{F}, 1 \le i \le r \ F \not\subseteq A_i\right\}.$$

For a subset P of [n], we define the S-weight of P to be

$$w_S(P) = \sum_{i \in P} s_i.$$

For a partition $\mathcal{P} = \{P_1, \ldots, P_l\}$ of [n], we also define the *r*-uniform Kneser-type hypergraph $\mathrm{KG}_S^r(\mathcal{F}, \mathcal{P})$ to be a hypergraph with the vertex set of those $A \in \mathcal{F}$ that have at most one element from each P_1, \ldots, P_l and the edge set of all multi-sets $\{A_1, \ldots, A_r\}$ of the vertices that are *S*-disjoint. We then have the following theorem.

Theorem 4.1. If the partition $\mathcal{P} = \{P_1, \ldots, P_l\}$ has the property that the S-weight of each partition piece is at most r, then one has

$$\chi(KG_S^r(\mathcal{F},\mathcal{P})) \ge \left\lceil \frac{ecd_S^r(\mathcal{F})}{r-1} \right\rceil$$

Proof. For each $1 \leq i \leq n$, we make s_i different copies of i, say $(i, 1), \ldots, (i, s_i)$ and make the set [n] into the bigger set $[\bar{n}]$. So we have a natural map $f : [\bar{n}] \to [n]$ that sends any copy of i to i. We define the lifted family $\bar{\mathcal{F}}$ to be all subsets A of $[\bar{n}]$ such that $f(A) \in \mathcal{F}$ and also all two-element subsets of $[\bar{n}]$ with two different copies of the same number. Finally, we define a partition $\bar{\mathcal{P}} = \{\bar{P}_1, \ldots, \bar{P}_l\}$ by replacing any element i in a partition piece with all of its s_i copies. Hence $|\bar{P}_i| = w_S(P_i) \leq r$. Now we claim that fdefines a hypergraph homomorphism from $\mathrm{KG}^r(\bar{\mathcal{F}}, \bar{\mathcal{P}})$ to $\mathrm{KG}^r_S(\mathcal{F}, \mathcal{P})$ and hence

$$\chi(\mathrm{KG}^r_S(\mathcal{F},\mathcal{P})) \ge \chi(\mathrm{KG}^r(\bar{\mathcal{F}},\bar{\mathcal{P}})).$$

The proof of the claim is straightforward, notice that the special two-element subsets of $\overline{\mathcal{F}}$, do not appear as vertices of this hypergraph. It remains to check that $\operatorname{ecd}_{S}^{r}(\mathcal{F}) \leq \operatorname{ecd}^{r}(\overline{\mathcal{F}})$, which will finish the proof of the theorem by applying Theorem 1.2. If $\{A_{1}, \ldots, A_{r}\}$ is an equitable disjoint family in $[\overline{n}]$ such that no element of $\overline{\mathcal{F}}$ is a subset of one of A_{1}, \ldots, A_{r} , then $f(A_{1}), \ldots, f(A_{r})$ is an S-disjoint equitable family of subsets of $[\overline{n}]$ with $|f(A_{i})| = |A_{i}|$ (note that because of the special two element subsets in $\overline{\mathcal{F}}$, each A_{i} must contain at most one copy from each element). Also, no $F \in \mathcal{F}$ is a subset of one of $f(A_{1}), \ldots, f(A_{r})$. This implies that $\operatorname{ecd}_{S}^{r}(\mathcal{F}) \leq \operatorname{ecd}^{r}(\overline{\mathcal{F}})$. The theorem is proved.

Remark 4.1. When \mathcal{P} is the trivial partition of [n] into singletons, this result extends the corresponding inequality

$$\chi(KG_S^r(\mathcal{F})) \ge \left\lceil \frac{ecd_S^r(\mathcal{F})}{r-1} \right\rceil$$

obtained by Abyazi Sani and Alishahi in [1] with the extra assumption that $s_i < \mu(r)$, where $\mu(r)$ is the largest prime factor of r.

5 Examples

In this section, we study the Kneser hypergraph of a special family introduced in [1] and its generalizations. For integers $n > k > a \ge 0$ and $k > s \ge 0$, define $\mathcal{H}(n, k, a, s)$ to be the family of all k-subsets $F \subseteq [n]$ with $F \not\subseteq_s \{n - a + 1, ..., n\}$ and let $\mathrm{KG}^r(n, k, a, s)$ be the r-uniform Kneser hypergraph with the vertex set $\mathcal{H}(n, k, a, s)$ and the edge set of all r-subsets $\{F_1, \ldots, F_r\}$ of vertices with pairwise intersection of at most s elements. The case s = 0, was considered by Abyazi Sani and Alishahi in [1] and was denoted by $\mathrm{KG}^r(n, k, a)$.

Remark 5.1. The pair $(\mathcal{H}(n, k, a, s), \mathcal{P})$ is $\lfloor s/2 \rfloor$ -good, if $n \geq rk$, $|P_i| \leq r, 1 \leq s < k$ and at least s + 1 of the non-empty partitions of \mathcal{P} have empty intersection with $A = \{n - a + 1, n - a + 2, ..., n\}$. The reason is as follows. Let $s' = \lfloor s/2 \rfloor$. Assume for a \mathcal{P} -admissible subset B and an element F in $\mathcal{H}(n, k, a, s)$ we have $F \subseteq_{s'} B$. Then there is a subset E of size at most s' such that $F \setminus E \subseteq B$. Since F has at least s + 1 elements outside of A, so $F \setminus E$ has $t \ge s + 1 - s'$ elements outside of A. Assume P_1, \ldots, P_{s+1} be the partitions with empty intersection with A. If $t \ge s + 1$, then one can add arbitrarily elements from different partitions that have empty intersection with $F \setminus E$ so it become a \mathcal{P} -admissible element $F' \in \mathcal{H}(n, k, a, s)$ with $F' \subseteq_{s'} B$. If t < s + 1, then $F \setminus E$ has non-empty intersection with at most t of P_1, \ldots, P_{s+1} , so we can use elements from those P_1, \ldots, P_{s+1} with empty intersection with $F \setminus E$ and if needed other partition parts to complete $F \setminus E$ to a \mathcal{P} -admissible $F' \in \mathcal{H}(n, k, a, s)$ such that $F' \subseteq_{s'} B$.

The following lemma is an extension of a computation made in [1] for $ecd^{r}(\mathcal{H}(n,k,a,0))$.

Lemma 5.1. Let n, k, r, s, and a be integers with $k, r \ge 2$ and $n \ge rk$, $0 \le s < k$, and n > a + s. Then, one has

$$ecd^{r}(\mathcal{H}(n,k,a,s),s) = \begin{cases} n - r(k-s-1) & a \le k-s-1\\ n - r(k-s-1) - \lfloor \frac{a}{k-s} \rfloor & k-s \le a \le r(k-s) - 2\\ n-a & a \ge r(k-s) - 1 \end{cases}$$

Proof. Let $A = \{n - a + 1, ..., n\}$. We prove each case separately.

- 1. In the first case, $\mathcal{H}(n, k, a, s)$ is $\binom{[n]}{k}$ of all k-subsets of [n] and it follows from Remark 1.1.
- 2. In the second case, let $X_0, X_1, ..., X_r$ be a partition of [n] such that as in the definition the generalized *r*-colorability defect, there are no $F \in \mathcal{H}(n, k, a, s)$ such that $F \subseteq_s X_i$ for some i = 1, ..., r. We show that $|X_i| \leq k - s$ for $1 \leq i \leq r$ and if $|X_i| = k - s$ then $X_i \subseteq A$. Assume that $|X_1| \geq k - s + 1$, and since the partition is equitable $|X_i| \geq k - s$ for $1 \leq i \leq r$. Hence, there exist $1 \leq i \leq r$ such that $X_i \not\subseteq A$. Let X'_i be a k - s subset of X_i such that $X'_i \not\subseteq A$. Since n > a + s there exist at least s elements in $[n] \setminus A$ so we can extend X'_i to a k-subset F such that $F \not\subseteq_s A$ and so $F \in \mathcal{H}(n, k, a, s)$, $F \subseteq_s X_i$ which violates the assumption on the partition. From the previous argument one can deduce the fact that $|X_i| = k - s$ can only happen when $X_i \subseteq A$. Based on these facts:

$$\operatorname{ecd}^{r}(\mathcal{H}(n,k,a,s),s) \ge n - (k-s)\lfloor \frac{a}{k-s} \rfloor - (k-s-1)(r - \lfloor \frac{a}{k-s} \rfloor)$$
$$= n - r(k-s-1) - \lfloor \frac{a}{k-s} \rfloor$$

This bound is sharp since you can find such an equitable partition by taking $\lfloor \frac{a}{k-s} \rfloor$ disjoint (k-s)-subsets of A as $X_1, ..., X_{\lfloor \frac{a}{k-s} \rfloor}$ and $r - \lfloor \frac{a}{k-s} \rfloor$ arbitrary disjoint (k-s-1)-subsets of the remaining elements as other X_i 's.

3. In the third case, If $X_0 = [n] \setminus A$ and X_1, \ldots, X_r be a equitable partition of A, then clearly there is no $F \in \mathcal{H}(n, k, a, s)$ such that $F \subseteq_s X_i$ for some $1 \leq i \leq r$. If $|X_0| < n - a$ then $|X_1 \cup \cdots \cup X_r| > a$. If $a \geq r(k - s)$, hence at least one X_i has a size of at least k - s + 1, which is not possible by the argument in the previous step. If a = r(k - s) - 1, then we must have $|X_i| = k - s$ for all i and hence $X_i \subseteq A$. This is not possible either, because it implies that $a \geq r(k - s)$. So ecd^r is n - a. **Theorem 5.1.** Let n, k, r, s and a be integers with $k, r \ge 2$, $n > a \ge 0$, $n \ge rk$, $0 \le s < k$, and $a \le r(k - s - 1)$. Then, one has

$$\chi(KG^{r}(n,k,a,s)) \ge \left\lceil \frac{n-r(k-\lfloor s/2 \rfloor - 1)}{r-1} \right\rceil$$

Proof. Let $A = \{n - a + 1, ..., n\}$. Take a partition $\mathcal{P} = \{P_1, ..., P_l\}$ of [n] such that $|P_i| = r$ for all $1 \leq i \leq k - s - 1$ and $|P_i| \leq r$ otherwise, and $A \subseteq \bigcup_{i=1}^{k-s-1} P_i$. Now, $\operatorname{KGr}(\binom{[n]}{k}, \mathcal{P}, s)$ is a sub-hypergraph of $\operatorname{KGr}(n, k, a, s)$, because if a k-subset F is \mathcal{P} -admissible then it contains at most (k - s - 1) elements from A and hence $F \not\subseteq_s A$. The result follows now from Theorem 1.5. Recall that $\operatorname{ecd}^r(\binom{[n]}{k}, s) = n - r(k - s - 1)$, and by Remark 1.2, the pair $\binom{[n]}{k}, \mathcal{P}$) is $\lfloor s/2 \rfloor$ -good. ■

Remark. The above theorem, for the case when s = 0, was conjectured in [1]. They showed that it is true when $a \leq 2(k-1)$. This was generalized by Aslam, Chen, Coldren, Frick, and Seitanrata in [6] for $a \leq b(r)(k-1)$, where b(r) for the prime decomposition $r = 2^{\alpha_0} p_1^{\alpha_1} \dots p_m^{\alpha_m}$ is defined to be $2^{\alpha_0} (p_1 - 1)^{\alpha_1} \dots (p_m - 1)^{\alpha_m}$. Our theorem hence, is a generalization of these results.

The following hypergraph is considered in [6]. Let $\mathrm{KG}^r(n, k, \mathcal{P})_{t-wide}$ be the subhypergraph of $\mathrm{KG}^r(n, k, \mathcal{P})$ induced by the vertices that are not contained in any one of the sets $\{i, i+1, ..., i+t-1\}$ for $i \in [n-t+1]$. The following theorem is proved in [6].

Theorem 5.2. Let $k \ge 1$ be an integer, $r \ge 2$ a prime, and $n \ge rk$ an integer. Let $\mathcal{P} = \{P_1, ..., P_l\}$ be a partition of [n] with $|P_i| \le r-1$. Let $t \le r(k-3)+2$. Then

$$\chi(KG^r(n,k,\mathcal{P})_{t-wide}) = \left\lceil \frac{n-r(k-1)}{r-1} \right\rceil.$$

In some special cases we can improve their result.

Theorem 5.3. Let $k \ge 1$ be an integer $r \ge 2$, and $n \ge rk$ an integer. Let $t \le r(k-2)+1$ then

$$\chi(KG^r(n,k)_{t-wide}) = \left| \frac{n-r(k-1)}{r-1} \right|.$$

Proof. Let $l = \lceil \frac{n}{r} \rceil$ and $\mathcal{P} = \{P_1, ..., P_l\}$ be a partition of [n] such that

$$P_i = \{(i-1)r + 1, ..., ir\}$$

for $1 \leq i \leq l-1$. Then, $\mathrm{KG}^r(n,k,\mathcal{P})$ is a sub-hypergraph of $\mathrm{KG}^r(n,k)_{t-wide}$, because suppose that a \mathcal{P} -admissible k-subset F is a subset of $X = \{i, i+1, \ldots, i+t-1\}$. Then, the smallest value that t can have is when $i \in F$ is the last element of some P_j and $(i+t-1) \in F$ is the first element of P_{j+k-1} and $P_{j+1}, \ldots, P_{j+k-2}$ are subsets of X, that is $t \geq r(k-2) + 2$, which contradicts the assumption on t. The result then follows from Theorem 1.2 and the standard coloring of the Kneser hypergraph $\mathrm{KG}^r(n,k)$.

Remark 5.2. The family of t-wide subsets are very interesting examples to compare the colorability defects for them. It is proved in [6] that the topological r-colorability defect

of this family for $t \leq r(k-3) + 2$ is at least n - r(k-1) but we will show that if $n > \max\{rt, r(k-1)\}$ then,

$$ecd^{r}\binom{[n]}{k}_{t-wide} = \begin{cases} n-r(k-1) & t \le k\\ n-rt & t > k \end{cases}$$

Therefore, there exist examples where the topological colorability defect is better than the equitable colorability defect.

Proof. In the first case, the family of *t*-wide *k*-subsets is the same as the family of all *k*-subsets and the result follows by Remark 1.1. In the second case, let $X_0, X_1, ..., X_r$ be a partition of [n] such that no $F \in {\binom{[n]}{k}}_{t-wide}$ is a subset of one of $X_1, ..., X_r$. Note that for all $1 \leq i \leq r$, one has $|X_i| \leq t$. Because otherwise, take a *k*-subset of X_i that contains its smallest and its biggest elements. This subset is a *t*-wide *k*-subset inside X_i and therefore violates the assumption on the partition. This shows $|X_0| \geq n - rt$. Finally, since the partition given by $X_i = \{(i-1)t+1, ..., it\}$ for $1 \leq i \leq r$ with $|X_0| = n - rt$ has the property that no *t*-wide *k*-subset is inside one of $X_1, ..., X_r$, the claim follows.

It is an interesting problem to see that if it is true that the topological r-colorability defect of Frick for a family of subsets is always greater than or equal to the equitable r-colorability defect.

References

- R. Abyazi-Sani, and M. Alishahi "A new lower bound for the chromatic number of general Kneser hypergraphs", Eur. J. Comb., Vol. 71, (2018) 229-245.
- [2] M. Alishahi and H. Hajiabolhassan. "On the chromatic number of general Kneser hypergraphs." J. of Comb. Theory, Series B 115 (2015): 186- 209.
- [3] N. Alon, P. Frankl, and L. Lovász, "The chromatic number of Kneser hypergraphs", Trans. Amer. Math. Soc. 298 (1986), no. 1, 359-370.
- [4] S. Azarpendar and A. Jafari, "On some topological and combinatorial lower bounds on the chromatic number of Kneser type hypergraphs", arXiv:2002.01748 (accepted for publication in J. of Comb. Theory, Series B).
- [5] H.R. Daneshpajouh, "On the chromatic number of generalized Kneser hypergraph", Eur. J. Comb., Vol. 81, (2019) 150-155.
- [6] J.Aslam, S. Chen, E. Coldren, F. Frick, L. Setiabrata. "On the generalized Erdös-Kneser conjecture: proofs and reductions", J. Combin Theory Ser. B 135 (2019) 227-237.
- [7] I. Kriz, "Equivariant cohomology and lower bounds for chromatic numbers". Trans. Amer. Math. Soc., 333(2), 567-577, 1992.
- [8] I. Kriz, A correction to: "Equivariant cohomology and lower bounds for chromatic numbers" [Trans. Amer. Math. Soc. 333 (1992), no. 2, 567-577. Trans. Amer. Math. Soc., 352(4), (2000), 1951-1952.
- [9] F. Meunier, "The chromatic number of almost stable Kneser hypergraphs", J. Combin. Theory Ser. A 118 (2011), no 6, 1820-1828.

- [10] K. S. Sarkaria. "A generalized Kneser conjecture.", J. Combinatorial Theory, Ser. B, 49, (1990) 236-240.
- [11] G. Ziegler, "Generalized Kneser coloring theorems with combinatorial proofs", Invent. Math. 147 (2002), no. 3, 671-691.
- [12] G. Ziegler, "Erratum: Generalized Kneser coloring theorems with combinatorial proofs", Invent. Math. 163 (2006), 227-228.