# Lower bounds for the chromatic number of certain Kneser-type hypergraphs 

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#### Abstract

Let $n \geq 1, r \geq 2$, and $s \geq 0$ be integers and $\mathcal{P}=\left\{P_{1}, \ldots, P_{l}\right\}$ be a partition of $[n]=\{1, \ldots, n\}$ with $\left|P_{i}\right| \leq r$ for $i=1, \ldots, l$. Also, let $\mathcal{F}$ be a family of nonempty subsets of $[n]$. The $r$-uniform Kneser-type hypergraph $\operatorname{KG}^{r}(\mathcal{F}, \mathcal{P}, s)$ is the hypergraph with the vertex set of all $\mathcal{P}$-admissible elements $F \in \mathcal{F}$, that is $\left|F \cap P_{i}\right| \leq 1$ for $i=1, \ldots, l$ and the edge set of all $r$-subsets $\left\{F_{1}, \ldots, F_{r}\right\}$ of the vertex set that $\left|F_{i} \cap F_{j}\right| \leq s$ for all $1 \leq i<j \leq r$. In this article, we extend the equitable $r$-colorability defect $\operatorname{ecd}^{r}(\mathcal{F})$ of Abyazi Sani and Alishahi to the case when one allows intersection among the vertices of an edge. It will be denoted by $\operatorname{ecd}^{r}(\mathcal{F}, s)$. We then, give (under certain assumptions) lower bounds for the chromatic number of $\mathrm{KG}^{r}(\mathcal{F}, \mathcal{P}, s)$ and some of its variants in terms of $\operatorname{ecd}^{r}(\mathcal{F},\lfloor s / 2\rfloor)$. This work generalizes many existing results in the literature of the Kneser hypergraphs. It generalizes the previous results of the current authors from the special family of all $k$-subsets of $[n]$ to a general family $\mathcal{F}$ of subsets.


## 1 Introduction

Let $n \geq 1$ and $r \geq 2$ be integers and $\mathcal{P}=\left\{P_{1}, \ldots, P_{l}\right\}$ be a partition of $[n]=\{1, \ldots, n\}$ with $\left|P_{i}\right| \leq r$ for $i=1, \ldots, l$. Let $\mathcal{F}$ be a family of non-empty subsets in [n]. The $r$ uniform Kneser-type hypergraph $\mathrm{KG}^{r}(\mathcal{F}, \mathcal{P})$ is the hypergraph with the vertex set of all $\mathcal{P}$-admissible elements $F \in \mathcal{F}$, that is $\left|F \cap P_{i}\right| \leq 1$ for $i=1, \ldots, l$, and the edge set of all $r$-subsets $\left\{F_{1}, \ldots, F_{r}\right\}$ of the vertex set that are pairwise disjoint. This hypergraph first was considered by Alishahi and Hajiabolhassan in [2]. It was later considered by Aslam, Chen, Coldren, Frick, and Setiabrata in [6]. For an integer $s \geq 0$, that is assumed to have the property ${ }^{1} s<|F|$ for all elements $F$ of $\mathcal{F}$, we may relax the condition of pairwise disjointness to $\left|F_{i} \cap F_{j}\right| \leq s$ and arrive at the definition of the $r$-uniform hypergraph $\mathrm{KG}^{r}(\mathcal{F}, \mathcal{P}, s)$. We are interested here to find lower bounds for the chromatic number $\chi\left(\operatorname{KG}^{r}(\mathcal{F}, \mathcal{P}, s)\right)$ of this hypergraph in terms of a generalization of the equitable $r$-colorability defect of Abyazi Sani and Alishahi [1]. This result is an extension of the previous results of the current authors in (4).

An equitable partition of a set $X$ is a partition of it into subsets $X_{i}$ for $i=1, \ldots, r$ such that $\left|\left|X_{i}\right|-\left|X_{j}\right|\right| \leq 1$ for all $1 \leq i \leq j \leq r$. The equitable $r$-colorability defect $\operatorname{ecd}^{r}(\mathcal{F})$ of a family $\mathcal{F}$ of non-empty subsets in $[n]$ is the minimum size of a subset $X_{0} \subseteq[n]$ so that there is an equitable partition

$$
[n] \backslash X_{0}=X_{1} \cup \cdots \cup X_{r}
$$

[^0]with the property that there are no elements $F \in \mathcal{F}$ and $i=1, \ldots, r$, with $F \subseteq X_{i}$. Abyazi Sani and Alishahi [1] proved the following generalization of the corresponding result of Kriz in [7] and [8] for the $r$-colorability defect.

Theorem 1.1. One has

$$
\chi\left(K G^{r}(\mathcal{F})\right) \geq\left\lceil\frac{e c d^{r}(\mathcal{F})}{r-1}\right\rceil
$$

Here $\mathrm{KG}^{r}(\mathcal{F})$ is the hypergraph with no partition condition, in other words, $\mathcal{P}$ is the partition of $[n]$ by the singletons. Our goal here, is to extend this result to the cases $\mathrm{KG}^{r}(\mathcal{F}, \mathcal{P})$ and $\mathrm{KG}^{r}(\mathcal{F}, \mathcal{P}, s)$.

For $s \geq 0$ and subsets $A$ and $B$ of $[n]$, we write $A \subseteq_{s} B$ if there is a set $E$ of size at most $s$, such that $A \backslash E \subseteq B$. The general $r$-equitable colorability defect $\operatorname{ecd}^{r}(\mathcal{F}, s)$ is the minimum size of a subset $X_{0} \subseteq[n]$, such that there is an equitable partition

$$
[n] \backslash X_{0}=X_{1} \cup \cdots \cup X_{r}
$$

with the property that there are no $F \in \mathcal{F}$ and $i=1, \ldots, r$, such that $F \subseteq_{s} X_{i}$.
Remark 1.1. Note that $\operatorname{ecd} d^{r}(\mathcal{F}, 0)$ is the original equitable r-colorability defect of Abyazi Sani and Alishahi. It is easy to see that for the family of all $k$-subsets of $[n]$, denoted by $\binom{[n]}{k}$, when $n \geq r(k-1)+1$ and $0 \leq s<k$, one has

$$
e c d^{r}\left(\binom{[n]}{k}, s\right)=n-r(k-s-1)
$$

We have the following results.
Theorem 1.2. Under the above condition on the partition $\mathcal{P}$, one has

$$
\chi\left(K G^{r}(\mathcal{F}, \mathcal{P})\right) \geq\left\lceil\frac{e c d^{r}(\mathcal{F})}{r-1}\right\rceil
$$

Theorem 1.3. Under the above condition on $s$, one has

$$
\chi\left(K G^{r}(\mathcal{F}, s)\right) \geq\left\lceil\frac{e c d^{r}(\mathcal{F},\lfloor s / 2\rfloor)}{r-1}\right\rceil
$$

Here the partition is understood to be trivial, in other words, by the singletons.
Unfortunately to give a unified theorem that deals with the case of $\mathrm{KG}^{r}(\mathcal{F}, \mathcal{P}, s)$ we need to either assume that the pair $(\mathcal{F}, \mathcal{P})$ satisfies an extra condition or modify the definition of the hypergraph into $\widetilde{\mathrm{KG}}^{r}(\mathcal{F}, \mathcal{P}, s)$ as follows.

Definition 1.1. The pair $(\mathcal{F}, \mathcal{P})$ is said to be $s$-good, if for any $\mathcal{P}$-admissible subset $A$ for which there exists $F \in \mathcal{F}$ so that $F \subseteq_{s} A$, one can find a $\mathcal{P}$-admissible element $F^{\prime} \in \mathcal{F}$ such that $F^{\prime} \subseteq_{s} A$.

Remark 1.2. Let us show that the pair $\left(\binom{[n]}{k}, \mathcal{P}\right)$ is s-good, if $n \geq r(k-1)+1,0 \leq s<k$, and $\left|P_{i}\right| \leq r$. Note that by the assumption on $n$, we have at least $k$ non-empty partition parts in $\mathcal{P}$. Now suppose $F \subseteq_{s} A$ for a $\mathcal{P}$-admissible subset $A$ and a $k$-subset $F$. If $|A| \geq k$ any $k$-subset $F^{\prime}$ of $A$ is $\mathcal{P}$-admissible. Hence we may assume, $k-s \leq|A| \leq k$ and therefore one can always add at most $s$ elements to $A$ from different partition parts with empty intersection with $A$, to make it into a $\mathcal{P}$-admissible $k$-subset $F^{\prime}$ with $F^{\prime} \subseteq{ }_{s} A^{\prime}$.

Without any assumptions on the partition and the family, we need to modify the definition of the hypergraph $\mathrm{KG}^{r}(\mathcal{F}, \mathcal{P}, s)$ as follows.
Definition 1.2. We let $\widetilde{K G}^{r}(\mathcal{F}, \mathcal{P}, s)$ be the r-uniform hypergraph with the vertex set of all elements $A$ of $\mathcal{F}$ such that

$$
\sum_{i=1}^{l} \max \left\{\left|A \cap P_{i}\right|-1,0\right\} \leq\lfloor s / 2\rfloor
$$

and the edge set of all $r$-subsets $\left\{A_{1}, \ldots, A_{r}\right\}$ of vertices such that $\left|A_{i} \cap A_{j}\right| \leq s$ for all $1 \leq i<j \leq r$.

Note that when $s=0$, the above condition is the same as $\mathcal{P}$-admissibility. Also if $\mathcal{P}$ is the trivial partition into singletons, this condition holds for all $A \in \mathcal{F}$. We have the following two results.

Theorem 1.4. Under the above assumptions, one has

$$
\chi\left(\widetilde{K G}^{r}(\mathcal{F}, \mathcal{P}, s)\right) \geq\left\lceil\frac{e c d^{r}(\mathcal{F},\lfloor s / 2\rfloor)}{r-1}\right\rceil .
$$

Theorem 1.5. If the pair $(\mathcal{F}, \mathcal{P})$ is $\lfloor s / 2\rfloor$-good, then

$$
\chi\left(K G^{r}(\mathcal{F}, \mathcal{P}, s)\right) \geq\left\lceil\frac{e c d^{r}(\mathcal{F},\lfloor s / 2\rfloor)}{r-1}\right\rceil .
$$

Note that theorem 1.4 implies as its special cases, theorems 1.2 and 1.3 ,
Remark 1.3. In [5], Daneshpajouh presents the following lower bound for the chromatic


$$
\chi\left(K G^{r}\left(\binom{[n]}{k}, s\right)\right) \geq\left\lceil\frac{n-r(k-s-1)}{r-1}\right\rceil .
$$

When $n \geq r(k-1)+1$, then $\left.\operatorname{ecd} d^{r}\binom{[n]}{k}, s\right)=n-r(k-s-1)$ and hence, this is a stronger lower bound than the one obtained from Theorem 1.3. It is feasible that the above theorems remain true if one replaces $\lfloor s / 2\rfloor$ with $s$.

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## 2 Reduction of Theorem 1.4 and Theorem 1.5

In this section, we prove the following lemma, which reduces the proof of Theorem 1.4 and Theorem [1.5, to the case when $r$ is a prime number. The proof is obtained by imitating a method used by Kriz in [8], who himself followed a similar method used by Alon, Frankl, and Lovász in [3].

Lemma 2.1. If Theorem 1.4 (resp. Theorem 1.5) is true for $r=r_{1}$ and $r=r_{2}$ then Theorem 1.4 (resp. Theorem (1.5) is true for $r=r_{1} r_{2}$.

Proof. Let $s^{\prime}=\lfloor s / 2\rfloor$ and $\mathcal{P}=\left\{P_{1}, \ldots, P_{l}\right\}$ be a partition of $[n]$ with $\left|P_{i}\right| \leq r_{1} r_{2}$. Also, let $\mathcal{P}^{\prime}=\left\{P_{1}^{\prime}, \ldots, P_{l^{\prime}}^{\prime}\right\}$ be a partition obtained from $\mathcal{P}$ by partitioning each $P_{i}$ into at most $r_{1}$ pieces of sizes less than or equal to $r_{2}$. For $X \subseteq[n]$, define

$$
\mathcal{F}(X, s)=\left\{A \subseteq X \mid \text { There exists } F \in \mathcal{F} \text { such that } A \subseteq F \subseteq_{s} A\right\}
$$

We also define a new family

$$
\mathcal{F}^{\prime}=\left\{X \subseteq[n] \mid \quad \operatorname{ecd}^{r_{1}}\left(\mathcal{F}\left(X, s^{\prime}\right)\right)>\left(r_{1}-1\right) t\right\}
$$

where $t=\chi\left(\widetilde{\mathrm{KG}}^{{ }^{1} r_{2}}(\mathcal{F}, \mathcal{P}, s)\right)$ (resp. $\quad t=\chi\left(\mathrm{KG}^{r_{1} r_{2}}(\mathcal{F}, \mathcal{P}, s)\right)$ ) and let $c$ be a proper coloring of its vertices into $\{1, \ldots, t\}$. Suppose $X \in \mathcal{F}^{\prime}$ is a vertex of $\mathrm{KG}^{r_{2}}\left(\mathcal{F}^{\prime}, \mathcal{P}^{\prime}\right)$, then for each $P_{i} \in \mathcal{P}$, one has $\left|X \cap P_{i}\right| \leq r_{1}$ so $\left.\mathcal{P}\right|_{X}:=\left\{P_{1} \cap X, \ldots, P_{l} \cap X\right\}$ is a partition of $X \underset{r_{1}}{\text { with each piece of size at most } r_{1} \text {. By the hypothesis of the lemma, for such an }}$ $X, \chi\left(\widetilde{\mathrm{KG}}^{r_{1}}\left(\mathcal{F}\left(X, s^{\prime}\right),\left.\mathcal{P}\right|_{X}, 0\right)\right)>t$. The induced coloring $c_{0}$ on the $\mathcal{P}$-admissible elements $A \in \mathcal{F}\left(X, s^{\prime}\right)$ is defined as follows. According to the definition, let $F \in \mathcal{F}$ be such that $F \subseteq_{s^{\prime}} A$, then $F$ is a vertex of $\widetilde{\mathrm{KG}}^{r_{1} r_{2}}(\mathcal{F}, \mathcal{P}, s)$ and define $c_{0}(A)=c(F)$, in the first case. In the case of Theorem 1.5, by the goodness assumption on the pair $(\mathcal{F}, \mathcal{P})$, we can find a $\mathcal{P}$-admissible $F^{\prime}$ such that $F^{\prime} \subseteq_{s^{\prime}} A$ and define $c_{0}(A)=c\left(F^{\prime}\right)$.
Since $c_{0}$ is not a proper coloring, it follows that one may find vertices

$$
B_{1}(X), \ldots, B_{r_{1}}(X)
$$

of $\widetilde{\mathrm{KG}}^{r_{1}}\left(\mathcal{F}\left(X, s^{\prime}\right),\left.\mathcal{P}\right|_{X}, 0\right)$ that are pairwise disjoint and have the same color. Define the coloring $c^{\prime}$ for $\mathrm{KG}^{r_{2}}\left(\mathcal{F}^{\prime}, \mathcal{P}^{\prime}\right)$ by $c^{\prime}(X)=c_{0}\left(B_{1}(X)\right)$. Note that for each $B_{i}(X)$, one has a vertex $F_{i}(X)$ of $\widetilde{\mathrm{KG}}^{r_{1} r_{2}}(\mathcal{F}, \mathcal{P}, s)$ (resp. of $\left.\mathrm{KG}^{r_{1} r_{2}}(\mathcal{F}, \mathcal{P}, s)\right)$ such that $F_{i}(X) \subseteq_{s^{\prime}} B_{i}(X)$, with $c\left(F_{1}(X)\right)=\cdots=c\left(F_{r_{1}}(X)\right)$. Then $c^{\prime}$ is a proper coloring since otherwise there exists pairwise disjoint vertices $X_{1}, \ldots, X_{r_{2}}$ with the same color, and hence the $r_{1} r_{2}$ subsets $F_{i}\left(X_{j}\right)$ of $\mathcal{F}$ have pairwise intersection of at most $s$ elements and the same color. This contradicts the properness of the coloring $c$. So by the hypothesis of the lemma, $\operatorname{ecd}^{r_{2}}\left(\mathcal{F}^{\prime}\right) \leq\left(r_{2}-1\right) t$. Hence, one may find $X_{0} \subseteq[n]$ of size at most $\left(r_{2}-1\right) t$ and an equitable partition

$$
[n] \backslash X_{0}=X_{1} \cup \cdots \cup X_{r_{2}}
$$

with the property that no $X \in \mathcal{F}^{\prime}$ is a subset of one of $X_{1}, \ldots, X_{r_{2}}$. So in particular for $1 \leq i \leq r_{2}, X_{i} \notin \mathcal{F}^{\prime}$ and hence $\operatorname{ecd}^{r_{1}}\left(\mathcal{F}\left(X_{i}, s^{\prime}\right)\right) \leq\left(r_{1}-1\right) t$. This implies the existence of a subset $X_{i, 0} \subseteq X_{i}$ of size at most $\left(r_{1}-1\right) t$ and an equitable partition

$$
X_{i} \backslash X_{i, 0}=X_{i, 1} \cup \cdots \cup X_{i, r_{1}}
$$

such that no $A \in \mathcal{F}\left(X_{i}, s^{\prime}\right)$ is a subset of one of $X_{i, 1}, \ldots, X_{i, r_{1}}$. We may assume that $\left|X_{i, 0}\right|=\left(r_{1}-1\right) t$, since if $\left|X_{i, 0}\right|<\left(r_{1}-1\right) t$, remove one element from an $X_{i, j}$ for $j=$ $1, \ldots, r_{1}$ with the largest size and add it to the $X_{i, 0}$ without violating any of the conditions. By repeating this process, we may assume $\left|X_{i, 0}\right|=\left(r_{1}-1\right) t$ for $i=1, \ldots, r_{2}$. If now $\left|X_{i, j}\right|-\left|X_{i^{\prime}, j^{\prime}}\right|>1$ for some $1 \leq i, i^{\prime} \leq r_{2}$ and $1 \leq j, j^{\prime} \leq r_{1}$, then it follows that $\left|X_{i}\right|-\left|X_{i^{\prime}}\right|>1$, which is a contradiction. The reason for this, is that if we let $a=\left|X_{i, j}\right|$, the minimum size that $X_{i}$ can have is $a+\left(r_{1}-1\right)(a-1)+t\left(r_{1}-1\right)$, and the maximum size that $X_{i^{\prime}}$ can have is $a-2+\left(r_{1}-1\right)(a-1)+\left(r_{1}-1\right) t$.

It follows that we have an equitable partition

$$
[n] \backslash X_{0}^{\prime}=X_{1,1} \cup \cdots \cup X_{1, r_{1}} \cup \cdots \cup X_{r_{2}, 1} \cup \cdots \cup X_{r_{2}, r_{1}}
$$

where

$$
X_{0}^{\prime}=X_{0} \cup X_{1,0} \cdots \cup X_{r_{2}, 0}
$$

is of size at most

$$
\left(r_{2}-1\right) t+r_{2}\left(r_{1}-1\right) t=\left(r_{1} r_{2}-1\right) t
$$

and this partition has the property that is no $F \in \mathcal{F}$ such that $F \subseteq_{s^{\prime}} X_{i, j}$ for some $i=1, \ldots, r_{2}$ and $j=1, \ldots, r_{1}$. Since otherwise, $A=F \cap X_{i, j} \in \mathcal{F}\left(X_{i}, s^{\prime}\right)$ and $A \subseteq X_{i, j}$, which is a contradiction. This shows that $\operatorname{ecd}^{r_{1} r_{2}}\left(\mathcal{F}, s^{\prime}\right)$ is less than or equal to $\left(r_{1} r_{2}-1\right) t$ or in other words, $t$ is greater than or equal to $\frac{\operatorname{ecd}^{r_{1} r_{2}}\left(\mathcal{F}, s^{\prime}\right)}{r_{1} r_{2}-1}$. This proves the lemma.

## 3 Proof of Theorem 1.4 and Theorem 1.5

To prove Theorems 1.4 and 1.5, hence we may suppose that $r=p$ is a prime number. We use $\mathbb{Z}_{p}$-Tucker lemma. We recall its statement from [9]. The simplicial complex $\mathrm{E}_{n-1}\left(\mathbb{Z}_{p}\right)$ has $\mathbb{Z}_{p} \times[n]$ as its vertices and all subsets $A \subseteq \mathbb{Z}_{p} \times[n]$ with pairwise different second components as faces. It has a free action of $\mathbb{Z}_{p}$ that acts on the first component of each vertex by multiplication. We take $\mathbb{Z}_{p}$ to be the multiplicative group of all $p$ th roots of unity.

Lemma 3.1. ( $\mathbb{Z}_{p}$-Tucker Lemma) Let $n, m>0$ and $m \geq \alpha \geq 0$ be integers and $p$ be a prime number. If $\lambda$ is a map from the non-empty faces of $E_{n-1}\left(\mathbb{Z}_{p}\right)$ to $\mathbb{Z}_{p} \times[m]$ with $\lambda(A)=\left(\lambda_{1}(A), \lambda_{2}(A)\right) \in \mathbb{Z}_{p} \times[m]$ that satisfies the following properties,

1. If $\omega \in \mathbb{Z}_{p}$ and $A$ is a non-empty face of $E_{n-1}\left(\mathbb{Z}_{p}\right)$, then $\lambda_{1}(\omega \cdot A)=\omega \cdot \lambda_{1}(A)$ and $\lambda_{2}(\omega \cdot A)=\lambda_{2}(A)$. That is $\lambda$ is $\mathbb{Z}_{p}$-equivariant.
2. If $A_{1} \subseteq A_{2}$ be non-empty faces of $E_{n-1}\left(\mathbb{Z}_{p}\right)$ and $\lambda_{2}\left(A_{1}\right)=\lambda_{2}\left(A_{2}\right) \leq \alpha$ then $\lambda_{1}\left(A_{1}\right)=$ $\lambda_{1}\left(A_{2}\right)$.
3. If $A_{1} \subseteq \cdots \subseteq A_{p}$ be non-empty faces of $E_{n-1}\left(\mathbb{Z}_{p}\right)$ and $\lambda_{2}\left(A_{1}\right)=\cdots=\lambda_{2}\left(A_{p}\right)>\alpha$ then $\lambda_{1}\left(A_{1}\right), \ldots, \lambda_{1}\left(A_{p}\right)$ are not pairwise distinct.
then

$$
\alpha+(m-\alpha)(p-1) \geq n
$$

Now let us present our proof for Theorem 1.4 (resp. Theorem 1.5).
Proof. Let $t=\chi\left(\widetilde{\mathrm{KG}}^{p}(\mathcal{F}, \mathcal{P}, s)\right.$ ) (resp. $\quad t=\chi\left(\mathrm{KG}^{p}(\mathcal{F}, \mathcal{P}, s)\right.$ ) for the case of Theorem 1.5) and let $c$ be a coloring of the vertices of this hypergraph with colors $\{1, \ldots, t\}$. Let $s^{\prime}=\lfloor s / 2\rfloor, \alpha=n-\operatorname{ecd}^{p}\left(\mathcal{F}, s^{\prime}\right)$ and $m=\alpha+t$. Also for simplicity choose a complete ordering on non-empty subsets of $[n]$, that has the property that if $|A|<|B|$ then $A<B$.

We define a $\mathbb{Z}_{p}$-equivariant map $\lambda$ from the non-empty faces of $\mathrm{E}_{n-1}\left(\mathbb{Z}_{p}\right)$ to $\mathbb{Z}_{p} \times[m]$ that satisfies the two properties of the $\mathbb{Z}_{p}$-Tucker lemma and hence

$$
\alpha+(m-\alpha)(p-1)=n-\operatorname{ecd}^{p}\left(\mathcal{F}, s^{\prime}\right)+(p-1) t \geq n
$$

and hence the result follows. For a non-empty face $A$ of $\mathrm{E}_{n-1}\left(\mathbb{Z}_{p}\right)$ and $i \in \mathbb{Z}_{p}$, let $A^{i}=\{1 \leq j \leq n \mid(i, j) \in A\}$. The definition of $\lambda(A)=\left(\lambda_{1}(A), \lambda_{2}(A)\right) \in \mathbb{Z}_{p} \times[m]$ is given
in two cases.
Case 1: If there is an element $F \in \mathcal{F}$ with $F \subseteq_{s^{\prime}} A^{i}$ for some $i \in \mathbb{Z}_{p}$ and

$$
\sum_{i=1}^{l} \max \left\{\left|F \cap P_{j}\right|-1,0\right\} \leq s^{\prime}
$$

(resp. $F$ is $\mathcal{P}$-admissible) for all $1 \leq j \leq l$, then choose the smallest such subset with respect to the complete ordering on subsets of $[n]$, say $F \subseteq_{s^{\prime}} A^{i}$ and define

$$
\lambda(A)=(i, c(F)+\alpha)
$$

We remark that since $|F|>s$, one can not have more than one $i \in \mathbb{Z}_{p}$ that $F \subseteq \subseteq_{s^{\prime}} A^{i}$.
Case 2: Otherwise, choose a non-empty subset $B \subseteq A$ such that for all $i \in \mathbb{Z}_{p}$ and $j=1, \ldots, l,\left|B^{i} \cap P_{j}\right| \leq 1$ and $\pi_{2}(B)$ is maximum with respect to the chosen complete order on subsets of $[n]$, this is clearly unique. Here $\pi_{2}: \mathbb{Z}_{p} \times[n] \rightarrow[n]$ is the projection onto the second component. Also, assume that

$$
\left|B^{i_{1}}\right|=\cdots=\left|B^{i_{h}}\right|<\left|B^{i_{h+1}}\right| \leq \cdots \leq\left|B^{i_{p}}\right|
$$

for some $1 \leq h \leq p$, where $h=p$ means that all the sizes are equal. Define

$$
\lambda_{2}(A)=p\left|B^{i_{1}}\right|+p-h
$$

Note that $\lambda_{2}(A) \leq \alpha$. This is because by removing elements from $B^{i_{h+1}} \ldots, B^{i_{p}}$ (if there are any) arbitrarily, we may assume that their sizes are $\left|B^{i_{1}}\right|+1$ to arrive at an equitable partition of a set of size $\lambda_{2}(A)$. If $\lambda_{2}(A)$ is greater than $n-\operatorname{ecd}^{p}\left(\mathcal{F}, s^{\prime}\right)$, then by the definition of $\operatorname{ecd}^{p}\left(\mathcal{F}, s^{\prime}\right)$ there is an element $F \in \mathcal{F}$ with $F \subseteq s_{s^{\prime}} B^{i_{k}}$ for some $k=1, \ldots, p$, and therefore

$$
\sum_{i=1}^{l} \max \left\{\left|F \cap P_{j}\right|-1,0\right\} \leq s^{\prime}
$$

(resp. $F$ can be chosen so that it is $\mathcal{P}$-admissible by the $s^{\prime}$-goodness assumption). This contradicts the fact that we are in the Case 2.

The definition of $\lambda_{1}(A)$ is more delicate. We define it in several sub-cases.
Case 2.1: If $h<p$, find $1 \leq h^{\prime}<p$ such that $h h^{\prime} \equiv 1 \bmod p$ and define

$$
\lambda_{1}(A)=\left(i_{1} \ldots i_{h}\right)^{h^{\prime}}
$$

Case 2.2: If $h=p$, find the smallest $1 \leq j \leq l$ that $\pi_{2}(B) \cap P_{j}$ is non-empty, and take the unique subset $B^{\prime} \subseteq B$ such that $\pi_{2}\left(B^{\prime}\right)=\pi_{2}(B) \cap P_{j}$. Let $\pi_{1}\left(B^{\prime}\right)=\left\{j_{1}, \ldots, j_{k}\right\}$, where $\pi_{1}$ is the projection onto the first component. Then we have again two sub-cases: Case 2.2.1: If $k<p$, choose $1 \leq k^{\prime}<p$ such that $k k^{\prime} \equiv 1 \bmod p$ and define:

$$
\lambda_{1}(A)=\left(j_{1} \ldots j_{k}\right)^{k^{\prime}}
$$

Case 2.2.2: If $k=p$, define $\lambda_{1}(A)$ to be the first component of the element of $B^{\prime}$ with the smallest second component.

It remains to check the properties of the $\mathbb{Z}_{p}$-Tucker lemma. First, $\lambda$ is $\mathbb{Z}_{p}$-equivariant in the Case 1. That is $\lambda_{1}(\omega \cdot A)=\omega \cdot \lambda_{1}(A)$ and $\lambda_{2}(\omega \cdot A)=\lambda_{2}(A)$ for any $\omega \in \mathbb{Z}_{p}$. This is because, if $F \subseteq A^{i}$ is the required subset for $A$ in case one then $F \subseteq(\omega A)^{\omega \cdot i}$ is the required subset for $\omega \cdot A$.
If $A_{1} \subseteq \cdots \subseteq A_{p}$ is a chain of non-empty faces of $\mathrm{E}_{n-1}\left(\mathbb{Z}_{p}\right)$ with $\lambda_{2}\left(A_{1}\right)=\cdots=\lambda_{2}\left(A_{p}\right)>$
$\alpha$, then we are in the Case 1. Hence with have vertices $F_{1}, \ldots, F_{p}$ of $\mathrm{KG}^{p}(\mathcal{F}, \mathcal{P}, s)$ with $F_{i} \subseteq_{s^{\prime}} A_{i}^{\lambda_{1}\left(A_{i}\right)}$ with $c\left(F_{1}\right)=\cdots=c\left(F_{p}\right)$. If $\lambda_{1}\left(A_{1}\right), \ldots, \lambda_{1}\left(A_{p}\right)$ are pairwise distinct, then since $A_{i}^{\lambda_{1}\left(A_{i}\right)} \cap A_{j}^{\lambda_{1}\left(A_{j}\right)}=\emptyset$ for $i \neq j$ then $\left|F_{i} \cap F_{j}\right| \leq 2 s^{\prime} \leq s$ and $\left\{F_{1}, \ldots, F_{p}\right\}$ will be a mono-chromic edge, which contradicts properness of $c$. Hence the third condition of the $\mathbb{Z}_{p}$-Tucker lemma holds.
To show that $\lambda$ is $\mathbb{Z}_{p}$-equivariant in Case 2, note that if $B \subseteq A$ is the required set for $A$, then $\omega \cdot B_{1} \subseteq \omega \cdot B_{2}$ is the required set in for $\omega \cdot A$, hence $\lambda_{2}(A)=\lambda_{2}(\omega \cdot A)$. Also, the corresponding $\left\{i_{1}, \ldots, i_{h}\right\}$ will be $\left\{\omega \cdot i_{1}, \ldots, \omega \cdot i_{h}\right\}$. In the Case 2.1, we have

$$
\lambda_{1}(\omega \cdot A)=\left(\left(\omega \cdot i_{1}\right) \ldots\left(\omega \cdot i_{h}\right)\right)^{h^{\prime}}=\omega^{h h^{\prime}} \cdot\left(i_{1} \ldots i_{h}\right)^{h^{\prime}}=\omega \cdot \lambda_{1}(A)
$$

In Case 2.2, we have $\omega \cdot B^{\prime}$ as the corresponding set for $\omega \cdot A$. So in both Cases 2.2.1 and 2.2.2 it follows that $\lambda_{1}(\omega \cdot A)=\omega \cdot \lambda_{1}(A)$. This proves that $\lambda$ is $\mathbb{Z}_{p}$-equivariant.

If $A_{1} \subseteq A_{2}$ are non-empty faces of $\mathrm{E}_{n-1}\left(\mathbb{Z}_{p}\right)$ with $\lambda_{2}\left(A_{1}\right)=\lambda_{2}\left(A_{2}\right) \leq \alpha$, then we are in the second case. With maximal subsets $B_{1} \subseteq A_{1}$ and $B_{2} \subseteq A_{2}$. Assume that

$$
\begin{aligned}
& \left|B_{1}^{i_{1}}\right|=\cdots=\left|B_{1}^{i_{h}}\right|<\left|B_{1}^{i_{h+1}}\right| \leq \cdots \leq\left|B_{1}^{i_{p}}\right| \\
& \left|B_{2}^{j_{1}}\right|=\cdots=\left|B_{2}^{j_{k}}\right|<\left|B^{j_{k+1}}\right| \leq \cdots \leq\left|B_{2}^{j_{p}}\right|
\end{aligned}
$$

for some $1 \leq h \leq p$ and $1 \leq k \leq p$. If $\lambda_{2}\left(A_{1}\right)=\lambda_{2}\left(A_{2}\right)$, then $\left|B_{1}^{i_{1}}\right|=\left|B_{2}^{j_{1}}\right|$ and $h=k$. Now since $B_{1} \subseteq A_{1} \subseteq A_{2}$, by the maximality of $B_{2}$, we have $\left|B_{1}^{i}\right| \leq\left|B_{2}^{i}\right|$. Therefore $\left\{i_{1}, \ldots, i_{h}\right\}=\left\{j_{1}, \ldots, j_{h}\right\}$. So in Case 2.1 we must have $\lambda_{1}\left(A_{1}\right)=\lambda_{1}\left(A_{2}\right)$.
If we are in Case 2.2, then $\left|B_{1}^{i}\right|=\left|B_{2}^{i}\right|$ for all $i \in \mathbb{Z}_{p}$ and hence $\left|B_{1}\right|=\left|B_{2}\right|$. This implies that the first $1 \leq j \leq l$ that $\pi_{2}\left(B_{1}\right) \cap P_{j}$ is non-empty is the same as the first $1 \leq j^{\prime} \leq l$ that $\pi_{2}\left(B_{2}\right) \cap P_{j^{\prime}}$ is non-empty. So by the maximality and equality of $\left|B_{1}\right|=\left|B_{2}\right|$, it follows that $\pi_{1}\left(B_{1}^{\prime}\right)=\pi_{1}\left(B_{2}^{\prime}\right)$. In the Case 2.2 .1 therefore $\lambda_{1}\left(A_{1}\right)=\lambda_{1}\left(A_{2}\right)$. Finally, in the Case 2.2 .2 since $\left|P_{j}\right| \leq p$, it follows that $B_{1}^{\prime}=B_{2}^{\prime}$ and hence the first component of the element with the smallest second component in both of them are the same, that is $\lambda_{1}\left(A_{1}\right)=\lambda_{1}\left(A_{2}\right)$. This finishes checking the conditions and hence the proof of the theorem is complete.

## 4 A generalization of a theorem of Abyazi Sani and Alishahi

In this section, using Theorem 1.2, we generalize Theorem 3 of Abyazi Sani and Alishahi in [1]. For an integer vector $S=\left(s_{1}, \ldots, s_{n}\right)$ with $0 \leq s_{i} \leq r$, the notion of an $S$-disjoint multi-set $\left\{A_{1}, \ldots, A_{r}\right\}$ of subsets of $[n]$ was considered by Sarkaria and Ziegler in [10], [11], and [12]. It means that for all $1 \leq i \leq n$, the number of $1 \leq j \leq r$ that $i \in A_{j}$ is at most $s_{i}$. This generalizes the notion of pairwise disjoint that is just $S=(1,1, \ldots, 1)$-disjoint. Ziegler [11] extended the $r$-colorability defect of a family $\mathcal{F}$ of subsets of $[n], \operatorname{cd}^{r}(\mathcal{F})$, to the $S$-disjoint $r$-colorability defect $\operatorname{cd}_{S}^{r}(\mathcal{F})$. This was also extended by Abyazi Sani and Alishahi [1] to the equitable $S$-disjoint $r$-colorability defect $\operatorname{ecd}_{S}^{r}(\mathcal{F})$ which is defined as follows. Let $\bar{n}=\sum_{i=1}^{n} s_{i}$. Then $\operatorname{ecd}_{S}^{r}(\mathcal{F})$ is defined by

$$
\bar{n}-\max \left\{\sum_{i=1}^{r}\left|A_{i}\right| \mid\left\{A_{1}, \ldots, A_{r}\right\} \text { equitable and } S \text {-disjoint } \forall F \in \mathcal{F}, 1 \leq i \leq r F \nsubseteq A_{i}\right\}
$$

For a subset $P$ of $[n]$, we define the $S$-weight of $P$ to be

$$
w_{S}(P)=\sum_{i \in P} s_{i}
$$

For a partition $\mathcal{P}=\left\{P_{1}, \ldots, P_{l}\right\}$ of $[n]$, we also define the $r$-uniform Kneser-type hypergraph $\mathrm{KG}_{S}^{r}(\mathcal{F}, \mathcal{P})$ to be a hypergraph with the vertex set of those $A \in \mathcal{F}$ that have at most one element from each $P_{1}, \ldots, P_{l}$ and the edge set of all multi-sets $\left\{A_{1}, \ldots, A_{r}\right\}$ of the vertices that are $S$-disjoint. We then have the following theorem.

Theorem 4.1. If the partition $\mathcal{P}=\left\{P_{1}, \ldots, P_{l}\right\}$ has the property that the $S$-weight of each partition piece is at most $r$, then one has

$$
\chi\left(K G_{S}^{r}(\mathcal{F}, \mathcal{P})\right) \geq\left\lceil\frac{e c d_{S}^{r}(\mathcal{F})}{r-1}\right\rceil .
$$

Proof. For each $1 \leq i \leq n$, we make $s_{i}$ different copies of $i$, say $(i, 1), \ldots,\left(i, s_{i}\right)$ and make the set $[n]$ into the bigger set $[\bar{n}]$. So we have a natural map $f:[\bar{n}] \rightarrow[n]$ that sends any copy of $i$ to $i$. We define the lifted family $\overline{\mathcal{F}}$ to be all subsets $A$ of $[\bar{n}]$ such that $f(A) \in \mathcal{F}$ and also all two-element subsets of $[\bar{n}]$ with two different copies of the same number. Finally, we define a partition $\overline{\mathcal{P}}=\left\{\bar{P}_{1}, \ldots, \bar{P}_{l}\right\}$ by replacing any element $i$ in a partition piece with all of its $s_{i}$ copies. Hence $\left|\bar{P}_{i}\right|=w_{S}\left(P_{i}\right) \leq r$. Now we claim that $f$ defines a hypergraph homomorphism from $\mathrm{KG}^{r}(\overline{\mathcal{F}}, \overline{\mathcal{P}})$ to $\mathrm{KG}_{S}^{r}(\mathcal{F}, \mathcal{P})$ and hence

$$
\chi\left(\operatorname{KG}_{S}^{r}(\mathcal{F}, \mathcal{P})\right) \geq \chi\left(\operatorname{KG}^{r}(\overline{\mathcal{F}}, \overline{\mathcal{P}})\right) .
$$

The proof of the claim is straightforward, notice that the special two-element subsets of $\overline{\mathcal{F}}$, do not appear as vertices of this hypergraph. It remains to check that $\operatorname{ecd}_{S}^{r}(\mathcal{F}) \leq \operatorname{ecd}^{r}(\overline{\mathcal{F}})$, which will finish the proof of the theorem by applying Theorem [1.2. If $\left\{A_{1}, \ldots, A_{r}\right\}$ is an equitable disjoint family in $[\bar{n}]$ such that no element of $\overline{\mathcal{F}}$ is a subset of one of $A_{1}, \ldots, A_{r}$, then $f\left(A_{1}\right), \ldots, f\left(A_{r}\right)$ is an $S$-disjoint equitable family of subsets of $[\bar{n}]$ with $\left|f\left(A_{i}\right)\right|=\left|A_{i}\right|$ (note that because of the special two element subsets in $\overline{\mathcal{F}}$, each $A_{i}$ must contain at most one copy from each element). Also, no $F \in \mathcal{F}$ is a subset of of one of $f\left(A_{1}\right), \ldots, f\left(A_{r}\right)$. This implies that $\operatorname{ecd}_{S}^{r}(\mathcal{F}) \leq \operatorname{ecd}^{r}(\overline{\mathcal{F}})$. The theorem is proved.

Remark 4.1. When $\mathcal{P}$ is the trivial partition of $[n]$ into singletons, this result extends the corresponding inequality

$$
\chi\left(K G_{S}^{r}(\mathcal{F})\right) \geq\left\lceil\frac{e c d_{S}^{r}(\mathcal{F})}{r-1}\right\rceil .
$$

obtained by Abyazi Sani and Alishahi in [1] with the extra assumption that $s_{i}<\mu(r)$, where $\mu(r)$ is the largest prime factor of $r$.

## 5 Examples

In this section, we study the Kneser hypergraph of a special family introduced in 11 and its generalizations. For integers $n>k>a \geq 0$ and $k>s \geq 0$, define $\mathcal{H}(n, k, a, s)$ to be the family of all $k$-subsets $F \subseteq[n]$ with $F \not \mathscr{E}_{s}\{n-a+1, \ldots, n\}$ and let $\mathrm{KG}^{r}(n, k, a, s)$ be the $r$-uniform Kneser hypergraph with the vertex set $\mathcal{H}(n, k, a, s)$ and the edge set of all $r$-subsets $\left\{F_{1}, \ldots, F_{r}\right\}$ of vertices with pairwise intersection of at most $s$ elements. The case $s=0$, was considered by Abyazi Sani and Alishahi in [1] and was denoted by $\mathrm{KG}^{r}(n, k, a)$.

Remark 5.1. The pair $(\mathcal{H}(n, k, a, s), \mathcal{P})$ is $\lfloor s / 2\rfloor$-good, if $n \geq r k,\left|P_{i}\right| \leq r, 1 \leq s<k$ and at least $s+1$ of the non-empty partitions of $\mathcal{P}$ have empty intersection with $A=$ $\{n-a+1, n-a+2, \ldots, n\}$. The reason is as follows. Let $s^{\prime}=\lfloor s / 2\rfloor$. Assume for $a$
$\mathcal{P}$-admissible subset $B$ and an element $F$ in $\mathcal{H}(n, k, a, s)$ we have $F \subseteq_{s^{\prime}} B$. Then there is a subset $E$ of size at most $s^{\prime}$ such that $F \backslash E \subseteq B$. Since $F$ has at least $s+1$ elements outside of $A$, so $F \backslash E$ has $t \geq s+1-s^{\prime}$ elements outside of $A$. Assume $P_{1}, \ldots, P_{s+1}$ be the partitions with empty intersection with $A$. If $t \geq s+1$, then one can add arbitrarily elements from different partitions that have empty intersection with $F \backslash E$ so it become a $\mathcal{P}$-admissible element $F^{\prime} \in \mathcal{H}(n, k, a, s)$ with $F^{\prime} \subseteq_{s^{\prime}} B$. If $t<s+1$, then $F \backslash E$ has non-empty intersection with at most $t$ of $P_{1}, \ldots, P_{s+1}$, so we can use elements from those $P_{1}, \ldots, P_{s+1}$ with empty intersection with $F \backslash E$ and if needed other partition parts to complete $F \backslash E$ to a $\mathcal{P}$-admissible $F^{\prime} \in \mathcal{H}(n, k, a, s)$ such that $F^{\prime} \subseteq_{s^{\prime}} B$.

The following lemma is an extension of a computation made in 1 for $\operatorname{ecd}^{r}(\mathcal{H}(n, k, a, 0))$.
Lemma 5.1. Let $n, k, r, s$, and $a$ be integers with $k, r \geq 2$ and $n \geq r k, 0 \leq s<k$, and $n>a+s$. Then, one has

$$
e c d^{r}(\mathcal{H}(n, k, a, s), s)= \begin{cases}n-r(k-s-1) & a \leq k-s-1 \\ n-r(k-s-1)-\left\lfloor\frac{a}{k-s}\right\rfloor & k-s \leq a \leq r(k-s)-2 \\ n-a & a \geq r(k-s)-1\end{cases}
$$

Proof. Let $A=\{n-a+1, \ldots, n\}$. We prove each case separately.

1. In the first case, $\mathcal{H}(n, k, a, s)$ is $\binom{[n]}{k}$ of all $k$-subsets of $[n]$ and it follows from Remark 1.1.
2. In the second case, let $X_{0}, X_{1}, \ldots, X_{r}$ be a partition of $[n]$ such that as in the definition the generalized $r$-colorability defect, there are no $F \in \mathcal{H}(n, k, a, s)$ such that $F \subseteq_{s} X_{i}$ for some $i=1, \ldots, r$. We show that $\left|X_{i}\right| \leq k-s$ for $1 \leq i \leq r$ and if $\left|X_{i}\right|=k-s$ then $X_{i} \subseteq A$. Assume that $\left|X_{1}\right| \geq k-s+1$, and since the partition is equitable $\left|X_{i}\right| \geq k-s$ for $1 \leq i \leq r$. Hence, there exist $1 \leq i \leq r$ such that $X_{i} \nsubseteq A$. Let $X_{i}^{\prime}$ be a $k-s$ subset of $X_{i}$ such that $X_{i}^{\prime} \nsubseteq A$. Since $n>a+s$ there exist at least $s$ elements in $[n] \backslash A$ so we can extend $X_{i}^{\prime}$ to a k-subset $F$ such that $F \not \mathbb{I}_{s} A$ and so $F \in \mathcal{H}(n, k, a, s), F \subseteq_{s} X_{i}$ which violates the assumption on the partition. From the previous argument one can deduce the fact that $\left|X_{i}\right|=k-s$ can only happen when $X_{i} \subseteq A$. Based on these facts:

$$
\begin{aligned}
\operatorname{ecd}^{r}(\mathcal{H}(n, k, a, s), s) & \geq n-(k-s)\left\lfloor\frac{a}{k-s}\right\rfloor-(k-s-1)\left(r-\left\lfloor\frac{a}{k-s}\right\rfloor\right) \\
& =n-r(k-s-1)-\left\lfloor\frac{a}{k-s}\right\rfloor
\end{aligned}
$$

This bound is sharp since you can find such an equitable partition by taking $\left\lfloor\frac{a}{k-s}\right\rfloor$ disjoint $(k-s)$-subsets of $A$ as $X_{1}, \ldots, X_{\left\lfloor\frac{a}{k-s}\right\rfloor}$ and $r-\left\lfloor\frac{a}{k-s}\right\rfloor$ arbitrary disjoint ( $k-s-1$ )-subsets of the remaining elements as other $X_{i}{ }^{\prime}$ s.
3. In the third case, If $X_{0}=[n] \backslash A$ and $X_{1}, \ldots, X_{r}$ be a equitable partition of $A$, then clearly there is no $F \in \mathcal{H}(n, k, a, s)$ such that $F \subseteq_{s} X_{i}$ for some $1 \leq i \leq r$. If $\left|X_{0}\right|<n-a$ then $\left|X_{1} \cup \cdots \cup X_{r}\right|>a$. If $a \geq r(k-s)$, hence at least one $X_{i}$ has a size of at least $k-s+1$, which is not possible by the argument in the previous step. If $a=r(k-s)-1$, then we must have $\left|X_{i}\right|=k-s$ for all $i$ and hence $X_{i} \subseteq A$. This is not possible either, because it implies that $a \geq r(k-s)$. So ecd ${ }^{r}$ is $n-a$.

Theorem 5.1. Let $n, k, r, s$ and $a$ be integers with $k, r \geq 2, n>a \geq 0, n \geq r k$, $0 \leq s<k$, and $a \leq r(k-s-1)$. Then, one has

$$
\chi\left(K G^{r}(n, k, a, s)\right) \geq\left\lceil\frac{n-r(k-\lfloor s / 2\rfloor-1)}{r-1}\right\rceil \text {. }
$$

Proof. Let $A=\{n-a+1, \ldots, n\}$. Take a partition $\mathcal{P}=\left\{P_{1}, \ldots, P_{l}\right\}$ of $[n]$ such that $\left|P_{i}\right|=r$ for all $1 \leq i \leq k-s-1$ and $\left|P_{i}\right| \leq r$ otherwise, and $A \subseteq \bigcup_{i=1}^{k-s-1} P_{i}$. Now, $\operatorname{KG}^{r}\left(\binom{[n]}{k}, \mathcal{P}, s\right)$ is a sub-hypergraph of $\operatorname{KG}^{r}(n, k, a, s)$, because if a $k$-subset $F$ is $\mathcal{P}$-admissible then it contains at most $(k-s-1)$ elements from $A$ and hence $F \not \mathbb{L}_{s} A$. The result follows now from Theorem 1.5. Recall that $\operatorname{ecd}^{r}\left(\binom{[n]}{k}, s\right)=n-r(k-s-1)$, and by Remark 1.2, the pair $\left(\binom{[n]}{k}, \mathcal{P}\right)$ is $\lfloor s / 2\rfloor$-good.

Remark. The above theorem, for the case when $s=0$, was conjectured in [1]. They showed that it is true when $a \leq 2(k-1)$. This was generalized by Aslam, Chen, Coldren, Frick, and Seitanrata in [6] for $a \leq b(r)(k-1)$, where $b(r)$ for the prime decomposition $r=2^{\alpha_{0}} p_{1}^{\alpha_{1}} \ldots p_{m}^{\alpha_{m}}$ is defined to be $2^{\alpha_{0}}\left(p_{1}-1\right)^{\alpha_{1}} \ldots\left(p_{m}-1\right)^{\alpha_{m}}$. Our theorem hence, is a generalization of these results.

The following hypergraph is considered in 6. Let $\mathrm{KG}^{r}(n, k, \mathcal{P})_{t-\text { wide }}$ be the subhypergraph of $\mathrm{KG}^{r}(n, k, \mathcal{P})$ induced by the vertices that are not contained in any one of the sets $\{i, i+1, \ldots, i+t-1\}$ for $i \in[n-t+1]$. The following theorem is proved in [6].

Theorem 5.2. Let $k \geq 1$ be an integer, $r \geq 2$ a prime, and $n \geq r k$ an integer. Let $\mathcal{P}=\left\{P_{1}, \ldots, P_{l}\right\}$ be a partition of $[n]$ with $\left|P_{i}\right| \leq r-1$. Let $t \leq r(k-3)+2$. Then

$$
\chi\left(K G^{r}(n, k, \mathcal{P})_{t-w i d e}\right)=\left\lceil\frac{n-r(k-1)}{r-1}\right\rceil
$$

In some special cases we can improve their result.
Theorem 5.3. Let $k \geq 1$ be an integer $r \geq 2$, and $n \geq r k$ an integer. Let $t \leq r(k-2)+1$ then

$$
\chi\left(K G^{r}(n, k)_{t-\text { wide }}\right)=\left\lceil\frac{n-r(k-1)}{r-1}\right\rceil \text {. }
$$

Proof. Let $l=\left\lceil\frac{n}{r}\right\rceil$ and $\mathcal{P}=\left\{P_{1}, \ldots, P_{l}\right\}$ be a partition of $[n]$ such that

$$
P_{i}=\{(i-1) r+1, \ldots, i r\}
$$

for $1 \leq i \leq l-1$. Then, $\operatorname{KG}^{r}(n, k, \mathcal{P})$ is a sub-hypergraph of $\mathrm{KG}^{r}(n, k)_{t-\text { wide }}$, because suppose that a $\mathcal{P}$-admissible $k$-subset $F$ is a subset of $X=\{i, i+1, \ldots, i+t-1\}$. Then, the smallest value that $t$ can have is when $i \in F$ is the last element of some $P_{j}$ and $(i+t-1) \in F$ is the first element of $P_{j+k-1}$ and $P_{j+1}, \ldots, P_{j+k-2}$ are subsets of $X$, that is $t \geq r(k-2)+2$, which contradicts the assumption on $t$. The result then follows from Theorem 1.2 and the standard coloring of the Kneser hypergraph $\mathrm{KG}^{r}(n, k)$.

Remark 5.2. The family of $t$-wide subsets are very interesting examples to compare the colorability defects for them. It is proved in [6] that the topological r-colorability defect
of this family for $t \leq r(k-3)+2$ is at least $n-r(k-1)$ but we will show that if $n>\max \{r t, r(k-1)\}$ then,

$$
\operatorname{ecd} d^{r}\left(\binom{[n]}{k}_{t-w i d e}\right)= \begin{cases}n-r(k-1) & t \leq k \\ n-r t & t>k\end{cases}
$$

Therefore, there exist examples where the topological colorability defect is better than the equitable colorability defect.

Proof. In the first case, the family of $t$-wide $k$-subsets is the same as the family of all $k$-subsets and the result follows by Remark 1.1, In the second case, let $X_{0}, X_{1}, \ldots, X_{r}$ be a partition of $[n]$ such that no $F \in\binom{[n]}{k}_{t-w i d e}$ is a subset of one of $X_{1}, \ldots, X_{r}$. Note that for all $1 \leq i \leq r$, one has $\left|X_{i}\right| \leq t$. Because otherwise, take a $k$-subset of $X_{i}$ that contains its smallest and its biggest elements. This subset is a $t$-wide $k$-subset inside $X_{i}$ and therefore violates the assumption on the partition. This shows $\left|X_{0}\right| \geq n-r t$. Finally, since the partition given by $X_{i}=\{(i-1) t+1, \ldots, i t\}$ for $1 \leq i \leq r$ with $\left|X_{0}\right|=n-r t$ has the property that no $t$-wide $k$-subset is inside one of $X_{1}, \ldots, X_{r}$, the claim follows.

It is an interesting problem to see that if it is true that the topological $r$-colorability defect of Frick for a family of subsets is always greater than or equal to the equitable $r$-colorability defect.

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[^0]:    ${ }^{1}$ Without this assumption, we will have a loop edge $\{F, \ldots, F\}$ and the chromatic number of the associated hypergraph is by convention infinity, so there is no need to give a lower bound.

