# The planar Cayley graphs are effectively enumerable II 

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#### Abstract

We show that a group admits a planar, finitely generated Cayley graph if and only if it admits a special kind of group presentation we introduce, called a planar presentation. Planar presentations can be recognised algorithmically. As a consequence, we obtain an effective enumeration of the planar Cayley graphs, yielding in particular an affirmative answer to a question of Droms et al. asking whether the planar groups can be effectively enumerated.


## 1 Introduction

In this paper we complete an effort, started in [12], and building upon 10, 11, the aim of which is to understand the planar Cayley graphs. In 12 we handled the special case of 3-connected Cayley graphs, and more generally, Cayley

[^0]graphs $G$ that admit a consistent embedding in $\mathbb{R}^{2}$, that is, an embedding the facial paths of which are preserved by the action on $G$ by its group (see Section 2.3 for a more detailed definition). It is shown in a follow-up paper in preparation [8] that, at least in the finitely generated case, the groups having such Cayley graphs are exactly those groups admitting a faithful, properly discontinuous action by homeomorphisms on a 2 -manifold contained in $\mathbb{S}^{2}$. It is shown in the same paper that there is a planar Cayley graph the group of which cannot act faithfully and properly discontinuously on $\mathbb{S}^{2}$. Therefore, the aforementioned groups form a proper subclass of the planar groups, i.e. the groups admitting a planar Cayley graph. In this paper we broaden the group presentations introduced in [12] so that we capture exactly the planar finitely generated Cayley graphs. In particular, we capture the planar groups, and we show that they can be effectively enumerated, answering a question of Droms et al. [4, 6].

The Cayley complex $X$ corresponding to a group presentation $\mathcal{P}=\langle\mathcal{S} \mid \mathcal{R}\rangle$ is the 2-complex obtained from the Cayley graph $G$ of $\mathcal{P}$ by glueing a 2-cell along each closed walk of $G$ induced by a relator $R \in \mathcal{R}$. We say that $X$ is almost planar, if it admits a map $\rho: X \rightarrow \mathbb{R}^{2}$ such that the 2-simplices of $X$ are nested in the following sense. We say that two 2 -simplices of $X$ are nested, if the images of their interiors are either disjoint, or one is contained in the other, or their intersection is the image of a 2 -cell bounded by two parallel edges corresponding to an involution $s \in \mathcal{S}$ We call the presentation $\mathcal{P}$ a planar presentation if its Cayley complex is almost planar. We will show that every planar, finitely generated Cayley graph admits a planar presentation. However, we prove something much stronger than that. We are going to introduce a specific type of planar presentation, called a general planar presentation, and show that every planar, finitely generated Cayley graph admits such a presentation and, conversely, every general planar presentation has a planar Cayley graph (Theorem4.10). This converse is the hardest result of this paper. Our main result is:

Theorem 1.1. A finitely generated group admits a planar Cayley graph if and only if it admits a general planar presentation.

The main idea of its proof is that if two relators in a presentation induce cycles whose interiors overlap but are not nested (in a sense similar to the nestedness of 2-simplices), then we replace a subword of one relator by a subword of the other to produce an equivalent presentation with less overlapping; our proof that a presentation with no such overlaps exists is based on a dual version of the machinery of Dunwoody cuts [2], but for cycles instead of cuts.

As a corollary of Theorem 1.1 we obtain that the planar Cayley graphs, and hence their groups, can be effectively enumerated (Theorem 6.3). This answers a question of Droms et al. [4, 6, M. Dunwoody (private communication) informs us that the fact that the planar groups can be effectively enumerated should also follow from his result [7, Theorem 3.8] with a little bit of additional work

[^1](the main issue here is whether the 'or a subgroup of index two' proviso can be dropped).

For more on the motivation of this work and the general background we refer the reader to [12]. We note that while it may help the reader to know [12], the present paper is self-contained.

### 1.1 Planar presentations

The formal definition of a general planar presentation is given in Section 6 and it is a slight generalisation of the notion of a generic planar presentation defined in Section 3. Here, we are going to sketch the most interesting special case of this concept, called a special planar presentation. Such presentations always exist for a 3-connected planar Cayley graph, or more generally, for a Cayley graph that can be embedded in the plane in such a way that its label-preserving automorphisms carry facial paths to facial paths.

We say that $\mathcal{P}=\langle\mathcal{S} \mid \mathcal{R}\rangle$ is a special planar presentation, if it can be endowed with a cyclic ordering $\sigma$-from now on called a spin - of the symmetrization $\mathcal{S}^{\prime}=\left\{s, s^{-1} \mid s \in \mathcal{S}\right\}$ of its generating set, with the following property. Suppose $W_{1}=s U t$ and $W_{2}=s^{\prime} U t^{\prime}$, where $s, s^{\prime}, t, t^{\prime} \in \mathcal{S}^{\prime}$, are two words, each contained in some rotation of a relator in $\mathcal{R}$ (possibly the same relator), where $U$ is any (possibly trivial) word with letters in $\mathcal{S}^{\prime}$. Then $\sigma$ allows us to say whether paths induced by these words $W_{1}, W_{2}$ would cross each other or not if we could embed the Cayley graph of $\mathcal{P}$ in the plane in such a way that for every vertex the cyclic ordering of the labels of its incident edges we observe coincides with $\sigma$. To make this more precise, we embed a tree consisting of a 'middle' path $P$ with edges labelled by the letters in $U$, and two leaves attached at each endvertex of $P$ labelled with $s, s^{\prime}, t, t^{\prime}$ as in Figure where the spin we use at each endvertex of $P$ is the one induced by $\sigma$ on the corresponding 3 -element subset of $\mathcal{S}^{\prime}$. There are essentially two situations that can arise, both shown in that figure. Naturally, we say that $W_{1}, W_{2}$ cross each other in the right-hand situation, and they do not in the left-hand one.


Figure 1: The definition of crossing; $W_{1}=s U t$ crosses $W_{2}=s^{\prime} U t^{\prime}$ in the right, but not in the left.

We then say that $\mathcal{P}$ is a special planar presentation, if there is a spin $\sigma$ on $\mathcal{S}^{\prime}$ such that no two words as above cross each other. Note that this is an abstract property of sets of words, and it is defined without reference to the Cayley graph of $\mathcal{P}$; in fact, it can be checked algorithmically. The essence of this paper is that this is enough to guarantee the planarity of the Cayley graph,
and that a converse statement holds.
This generalises an idea from [9, where it was shown that every planar discontinuous group admits a special planar presentation where every relator is facial, i.e. it crosses no other word (where we consider words that are not necessarily among our relators).

Our actual definition of a special planar presentation, given in Section 3.1 is in fact a bit more general than the above sketch. Consider for example the Cayley graph of the presentation $\left\langle a, b \mid a^{n}, b^{2}, a b a^{-1} b\right\rangle$. Its Cayley graph is a prism graph with an essentially unique embedding in $\mathbb{R}^{2}$. Note that the spin of half of its vertices is the reverse of the spin of other half. This is a general phenomenon: every 3 -connected Cayley graph has an essentially unique embedding, and in that embedding all vertices have the same spin up to reflection. However, for every generator $s$, either the two end-vertices of all edges labelled $s$ have the same spin, or they always have reverse spins. This yields a classification of generators into spin-preserving and spin-reversing ones, and our definition of a special planar presentation takes this into account; still, everything can be checked algorithmically.

The situation becomes much more complex however if one wants to consider planar Cayley graphs that are not 3 -connected. Such graphs do not always have an embedding with all vertices having the same spin up to reflection; perhaps the simplest such example is the one of Figure 2.


Figure 2: A 2-connected planar Cayley graph from [6], obtained by amalgamating two 6 -element groups along an involution, which does not admit a consistent embedding.

In order to capture such Cayley graphs we had to come up with the notion we call a general planar presentation (defined in Section 6), which in particular translates, into abstract, algorithmically checkable, properties of words as above situations as in Figure 2 where a certain generator $s$ with $s^{2}=1$ separates the graph into two parts, and behaves in a spin-preserving way in one part and in a spin-reversing way in the other part. That such general planar presentations always give rise to planar Cayley graphs is the hardest result of this paper, many of its complications arising from the fact that given a general planar presentation with such a 'separating' generator $s$, it is impossible to predict whether $s=1$, which would imply that our Cayley graph does not quite have the structure
anticipated by the presentation. The situation is complicated further by the fact that separating generators need not be involutions; an example is given in Figure 3 .


Figure 3: An (infinite) planar Cayley graph, corresponding to the presentation $\left\langle a, b, c, d, f, g \mid a^{2}, c^{2}, d^{2}, f^{2}, g^{2},(a f)^{2},(a g)^{2}, a b a b^{-1} g b f b^{-1}, c b d b^{-1}\right\rangle$, with a separating edge $b$ which is not an involution.

This paper is structured as follows. After some general definitions in Section 2, we introduce generic planar presentations in Section 3 and show that every Cayley graph of every generic planar presentation is planar in Section 4 In Section 5we prove the reverse direction, i. e. that every planar Cayley graph admits a generic planar presentation. In Section 6 we slightly generalise from generic to general planar presentations, and put those facts together to obtain the results stated above. We finish with some open problems in Sections 6 and 7

## 2 Definitions

### 2.1 Cayley graphs and group presentations

We will follow the terminology of [3 for graph-theoretical terms and that of 1 ] and [17] for group-theoretical ones. Let us recall the definitions most relevant for this paper.

A group presentation $\langle\mathcal{S} \mid \mathcal{R}\rangle$ consists of a set $\mathcal{S}$ of distinct symbols, called the generators and a set $\mathcal{R}$ of words with letters in $\mathcal{S} \cup \mathcal{S}^{-1}$, where $\mathcal{S}^{-1}$ is the set of symbols $\left\{s^{-1} \mid s \in \mathcal{S}\right\}$, called relators. Each such group presentation uniquely determines a group, namely the quotient group $F_{\mathcal{S}} / N$ of the (free) group $F_{\mathcal{S}}$ of words with letters in $\mathcal{S} \cup \mathcal{S}^{-1}$ over the (normal) subgroup $N=N(\mathcal{R})$ generated by all conjugates of elements of $\mathcal{R}$.

The Cayley graph $\operatorname{Cay}(\mathcal{P})=\operatorname{Cay}\langle\mathcal{S} \mid \mathcal{R}\rangle$ of a group presentation $\mathcal{P}=$ $\langle\mathcal{S} \mid \mathcal{R}\rangle$ is an edge-coloured directed graph $G=(V, E)$ constructed as follows. The vertex set of $G$ is the group $\Gamma=F_{\mathcal{S}} / N$ corresponding to $\mathcal{P}$. The set of colours we will use is $\mathcal{S}$. For every $g \in \Gamma, s \in \mathcal{S}$ join $g$ to $g s$ by an edge coloured
$s$ directed from $g$ to $g s$. Note that $\Gamma$ acts on $G$ by multiplication on the left; more precisely, for every $g \in \Gamma$ the mapping from $V(G)$ to $V(G)$ defined by $x \mapsto g x$ is an automorphism of $G$, that is, an automorphism of $G$ that preserves the colours and directions of the edges. In fact, $\Gamma$ is precisely the group of such automorphisms of $G$. Any presentation of $\Gamma$ in which $\mathcal{S}$ is the set of generators will also be called a presentation of $\operatorname{Cay}(\mathcal{P})$.

Note that some elements of $\mathcal{S}$ may represent the identity element of $\Gamma$, and distinct elements of $\mathcal{S}$ may represent the same element of $\Gamma$; therefore, $\operatorname{Cay}(\mathcal{P})$ may contain loops and parallel edges of the same colour.

If $s \in \mathcal{S}$ is an involution, i.e. $s^{2}=1$, then every vertex of $G$ is incident with a pair of parallel edges coloured $s$ (one in each direction). If $s^{2}$ is a relator in $\mathcal{R}$, we will follow the convention of replacing this pair of parallel edges by a single, undirected edge. This convention is common in the literature [16], and is convenient when studying planar Cayley graphs.

If $G$ is a Cayley graph, then we use $\Gamma(G)$ to denote its group.
If $R$ is any (finite or infinite) word with letters in $\mathcal{S} \cup \mathcal{S}^{-1}$, and $g$ is a vertex of $G=C a y\langle\mathcal{S} \mid \mathcal{R}\rangle$, then starting from $g$ and following the edges corresponding to the letters in $R$ in order we obtain a walk $W$ in $G$. We then say that $W$ is induced by $R$ at $g$, and we will sometimes denote $W$ by $g R$; note that for a given $R$ there are several walks in $G$ induced by $R$, one for each starting vertex $g \in V(G)$.

Let $H_{1}(G)$ denote the first simplicial homology group of $G$ over $\mathbb{Z}$. We will use the following well-known fact which is easy to prove.

Lemma 2.1. Let $G=C a y\langle\mathcal{S} \mid \mathcal{R}\rangle$ be a Cayley graph. Then the (closed) walks in $G$ induced by relators in $\mathcal{R}$ generate $H_{1}(G)$.

### 2.2 Graph-theoretical concepts

Let $G=(V, E)$ be a connected graph fixed throughout this section. Two paths in $G$ are independent, if they do not meet at any vertex except perhaps at common endpoints. If $P$ is a path or cycle we will use $|P|$ to denote the number of vertices in $P$ and $\|P\|$ to denote the number of edges of $P$. Let $x P y$ denote the subpath of $P$ between its vertices $x$ and $y$.

A hinge of $G$ is an edge $e=x y$ such that the removal of the pair of vertices $x, y$ disconnects $G$. A hinge should not be confused with a bridge, which is an edge whose removal separates $G$ although its endvertices are not removed.

The set of neighbours of a vertex $x$ is denoted by $N(x)$.
$G$ is called $k$-connected if $G-X$ is connected for every set $X \subseteq V$ with $|X|<k$. Note that if $G$ is $k$-connected then it is also $(k-1)$-connected. The connectivity $\kappa(G)$ of $G$ is the greatest integer $k$ such that $G$ is $k$-connected.

A 1-way infinite path is called a ray. Two rays are equivalent if no finite set of vertices separates them. The corresponding equivalence classes of rays are the ends of $G$. A graph is multi-ended if it has more than one end. Note that given any two finitely generated presentations of the same group, the corresponding

Cayley graphs have the same number of ends. Thus this number, which is known to be one of $0,1,2, \infty$, is an invariant of finitely generated groups.

A double ray is a directed 2-way infinite path.
The set of all finite sums of (finite) cycles forms a vector space over $\mathbb{F}_{2}$, the cycle space of $G$.

### 2.3 Embeddings in the plane

An embedding of a graph $G$ will always mean a topological embedding of the corresponding 1-complex in the euclidean plane $\mathbb{R}^{2}$; in simpler words, an embedding is a drawing in the plane with no two edges crossing.

A face of an embedding $\sigma: G \rightarrow \mathbb{R}^{2}$ is a component of $\mathbb{R}^{2} \backslash \sigma(G)$. The boundary of a face $F$ is the set of vertices and edges of $G$ that are mapped by $\sigma$ to the closure of $F$. The size of $F$ is the number of edges in its boundary. Note that if $F$ has finite size then its boundary is a cycle of $G$.

A walk in $G$ is called facial with respect to $\sigma$ if it is contained in the boundary of some face of $\sigma$.

An embedding of a Cayley graph is called consistent if, intuitively, it embeds every vertex in a similar way in the sense that the group action carries faces to faces. Let us make this more precise. Given an embedding $\sigma$ of a Cayley graph $G$ with generating set $S$, we consider for every vertex $x$ of $G$ the embedding of the edges incident with $x$, and define the spin of $x$ to be the cyclic order of the set $L:=\left\{x y^{-1} \mid y \in N(x)\right\}$ in which $x y_{1}^{-1}$ is a successor of $x y_{2}^{-1}$ whenever the edge $x y_{2}$ comes immediately after the edge $x y_{1}$ as we move clockwise around $x$. Note that the set $L$ is the same for every vertex of $G$, and depends only on $S$ and on our convention on whether to draw one or two edges per vertex for involutions. This allows us to compare spins of different vertices. Call an edge of $G$ spin-preserving if its two endvertices have the same spin in $\sigma$, and call it spin-reversing if the spin of one of its endvertices is the reverse of the spin of its other endvertex. Call a colour in $S$ consistent if all edges bearing that colour are spin-preserving or all edges bearing that colour are spin-reversing in $\sigma$. Finally, call the embedding $\sigma$ consistent if every colour is consistent in $\sigma$. Note that if $\sigma$ is consistent, then there are only two types of spin in $\sigma$, and they are the reverse of each other.

The following classical result was proved by Whitney [21, Theorem 11] for finite graphs and by Imrich (15) for infinite ones.

Theorem 2.2. Let $G$ be a 3-connected graph embedded in the sphere. Then every automorphism of $G$ maps each facial path to a facial path.

This implies in particular that if $\sigma$ is an embedding of the 3-connected Cayley graph $G$, then the cyclic ordering of the colours of the edges around any vertex of $G$ is the same up to orientation. In other words, at most two spins are allowed in $\sigma$. Moreover, if two vertices $x, y$ of $G$ that are adjacent by an edge, bearing a colour $b$ say, have distinct spins, then any two vertices $x^{\prime}, y^{\prime}$ adjacent by a $b$-edge also have distinct spins. We just proved

Lemma 2.3. Let $G$ be a 3-connected planar Cayley graph. Then every embedding of $G$ is consistent.

Cayley graphs of connectivity 2 do not always admit a consistent embedding [6]. However, in the cubic case they do; see [11].

An embedding is Vertex-Accumulation-Point-free, or accumulation-free for short, if the images of the vertices have no accumulation point in $\mathbb{R}^{2}$.

A crossing of a path $X$ by a path or walk $Y$ in a plane graph is a subwalk $Q=e \grave{Q} f$ of $Y$ where the end-edges $e, f$ of $Q$ are incident with $X$ on opposite sides of $X$ (but not contained in $X$ ) and (the image of) $Q$ is contained in $X$ (Figure (4). Note that if $Q$ is a crossing of $X$ by $Y$, then $X$ contains a crossing $Q^{\prime}=g Q h$ of $Y$ by $X$, which we will call the dual crossing of $Q$.


Figure 4: A crossing of $X$ by $Y$.
For a closed walk $W$ and $n \in \mathbb{N}$, let $W^{n}$ be the $n$-times concatenation of $W$. Two closed walks $R$ and $W$ cross if there are $i, j \in \mathbb{N}$ such that $R^{i}$ contains a crossing of a subwalk of $W^{j}$. They are nested if they do not cross.

### 2.4 Fundamental groups of planar graphs

Let $G$ be a graph. The sum of two walks $W_{1}, W_{2}$ where $W_{1}$ ends at the starting vertex of $W_{2}$ is their concatenation. Let $W=x_{1} x_{2} \ldots x_{n}$ be a walk. Its inverse is $x_{n} \ldots x_{1}$. If $x_{i-1}=x_{i+1}$ for some $i$, we call the walk $W^{\prime}:=$ $x_{1} \ldots x_{i-1} x_{i+2} \ldots x_{n}$ a reduction of $W$. Conversely, we add the spike $x_{i-1} x_{i} x_{i+1}$ to $W^{\prime}$ to obtain $W$. If $W$ is a closed walk, we call $x_{i} \ldots x_{n} x_{1} \ldots x_{i-1}$ a rotation of $W$.

Let $\mathcal{V}$ be a set of closed walks. The smallest set $\overline{\mathcal{V}} \supseteq \mathcal{V}$ of closed walks that is invariant under taking sums, reductions and rotations and under adding spikes is the set of closed walks generated by $\mathcal{V}$. We also say that any $V \in \overline{\mathcal{V}}$ is generated by $\mathcal{V}$. A closed walk is indecomposable if it is not generated by closed walks of strictly smaller length. Note that no indecomposable closed walk $W$ has a shortcut, i.e. a (possibly trivial) path between any two of its vertices that has smaller length than any subwalk of any rotation of $W$ between them. In particular, indecomposable closed walks induce cycles.

For any $\eta \in \pi_{1}(G)$, let $W_{\eta} \in \eta$ be the unique reduced closed walk in $\eta$ and $W_{\eta}^{\circ}$ be its (unique) cyclical reduction. For $\mathcal{V} \subseteq \pi_{1}(G)$, set

$$
\mathcal{V}^{\circ}:=\left\{W_{\eta}^{\circ} \mid \eta \in \mathcal{V}\right\}
$$

By $\mathcal{W}(G)$ we denote the set of all closed walks in $G$.

The following theorem is an immediate consequence of [14, Theorem 6.2], which is a generalisation of the main theorem of 13$]$.

Theorem 2.4. Let $G$ be a planar locally finite 3 -connected graph and $\Gamma$ a group acting on $G$. Then $\pi_{1}(G)$ has a generating set $\mathcal{V}$ such that $\mathcal{V}^{\circ}$ is a $\Gamma$-invariant nested generating set for $\mathcal{W}(G)$ that consists of indecomposable closed walks.

## 3 Planar presentations

In this section we introduce our notion of planar presentation, which is the central definition of this paper. For the convenience of the reader, we start by recalling the definition of a special planar presentation from [12]. We then define the more involved generic planar presentations in Section 3.2,

### 3.1 Special planar presentations

The intuition behind special planar presentations comes from the notion of a consistent embedding given above: a planar presentation is a group presentation endowed with some additional data (forming what we will call an embedded presentation) which describe the local structure of a consistent embedding of the corresponding Cayley graph, that is, the spin and the information of which generators preserve or reflect it.

Given a group presentation $\mathcal{P}=\langle\mathcal{S} \mid \mathcal{R}\rangle$, where $\mathcal{S}$ is finite, or countably infinite, we will distinguish between two types of generators $s$ : those for which we have $s^{2}$ as a relator in $\mathcal{R}$ and the rest. The reasons for this distinction will become clear later. Generators $t$ for which the relation $t^{2}$ is provable but not explicitly part of the presentation might exist, but do not cause us any concerns. Given a group presentation $\mathcal{P}=\langle\mathcal{S} \mid \mathcal{R}\rangle$, we thus let $\mathcal{I}=\mathcal{I}(\mathcal{P})$ denote the set of elements $s \in \mathcal{S}$ such that $\mathcal{R}$ contains the relator $s^{2}$ or $s^{-2}$.

Let $\mathcal{S}^{\prime}=\mathcal{S} \cup(\mathcal{S} \backslash \mathcal{I})^{-1}$. For example, if $\mathcal{P}=\left\langle a, b, c \mid a^{2}, b^{2}\right\rangle$, then $\mathcal{S}^{\prime}=$ $\left\{a, b, c, c^{-1}\right\}$.

A spin on $\mathcal{P}=\langle\mathcal{S} \mid \mathcal{R}\rangle$ is a cyclic ordering of $\mathcal{S}^{\prime}$ (to be thought of as the cycling ordering of the edges that we expect to see around each vertex of our Cayley graph once we have proved that it is planar)

An embedded presentation is a triple $\mathcal{P}, \sigma, \tau$ where $\mathcal{P}=\langle\mathcal{S} \mid \mathcal{R}\rangle$ is a group presentation, $\sigma$ is a spin on $\mathcal{P}$, and $\tau$ is a function from $\mathcal{S}$ to $\{0,1\}$ (encoding the information of whether each generator is spin-preserving or spin-reversing).

To every embedded presentation $\mathcal{P}, \sigma, \tau$ we can associate a tree $\mathbb{T}$ with an accumulation-free embedding in $\mathbb{R}^{2}$. As a graph, we let $\mathbb{T}$ be Cay $\left\langle\mathcal{S} \mid s^{2}, s \in \mathcal{I}\right\rangle$. Easily, we can embed $\mathbb{T}$ in $\mathbb{R}^{2}$ in such a way that for every vertex $v$ of $\mathbb{T}$, one of the two cyclic orderings of the colours of the edges of $v$ inherited by the embedding coincides with $\sigma$ and moreover, for every two adjacent vertices $v, w$ of $\mathbb{T}$, the clockwise cyclic ordering of the colours of the edges of $v$ coincides with that of $w$ if and only if $\tau(a)=0$ where $a$ is the colour of the $v-w$ edge.
(If $\tau(a)=1$, then the clockwise ordering of $v$ coincides with the anti-clockwise ordering of $w$.)

Given a word $W$, we let $W^{\infty}$ be the 2-way infinite word obtained by concatenating infinitely many copies of $W$. We say that two words $W, Z \in \mathcal{R}$ cross, if there is a 2 -way infinite path $R$ of $\mathbb{T}$ induced by $W^{\infty}$ and a 2-way infinite path $L$ induced by $Z^{\infty}$ such that $L$ meets both components of $\mathbb{R}^{2} \backslash R$. Note that, if two non-trivial words form closed walks in the Cayley graph, then the words cross if and only if the closed walks cross.

For example, consider the presentation $\mathcal{P}=\left\langle n, e, s, w \mid n^{2}, e^{2}, s^{2}, w^{2}\right\rangle$, the spin $n, e, s, w, n$ (read 'north, east, south, west'), and $\tau$ identically 0 . Then any word containing $n s$ as a subword crosses any word containing $e w$. The word nesw however crosses no other word, and indeed adding that word to the above presentation yields a planar Cayley graph: the square grid.

Definition 3.1. A special planar presentation is an embedded presentation $(\mathcal{P}, \sigma, \tau)$ such that
(sP1) no two relators $W, Z \in \mathcal{R}$ cross, and
( sP 2 ) for every relator $R$, the number of occurrences of letters $s$ in $R$ with $\tau(s)=1$ (i.e. spin-reversing letters) is even; here, the symbol $s^{n}$ counts as $|n|$ occurrences of $s$.

Requirement ( sP 2) is necessary, as the spin of the starting vertex of a cycle must coincide with that of the last vertex.

In 12 we proved the following results about special planar presentations.
Theorem 3.2 ([12, Theorem 3.3]). Every planar, locally finite, 3-connected Cayley graph admits a special planar presentation.

Theorem 3.3 ([12, Theorem 4.2]). If $(\mathcal{P}, \sigma, \tau)$ is a special planar presentation, then its Cayley graph $\operatorname{Cay}(\mathcal{P})$ is planar. Moreover, $\operatorname{Cay}(\mathcal{P})$ admits a consistent embedding, with spin $\sigma$ and spin-behaviour of generators given by $\tau$.

### 3.2 General planar presentations

We now extend the above definition of a planar presentation, to a more general one, the advantage of which is that it can capture Cayley graphs with 2-separators that do not admit consistent embeddings, which will allow us to extend Theorem 3.2 and Theorem 3.3 to all planar Cayley graphs.

Let again $\mathcal{P}=\langle\mathcal{S} \mid \mathcal{R}\rangle$ be a group presentation, and define $\mathcal{S}^{\prime}$ as above.
A spin structure $\mathcal{C}$ on $\mathcal{P}$ consists of a cover $B_{1}, \ldots, B_{k}$ of $\mathcal{S}^{\prime}$ (i.e. $\bigcup_{i} B_{i}=\mathcal{S}^{\prime}$ ) with the following properties
(S1) for every generator $b$, the number of $B_{i}$ 's containing $b$ equals the number of $B_{i}$ 's containing $b^{-1}$, and
(S2) the auxiliary graph $X$ on $\mathcal{C} \cup \mathcal{S}^{\prime}$ with $s \sim B_{i}$ whenever $s \in B_{i}$, is a tree.
(It will become clear later that a special planar presentation is a special case of a general one when $\mathcal{C}$ consists of a single set coinciding with $\mathcal{S}^{\prime}$.)

The hinges of this spin structure are the elements of $\mathcal{S}^{\prime}$ that have degree at least 2 in $X$; in other words, $h \in \mathcal{S}^{\prime}$ is a hinge if $h \in B_{i} \cap B_{j}$ for some $i \neq j$. Hinges of a spin structure correspond to edges of our Cayley graph $G$ whose two endvertices separate $G$.

For example, $a, b$ are the hinges of the presentation

$$
\left\langle a, b, c, d, f, g \mid a^{2}, c^{2}, d^{2}, f^{2}, g^{2},(a f)^{2},(a g)^{2}, a b a b^{-1} g b f b^{-1}, c b d b^{-1}\right\rangle
$$

given in Figure 3, and $b$ is the only hinge in Figure 2, The tree $X$ of condition (S2) corresponding to the presentation of Figure 3 is shown in Figure 5] Figure 6 shows the corresponding tree $X$ that would result if we amalgamated the above group with two more groups each of which being isomorphic to the subgroup generated by $b, c, d$ along the subgroup spanned by $b$.


Figure 5: An example: the tree $X$ of condition (s2) corresponding to Figure 3


Figure 6: The tree $X$ of condition (s2) corresponding to a variant of Figure 3

Condition ( S 2 ) has the following important consequences:

$$
\begin{equation*}
B_{i} \cap B_{j} \text { is either empty or a singleton for every } i \neq j, \tag{1}
\end{equation*}
$$

because if $h, g \in B_{i} \cap B_{j}$ then $h, g, B_{i}, B_{j}$ span a 4 -cycle in $X$, which cannot happen when $X$ is a tree, and

$$
\begin{align*}
& \text { every } B_{i} \text { contains at least one hinge unless } k=1 \text {, i.e. } \mathcal{C} \text { is the } \\
& \text { singleton }\left\{\mathcal{S}^{\prime}\right\}, \tag{2}
\end{align*}
$$

because if each neighbour of $B_{i}$ in $X$ has degree 1, then $B_{i}$ and its neighbours form a component of $X$.

A generic embedded presentation is a quintuple $\mathcal{P}, \mathcal{C}, \sigma, \tau, \mu$ as follows; $\mathcal{P}$ is a group presentation and $\mathcal{C}$ a spin structure on $\mathcal{P}$ as above; $\sigma$ is a function of $i \in\{1, \ldots, k\}$ assigning a spin (i.e. a cyclic ordering) to each $B_{i} \in \mathcal{C}$;
$\tau: \mathcal{S} \times\{1, \ldots, k\} \rightarrow\{0,1\}$ encodes the information of whether each generator is spin-preserving or spin-reversing in each $B_{i}$ it participates in (if $s \in \mathcal{S} \backslash B_{i}$, then the value of $\tau(s, i)$ will be irrelevant in the sequel); and for every $b \in \mathcal{S}$, and every $i$ for which $b \in B_{i}, \mu(b, i)$ is a $B_{j}$ such that $b^{-1} \in B_{j}$, and $\mu(b, i) \neq \mu(b, m)$ for $m \neq i$. This $\mu$ encodes the information of which pairs of $B_{i}$ incident with the two endvertices of a given hinge belong to the same block of $G$. The use of $\mathcal{S}$ rather than $\mathcal{S}^{\prime}$ in the definition of $\mu$ and $\tau$ is intended: the values we assign to each $b \in S$ give us enough information about how to treat $b^{-1}$.

For the time being, the data $\sigma, \tau, \mu$ are abstract objects describing the intended structure and embedding of our Cayley graph given by $\mathcal{P}$. But we will indeed prove that if these data satisfy certain conditions, then the Cayley graph is indeed planar and can be embedded in the intended way.

As an example, the presentation $\left\langle\mathcal{S} \mid b^{2}, a^{3}, c^{3}, a b a^{-1} b, c b c b\right\rangle$ of the graph of Figure 2 can be endowed with the following data. The spin structure $\mathcal{C}$ consists of two sets $B_{1}=\left\{b, c, c^{-1}\right\}, B_{2}=\left\{b, a, a^{-1}\right\}$. We can then let $\sigma(1)=\left(b, c, c^{-1}\right)$, $\sigma(2)=\left(b, a^{-1}, a\right)$-but any other $\sigma$ would do in this case as there are only two cyclic orderings of a set of three elements, and they are the reflection of each other- $\tau(b, 1)=0, \tau(b, 2)=1$-this is the most interesting aspect of this graph: any $b$ edge is spin-preserving in one of its incident blocks and spinreversing in the other- and $\mu(b, 1)=B_{1}, \mu(b, 2)=B_{2}$-because $b$ stabilises the two components into which it splits the graph.

Our general definition of a planar presentation will be very similar to that of Section 3.1, and still based on the idea of non-crossing relators. One difference is that we have to embed the tree $\mathbb{T}=C a y\left\langle\mathcal{S} \mid s^{2}, s \in \mathcal{I}\right\rangle$ in $\mathbb{R}^{2}$ more carefully: rather than demanding every vertex to have the same cyclic ordering of its incident colours in the embedding, which would in general make it impossible to adhere to the spin-behaviour encoded by $\tau$, we embed $\mathbb{T}$ (accumulation-free) in $\mathbb{R}^{2}$ in such a way that the following two conditions are satisfied. Given a vertex $x \in V(\mathbb{T})$ and $B_{i} \in \mathcal{C}$, we write $B_{i}(x)$ for the edges of $x$ with labels in $B_{i}$.
(B1) $\sigma$ is respected, i.e. for every vertex $x \in V(\mathbb{T})$, and every $B_{i} \in \mathcal{C}$, the cyclic ordering induced on $B_{i}(x)$ by our embedding coincides with $\sigma(i)$ up to reflection. Moreover, the edges of $B_{i}(x)$ are consecutive in our embedding.
(B2) $\tau$ is respected, i.e. for every edge $e=v w$ of $\mathbb{T}$, and every $i$ such that the label $s$ of $e$ is in $B_{i} \in \mathcal{C}$, we have $1_{\sigma(i)}\left(B_{i}(v)\right)=1_{\sigma(j)}\left(B_{j}(w)\right)$ if and only
if $\tau(s, i)=0$, where $B_{j}=\mu(s, i)$ and $1_{\sigma(i)}\left(B_{i}(v)\right)$ is 1 if the clockwise cyclic ordering of the colours of the edges of $B_{i}(v)$ coincides with $\sigma(i)$ and 0 otherwise.

We repeat the definition of crossing from Section 3.1verbatim: given a word $W$, we let $W^{\infty}$ be the 2-way infinite word obtained by concatenating infinitely many copies of $W$. We say that two words $W, Z \in \mathcal{R}$ cross, if there is a 2 -way infinite path $R$ of $\mathbb{T}$ induced by $W^{\infty}$ and a 2 -way infinite path $L$ induced by $Z^{\infty}$ such that $L$ meets both components of $\mathbb{R}^{2} \backslash R$.

The second and final difference of our generalised definition of a planar presentation compared to that of Section 3.1 will be an additional condition reflecting the idea that in a planar Cayley graph of connectivity 2, we can choose the relators in such a way that each cycle they induce is contained in a block. Recalling that our spin structure $\mathcal{C}$ is intended to capture the decomposition into blocks, the following definition should not be too surprising.

We say that a relator $R$ is blocked with respect to $\mathcal{C}$, if it satisfies the following two properties. Firstly, for every two (possibly equal) consecutive letters st appearing in $R^{\infty}$ or $\left(R^{-1}\right)^{\infty}$, there is some $B_{i} \in \mathcal{C}$ containing both $s^{-1}, t$. Secondly, for every three consecutive letters $s b t$, where $b$ is a hinge, appearing in $R^{\infty}$ or $\left(R^{-1}\right)^{\infty}$, if $B_{i}$ is the unique element of $\mathcal{C}$ containing $s^{-1}, b$, then $\mu(b, i)$ contains both $b^{-1}, t$, unless $s=b=t$ and $b^{2} \in \mathcal{R}$; here, the existence of such a $B_{i}$ is guaranteed by the previous requirement, and its uniqueness is a consequence of (1) in the definition of a spin structure.

Definition 3.4. A generic planar presentation is a generic embedded presentation such that
(P1) every relator in $\mathcal{R}$ is blocked with respect to $\mathcal{C}$;
(P2) no two relators $W, Z \in \mathcal{R}$ cross;
(P3) for every relator $R$, the number of occurrences of letters $t$ in $R$ with $\tau(t, i)=1$ (i.e. spin-reversing letters), where $i$ is the unique value for which $s^{-1}, t \in B_{i}$ for the letter $s$ preceding $t$ in $R$, is even ${ }^{2}$; here, the symbol $s^{n}$ counts as $|n|$ occurrences of $s$;
(P4) no relator is a sub-word of a rotation of another relator.
Note that a planar presentation as defined in Section 3.1 is a special case of a generic one when $\mathcal{C}$ consists of a single set coinciding with $\mathcal{S}^{\prime}$.

In Section 6 we will slightly generalise the notion of a generic planar presentation further, by allowing the removal of certain redundancies, to obtain the notion of general planar presentation discussed in the introduction.

[^2]
## 4 Proof of planarity of the Cayley graph of a generic planar presentation

In this section we prove that the Cayley graph defined by any generic planar presentation is planar (Theorem 4.10).

For a hinge $h \in \mathcal{S}$, we let $\mathcal{C}(h):=\left\{B_{i} \in \mathcal{C} \mid h \in B_{i}\right\}$ and let $N(h)$ be the cardinality $|\mathcal{C}(h)|$. Note that $|\mathcal{C}(h)|=\operatorname{deg}_{X}(h)$, where the tree $X$ is as in (S2) of the definition of a spin structure.

Every hinge $b=x y \in E(\mathbb{T})$ of $\mathbb{T}$ labelled $h$ naturally splits $\mathbb{T}$ into $N(h)$ subtrees: each of these subtrees contains $b$, it contains all edges of $x$ with labels in a component of $X-h$ containing some $B_{i} \in \mathcal{C}(h)$ and no other edges of $x$, and it contains those edges of $y$ with labels in the component of $X-h^{-1}$ containing $\mu(h, i)$ and no other edges of $y$; moreover, each such subtree is maximal with these properties. Let $S e p_{b}=\left\{T_{1}, T_{2}, \ldots, T_{N(h)}\right\}$ denote the set of those subtrees, and note that $\bigcap S e p_{b}=\{b\}$.

Definition 4.1. A pre-block of $\mathbb{T}$ is a maximal subtree $A \subseteq \mathbb{T}$ not separated by any $S e p_{b}$; that is, for every hinge $b$ of $\mathbb{T}, A$ is contained in some element of $S e p_{b}$.

Alternatively, we can define a pre-block as a maximal subtree of $\mathbb{T}$ such that for every $x, y \in V(A)$, if we let $s_{1} s_{2} \ldots s_{k}$ denote the word (with letters in $\mathcal{S}$ ) read along the $x-y$ path, then $s_{j-1}^{-1}, s_{j}$ lie in a common element of $\mathcal{C}$ for every $j>1$, and whenever $s_{j}$ is a hinge, and $s_{j-1}^{-1}, s_{j} \in B_{i} \in \mathcal{C}$, then $s_{j}^{-1}, s_{j+1} \in \mu\left(s_{j}, i\right)$.

### 4.1 The embedding $\rho$ of $\mathbb{T}$

Recall that our proof of Theorem 3.3 starts with an embedding of the corresponding tree $\mathbb{T}$ respecting the spin data. In our new setup of a generic embedded presentation our spin data give us some restrictions but do not uniquely determine an embedding of $\mathbb{T}$, and in fact we have to be careful with our choices in order for the proof in subsection 4.2 to work.

Recall that our generic embedded presentation consists of the data $\mathcal{P}, \mathcal{C}, \sigma$, $\tau, \mu$. For $B \in \mathcal{C}$ and a vertex $x \in V(\mathbb{T})$, recall that $B_{i}(x)$ denotes the edges going out of $x$ whose labels are in $B$. We claim that there is an embedding $\rho: \mathbb{T} \rightarrow \mathbb{R}^{2}$ satisfying all of the following (the first two were also used in the definition of crossing relators in Section (3.2).
$(\rho 1) \sigma$ is respected, i.e. for every vertex $x \in V(\mathbb{T})$, and every $B_{i} \in \mathcal{C}$, the cyclic ordering induced on $B_{i}(x)$ by $\rho$ coincides with $\sigma(i)$ up to reflection. Moreover, the edges of $B_{i}(x)$ are consecutive in the spin of $x$ induced by $\sigma$.
( $\rho 2$ ) $\tau$ is respected, i.e. for every edge $e=v w$ of $\mathbb{T}$, and every $i$ such that the label $s$ of $e$ is in $B_{i} \in \mathcal{C}$, we have $1_{\sigma(i)}\left(B_{i}(v)\right)=1_{\sigma(j)}\left(B_{j}(w)\right)$ if and only if $\tau(s, i)=0$, where $B_{j}=\mu(s, i)$ and $1_{\sigma(i)}\left(B_{i}(v)\right)$ is 1 if the clockwise cyclic ordering of the colours of the edges of $B_{i}(v)$ coincides with $\sigma(i)$ and 0 otherwise.
( $\rho 3) \mu$ is respected: let $b \in E(\mathbb{T})$ be a hinge, and $U, W$ two paths containing $b$ contained in distinct pre-blocks containing $b$. Then $U, W$ do not cross each other (at $b$ ).
( $\rho 4$ ) If $x, y$ belong to the same $N(\mathcal{R})$-orbit (where $N(\mathcal{R})$ is the normal subgroup generated by $\mathcal{R}$ as in Section 2.1), and $b$ is a hinge at $x$ with label in $h \in \mathcal{I}$, and $h \neq 1$, then the local spin at $x$ with respect to $b$ coincides up to reflection with the local spin at $y$ with respect to the corresponding hinge labelled $h$.

Here, the local spin with respect to a generator $h \in \mathcal{S}^{\prime}$ at a vertex $x$ is the cyclic ordering on $N_{X}(h)$ induced by the embedding, where $X$ denotes the tree from Section 3.2 .

If $G$ is a planar Cayley graph, then the results of Section 5.2 imply that if we embed the universal cover $\mathbb{T}$ of $G$ into $\mathbb{R}^{2}$ in a way that locally imitates an embedding of $G$, then all above properties are satisfied.

An open star is a subspace of a graph consisting of a single vertex and all open half-edges incident with it. A star is the union of an open star with some of the midpoints in its closure.

Properties ( $A 1)$ to ( $\& 3$ ) are not hard to satisfy: we can embed $\mathbb{T}$ by starting with the star $E(o)$ and then recursively attaching the star $E(v)$ of a new vertex to the subtree embedded so far, and it is always possible to embed $E(v)$ without violating any of $(\rho \mathbb{1})-(\rho \sqrt{3})$. In fact we could have several ways to extend the current embedding to $E(v)$, arising by 'permuting' those $B_{i}(v), 1 \leq i \leq k$ that do not contain the edge of $v$ embedded before, and by 'reflecting' any such $B_{i}(v)$. These choices are in direct analogy to the flexibility we have in the embedding of any planar Cayley graph of connectivity 2 : permuting the $B_{i}(v)$ corresponds to 'activating' a hinge $b$ incident with $v$ to exchange the order in which blocks separated by $b$ are embedded. Reflecting a $B_{i}(v)$ corresponds to flipping such a block around.

These choices mean that ( $/ 4$ 4) will be violated unless we make them carefully. To achieve this, recall from (S2) of Section 3.2 that the auxiliary graph $X$ on $\mathcal{C} \cup \mathcal{S}^{\prime}$ with $s \sim B_{i}$ whenever $s \in B_{i}$, is a tree. Let $X^{\ell}$ denote the tree obtained from $X$ by attaching to each vertex $v$ in $\mathcal{S}^{\prime} \subset V(X)$ a new leaf, which leaf we denote by $\ell(v)$.

Fix an embedding $\chi: X^{\ell} \rightarrow \mathbb{R}^{2}$ of that tree with the following two properties. Firstly, the spin of any vertex $B \in \mathcal{C}$ of $X^{\ell}$ coincides with $\sigma(B)$ up to reflection.

Recall that $N(v)=N_{G}(v)$ denotes the neighbourhood of $v$ in a graph $G$. For every hinge $h \in \mathcal{S} \backslash \mathcal{I}$, note that $\mu(h, \cdot)$ defines a bijection between $N_{X}(h)$ and $N_{X}\left(h^{-1}\right)$ by the definition of $\mu$. We extend that bijection to $N_{X^{\ell}}(h)$ and $N_{X^{\ell}}\left(h^{-1}\right)$ by mapping $\ell(h)$ to $\ell\left(h^{-1}\right)$. The second property we impose on $\chi$ is that the spin it induces on $N_{X^{\ell}}(h)$ coincides up to reflection with the $\mu$-image of that spin induced by $\chi$ on $N_{X^{\ell}}\left(h^{-1}\right)$, and this holds for every such $h$.

For an involution hinge $h \in \mathcal{I}, \mu(h, \cdot)$ still defines a bijection between $N_{X}(h)$ and $N_{X}\left(h^{-1}\right)=N_{X}(h)$, and we do not impose any requirement on $\chi$ as we did for $h \in \mathcal{S} \backslash \mathcal{I}$. Instead, we let $\chi$ embed $N_{X^{\ell}}(h)$ with an arbitrary spin $\phi=\phi(h)$, and define

Definition 4.2. The dual spin of $\phi$ is the cyclic ordering on $N_{X^{\ell}}(h)$ obtained by composing $\phi$ with $\mu(h, \cdot)$.

To satisfy ( $\rho 4$ ), we will construct $\rho$ in such a way that the local spin with respect to $h$ at every vertex in a given $N(\mathcal{R})$-orbit either always coincides with $\phi$ or it always coincides with the dual of $\phi$. We remark that we cannot construct $\rho$ algorithmically since we cannot predict which vertices of $\mathbb{T}$ are in the same $N(\mathcal{R})$-orbit; we can only prove the existence of such a $\rho$ abstractly.

We think of this $\chi$ as providing instructions about how to construct $\rho$. As an example, if the set $\mathcal{I}$ of involutions in $\mathcal{S}$ is empty, then every vertex of $\mathbb{T}$ will have the same spin up to reflection in $\rho$, and that spin can be read from $\chi$ by contracting all non-leaves of $X^{\ell}$ into a single vertex; that vertex has the right spin in the resulting star.

Let $o=x_{1}, x_{2}, \ldots$ be an enumeration of $V(\mathbb{T})$ such that $\left\{x_{1}, \ldots, x_{k}\right\}$ spans a connected subgraph for all $k$. We will construct $\rho$ by embedding the $x_{i}$ one at a time as indicated above. To begin with, we embed one edge $e_{0}$ incident with $x_{1}=o$ in the 0 th step. From now on, each step $i$ begins with some vertices being embedded fully, i.e. with all incident edges, and some vertices having exactly one of their edges embedded in the current embedding $\rho_{i-1}$ of some subtree of $\mathbb{T}$. Let $j$ be the smallest index such that $x_{j}$ has exactly one of its edges $e_{i}$ embedded in $\rho_{i-1}$. We may assume without loss of generality that $j=i$ by changing our enumeration.

We extend $\rho_{i-1}$ to $\rho_{i}$ by embedding the remaining edges incident with $x_{i}$. This will be done by the performing the following recursive procedure on $X^{\ell}$ to obtain an embedded star $S_{i}$ with its edges labelled by $\mathcal{S}^{\prime}$, and then embedding $N_{\mathbb{T}}\left(x_{i}\right)$ with the same spin as $S_{i}$.

To begin with, let $\ell$ be the unique leaf of $X^{\ell}$ such that $\ell=\ell(s)$ for the label $s \in \mathcal{S}$ of the edge $e_{i}$ considered as outgoing from $x_{i}$. We distinguish the following cases.

Case 1: If $s \notin \mathcal{I}$, and $s$ is a hinge, then we embed the star $N(s)$ of $s$ in $X^{\ell}$ into $\mathbb{R}^{2}$ so that the spin of $s$ in this embedding coincides with the spin of $s$ in $\chi$ up to reflection; there are exactly two possibilities for this -because of reflection - and we choose the unique one guaranteeing ( $/ 3$ 3): unless we are in step $i=1$, in which case we just embed $N(s)$ with the spin of $s$ in $\chi$ without reflection, the other endvertex $x$ of $e_{i}$ has already been fully embedded, and the local spin with respect to $e_{i}$ (which now label $s^{-1}$ as seen from $x$ ) at $x$ coincides up to reflection with that induced on $N\left(s^{-1}\right)$ by $\chi$ by induction hypothesis. We use the possibility to reflect or not in order to guarantee that the clockwise ordering of the $B_{i}$ in $N(s)$ coincides with the counterclockwise ordering of the $\mu(s, i)$ induced by the spin of $x$ in the embedding $\rho_{i-1}$.

Case 2: If $s \notin \mathcal{I}$, and $s$ is not a hinge, then it has exactly two neighbours in $N(s)$ ( $\ell(s)$ and the unique $B \in \mathcal{C}$ containing $s$ ), and so reflection does not change the spin; we just embed $N(s)$ in the unique possible way.

Case 3: If $s \in \mathcal{I}$, and $s$ is not a hinge, then again we just embed $N(s)$ in the unique possible way.

Case 4: Finally, if $s \in \mathcal{I}$, and $s$ is a hinge, then we follow a similar approach to the $s \notin \mathcal{I}$ case, except that we now do not insist that the spin of $s$ in the embedding of $N(s)$ we produce coincides with the spin of $s$ in $\chi$ up to reflection; we just make sure that ( $\kappa \sqrt{3}$ ) is satisfied, by embedding $N(s)$ so that the clockwise ordering of the $B_{i}$ in $N(s)$ coincides with the counterclockwise ordering of the $\mu(s, i)$ induced by the spin of $x$ in the embedding $\rho_{i-1}$; again this is well-defined unless we are in step $i=1$, in which case we just embed $N(s)$ with the spin of $s$ in $\chi$.

Once $N(s)$ is embedded as above, we set $X_{0}^{\ell}:=N(s)$ and proceed by the following recursive procedure, which produces embeddings of an increasing sequence $X_{1}^{\ell}, \ldots, X_{k}^{\ell}\left(=X^{\ell}\right)$ of subtrees of $X^{\ell}$ to embed the rest of $X^{\ell}$.

For $j=1,2, \ldots$, pick a leaf $v_{j}$ of $X_{j-1}^{\ell}$ which is not a leaf of $X^{\ell}$; if no such leaf exists then $X_{j-1}^{\ell}=X^{\ell}$ and we stop. Then we extend the current embedding of $X_{j-1}^{\ell}$ by embedding $N\left(v_{j}\right)$ in such a way that the spin of $v_{j}$ coincides up to reflection with that induced by $\chi$, unless $v_{j} \in \mathcal{I} \subseteq \mathcal{S}$ and $v_{j} \neq 1$, in which case we do the following. Let $y_{i}=x_{i} v_{j}$ be the vertex of $\mathbb{T}$ joined to $x_{i}$ by the edge labelled $v_{j}$. If no vertex of $\mathbb{T}$ from the $N(\mathcal{R})$-orbit of $x_{i}$ or $y_{i}$ has been embedded yet by $\rho_{i}$, then we embed $N\left(v_{j}\right)$ with local spin given by $\chi$. If some vertex of $\mathbb{T}$ from the $N(\mathcal{R})$-orbit of $x_{i}$ has already been embedded by $\rho_{i}$, we embed $N\left(v_{j}\right)$ with same spin up to reflection as we used so far for all $x_{j}, j<i$, that are $N(\mathcal{R})$-equivalent to $x_{i}$; (we make this choice in order to satisfy ( $\left.\kappa \sqrt[4]{4}\right)$ ). Otherwise, we embed $N\left(v_{j}\right)$ with the dual spin -recall Definition 4.2- up to reflection of the spin we used so far for all $x_{j}, j<i$ that are $N(\mathcal{R})$-equivalent to $y_{i}$. Note that these choices ensure that $N\left(v_{j}\right)$ is embedded with the same spin up to reflection -namely, either that induced by $\chi$ or its dual- for all vertices in an $N(\mathcal{R})$-orbit, where we use the fact that, as $v_{j} \neq 1, x_{i}$ and $y_{i}$ are never in the same orbit.

In all cases, we still have the option of reflecting. If $v_{j} \in N(s)$, which means that $v_{j} \in \mathcal{C}$ and $v_{j}$ contains the label $s$ of $e_{i}$, then we have to worry about satisfying ( $/ 24$; but one of the two choices we have due to the option of reflecting will satisfy ( $/ 2$ ) for $e=e_{i}$ and $B_{i}=v_{j}$ and we make that choice. (If $v_{j} \notin N(s)$ then we do not worry about $\mu$ and $\tau$; the other endvertices of the edges incident with $x_{i}$ will make sure that this data is respected, just as we were careful above when embedding $N(s)$ for the label $s$ of $e_{i}$.)

Let $X_{j}^{\ell}:=X_{j-1}^{\ell} \cup N\left(v_{j}\right)$.
The procedure finishes when all of $X^{\ell}$ has been embedded. Then, we contract all non-leafs of $X^{\ell}$ to obtain the desired embedded star $S_{i}$ out of that embedding. Finally, we embed $N_{\mathbb{T}}\left(x_{i}\right)$ with the same spin as $S_{i}$ to extend $\rho_{i-1}$ to $\rho_{i}$.

Let $\rho=\bigcup \rho_{i}$ be the limit of the $\rho_{i}$. We claim that $\rho$ satisfies conditions ( $\kappa 1$ ) $-(\rho 4)$. Indeed, if any of them is violated, then there is a first step in the above procedures violating it. But we designed all steps so that none of those conditions are violated: condition (A1) is never violated because we chose $\chi$ so that the spin of every $B_{i} \in \mathcal{C}$ coincides with $\sigma(i)$ up to reflection, which implies that the corresponding edges of $x_{i}$ appear in that cyclic order up to reflection in
$S_{i}$, and therefore in $\rho$, by the construction of the embedded star $S_{i}$. Condition ( $\sqrt{2}$ ) is never violated because of the way we embedded $N\left(v_{j}\right)$ for $v_{j} \in N(s)$ in the construction of $S_{i}$. Condition ( $\beta 3$ ) is never violated because of the way we embedded $N(s)$ in the first step of the construction of $S_{i}$. Finally, condition ( $\kappa 4$ ) is never violated because of the way we embedded $N\left(v_{j}\right)$ for $v_{j} \in \mathcal{I}$ in the construction of $S_{i}$.

In fact, we obtain a slightly stronger property than ( $\kappa 44$, and this will be useful later:

Condition ( $(4)$ remains true if we define local spin using $X^{\ell}$ instead of $X$.

### 4.2 Planarity of blocks

A block of $G$ is an image $\pi([A])$ under the covering map $\pi$, where $A$ denotes a pre-block of $\mathbb{T}$ and $[A]:=\left\{x \in V(\mathbb{T}) \mid x \simeq_{N} y\right.$ for some $\left.y \in A\right\}$ denotes its $N(\mathcal{R})$-equivalence class.

Note that every block of $G$ is connected: given vertices $x, z$ in a block $K=$ $\pi([A])$, we can find $x^{\prime}, z^{\prime} \in A$ (and not just in the $N(\mathcal{R})$-orbit of $A$ ) with $\pi\left(x^{\prime}\right)=x, \pi\left(z^{\prime}\right)=z$, and so the $x^{\prime}-z^{\prime}$ path $P$ in $A$ yields the $x-z$ path $\pi(P)$ in $K$.

Lemma 4.3. Every block of $G$ is planar.
In fact, we will prove a stronger statement similar to Theorem 3.3 ([12, Theorem 4.2]), namely, that every block admits an embedding into $\mathbb{R}^{2}$ respecting $\sigma$ and $\tau$.

The proof of this follows the lines of our proof of the planarity of $G$ in the consistent case ([12, Theorem 4.2]), and we assume that the reader has already understood that proof. Here we will point out the differences.

Let $K$ be a block of $G$. Let $D$ be a fundamental domain of $K$ in $\mathbb{T}$; that is, $D$ is a subset of $\mathbb{T}$ containing exactly one point from each $N(\mathcal{R})$-orbit $O$ such that $\pi(O) \in K$. With the same argument as in [12, Lemma 4.1] we may assume that $D$ is connected since $K$ is. Moreover, we may assume without loss of generality that $D$ is a union of stars. Thus the closure $\bar{D}$ of $D$ in $\mathbb{T}$ is still the union of $D$ with all midpoints of edges that have exactly one half-edge in $D$, and $K$ can be obtained from $\bar{D}$ by identifying pairs of $N(\mathcal{R})$-equivalent midpoints. As in the proof of [12, Theorem 4.2], we will prove that any two pairs of $\operatorname{such} N(\mathcal{R})$ equivalent midpoints are nested, where we say that two pairs of midpoints $x, x^{\prime}$ and $y, y^{\prime}$ in $\bar{D} \backslash D$ are nested, if the $x-x^{\prime}$ path in $D$ does not cross the $y-y^{\prime}$ path, where we define crossing similarly to Section 2.3.

In order to guarantee this nestedness, we will have to embed $\mathbb{T}$ appropriately; in our general setup, $\mathbb{T}$ cannot be embedded consistently as in the case of special planar presentations, and this is why we are now only trying to prove the planarity of a block, and not of all of $G$ at once.

For a relator $W$, we use $W_{o}$ to denote the closed walk $o_{G} W$ in $G$ induced by $W$ at $o_{G}$, and let $\mathbb{T}_{W}:=\pi^{-1}\left(W_{o}\right)$, which is a union of a set of double-rays of $\mathbb{T}$, which set we denote by $\mathbb{T}\left[W_{o}\right]$.

Recall we have chosen an embedding $\rho$ of $\mathbb{T}$ in Section 4.1. For a pre-block $C$ of $\mathbb{T}$, we define a super-face of $C$ to be a face of the embedding $\sigma(C)$ of $C$ inherited by $\rho$. The super-faces of $\mathbb{T}$ are the super-faces of all of the pre-blocks of $\mathbb{T}$. Note that a super-face can contain several faces of $\mathbb{T}$.

The dual graph $\mathbb{T}^{*}$ of $\mathbb{T}$ is the graph whose vertex set is the set of faces of $\mathbb{T}$, and two faces of $\mathbb{T}$ are joined with an edge $e^{*}$ of $\mathbb{T}^{*}$ whenever their boundaries share an edge $e$ of $\mathbb{T}$. For two faces $F, H$ of $\mathbb{T}$ and an $F-H$ path $P_{F H}$ in $\mathbb{T}^{*}$, let $C r\left(\mathbb{T}\left[W_{o}\right], P_{F H}\right)$ denote the number of crossings of $\mathbb{T}\left[W_{o}\right]$ by $P_{F H}$; to make this more precise, for a double-ray $T$ in $\mathbb{T}\left[W_{o}\right]$, we write $\operatorname{cr}\left(T, P_{F H}\right)$ for the number of edges $e$ in $T$ such that $P_{F H}$ contains $e^{*}$, and we let $\operatorname{Cr}\left(\mathbb{T}\left[W_{o}\right], P_{F H}\right):=$ $\sum_{T \in \mathbb{T}\left[W_{o}\right]} c r\left(T, P_{F H}\right)$. We claim that
for every two faces $F, H$ of $\mathbb{T}$, the parity of the number of crossings $\operatorname{Cr}\left(\mathbb{T}\left[W_{o}\right], P_{F H}\right)$ is independent of the choice of the path $P_{F H}$.

To see this, note that if $C$ is a cycle in $\mathbb{T}^{*}$, then $\operatorname{Cr}\left(\mathbb{T}\left[W_{o}\right], C\right)$-defined similarly to $\operatorname{Cr}\left(\mathbb{T}\left[W_{o}\right], P_{F H}\right)$ - is even because the embedding of $\mathbb{T}$ is accumulation-free and so any ray entering the bounded side of $C$ has to exit it again. This immediately implies (4).

We will define our relation $\sim_{K}$, or just $\sim$ if $K$ is fixed, on the set of superfaces of pre-clusters contained in $\pi^{-1}(K)$. Given two super-faces $F, H$ lying in pre-clusters contained in $\pi^{-1}(K)$, let $\mathbb{T}\left[W_{o}\right]_{K}$ denote the subset of $\mathbb{T}\left[W_{o}\right]$ contained in $\pi^{-1}(K)$. Now pick two faces $F^{\prime} \subseteq F, H^{\prime} \subseteq H$ contained in the super-faces $F, H$, and write $F \sim H$ if for each $F^{\prime}-H^{\prime}$ path $P_{F^{\prime} H^{\prime}}$ in $\mathbb{T}^{*}$, the number of crossings $C r\left(\mathbb{T}\left[W_{o}\right]_{K}, P_{F^{\prime} H^{\prime}}\right)$ of $\mathbb{T}\left[W_{o}\right]_{K}$ by $P_{F^{\prime} H^{\prime}}$ is even. Since $C r\left(\mathbb{T}\left[W_{o}\right]_{K}, P_{F^{\prime} H^{\prime}}\right)$ is independent of the choice of $P_{F^{\prime} H^{\prime}}$ by (4), it is also independent of the choice of $F^{\prime}, H^{\prime}$, because if $F^{\prime \prime}$ is another face contained in $F$, then the $F^{\prime}-F^{\prime \prime}$ path of $\mathbb{T}^{*}$ contained inside $F$ crosses no element of $\mathbb{T}\left[W_{o}\right]_{K}$, because a super-face of any pre-cluster $C$ in $\pi^{-1}(K)$ meets no element of $\mathbb{T}\left[W_{o}\right]_{K}$ by the definitions.

### 4.2.1 The bipartitions $\{I, O\}$

An important part of our planarity proof in the consistent case was that $\sim$ was invariant under the action of $N(\mathcal{R})$, see [12, Lemma 4.4]. Below (Lemma 4.8) we prove an analogous statement for the general case, namely that the restriction of $\sim$ to the super-faces of the pre-blocks in $\pi^{-1}(K)$ is $N(\mathcal{R})$-invariant.

The rest of our proof is almost identical to that of [12, Theorem 4.2], except that we are now working with the block $K$ of $G$ rather than the whole graph.

The equivalence relation $\sim$, now restricted on the set of super-faces $\mathcal{F}$ of $\pi^{-1}(K)$, uniquely determines a bipartition $\{I, O\}$ on $\mathcal{F}$ by choosing one superface $F \in \mathcal{F}$ and letting $I:=\{H \in \mathcal{F} \mid H \sim F\}$ and $O:=\mathcal{F} \backslash I$.

Next, we adapt the material of [12, Section 4.3.1] to our new setup. For every super-face $F$ in $\pi^{-1}(K)$, glue a copy of the domain $\bar{F} \subset \mathbb{R}^{2}$ to $K$ by identifying each point of $\partial F$ with $\pi(\partial F)$. If $F, F^{\prime}$ are equivalent face boundaries, in other words, if $\pi(\partial F)=\pi\left(\partial F^{\prime}\right)$, then we identify the corresponding 2-cells glued
onto $K$. Let $K^{2}$ denote the set of these 2-cells, and let $\bar{K}=K \cup K^{2}$ denote the 2-complex consisting of $K$ and these 2-cells.

Lemma 4.8 now means that if $Z$ is a closed walk of $G$ (here we really mean $G$ and not just $K)$ induced by a relator, then $\{I, O\}$ induces a bipartition $\pi[I], \pi[O]$ of $K^{2}$. Let us still denote this bipartition of $K^{2}$ by $B_{Z}$.

We extend that bipartition to an arbitrary cycle in $K$ : given a cycle $C$ of $K$, we choose a 'proof' $P$ of $C$; that is, a sequence of closed walks $W_{i}, 1 \leq i \leq k$ of $G$ induced by rotations of relators such that $C=\sum_{1 \leq i \leq k} W_{i}$. The existence of such a sequence $\left(W_{i}\right)$ is not affected by the fact that we are focusing on a subgraph $K$; the $W_{i}$ are allowed to be arbitrary relators. For every $W_{i}$, let $I_{W_{i}}, O_{W_{i}}$ denote the two sides of the bipartition $B_{W_{i}}$ of $K^{2}$ from above, and define the bipartition $B_{C}:=\left\{I_{C}, O_{C}\right\}$ of $K^{2}$ by $I_{C}:=\triangle_{i} I_{W_{i}}$ and $O_{C}:=$ $G^{2} \triangle I_{C}$.

While in the definition of $B_{C}$ it appear that it depends on the proof $P$, it actually does not as we shall see later. Until then, we denote it by $B_{C}(P)$ to make it clear that it depends on $P$. Our next aim is to show that, in a certain way, $B_{C}(P)$ behaves like the bipartition of the faces of a plane graph induced by a cycle $C$ : to move between the two sides, one has to cross an edge of $C$. This is achieved by Lemma 4.5 below, for the proof of which we need the following.

Lemma 4.4. Let $e$ be a directed edge of $K$, let $W \in \mathcal{R}$ be a relator which is not of the form $b^{2}=1$ for $b \in \mathcal{S}$, and let $o_{K} W$ be the closed walk of $K$ rooted at some vertex $o_{K}$ of $K$ induced by $W$. Then the number of double-rays in $\mathbb{T}\left[W_{o}\right]$ containing e equals the number of times that $o_{K} W$ traverses $\pi(e)$.

Proof. If $o_{K} W$ does not traverse $\pi(e)$ then $\mathbb{T}\left[W_{o}\right]$ avoids $e$ and we are done. So suppose that $o_{K} W$ does traverse $\pi(e)$. Let $o_{K} W^{\infty}$ denote the two-way infinite walk on $K$ obtained by repeating $o_{K} W$ indefinitely. Let $T \in \mathbb{T}\left[W_{o}\right]$ be the lift of $o_{K} W^{\infty}$ to $\mathbb{T}$ (via $\pi^{-1}$ ) sending $\pi(e)$ to $e$, and note that $T$ is a double-ray containing $e$. Let $Q$ be the subpath of $T$ that starts with $e$ and finishes when a rotation of the word $W$ is completed. By the definition of $\mathbb{T}\left[W_{o}\right]$, there is a 1-1 correspondence between the elements of $\mathbb{T}\left[W_{o}\right]$ containing $e$ and the directed edges $e^{\prime}$ in $Q$ that are $N(\mathcal{R})$-equivalent to $e$ : each such element of $\mathbb{T}\left[W_{o}\right]$ can be obtained by translating $T$ by the automorphism of $\mathbb{T}$ sending $e^{\prime}$ to $e$.

Now note that $o_{K} W$ traverses $\pi(e)$ whenever its lift $T$ traverses one of those $e^{\prime}$. Combined with the above observations this proves our assertion.

Lemma 4.5. For every $e \in E(K)$, the bipartition $B_{C}(P)$ separates 2-cells of $e$ if and only if $e \in C$.

Proof. Let $I, O$ be the two elements of $B_{C}(P)$ as defined above. Then, letting $1_{F \in I}$ denote the indicator function of $F \in I$, we have

$$
1_{F \in I}=N_{F}:=\left|\left\{W_{i} \mid F \in I_{W_{i}}\right\}\right| \quad(\bmod 2)
$$

and similarly

$$
1_{H \in I}=N_{H}:=\left|\left\{W_{i} \mid H \in I_{W_{i}}\right\}\right| \quad(\bmod 2)
$$

But

$$
N_{F}+N_{H}=\mid\left\{W_{i} \mid W_{i} \text { separates } F \text { from } H\right\} \mid \quad(\bmod 2)
$$

by the construction of $I, O$. We claim that $\mid\left\{W_{i} \mid W_{i}\right.$ separates $F$ from $\left.H\right\} \mid$ is odd if and only if $e \in E(C)$. Indeed, $B_{W_{i}}$ separates $F$ from $H$ exactly when $W_{i}$ traverses $e$ an odd number of times by
for every edge $e$ of $\mathbb{T}$, the two faces $F, H$ of $e$ lie in distinct elements of $\{I, O\}$ if and only if $e \in \mathbb{T}_{W}$ and $e$ lies in an odd number of elements of

$$
\begin{equation*}
\mathbb{T}\left[W_{o}\right] \tag{5}
\end{equation*}
$$

and Lemma 4.4 and $e$ is in $C$ exactly when there is an odd number of $W_{i}$ that traverse $e$ an odd number of times.

Since that number is even if $e \notin E(C)$ and odd otherwise, our last congruence yields $N_{F}+N_{H}=1(\bmod 2)$ if and only if $e \in E(C)$. Therefore, the previous congruences imply that $1_{F \in I}=1_{H \in I}$ if $e \notin E(C)$ and $1_{F \in I} \neq 1_{H \in I}$ if $e \in E(C)$, which is our claim.

Lemma 4.5 implies in particular that $B_{C}(P)$ is characterised by $C$ alone and is therefore independent of $P$, since $\bar{K}$ was defined without reference to $P$. Thus we can denote it by just $B_{C}$ from now on.

In the following, we use again the definition of a crossing from Section 2.3
Lemma 4.6. Let $C^{\prime}$ be a finite path of $\mathbb{T}$ such that $C:=\pi\left(C^{\prime}\right)$ is a cycle of $K$, and let $Q=e Q f$ be a crossing of $C^{\prime}$ in $\mathbb{T}$. Then $B_{C}$ separates the 2-cells incident with $\pi(e)$ from the 2-cells incident with $\pi(f)$. Moreover, if $Q_{2}$ is a path of $\mathbb{T}$ such that $\pi\left(Q_{2}\right)$ is a cycle of $K$, then $Q_{2}$ crosses $C^{\prime}$ an even number of times.

Proof. Let $F$ be a face incident with the first edge $e$ of $Q$, and let $H$ be a face incident with the last edge $f$ of $Q$. By the definition of a crossing, we can find a finite sequence $(F=) F_{1}, \ldots, F_{k}(=H)$ of faces of $\mathbb{T}$ such that each $F_{i}$ shares an edge $e_{i}$ with $F_{i+1}$ and exactly one of the $e_{i}$ lies in $C^{\prime}$ : we can visit all faces incident with $Q$ until we reach $H$. By Lemma 4.5 and Lemma4.8, $B_{C}$ separates $\pi\left(F_{1}\right)$ from $\pi\left(F_{k}\right)$. This proves our first assertion.

For the second assertion, note that $\pi\left(Q_{2}\right)$ can be written as a concatenation of subarcs $C_{1} D_{1} C_{2} D_{2} \ldots C_{k}=C_{1}$ where each $C_{i}$ lifts to a crossing of $C^{\prime}$ by $Q_{2}$ and each $D_{i}$ avoids $C$ and shares exactly one end-edge with each of $C_{i}$ and $C_{i+1}$. We proved above that the 2-cells incident with end-edges of each $C_{i}$ are separated by $B_{C}$. The same arguments imply that the 2-cells incident with endedges of each $D_{i}$ are not separated by $B_{C}$. Since $\pi\left(Q_{2}\right)$ is a cycle, this implies that $Q_{2}$ crosses $C^{\prime}$ an even number of times.

As in the end of the proof of Theorem 3.3, the last lemma says that any two cycles of $K$ cross each other an even number of times, and therefore any two pairs of identified points of $\bar{D}$ are nested.

This completes the proof of Lemma 4.3, except that we still have to prove the two lemmas we used above:

Lemma 4.7. For $b \in \mathcal{I}$ with $b=1$, and any relator $W$ in $\mathcal{R}$, the number of elements of $\mathbb{T}\left[W_{o}\right]$ containing any edge $e$ labelled by $b$ is even.

Proof. Let $T$ be an element of $\mathbb{T}\left[W_{o}\right]$ containing $e$. The automorphism $\beta$ of $\mathbb{T}$ exchanging the two endvertices of $e$ maps $T$ to an element $T^{\prime}$ of $\mathbb{T}\left[W_{o}\right]$ because $b=1$ and so the two end-vertices of $e$ are $N(\mathcal{R})$-equivalent. Note that $T \neq T^{\prime}$ even if $T, T^{\prime}$ contain the same vertices, because they have opposite directions (remember that double-rays are directed by definition). Note that $\beta\left(T^{\prime}\right)=T$. Therefore, $\beta$ establishes a bijection without fixed points on the elements of $\mathbb{T}\left[W_{o}\right]$ containing $e$, which means that the number of those elements is even.

Lemma 4.8. For every block $K$ of $G$, the restriction of $\sim_{K}$ to the super-faces of $\pi^{-1}(K)$ is invariant under the action of $N(\mathcal{R})$ on $\mathbb{T}$.

Proof. We will adapt the proof of [12, Lemma 4.4]. Since $K$ is fixed, let us just write $\sim$ instead of $\sim_{K}$.

We need to prove that if $F, H$ are super-faces of $\pi^{-1}(K)$ in the same orbit of $N(\mathcal{R})$, then $F \sim H$. Again, we may assume that there are vertices $x, y$ in the boundaries of $F, H$ respectively, such that $y=x w R w^{-1}$ for some word $w$ and some relator $R \in \mathcal{R}$ : by the definition of the normal closure $N(\mathcal{R})$, if we can prove $F \sim H$ in this case, we can prove $F \sim H$ for every two $F, H$ in the same orbit of $N(\mathcal{R})$.

Let $\alpha_{F H}$ be the automorphism of $\mathbb{T}$ mapping $x$ to $y$.
Decompose the path $Q:=x w R w^{-1}$ into (inclusion-)maximal subpaths contained in a pre-block. Then we can write

$$
Q=P_{1} \cup P_{2} \cup \ldots \cup P_{k}\left(=P_{k}^{\prime}\right) \cup P_{k-1}^{\prime} \cup \ldots \cup P_{1}^{\prime}
$$

where the $P_{i}, P_{i}^{\prime}$ are those maximal subpaths, $P_{i}^{\prime}$ is $N(\mathcal{R})$-equivalent to $P_{i}$ for every $i<k$, and $P_{k}$ contains the subpath of $Q$ induced by $R$ (such a $P_{k}$ exists because every relator $R$ is blocked). Note that the intersection of any two subsequent $P_{i}$ or $P_{i}^{\prime}$ is either a hinge separating the corresponding pre-blocks, or a single vertex incident with such a hinge.

Since we are free to choose any $F-H$ walk $P_{F H}$ in $\mathbb{T}^{*}$ to decide whether $F \sim H$, we will choose a convenient one, which we construct now.

Recall that every $P_{i}, i>1$ starts and ends at hinges, which we will call $h_{i-1}, h_{i}$, separating its pre-block from the pre-blocks containing $P_{i-1}, P_{i+1}$ respectively; here $h_{i-1}, h_{i}$ may or may not be contained in $P_{i}$ as end-edges.

Let $C_{i}$ be the pre-block containing $P_{i}$ and let $C_{i}^{\prime}$ be the pre-block contain$\operatorname{ing} P_{i}^{\prime}$.

Let $\Pi_{i}, k>i>1$, be an (inclusion-)minimal path in $\mathbb{T}^{*}$ joining a superface incident with $h_{i-1}$ to a super-face incident with $h_{i}$-where we say that a super-face $F$ is incident with an edge if the boundary of $F$ contains that edgesuch that all vertices of $\Pi_{i}$ are faces sharing a vertex with $P_{i}$, and $\Pi_{i}$ does not intersect $P_{i}$ (at a midpoint of any edge); see Figure 7. Define $\Pi_{i}^{\prime}$ similarly using $P_{i}^{\prime}$ instead of $P_{i}$. Note that there are exactly two such paths $\Pi_{i}$ to choose from, one on either side of $P_{i}$; it doesn't matter much which of the two we will choose,
but let us make 'the same' choice for both $\Pi_{i}$ and $\Pi_{i}^{\prime}$; more precisely, we ensure that
$\Pi_{i}$ crosses an edge $e$ of $C_{i}$ (incident with $P_{i}$ ) if and only if $\Pi_{i}^{\prime}$ crosses the edge $\alpha_{F H}(e)$ of $C_{i}^{\prime}$.

This is possible because $\rho$ embeds $C_{i}$ the same way as $C_{i}^{\prime}$ up to reflection, and $\Pi_{i}$ is uniquely determined once we choose which of the two super-faces of $C_{i}$ incident with $h_{i}$ we want it to contain; by choosing $\Pi_{i}^{\prime}$ to contain the corresponding super-face incident with $h_{i}^{\prime}$, our claim is satisfied. Note that $\Pi_{i}$ does not cross $h_{i}$, because if it did we could shorten it.

For $i=1$ we let $\Pi_{1}$ be a minimal path in $\mathbb{T}^{*}$ joining $F$ to a super-face incident with $h_{1}$, and otherwise be defined similarly to $\Pi_{i}, k>i>1$. Define $\Pi_{1}^{\prime}$ similarly. Finally, let $\Pi_{k}=\Pi_{k}^{\prime}$ be a minimal path in $\mathbb{T}^{*}$ joining a super-face incident with $h_{k-1}$ to a super-face incident with $\alpha_{F H}\left(h_{k-1}\right)$ without crossing $P_{k}$.

Let $\sqcup_{i}, k>i \geq 1$ be a path in $\mathbb{T}^{*}$ joining the last vertex of $\Pi_{i}$ to the first vertex of $\Pi_{i+1}$ such that all vertices of $\sqcup_{i}$ are faces sharing a vertex with $P_{i} \cap P_{i+1}$, and define $\sqcup_{i}^{\prime}$ similarly for $\Pi_{i}^{\prime}, \Pi_{i+1}^{\prime}$; there are several choices for this $\sqcup_{i}$, so let us make it uniquely determined: if $P_{i} \cap P_{i+1}$ is a single vertex, then there are two candidates, and we always choose the one crossing $h_{i}$. If $P_{i} \cap P_{i+1}$ is the hinge $h_{i}$, then there are up to four choices, and we choose the one that crosses $h_{i}$ and is contained in the two super-faces of $C_{i}$ incident with $h_{i}$ and in the two super-faces of $C_{i+1}$ incident with $h_{i}$.

It follows from the choice of $\sqcup_{i}$ that it behaves well with respect to elements of $\mathcal{C}$ :

If $\sqcup_{i}$ meets an edge in $B_{i}(v) \backslash\left\{h_{i}\right\}$ (where $B_{i} \in \mathcal{C}$ ) where the vertex $v$ is incident with $h_{i}$, then $\sqcup_{i}$ meets every edge of $B_{i}(v)$.

A similar but slightly stronger is true for $\Pi_{i}$ :
If $\Pi_{i}$ meets an edge lying inside some super-face of $C_{i}$, then $\Pi_{i}$ visits all faces incident with $P_{i}$ inside that super-face.

Indeed, $\Pi_{i}$ is by definition a minimal path joining certain super-faces of $C_{i}$; therefore, it crosses any super-face either completely or at a single boundary edge.

Finally, we obtain $P_{F H}$ by concatenating all the $\Pi_{i}, \sqcup_{i}, \Pi_{i}^{\prime}$ and $\sqcup_{i}^{\prime}$ :

$$
P_{F H}:=\Pi_{1} \cup \sqcup_{1} \cup \Pi_{2} \ldots \cup \sqcup_{k-1} \cup \Pi_{k}\left(=\Pi_{k}^{\prime}\right) \cup \sqcup_{k-1}^{\prime} \ldots \cup \sqcup_{1}^{\prime} \cup \Pi_{1}^{\prime} .
$$

We need to check that $\operatorname{Cr}\left(\mathbb{T}\left[W_{o}\right]_{K}, P_{F H}\right)$ is even. We will do so by showing that the contributions of the $\Pi_{i}$ to $\operatorname{Cr}\left(\mathbb{T}\left[W_{o}\right]_{K}, P_{F H}\right)$ cancel with those of the $\Pi_{i}^{\prime}$, and the contributions of the $\sqcup_{i}$ cancel with those of the $\sqcup_{i}^{\prime}$.

Let $T$ be an element of $\mathbb{T}\left[W_{o}\right]_{K}$ with odd $\operatorname{cr}\left(T, P_{F H}\right)$, i.e. with an odd number of crossings of $T$ by $P_{F H}$; only such $T$ matter. Let $T^{\prime}:=\alpha_{F H}(T)$.

Let us first consider the total number of crossings of such $T$ by the subpaths $\Pi_{i}, \Pi_{i}^{\prime}, i<k$, of $P_{F H}$.

If $T$ is contained in $C_{i}$, then $\operatorname{cr}\left(T, \Pi_{i}\right)=\operatorname{cr}\left(T^{\prime}, \Pi_{i}^{\prime}\right)$ by (16).


Figure 7: The path $P_{F H}$ (dashed) in the proof of Lemma 4.8 with the paths $\Pi_{i}, \Pi_{i}^{\prime}$, $\sqcup_{i}, \sqcup_{i}^{\prime}$.

If $T$ is not contained in $C_{i}$, then $\Pi_{i}$ crosses $T$ an even number of times ( 0 or 2): this is easy to see when $T \cap P_{i}$ is a single vertex $v$ by applying (8) to that vertex. The situation is slightly subtler when $T \cap P_{i}$ is a hinge $g$-no other option is possible as distinct pre-blocks intersect at an edge at most by construction. In this case, we remark that the pre-block $D$ containing $T$ lies in some super-face of $C_{i}$ by the construction of $\rho$, and again $\Pi_{i}$ must cross all faces incident with $g$ inside that super-face by (8), therefore crossing both edges of $T$ incident with $g$.

Finally, it is not hard to see that $\Pi_{k}=\Pi_{k}^{\prime}$ has an even contribution to $C r\left(\mathbb{T}\left[W_{o}\right]_{K}, P_{F H}\right)$.

These facts combined show that $\sum_{T \in \mathbb{T}\left[W_{o}\right]_{K}} \operatorname{cr}\left(T, \bigcup_{i} \Pi_{i}\right)$ is even.
Next, we consider the total number of crossings of such $T$ by the subpaths $\sqcup_{i}, \sqcup_{i}^{\prime}$. Suppose $\operatorname{cr}\left(T, \sqcup_{i}\right)$ is odd. Then it must equal 1 as $\sqcup_{i}$ is too short to cross a double-ray three times, where we used property ( $\alpha \sqrt[3]{ }$ ) of our embedding $\rho$ that pre-blocks do not cross each other.

Let $v_{i}$ be the last vertex of $P_{i}$ and $v_{i}^{\prime}$ the last vertex of $P_{i}^{\prime}$. If the local spin at $v_{i}$ with respect to $h_{i}$ coincides up to reflection with the local spin at $v_{i}^{\prime}$ with respect to $h_{i}^{\prime}$, then $\operatorname{cr}\left(T, \sqcup_{i}\right)=\operatorname{cr}\left(T^{\prime}, \sqcup_{i}^{\prime}\right)$ (here, local spin refers to $X^{\ell}$ rather than $X$; recall (3)). Therefore, the total contribution of the pair $T, T^{\prime}$ to $C r\left(\mathbb{T}\left[W_{o}\right]_{K}, P_{F H}\right)$ is even and can be ignored.

If those local spins do not coincide up to reflection, then by the choice of $\rho$ ( $\alpha 4$ ), the label of $h_{i}$ is an involution $b \in \mathcal{I}$ with $b=1$. In this case however, Lemma 4.7 applies, yielding that the set $H$ of elements of $\mathbb{T}\left[W_{o}\right]_{K}$ containing $h_{i}$ is even. We claim that $T \in H$ (i.e. $h_{i} \subset T$ ): this follows from $\operatorname{cr}\left(T, \sqcup_{i}\right)=1$, the fact that $\sqcup_{i}$ only contains faces of $\mathbb{T}$ incident with $h_{i}$ by its construction,
and (77). Moreover, (7) also implies that $\operatorname{cr}\left(R, \sqcup_{i}\right)=1$ for every other $R \in H$. But as $|H|$ is even, the total contributions $\sum_{R \in H} c r\left(R, \sqcup_{i}\right)$ of its elements are even and can be ignored as well.

Summing up, we proved that both

$$
\sum_{T \in \mathbb{T}\left[W_{o}\right]_{K}} c r\left(T, \bigcup_{i} \Pi_{i}\right) \quad \text { and } \quad \sum_{T \in \mathbb{T}\left[W_{o}\right]_{K}} c r\left(T, \bigcup_{i} \sqcup_{i}\right)
$$

are even. Therefore $\operatorname{Cr}\left(\mathbb{T}\left[W_{o}\right]_{K}, P_{F H}\right)$ is even as well, since it is the sum of those two sums by definition.

### 4.3 From the planarity of blocks to the planarity of $G$

The main aim of this section is to prove
Lemma 4.9. Every hinge of $G$ separates its incident blocks.
Proof. The statement is equivalent to the statement that every cycle of $G$ crosses each hinge $b$ an even number of times, where the number of crosses of $b$ by $C$ is the maximum number of edge disjoint subpaths $P_{i}$ of $C$ such that $b$ separates each $P_{i}$ into two (possibly trivial, but non-empty) subpaths that lie in distinct blocks.

To prove the latter, let $C=c_{0} c_{1} \ldots c_{k}$ with $c_{k}=c_{0}$ be a cycle, and let $L=t_{0} t_{1} \ldots t_{k}$ be a lift of $C$ to $\mathbb{T}$ via $\pi^{-1}$. Fix a hinge $b$. We may assume without loss of generality that $c_{0}$ is not a vertex of $b$. Let $P=w_{1} R_{1} w_{1}^{-1} \ldots w_{k} R_{k} w_{k}^{-1}$ be a proof of $C$ in our presentation.

Since $c_{0} \notin b$ and since the end vertices of $w_{i} R_{i} w_{i}^{-1}$ are $N(\mathcal{R})$-equivalent to $c_{0}$, any crossings of $b$ by $P$ occur inside the subpaths $w_{i} R_{i} w_{i}^{-1}$ and not when switching from $w_{i-1}$ to $w_{i}$. We have no crossings of $b$ inside any $R_{i}$ because our relators are blocked. Moreover, any crossings of $b$ inside a $w_{i}$ are paired up by crossings of $b$ inside $w_{i}^{-1}$. Thus the number of crossings of $b$ by $P$, and hence by $C$, is even.

This, combined with the planarity of blocks we proved in the previous section, easily implies the planarity of $G$ :

Theorem 4.10. Let $G$ be the Cayley graph of a generic planar presentation. Then $G$ is planar.

Proof. Combining Lemma 4.3 with Lemma 4.9 easily yields that $G$ is planar. Indeed, we can embed $G$ one block at a time: since incident blocks share a hinge only by Lemma 4.9 if we have already embedded a block $A$ meeting a block $B$ at a hinge $b$, then it is easy to embed $B$ inside one of the two faces (we are free to choose) of the current embedding whose boundary contains $b$.

## 5 Every planar Cayley graph admits a generic planar presentation

In this section we prove the converse of Theorem4.10, namely that every planar Cayley graph admits a generic planar presentation.

We start by showing that every planar Cayley graph of connectivity 1 can be extended into a 2 -connected one using redundant generators; see Lemma 5.1 below. We then show that every 2-connected planar Cayley graph admits a generic planar presentation in Section 5.2

### 5.1 Planar Cayley graphs of connectivity 1

Lemma 5.1. Every planar, locally finite, Cayley graph of connectivity 1 can be extended into a planar 2-connected, locally finite, Cayley graph by adding redundant generators.

Proof. We proceed by induction on the number of blocks incident with the vertex $o$, where a block means a maximal 2-connected subgraph in this subsection. Pick two such blocks $B, C$, an edge from $B$ corresponding to some generator $b$, and an edge from $C$ corresponding to some generator $c$. Introduce a new redundant generator $x$ and the relation $x=b^{-1} c$. Clearly, the resulting Cayley graph $G^{\prime}$ obtained from the original Cayley graph $G$ by adding the generator $x$ has less blocks incident with $o$ than $G$.

We claim that $G^{\prime}$ is still planar. If none of $b^{2}$ or $c^{2}$ is a relator, then this is an easy exercise, based on the observation that $G$ can be embedded in such a way that for every vertex $v$, the edges labelled $b$ and $c$ emanating from $v$ lie in a common face boundary.

If however $b^{2}$, say, is a relator, then it is a bit harder to avoid that the two $x$ edges emanating out of $o$ and $o b$ cross in our embedding. Still, the following observation will help us embed $G^{\prime}$ in this case (and it is also applicable to the case where none of $b^{2}$ or $c^{2}$ is a relator). A good example to bear in mind throughout the rest of the proof is where $G$ is the Cayley graph Cay $\left\langle b, c \mid b^{2}, c^{2}\right\rangle$ of the free product of two copies of $\mathbb{Z} / 2 \mathbb{Z}$, and $x=b c$.

> Let $H_{0}$ be the graph consisting of a single vertex, and suppose that for every $i \in \mathbb{N}$, the graph $H_{i}$ is obtained from $H_{i-1}$ by attaching a planar graph $P_{i}$ to $H_{i-1}$ by identifying some vertex $p_{i} \in V\left(P_{i}\right)$ with some vertex $h_{i} \in V\left(H_{i-1}\right)$, and possibly joining a neighbour $p_{i}^{\prime}$ of $p_{i}$ to a neighbour $h_{i}^{\prime}$ of $h_{i}$ with an edge. Then $\bigcup_{i \geq 0} H_{i}$ is planar.

To prove this, we first use induction to show that $H_{i}$ is planar: given an embedding of $H_{i-1}$, observe that $p_{i}^{\prime}, p_{i}$ lie in a common face $F_{i}$ since they are neighbours. Likewise, $h_{i}^{\prime}$, $h_{i}$ lie in a common face of $P_{i}$, and we may assume that that face is the outer face by embedding $P_{i}$ appropriately. We now embed $H_{i}$ by drawing $P_{i}$ inside $F_{i}$ and, if there is a $h_{i}^{\prime}-p_{i}^{\prime}$ edge in $H_{i}$, joining $h_{i}^{\prime}$ to $p_{i}^{\prime}$ with an arc in $F_{i}$ that avoids the rest of the graph.

The fact that $\bigcup_{i \geq 0} H_{i}$ is planar now follows from a standard compactness argument.

To complete our proof, we will show that our $G^{\prime}$ can be constructed as described in (9).

Indeed, let $\mathcal{H}$ be the set of blocks (i.e. maximal 2-connected subgraphs) of $G$, and let $H_{1}, H_{2}, \ldots$ be an enumeration of $\mathcal{H}$ such that for $i>1, H_{i}$ is incident with some $H_{j}$ for $j<i$. Then $G^{\prime}$ has the claimed structure, with the $x$-edges playing the role of the $h_{i}^{\prime}-p_{i}^{\prime}$ edges.

### 5.2 Cayley graphs of connectivity 2

In this section, we will complete the proof of our main theorem by showing that every locally finite 2-connected planar Cayley graphs admits a generic planar presentation.

A cut in a graph $G$ is a set of vertices $C$ spanning a connected subgraph of $G$, such that the boundary

$$
\partial C:=\{x \in V(G) \backslash C \mid x \text { has a neighbour in } C\}
$$

of $C$ is finite and $C \cup \partial C \neq V(G)$. The order of $C$ is the cardinality of $\partial C$.
We call two cuts $C, D$ nested if, setting $C^{*}:=V(G) \backslash C$ and $D^{*}:=V(G) \backslash D$, one of the four relations holds:

$$
C \subseteq D, \quad C \subseteq D^{*}, \quad C^{*} \subseteq D, \quad C^{*} \subseteq D^{*}
$$

We call a set of cuts nested, if every two of its elements are nested.
Definition 5.2. Given a nested set $\mathcal{C}$ of cuts, a block is a maximal subgraph $H$ such that for every cut $C$, we have either $V(H) \subseteq C \cup \partial C$ or $V(H) \subseteq C^{*}$ but not both.

To obtain a torso of a block $H$ from $H$ we add all edges $x y$ such that $\{x, y\} \subseteq V(H)$ is a boundary of a cut in $\mathcal{C}$.

Tutte [20] showed that every finite 2 -connected graph $G$ has an $\operatorname{Aut}(G)$ invariant nested set $\mathcal{C}$ of cuts of order 2 whose torsos are either 3 -connected or cycles. This theorem also holds for locally finite graphs, see Droms et al. [5]. Nevertheless, we will refer to it as Tutte's theorem. To each such nested set of cuts, there is an associated tree $T$ that admits a bijection from $V(T)$ to the blocks and boundaries of cuts in $\mathcal{C}$ such that, for any $t_{1}, t_{2} \in V(T)$ and any $t$ on the unique $t_{1}-t_{2}$ path in $T$, the image of $t$ separates the images of $t_{1}$ and $t_{2} I^{3}$ We call this tree $T$ the decomposition tree of the set of cuts.

A 2-separator is the boundary of a cut of order 2. Lemma 5.3 allows us to assume that all 2 -separators of $G$ are joined by an edge, i.e. they are hinges in

[^3]the sense of Section 3.2 Given two Cayley graphs $G, H$, we call $G$ a Tietzesupergraph of $H$ if there are presentations $\left\langle\mathcal{S}_{G} \mid \mathcal{R}_{G}\right\rangle$ of $\Gamma(G)$ and $\left\langle\mathcal{S}_{H} \mid \mathcal{R}_{H}\right\rangle$ of $\Gamma(H)$ with $G=C a y\left\langle\mathcal{S}_{G} \mid \mathcal{R}_{G}\right\rangle$ and $H=C a y\left\langle\mathcal{S}_{H} \mid \mathcal{R}_{H}\right\rangle$ and with $\mathcal{S}_{G} \supseteq \mathcal{S}_{H}$ and $\mathcal{R}_{G} \supseteq \mathcal{R}_{H}$.

Lemma 5.3. Every planar 2-connected Cayley graph $G$ has a planar Tietzesupergraph $H$ in which every pair of vertices that separates $H$ is connected by an edge. In addition, the new edges are labelled by a new redundant generator. (Moreover, if $G$ is locally finite, then so is $H$.)

Proof. To begin with, pick a $\Gamma(G)$-invariant nested set $\mathcal{C}$ of cuts of order 2. This set exists due to Tutte's theorem mentioned above. For every pair of nonadjacent vertices $x, y$ such that one component of $G-\{x, y\}$ lies in $\mathcal{C}$, we add a new redundant generator $a$ and relation $a=x^{-1} y$. Let us show that the nestedness of $\mathcal{C}$ implies that we do not lose planarity.

Note that every 2 -separator lies on the boundary of some face. So if we join $x_{1}$ and $y_{1}$ by a new edge and also want to join $x_{2}$ and $y_{2}$, then the only reason why we cannot do this is because the edge $x_{1} y_{1}$ separates the face on whose boundary the vertices $x_{2}$ and $y_{2}$ lie. So, originally, all four vertices $x_{1}, x_{2}, y_{1}, y_{2}$ are distinct and lie on a boundary $C$ of some face $F$ in this order (either clockwise or anticlockwise). For $i=1,2$, let $P_{i}$ be an $x_{i}-y_{i}$ path whose inner vertices lie in a component of $G-\left\{x_{i}, y_{i}\right\}$ that avoids $x_{j}$ and $y_{j}$ for $j \neq i$. As the two paths $P_{i}$ lie outside of $F$, the path $P_{2}$ connects a vertex in the inner face of $P_{1}+y_{1} x_{1}$ to one in its outer face, which is impossible due to the Jordan curve theorem. This proves that we can indeed add the aforementioned redundant generators and relations without losing planarity.

Since every vertex has only finitely many neighbours and every two of them can be separated by only finitely many 2 -separators (see e.g. [19, Proposition $4.2]$ ), the resulting Cayley graph $G^{\prime}$ is still locally finite.

Call a graph well-separated if it is 2-connected and every 2-separator is joined by an edge.

Theorem 5.4. Every planar, locally finite, well-separated Cayley graph $G$ with $\kappa(G)=2$ admits a generic planar presentation.

Proof. Let $\mathcal{C}$ be a $\Gamma(G)$-invariant nested set of cuts of order 2 as in Tutte's Theorem. Let $\mathcal{B}_{o}$ be the set of blocks (in the sense of Definition5.2) that contain the vertex $o$. For $B \in \mathcal{B}_{o}$, let $S_{B}$ be the set of those generators $s \in \mathcal{S} \cup \mathcal{S}^{-1}$ such that the edge with label $s$ starting at $o$ lies in $B$. Then $\mathcal{S} \cup \mathcal{S}^{-1}$ is covered by the set of $S_{B}$. We fix an embedding $\rho$ of $G$ in $\mathbb{R}^{2}$, and endow every $S_{B}$ with the cyclic order induced by $\rho$ at $o$. Let $\mathcal{B}_{o}^{\prime} \subseteq \mathcal{B}_{o}$ be maximal such that no two distinct $B, B^{\prime} \in \mathcal{B}_{o}$ are of the form $B=g\left(B^{\prime}\right)$ for any $g \in \Gamma(G)$. We can apply Theorem 2.4 to each $B \in \mathcal{B}_{o}^{\prime}$ to obtain a set $\mathcal{D}_{B} \subseteq \pi_{1}(B)$ that generates $\pi_{1}(B)$, and such that $\mathcal{D}_{B}^{\circ}$ is a nested set of indecomposable closed walks that is
invariant under the stabiliser of $B$ in $\Gamma(G)$. Then it is easy to see that

$$
\mathcal{D}:=\bigcup_{\substack{B \in \mathcal{B}^{\prime} \\ g \in \Gamma(G)}} g\left(\mathcal{D}_{B}\right)
$$

generates $\pi_{1}(G)$. Let $\mathcal{R}_{\mathcal{D}}$ be the set of words corresponding to closed walks in $\mathcal{D}^{\circ}$. Easily, $\left\langle\mathcal{S} \mid \mathcal{R}_{\mathcal{D}}\right\rangle$ is a presentation of $\Gamma(G)$. Once more, we use Tietzetransformations to obtain a finite subset $\mathcal{R} \subseteq \mathcal{R}_{\mathcal{D}}$ with $\left\langle\mathcal{S} \mid \mathcal{R}_{\mathcal{D}}\right\rangle=\langle\mathcal{S} \mid \mathcal{R}\rangle$, which is possible as $\Gamma(G)$ is finitely presented (Droms [4, Theorem 5.1]). To see that the set $\mathcal{C}:=\left\{B_{1}, \ldots, B_{n}\right\}:=\left\{S_{B} \mid B \in \mathcal{B}_{o}\right\}$ is a spin structure of $\mathcal{P}:=\langle\mathcal{S} \mid \mathcal{R}\rangle$, it remains to show that the graph $\mathcal{T}:=\left(\mathcal{C} \cup \mathcal{S}^{\prime}, \mathcal{E}\right)$, where $x y \in \mathcal{E}$ if and only if $x \in y$ or $y \in x$, is a tree.

Let us suppose that $\mathcal{T}$ is not a tree. Obviously, $\mathcal{T}$ is connected. So it contains some cycle $S_{1} s_{1} \ldots S_{m} s_{m} S_{1}$ with $S_{i} \in \mathcal{C}$ and $s_{i} \in \mathcal{S}^{\prime}$. For each $i \leq m$, let $B\left(S_{i}\right) \in \mathcal{B}_{o}$ be such that $S_{i}=S_{B\left(S_{i}\right)}$. As each element of $\mathcal{B}_{o}$ is a block, there is some path $P_{i}$ in $S_{i}$ connecting the end vertices of $s_{i-1}$ and $s_{i}$ distinct from $o$ (with $s_{0}=s_{m}$ ). The concatenation of all these paths $P_{i}$ is a cycle $C$ in $G$ that crosses all hinges $s_{i}$ precisely once as $S_{i} \neq S_{i+1}$ (with $S_{m+1}=S_{1}$ ). But this is not possible as each cycle, and hence also $C$, must lie in a unique block of $G$.

For $i \leq n$, let $B(i)$ be that element of $\mathcal{B}_{o}$ with $S_{B(i)}=B_{i}$. For every hinge $b \in \mathcal{S}$ incident with $o$ and every $i \leq n$ with $b \in B_{i}$, let $\mu(b, i)$ be that $B_{j}$ with $b(B(i))=B(j)$. So we have $b^{-1} \in B_{j}$. Let $\sigma(i)$ be the spin of $B_{i}$ at $o$. To define whether every generator is spin-preserving or spin-reversing in each element of the spin-structure (it participates in), we remember that the blocks -being either 3 -connected or cycles - have a unique embedding in the plane. So for $s \in \mathcal{S}$ and $i \leq n$, we define $\tau(s, i)$ to be 0 if $s$ is spin-preserving in $B(i)$ and 1 otherwise. (Note that $\tau$ is also defined if $s \notin B_{i}$.) Clearly, $(\mathcal{P}, \mathcal{C}, \sigma, \mu, \tau)$ is a generic embedded presentation.

As every element of $\mathcal{D}^{\circ}$ lies in a unique block, every $R \in \mathcal{R}$ is blocked with respect to $\mathcal{C}$ by definition, and the number of spin-reversing generators in $R$ is even. As $\mathcal{D}$ is nested, it is easy to check that no two relators cross. The fact that no cycle is a subgraph of any other cycle implies that no relator is a sub-word of a rotation of another relator, and hence our generic embedded presentation is a generic planar presentation.

With an argument similar to the proof of [12, Corollary 3.4], we obtain:
Corollary 5.5. Every planar well-separated Cayley graph $G$ with $\kappa(G)=2$ is the 1-skeleton of an almost planar Cayley complex of $\Gamma(G)$.

Proof. Since $G$ is planar, there is an embedding $\rho^{\prime}: G \rightarrow \mathbb{R}^{2}$ by definition. We will extend $\rho^{\prime}$ to the desired map $\rho$ from the Cayley complex $X$ of $\Gamma(G)$ with respect to the presentation $\langle\mathcal{S} \mid \mathcal{R}\rangle$ from above. For this, given any 2-cell $Y$ of $X$ with boundary cycle $C$, we embed $Y$ in the finite component of $\mathbb{R}^{2} \backslash C$. It is a straightforward consequence of the nestedness of $\mathcal{D}$ that the resulting map $\rho$ has the desired property.

### 5.3 Consistent embeddings lead to special planar presentations

In the previous section, we have seen that 2-connected planar Cayley graphs admit generic planar presentations. However, if the Cayley graph has a consistent embedding, we obtain a bit more even for 1-connected graphs:

Theorem 5.6. Every planar Cayley graph with a consistent embedding admits a special planar presentation.

Proof. Let $G$ be such a graph. First note that, by repeating the arguments of the proof of Lemma 5.3, we can join the two vertices of any 2 -separator $\{x, y\}$ by a new edge whenever $x y \notin E(G)$ and $G-\{x, y\}$ has two components $C$ with $\partial C=\{x, y\}$, while keeping the embedding consistent. So we may assume that every maximal 2 -connected subgraph of $G$ is well-separated.

Let $\mathcal{B}$ be a set of blocks of the maximal 2-connected subgraphs of $G$ consisting of one block from each $\Gamma(G)$-orbit. As before, Theorem 2.4 gives us for each $B \in$ $\mathcal{B}$ a set $\mathcal{D}_{B}$ that generates $\pi_{1}(B)$ such that $\mathcal{D}_{B}^{\circ}$ is a nested set of indecomposable closed walks that is invariant under the stabiliser in $\Gamma(G)$ of $B$. Let $\mathcal{R}_{B}$ be the set of words corresponding to the elements of $\mathcal{D}_{B}^{\circ}$. As above, Tietze-transformations give us a finite $\mathcal{R} \subseteq \bigcup_{B \in \mathcal{B}} \mathcal{R}_{B}$ such that $\mathcal{P}=\langle\mathcal{S} \mid \mathcal{R}\rangle$ is a finite presentation of $\Gamma(G)$, where $\mathcal{S}$ is the generating set of $G$.

If we let $\sigma$ be the spin of one fixed vertex $x$ and $\tau(s)=0$ if the edge from $x$ labelled $s$ is spin-preserving and $\tau(s)=1$ otherwise, then $(\mathcal{P}, \sigma, \tau)$ is a special planar presentation of $\Gamma(G)$. Indeed, nestedness of the closed walks in $\mathcal{D}_{B}^{\circ}$ implies that the corresponding words are non-crossing, the fact that they are indecomposable implies that no relator is a subword of any other relator, and the embedding implies that every relator contains an even number of spin-reversing letters.

## 6 Conclusions

We now put the above results together to prove the statements of the introduction. Because of the redundant generators used in Lemmas 5.1 and 5.3, we need to generalise our notion of planar presentation slightly. We say that $s \in \mathcal{S}$ is an obviously redundant generator of a presentation $\langle\mathcal{S} \mid \mathcal{R}\rangle$, if there is exactly one relator $W_{s} \in \mathcal{R}$ in which $s$ appears, and $s$ appears exactly once in $W$. A general planar presentation is a presentation obtained from a generic planar presentation by recursively removing zero or more obviously redundant generators $s$ along with the corresponding relator $W_{s}$. The last two sections prove the two directions of Theorem 1.1:

Proof of Theorem [1.1. If $G$ is a finitely generated planar Cayley graph, then by Lemmas 5.1 and 5.3 we may find a Tietze-supergraph that is is 2-connected and well-separated. Theorem 5.4 then yields a generic planar presentation, from which we can remove any generators that were not present in $G$ to obtain a general planar presentation of $G$, which proves the forward direction.

For the backward direction, if $G$ admits a general planar presentation, then some supergraph $G^{\prime}$ admits a generic planar presentation, and is thus planar by Theorem 4.10. Since planarity is preserved under deleting edges, so is $G$.

A similar result holds when we insist that there is a consistent embedding, and we can even allow our Cayley graphs to have infinitely many generators:

Theorem 6.1. A Cayley graph admits a consistent embedding in the plane if and only if it admits a special planar presentation.

The two directions of Theorem 6.1 are given by Theorem 5.6 and Theorem 3.3

Next, we use our presentations to obtain effective enumerations.
Theorem 6.2. The Cayley graphs that admit a consistent embedding in the plane are effectively enumerable.

Proof. By Theorem 6.1, it suffices to produce an effective enumeration of the special planar presentations. For this, it suffices to produce an enumeration of the embedded presentations, and output those embedded presentations that satisfy the three conditions in the definition of a special planar presentation (Definition 3.1); it is easy to see that these conditions can be checked algorithmically.

Theorem 6.3. The planar, locally finite Cayley graphs are effectively enumerable.

Proof. Similarly to the proof of Theorem 6.2, we remark that any effective enumeration of the general planar presentations gives rise to an effective enumeration of the planar Cayley graphs by Theorem 1.1.

To effectively enumerate the general planar presentations, we start with an enumeration of the generic embedded presentations, and output those that satisfy the four conditions of Definition 3.4 which can be checked algorithmically. Having thus effectively enumerated the generic planar presentations, we remove any obviously redundant generators to effectively enumerate the general planar presentations: for each output $G=\langle\mathcal{S} \mid \mathcal{R}\rangle$, check for every $s \in \mathcal{S}$ whether $s$ is an obviously redundant generator. For every such $s$ found, output the presentation $G^{\prime}:=\left\langle\mathcal{S} \backslash\{s\} \mid \mathcal{R} \backslash\left\{W_{s}\right\}\right\rangle$. Then, recursively apply the same check to $G^{\prime}$, removing any obviously redundant generators of that presentation and so on.

We conclude with some related questions concerning embeddings of Cayley complexes. Let $C C(\mathcal{P})$ denote the Cayley complex of a presentation $\mathcal{P}$. Call a map $\rho: C C(\mathcal{P}) \rightarrow \mathbb{R}^{2}$ consistent if its restriction to $\operatorname{Cay}(\mathcal{P})$ is consistent. Call $\rho$ nested if it witnesses the fact that $C C(\mathcal{P})$ is almost planar, i.e. if the images under $\rho$ of the interiors of any two 2-cells are either disjoint, or one is contained in the other.

The following might be interesting as it exhibits a geometric property of Cayley complexes which can be decided by an algorithm.

Theorem 6.4. There is an algorithm that given a presentation $\mathcal{P}=\langle\mathcal{S} \mid \mathcal{R}\rangle$ decides whether $C C(\mathcal{P})$ admits a nested, consistent map into $\mathbb{R}^{2}$.
Proof. We claim that $C C(\mathcal{P})$ admits a nested, consistent map into $\mathbb{R}^{2}$ if and only if there is a spin $\sigma$ on $\mathcal{S}$ and a 'spin-behaviour' function $\tau$ from $\mathcal{S}$ to $\{0,1\}$ such that the triple $(\mathcal{P}, \sigma, \tau)$ is a special planar presentation.

To prove the backward direction, note that if $\mathcal{P}, \sigma, \tau$ is a special planar presentation, then $\operatorname{Cay}(\mathcal{P})$ admits a consistent embedding $\rho$ into $\mathbb{R}^{2}$ by Theorem 3.3. Extend this embedding into a map $\rho^{\prime}$ from $C C(\mathcal{P})$ to $\mathbb{R}^{2}$ by mapping each 2 -cell inside the closed curve to which $\rho$ maps its boundary. Then $\rho^{\prime}$ is nested because no two words in $\mathcal{R}$ cross each other by the definition of a special planar presentation.

For the forward direction, given such a map $\rho: C C(\mathcal{P}) \rightarrow \mathbb{R}^{2}$, we can read the spin data $\sigma, \tau$ from $\rho$ since $\rho$ is consistent. Then $\mathcal{P}, \sigma, \tau$ is an embedded presentation. To prove that it is a special planar presentation it remains to show that no two words in $\mathcal{R}$ cross each other, which follows immediately from the nestedness of $\rho$.

By using general planar presentations instead of special ones, Theorem 6.4 can be generalised to yield a further decidable property of Cayley complexes, but instead of maps into $\mathbb{R}^{2}$ we have to consider maps into larger spaces obtained by glueing copies of $\mathbb{R}^{2}$ along (possibly closed) bounded simple curves - to which we map the hinges of our Cayley graphs - in a tree like fashion. We leave the details to the interested reader.

Our results do not yet answer the following
Problem 6.5. Is there an algorithm that given a presentation $\mathcal{P}=\langle\mathcal{S} \mid \mathcal{R}\rangle$ decides whether $C C^{\prime}(\mathcal{P})$ is planar?

In this problem $C C^{\prime}(\mathcal{P})$ denotes the complex obtained from $C C(\mathcal{P})$ by removing redundant 2-cells, that is, if a set of 2-cells have the same boundary, we remove all but one of them. Some authors still call $C C^{\prime}(\mathcal{P})$ the Cayley complex of $\mathcal{P}$. (In Theorem 6.4 it does not make a difference whether we consider $C C(\mathcal{P})$ or $C C^{\prime}(\mathcal{P})$.)

We remark that it is not true that $C C(\mathcal{P})$ is planar if and only if $\mathcal{P}$ is a facial presentation in the sense of [9]; the presentation $\mathcal{P}=\left\langle a, b \mid a^{2}, b^{3}, a b^{-1}\right\rangle$ if facial, but its Cayley complex consists of a single vertex, two loops, a 2 -cell winding twice around a loop, and a 2 -cell winding three times around the other loop.

Having studied embeddings of Cayley complexes in $\mathbb{R}^{2}$, the following suggests itself
Problem 6.6. Which groups admit a Cayley complex embeddable in $\mathbb{R}^{3}$ ?

## 7 Further remarks

We proved that every planar Cayley graph $G$ admits a planar presentation such that every relator induces a cycle of $G$ (rather than an arbitrary closed walk
with repetitions of vertices). It would be interesting if we could strengthen the definition of a planar presentation in such a way that this is always the case in the resulting planar Cayley graph. Some strengthening will be necessary as shown by the example $\mathcal{P}=\left\langle a, b \mid a^{2}, b^{3}, a b^{-1}\right\rangle$ from the previous section. This is a planar presentation - even stronger, every relator is facial- but it is easy to see that its group is the group of one element. Our optimism that this may be possible stems from the fact that it was possible in the cubic case [10].

If we could do this, then it would probably help to prove that the planar Cayley graphs are effectively constructible:

Conjecture 7.1. There is an algorithm that given a general planar presentation $\mathcal{P}$, and $n \in \mathbb{N}$, outputs the ball of radius $n$ in the Cayley graph of $\mathcal{P}$.

This was proved in 10 in the cubic case.
A further interesting question, also asked in [10], is whether for every $n \in \mathbb{N}$ there is an upper bound $f(n)$, such that every $n$-regular planar Cayley graph admits a planar presentation with at most $f(n)$ relators. This would strengthen Droms' result [4, Theorem 5.1] that planar groups are finitely presented.

We conclude with a rather unrelated observation. It is known that the fundamental group of a finite graph of groups with residually finite vertex groups and finite edge groups is residually finite [18, II.2.6.12]. Combining this with Dunwoody's result mentioned in the introduction, we obtain the following corollary, to which this paper has no contribution

Corollary 7.2. Every planar group is residually finite.

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[^1]:    ${ }^{1}$ The third option can be dropped by considering the modified Cayley complex in the sense of [16], i.e. by representing involutions in $\mathcal{S}$ by single, undirected edges.

[^2]:    ${ }^{2}$ The existence and uniqueness of this $B_{i}$ is a consequence of ( F 1 ; see the definition of 'blocked'.

[^3]:    ${ }^{3}$ Readers that are familiar with tree-decompositions of graphs might notice that this just says that for every nested set of cuts, we find a tree-decomposition of the graph whose parts are the blocks and boundaries of cuts.

