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November 12, 2019

Abstract

We prove that the parameter x of a tight set \mathcal{T} of a hyperbolic quadric $Q^+(2n+1,q)$ of an odd rank n+1 satisfies $\binom{x}{2} + w(w-x) \equiv 0 \mod q+1$, where w is the number of points of \mathcal{T} in any generator of $Q^+(2n+1,q)$. As this modular equation should have an integer solution in w if such a \mathcal{T} exists, this condition rules out roughly at least one half of all possible parameters x. It generalizes a previous result by the author and K. Metsch shown for tight sets of a hyperbolic quadric $Q^+(5,q)$ (also known as Cameron-Liebler line classes in PG(3,q)).

1 Introduction

Let PG(n,q) denote the projective space of dimension n with underlying vector space $V := \mathbb{F}_q^{n+1}$ over the finite field \mathbb{F}_q with q elements. For a non-degenerate quadratic (or reflexive sesquilinear) form f on V, the **classical polar space** P associated with f is the incidence structure formed by the totally singular (or totally isotropic, respectively) subspaces of f and their incidence is defined by symmetrized containment [6]. We consider the elements of P as subspaces of PG(n,q), so they are projective points, lines, A maximal subspace of P has dimension r-1, where r is the Witt index of f, also called the **rank** of P, and such a subspace is called a **generator**.

We will consider the polar space $Q^+(2n+1,q)$ of rank n+1 defined by a hyperbolic quadric f, i.e., the set of projective points of PG(2n+1,q) satisfying $f(\mathbf{x}) = 0$, where

$$f(\mathbf{x}) := f(x_0, \dots, x_{2n+1}) = x_0 x_1 + \dots + x_{2n} x_{2n+1}, \ \mathbf{x} \in V.$$

The associated bilinear form $b(\mathbf{x}, \mathbf{y}) := f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x}) - f(\mathbf{y})$ defines the **polarity** \perp of PG(2n + 1, q). Two points of the polar space are collinear if and only if $b(\mathbf{x}, \mathbf{y}) = 0$. For a point P of the quadric, denote by P^{\perp} the set of points collinear with P, which form the (tangent) hyperplane of P. Note that $P \in P^{\perp}$, and, for a point set (or a subspace) S, let S^{\perp} denote $\cap_{P \in S} P^{\perp}$.

The notion of tight sets was introduced by Payne [22] for generalized quadrangles (which include the classical polar spaces of rank 2), and it was extended to polar spaces of higher rank by Drudge [13]. Tight sets are extremal sets of points in the following sense. It was shown [13, Theorem 8.1] that, for a set \mathcal{T} of points of a finite polar space P of rank r over \mathbb{F}_q , the average number κ of points of \mathcal{T} collinear to a given point of \mathcal{T} is bounded above by

$$\kappa \le |\mathcal{T}| \frac{q^{r-1} - 1}{q^r - 1} + q^{r-1},\tag{1.1}$$

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and if equality attains, then \mathcal{T} is said to be **tight**. Moreover, in this case $|\mathcal{T}| = x \frac{q^{r-1}}{q-1}$ holds for some non-negative integer x, which is called the **parameter** of the tight set \mathcal{T} . The complement of an x-tight set is a $(q^{r-1} + 1 - x)$ -tight set, so that we may assume $x \leq \frac{q^{r-1}+1}{2}$.

The question is to determine for which parameters x an x-tight set exists, and to classify the examples admitting a given parameter x. The answer fundamentally depends on the type and the rank of P, see [2, 3, 12, 20, 21] for some recent results. Besides that, geometric properties and characterizations of tight sets are of interest, as they nicely interact with related structures of polar spaces such as m-ovoids. This fact can be explained from the point of view of algebraic graph theory [2], regardless of the type of a polar space.

Note that a disjoint union of x generators of P, which are the only tight sets with parameter 1 [13, Theorem 9.1], is itself a tight set with parameter x. As for the hyperbolic polar spaces $Q^+(2n+1,q)$, their tight sets in the partial case n = 2 appeared in a different context in a paper by Cameron and Liebler [5] as classes of lines in PG(3,q) satisfying certain geometric conditions. While their equivalence was observed by Drudge [13] via the Klein-correspondence, they have been studied for almost three decades under the name of Cameron-Liebler line classes [23].

Further, Drudge [13, Corollary 9.1] proved that, for $q \ge 4$, any x-tight set of $Q^+(2n+1,q)$ with $x \le \sqrt{q}$ is the disjoint union of x generators (hence, none such exist if x > 2 and the rank n + 1 is odd). This bound was improved several times [2, 11] and finally Beukemann and Metsch [4] proved that the same conclusion holds if $1 \le n \le 3$ and $x \le q$, or if $n \ge 4$, $q \ge 71$ and $x \le q - 1$.

A much stronger bound is shown [19] for the hyperbolic polar space $Q^+(5,q)$ of rank 3: the parameter x of a tight set that is not a disjoint union of generators should satisfy $x \ge cq^{4/3}$ with some constant c. Moreover, the following result was obtained in [16].

Result 1.1. Let \mathcal{T} be a tight set with parameter x of $Q^+(5,q)$. Then, for every generator G of $Q^+(5,q)$, the number $w := |G \cap \mathcal{T}|$ satisfies

$$\binom{x}{2} + w(w-x) \equiv 0 \mod q+1.$$
(1.2)

Thus, if $Q^+(5,q)$ has a tight set with parameter x, then Eq. (1.2) has a solution in w from the set $\{0, 1, \ldots, q\}$. It was shown in [16] that it implies a strong non-existence result for tight sets of $Q^+(5,q)$, as, for any given q, it rules out roughly at least half of the possible parameters x from the set $\{3, \ldots, \frac{q^2+1}{2}\}$.

The main result of the present paper generalizes Result 1.1 to the hyperbolic polar space $Q^+(2n+1,q)$ of an arbitrary rank n+1 as follows.

Theorem 1.2. Let \mathcal{T} be a tight set with parameter x of $Q^+(2n+1,q)$. Then, for every generator G of $Q^+(2n+1,q)$, the number $w := |G \cap \mathcal{T}|$ satisfies

$$\binom{x}{2} + w(w-x) \equiv 0 \mod q+1, \tag{1.3}$$

if n is even, and

$$w(w-x) \equiv 0 \mod q+1,\tag{1.4}$$

if n is odd.

The argument from [16, Section 3] shows that Eq. (1.3) rules out roughly at least half of the possible parameters x from the set $\{3, \ldots, \frac{q^n+1}{2}\}$. Unfortunately, if n is odd, it seems that Eq.

(1.4) does not impose any restrictions on x, which is consistent with the existence of large sets of disjoint generators in $Q^+(2n+1,q)$ in this case.

The proof of Theorem 1.2 generalizes a technique developed in [16, 17]. Suppose that \mathcal{T} is a tight set with parameter x of $Q := Q^+(2n+1,q)$. It follows [13, Theorem 8.1] that, for any point P of Q, one has

$$|P^{\perp} \cap \mathcal{T}| = q^n |\{P\} \cap \mathcal{T}| + x\theta_{n-1}, \qquad (1.5)$$

where, for an integer $k \ge -1$, we define $\theta_k := (q^{k+1} - 1)/(q - 1)$. Observe that θ_k is the number of points in the k-dimensional projective space over \mathbb{F}_q , and $|\mathcal{T}| = x\theta_n$.

The following result [4, Lemma 2.1] generalizes Eq. (1.5) to the subspaces of PG(2n+1,q), and it will play a crucial role in the proof.

Result 1.3. Every s-dimensional subspace $S, s \leq n$, of the ambient space PG(2n + 1, q) satisfies the equality

$$|S^{\perp} \cap \mathcal{T}| = q^{n-s} |S \cap \mathcal{T}| + x \theta_{n-s-1}.$$

Drudge [13, Theorem 8.1(3)] proved a partial case of Result 1.3 when S is a line not contained in Q (i.e., S intersects Q in two non-collinear points). For a point $P_0 \in Q \setminus \mathcal{T}$, applying Result 1.3 in different settings, we count the number of pairs (P_1, P_2) of collinear points $P_1 \in P_0^{\perp} \cap \mathcal{T}$, $P_2 \in (Q \setminus P_0^{\perp}) \cap \mathcal{T}$ in two ways, which gives a certain equation. We then analyze this equation modulo q + 1.

Note that via the field reduction [18] one can construct tight sets of $Q^+(6b-1,q)$ from a variety of those of $Q^+(5,q^b)$ that have been discovered recently [7, 8, 9, 10, 14, 15]. Thus, $Q^+(7,q)$ and $Q^+(9,q)$ seem to be the first unexplored cases, where we are not aware of any non-trivial tight sets.

2 The proof of Theorem 1.2

We first recall some well-known properties of a hyperbolic polar space [6].

Result 2.1. Let Q be a hyperbolic polar space $Q^+(2n+1,q)$ or rank n+1.

- (a) For every pair P_1, P_2 of non-collinear points of \mathbb{Q} , $\{P_1, P_2\}^{\perp}$ is a hyperbolic quadric $\mathbb{Q}^+(2n-1,q)$ of rank n.
- (b) For every s-subspace $S \subset Q$, the quotient space S^{\perp}/S is a hyperbolic quadric of rank n s (over the same field).
- (c) The number of points of Q is $k_n := (q^n + 1)\theta_n$, of which $k_{2,n} := q^{2n}$ are not collinear to a given point.
- (d) For every pair of distinct collinear points of Q, the number of points of Q that are collinear to only one of them equals $b_n := q^{2n-1}$.

Lemma 2.2. Let Q be a hyperbolic polar space $Q^+(2n-1,q)$ or rank n. For a point P_0 of Q, let P and P' be two distinct points of $P_0^{\perp} \setminus \{P_0\}$, and $\ell := \langle P, P' \rangle$.

- (a) If $\ell \subset \mathbb{Q}$ with $P_0 \in \ell$, then $P^{\perp} \cap P'^{\perp} \subseteq P_0^{\perp}$.
- (b) If $\ell \subset \mathbb{Q}$ with $P_0 \notin \ell$, then $P^{\perp} \cap P'^{\perp}$ contains precisely $\lambda_n := q^{2n-4}$ points of $\mathbb{Q} \setminus P_0^{\perp}$.
- (c) If $\ell \not\subset \mathbf{Q}$, then $P^{\perp} \cap P'^{\perp}$ contains precisely λ_n points of $\mathbf{Q} \setminus P_0^{\perp}$.

Proof. Statement (a) is obvious. To prove (b), we observe that ℓ^{\perp}/ℓ is a hyperbolic quadric of rank n-2, whose points are the planes on ℓ , and therefore the number of points in $P^{\perp} \cap P'^{\perp}$ equals $\theta_1 + k_{n-3}(\theta_2 - \theta_1)$, of which θ_2 points are contained in the plane $\langle P_0, \ell \rangle$ and $k_{n-4}(\theta_3 - \theta_2)$ points correspond to $\langle P_0, \ell \rangle^{\perp}/\langle P_0, \ell \rangle$. Thus, the number of points in $(P^{\perp} \cap P'^{\perp}) \setminus P_0^{\perp}$ equals:

$$\theta_1 + k_{n-3}(\theta_2 - \theta_1) - \theta_2 - k_{n-4}(\theta_3 - \theta_2) = q^{2n-4}$$

which shows (b). The proof of (c) is similar.

We will need the following technical lemma. For a point P of a hyperbolic quadric, define L(P) to be the set of lines on the quadric through P.

Lemma 2.3. Let Q be a hyperbolic quadric $Q^+(2n-1,q)$ of rank n. Suppose that μ is an integervalued function defined on the set of points of Q such that, for a positive integer x, the following properties hold.

(*) For every point P of Q:

$$\sum_{P_1 \in P^{\perp}} \mu(P_1) = x\theta_{n-2} + q^{n-1}\mu(P).$$

(**) For every pair P_1, P_2 of non-collinear points of Q:

$$\sum_{P' \in \{P_1, P_2\}^{\perp}} \mu(P') = x\theta_{n-3} + q^{n-2}(\mu(P_1) + \mu(P_2)).$$

Then, for an arbitrary point P_0 of Q, one has:

$$\sum_{P \in \mathbf{Q}} \mu(P)^2 = \mu(P_0)^2 + (x - \mu(P_0))^2 + (q + 1) \cdot \sum_{P_1 \in P_0^{\perp} \setminus \{P_0\}} \mu(P_1)^2 - \sum_{\ell \in \mathbf{L}(P_0)} \left(\sum_{P_2 \in \ell \setminus \{P_0\}} \mu(P_2)\right)^2.$$

Proof. For a point $P_0 \in Q$, one can write:

$$\sum_{P \in \mathbf{Q}} \mu(P)^2 = \mu(P_0)^2 + \sum_{P_1 \in P_0^{\perp} \setminus \{P_0\}} \mu(P_1)^2 + \sum_{P_2 \in \mathbf{Q} \setminus P_0^{\perp}} \mu(P_2)^2.$$
(2.1)

For a point $P_2 \in \mathsf{Q} \setminus P_0^{\perp}$, Property (**) implies that:

$$\mu(P_2) = \frac{1}{q^{n-2}} \Big(\sum_{P' \in \{P_0, P_2\}^{\perp}} \mu(P') - q^{n-2} \mu(P_0) - x\theta_{n-3} \Big),$$

which allows to rewrite the last sum in Eq. (2.1) as follows:

$$\sum_{P_2 \in \mathbb{Q} \setminus P_0^{\perp}} \mu(P_2)^2 = \frac{1}{q^{2n-4}} \sum_{P_2 \in \mathbb{Q} \setminus P_0^{\perp}} \left(\sum_{P' \in \{P_0, P_2\}^{\perp}} \mu(P') - q^{n-2} \mu(P_0) - x \theta_{n-3} \right)^2$$
$$= \frac{1}{q^{2n-4}} \sum_{P_2 \in \mathbb{Q} \setminus P_0^{\perp}} \left(\left(\sum_{P' \in \{P_0, P_2\}^{\perp}} \mu(P') \right)^2 - 2 \left(\sum_{P' \in \{P_0, P_2\}^{\perp}} \mu(P') \right) \left(q^{n-2} \mu(P_0) + x \theta_{n-3} \right)^2 \right)$$
$$+ \left(q^{n-2} \mu(P_0) + x \theta_{n-3} \right)^2 \right).$$

Further, we observe that

$$\sum_{P_2 \in \mathbb{Q} \setminus P_0^{\perp}} \sum_{P' \in \{P_0, P_2\}^{\perp}} \mu(P') = b_{n-1} \cdot \sum_{P_1 \in P_0^{\perp} \setminus \{P_0\}} \mu(P_1) \text{ [holds by Result 2.1(d)]}$$
$$= q^{2n-3} ((q^{n-1}-1)\mu(P_0) + x\theta_{n-2}) \text{ [holds by Property (*)]},$$
and
$$\sum_{P_2 \in \mathbb{Q} \setminus P_0^{\perp}} 1 = k_{2,n-1} \text{ [holds by Result 2.1(b),(c)]}$$
$$= q^{2n-2}.$$

Thus, we obtain:

$$\sum_{P_2 \in \mathbb{Q} \setminus P_0^{\perp}} \mu(P_2)^2 = \frac{1}{q^{2n-4}} \sum_{P_2 \in \mathbb{Q} \setminus P_0^{\perp}} \left(\sum_{P' \in \{P_0, P_2\}^{\perp}} \mu(P') \right)^2 + q^2 \left(q^{n-2} \mu(P_0) + x \theta_{n-3} \right)^2 - 2q \left(q^{n-2} \mu(P_0) + x \theta_{n-3} \right) \left((q^{n-1} - 1) \mu(P_0) + x \theta_{n-2} \right),$$

where we shall evaluate the first double sum by using Lemma 2.2. Indeed, for any pair P, P' of points of $P_0^{\perp} \setminus \{P_0\}$ such that $P_0 \notin \langle P, P' \rangle$, there are precisely λ_n points $P_2 \in \mathbb{Q} \setminus P_0^{\perp}$ with $P, P' \in \{P_0, P_2\}^{\perp}$. Thus, it follows that

$$\sum_{P_2 \in \mathbb{Q} \setminus P_0^{\perp}} \left(\sum_{P' \in \{P_0, P_2\}^{\perp}} \mu(P') \right)^2 = b_{n-1} \cdot \sum_{P_1 \in P_0^{\perp} \setminus \{P_0\}} \mu(P_1)^2 + \lambda_n \cdot \sum_{\ell \in \mathbb{L}(P_0)} \sum_{P \in \ell \setminus \{P_0\}} \mu(P) \sum_{P' \in P_0^{\perp} \setminus \ell} \mu(P'),$$

where the last triple sum can be rewritten as follows:

$$\sum_{\ell \in \mathsf{L}(P_0)} \sum_{P \in \ell \setminus \{P_0\}} \mu(P) \sum_{P' \in P_0^{\perp} \setminus \ell} \mu(P') = \sum_{\ell \in \mathsf{L}(P_0)} \sum_{P \in \ell \setminus \{P_0\}} \mu(P) \Big(\sum_{P' \in P_0^{\perp} \setminus \{P_0\}} \mu(P') - \sum_{P'' \in \ell \setminus \{P_0\}} \mu(P'') \Big)$$
$$\begin{bmatrix} \text{by } \sum_{\ell \in \mathsf{L}(P_0)} \sum_{P \in \ell \setminus \{P_0\}} \mu(P) = \sum_{P_1 \in P_0^{\perp} \setminus \{P_0\}} \mu(P_1) \end{bmatrix}$$
$$= \Big(\sum_{P_1 \in P_0^{\perp} \setminus \{P_0\}} \mu(P_1) \Big)^2 - \sum_{\ell \in \mathsf{L}(P_0)} \Big(\sum_{P \in \ell \setminus \{P_0\}} \mu(P) \Big)^2$$
$$= \Big((q^{n-1} - 1)\mu(P_0) + x\theta_{n-2} \Big)^2 - \sum_{\ell \in \mathsf{L}(P_0)} \Big(\sum_{P \in \ell \setminus \{P_0\}} \mu(P) \Big)^2.$$

Putting it all together and simplifying, we obtain:

$$\begin{split} \sum_{P \in \mathbf{Q}} \mu(P)^2 &= \mu(P_0)^2 + \sum_{P_1 \in P_0^{\perp} \setminus \{P_0\}} \mu(P_1)^2 + \sum_{P_2 \in \mathbf{Q} \setminus P_0^{\perp}} \mu(P_2)^2 \\ &= \mu(P_0)^2 + (q+1) \cdot \sum_{P_1 \in P_0^{\perp} \setminus \{P_0\}} \mu(P_1)^2 - \sum_{\ell \in \mathsf{L}(P_0)} \left(\sum_{P \in \ell \setminus \{P_0\}} \mu(P)\right)^2 \\ &+ \left((q^{n-1}-1)\mu(P_0) + x\theta_{n-2}\right)^2 + q^2 \left(q^{n-2}\mu(P_0) + x\theta_{n-3}\right)^2 \\ &- 2q \left((q^{n-1}-1)\mu(P_0) + x\theta_{n-2}\right) \left(q^{n-2}\mu(P_0) + x\theta_{n-3}\right) \\ &= \mu(P_0)^2 + \left(x - \mu(P_0)\right)^2 + (q+1) \sum_{P_1 \in P_0^{\perp} \setminus \{P_0\}} \mu(P_1)^2 - \sum_{\ell \in \mathsf{L}(P_0)} \left(\sum_{P \in \ell \setminus \{P_0\}} \mu(P)\right)^2, \end{split}$$

and the lemma follows.

One can see that a function μ satisfying the condition of Lemma 2.3 generalizes the notion of tight sets. The proof of the following lemma justifies this by showing that μ is a weighted tight set [1] and, moreover, given such a function μ one can construct a weighted set for hyperbolic quadrics of smaller rank.

Lemma 2.4. Let Q be a hyperbolic quadric $Q^+(2n-1,q)$ of rank n. Suppose that μ is an integervalued function defined on the set of points of Q such that, for a positive integer x, μ satisfies Property (*) of Lemma 2.3. Then the following holds.

- (1) μ satisfies Property (**) of Lemma 2.3.
- (2) For a point $P_0 \in \mathbb{Q}$, a function $\tilde{\mu}$ on the points of the hyperbolic quadric P_0^{\perp}/P_0 defined by:

$$\widetilde{\mu}(\ell) = \sum_{P \in \ell \setminus \{P_0\}} \mu(P), \quad (\ell \in \mathsf{L}(P_0)),$$

satisfies Property (*) with $\tilde{x} = (q-1)\mu(P_0) + x$.

Proof. First observe that by Property (*) and Result 2.1(b),(c), we have

$$\sum_{P \in \mathbf{Q}} \mu(P) = \frac{1}{q^{n-1}} \Big(\sum_{P \in \mathbf{Q}} \sum_{P_1 \in P^\perp} \mu(P_1) \Big) - k_{n-1} \frac{x\theta_{n-2}}{q^{n-1}} \\ = \frac{1 + qk_{n-2}}{q^{n-1}} \Big(\sum_{P \in \mathbf{Q}} \mu(P) \Big) - k_{n-1} \frac{x\theta_{n-2}}{q^{n-1}},$$

which implies that $M := \sum_{P \in \mathbf{Q}} \mu(P) = x \theta_{n-1}$.

(1) Recall that the collinearity graph of Q, in which two distinct points are adjacent whenever they are collinear, is strongly regular. Therefore its symmetric (0, 1)-adjacency matrix is diagonalizable with three distinct eigenvalues, namely, $\sigma_0 = qk_{n-2}$ with multiplicify one (and the all-one eigenvector), $\sigma_1 = q^{n-1} - 1$ and $\sigma_2 = -(q^{n-2} + 1)$, see [13, Lemma 8.3].

It now follows from Property (*) that a vector $\mathbf{v} \in \mathbb{R}^{\mathbb{Q}}$ defined by $\mathbf{v}(P) = \mu(P) + \frac{x\theta_{n-2}}{q^{n-1}-1-qk_{n-2}}$ is an eigenvector for σ_1 , as it satisfies

$$\mathbf{v}(P) = (q^{n-1} - 1) \cdot \sum_{P_1 \in P^{\perp} \setminus \{P\}} \mathbf{v}(P_1) \text{ for any } P \in \mathsf{Q}$$

Given two non-collinear points P, P' of Q, it follows from the proof of [13, Theorem 8.1(3)] that

$$\mathbf{w}_{P,P'} = \mathbf{e}_{\{P,P'\}^{\perp}} - \frac{\theta_{n-3}}{\theta_{n-1}} \mathbf{e}_{\mathsf{Q}} - q^{n-2} (\mathbf{e}_{\{P\}} + \mathbf{e}_{\{P'\}}),$$

where $\mathbf{e}_X \in \mathbb{R}^{\mathsf{Q}}$ denotes the characteristic vector of a point set X, is an eigenvector for σ_2 .

Since the eigenspaces associated to σ_1 and σ_2 are orthogonal, it follows that **v** is perpendicular with $\mathbf{w}_{P,P'}$. Evaluating $\langle \mathbf{v}, \mathbf{w}_{P,P'} \rangle = 0$ and simplifying, we find that Property (**) follows.

(2) We follow the proof of [4, Lemma 2.1]. Fix a line $\ell_0 \in \mathsf{L}(P_0)$. If we consider the subspaces P^{\perp} for the q + 1 points P of ℓ_0 , then the points of ℓ_0^{\perp} lie in each such subspace P^{\perp} , whereas every

point outside ℓ_0^{\perp} lies in exactly one of these. By double counting, we find:

$$q\left(\sum_{P \in \ell_0^{\perp}} \mu(P)\right) + M = \sum_{P_1 \in \ell_0} \sum_{P_2 \in P_1^{\perp}} \mu(P_2)$$

=
$$\sum_{P_1 \in \ell_0} \left(q^{n-1}\mu(P_1) + x\theta_{n-2}\right)$$

=
$$q^{n-1}\tilde{\mu}(\ell_0) + q^{n-1}\mu(P_0) + (q+1)x\theta_{n-2}.$$

where on the left-hand side we have:

$$\sum_{P \in \ell_0^\perp} \mu(P) = \mu(P_0) + \widetilde{\mu}(\ell_0) + \sum_{\ell_1 \in (\mathsf{L}(P_0) \setminus \{\ell_0\}) \cap \ell_0^\perp} \widetilde{\mu}(\ell_1),$$

and simplifying gives the desired property (*) for $\tilde{\mu}$.

For the rest of the proof, we assume that \mathcal{T} is a tight set with parameter x of a hyperbolic quadric $Q = Q^+(2n+1,q)$ of rank n+1. Fix a point $P_0 \in Q \setminus \mathcal{T}$ and, for every line ℓ of $L(P_0)$, put $m_{\ell} := |\ell \cap \mathcal{T}|.$

Lemma 2.5. The following holds:

$$\sum_{\ell \in \mathsf{L}(P_0)} m_{\ell}^2 = x(\theta_{n-1} - 1 + x).$$
(2.2)

Proof. We prove the result by double counting the number E of pairs (P_1, P_2) where $P_1 \in P_0^{\perp} \cap \mathcal{T}$, $P_2 \in \mathcal{T} \setminus P_0^{\perp}$ and $P_1 \in P_2^{\perp}$.

Observe that, by Eq. (1.5), there are precisely $|\mathcal{T} \setminus P_0^{\perp}| = |\mathcal{T}| - |P_0^{\perp} \cap \mathcal{T}| = xq^n$ points P_2 of \mathcal{T} that are not collinear to P_0 . By Result 1.3 applied to a line $\langle P_0, P_2 \rangle \not\subset \mathsf{Q}, P_2 \in \mathcal{T}$, each of them is collinear to

$$|\langle P_0, P_2 \rangle^{\perp} \cap \mathcal{T}| = q^{n-1} |\langle P_0, P_2 \rangle \cap \mathcal{T}| + x\theta_{n-2} = q^{n-1} + x\theta_{n-2}$$

points $P_1 \in P_0^{\perp} \cap \mathcal{T}$, as $\langle P_0, P_2 \rangle \cap \mathcal{T} = \{P_2\}$. Thus, E equals $xq^n(q^{n-1} + x\theta_{n-2})$. On the other hand, for a point $P_1 \in P_0^{\perp} \cap \mathcal{T}$, the number of points $P_2 \in \mathcal{T} \setminus P_0^{\perp}$ that are collinear to P_1 equals

$$|(P_1^{\perp} \setminus P_0^{\perp}) \cap \mathcal{T}| = |P_1^{\perp} \cap \mathcal{T}| - |\langle P_0, P_1 \rangle^{\perp} \cap \mathcal{T}| = q^n + q^{n-1}(x - m_\ell)$$

where we apply Result 1.3 to the line $\ell := \langle P_0, P_1 \rangle$, and observe that this number does not depend on the particular choice of a point P_1 of $\ell \cap \mathcal{T}$. Therefore, E equals

$$\sum_{\ell \in \mathsf{L}(P_0)} m_\ell(q^n + q^{n-1}(x - m_\ell)),$$

and note that by Eq. (1.5) and $P_0 \notin \mathcal{T}$, we have:

$$|P_0^{\perp} \cap \mathcal{T}| = \sum_{\ell \in \mathsf{L}(P_0)} m_{\ell}$$
$$= x \theta_{n-1}.$$

Thus, we obtain that

$$\sum_{\ell \in \mathsf{L}(P_0)} m_{\ell}(q^n + q^{n-1}(x - m_{\ell})) = xq^n(q^{n-1} + x\theta_{n-2})$$

which simplifies to

$$\sum_{\ell \in \mathsf{L}(P_0)} m_{\ell}^2 = x(\theta_{n-1} - 1 + x),$$

and the lemma follows.

In the following lemma, using Lemma 2.3, we obtain another result for the left-hand side of Eq. (2.2). Let $c \in \{0, 1\}$ be defined such that $n \equiv c \pmod{2}$.

Lemma 2.6. For any generator G of Q on P_0 with $w := |G \cap \mathcal{T}|$, the following equality holds:

$$\sum_{\ell \in \mathsf{L}(P_0)} m_{\ell}^2 \equiv x(\theta_{n-1} - c) + (-1)^c \cdot 2w(x - w) + cx^2 \mod 2(q+1).$$

Proof. Recall that P_0^{\perp}/P_0 is a hyperbolic quadric of rank n, whose points are the lines of $L(P_0)$. Let μ be a function defined on $L(P_0)$ such that μ satisfies the condition of Lemma 2.3. By Lemma 2.3, for an arbitrary line $\ell_0 \in L(P_0)$ and the set $P(\ell_0)$ of planes of Q on the line ℓ_0 , we obtain that

$$\sum_{\ell \in \mathsf{L}(P_0)} \mu(\ell)^2 = \mu(\ell_0)^2 + \left(x - \mu(\ell_0)\right)^2 + (q+1) \sum_{\ell_1 \in \mathsf{L}(P_0) \cap (\ell_0^{\perp} \setminus \{\ell_0\})} \mu(\ell_1)^2 - \sum_{\pi \in \mathsf{P}(\ell_0)} \left(\sum_{\ell' \in \mathsf{L}(P_0) \cap (\pi \setminus \{\ell_0\})} \mu(\ell')\right)^2,$$

which is congruent modulo 2(q+1) to

$$\mu(\ell_0)^2 + (x - \mu(\ell_0))^2 + (q + 1)((q^{n-1} - 1)\mu(\ell_0) + x\theta_{n-2}) - \sum_{\pi \in \mathsf{P}(\ell_0)} \left(\sum_{\ell' \in \mathsf{L}(P_0) \cap (\pi \setminus \{\ell_0\})} \mu(\ell')\right)^2, \quad (2.3)$$

as modulo 2 we have $m^2 \equiv m$ for every integer m and $\sum_{\ell_1 \in \mathsf{L}(P_0) \cap (\ell_0^{\perp} \setminus \{\ell_0\})} \mu(\ell_1) = (q^{n-1} - 1)\mu(\ell_0) + x\theta_{n-2}$ by Property (*).

Further, consider a hyperbolic quadric ℓ_0^{\perp}/ℓ_0 , which has rank n-1 and whose points are the planes of $\mathsf{P}(\ell_0)$, and, as in Lemma 2.4(2), define a function $\tilde{\mu}$ by

$$\widetilde{\mu}(\pi) := \sum_{\ell' \in \mathsf{L}(P_0) \cap (\pi \setminus \{\ell_0\})} \mu(\ell')$$

on the set $P(\ell_0)$. By Lemma 2.4, $\tilde{\mu}$ satisfies the condition of Lemma 2.3 for the hyperbolic quadric ℓ_0^{\perp}/ℓ_0 , and therefore the last double sum in Eq. (2.3) can be evaluated by induction on n.

We now define μ by $\mu(\ell) := m_{\ell}$ for a line $\ell \in L(P_0)$. By Result 1.3, μ satisfies the condition of Lemma 2.3. For n + 1 = 3, it follows from the proof of [16, Theorem 3.1] that

$$\sum_{\ell \in \mathsf{L}(P_0)} \mu(\ell)^2 \equiv x(q+1) + 2w(x-w) \mod 2(q+1),$$

where $w = \sum_{\ell \in \mathsf{L}(P_0) \cap \pi} \mu(\ell)$, i.e., the number of points of \mathcal{T} in any plane π on P_0 .

Therefore, for n + 1 = 4, from Eq. (2.3) we obtain that

$$\sum_{\ell \in \mathsf{L}(P_0)} m_{\ell}^2 \equiv m_{\ell_0}^2 + (x - m_{\ell_0})^2 + (q + 1)((q^2 - 1)m_{\ell_0} + x\theta_1) - \sum_{\pi \in \mathsf{P}(\ell_0)} \widetilde{\mu}(\pi)^2$$
$$\equiv m_{\ell_0}^2 + (x - m_{\ell_0})^2 + (q + 1)((q^2 - 1)m_{\ell_0} + x\theta_1) - (\widetilde{x}(q + 1) + 2\widetilde{w}(\widetilde{x} - \widetilde{w})) \mod 2(q + 1)$$

where $\widetilde{w} := \sum_{\pi \in \mathsf{P}(\ell_0) \cap G} \widetilde{\mu}(\pi) = |(G \setminus \{\ell_0\}) \cap \mathcal{T}|$ so that $w = \widetilde{w} + m_{\ell_0}$ for any 3-dimensional space G of Q on ℓ_0 , and this simplifies to

$$\sum_{\ell \in \mathsf{L}(P_0)} m_{\ell}^2 \equiv x(q^2 + q + x) - 2w(x - w) \mod 2(q + 1).$$

Arguing in the same manner for n + 1 > 4 by induction, a routine (but tedious) check shows that

$$\sum_{\ell \in \mathsf{L}(P_0)} m_{\ell}^2 \equiv x(\theta_{n-1} - 1 + x) - 2w(x - w) \mod 2(q+1).$$

if n is odd, and

$$\sum_{\ell \in \mathsf{L}(P_0)} m_{\ell}^2 \equiv x \theta_{n-1} + 2w(x-w) \mod 2(q+1).$$

if n is even, which shows the lemma.

We are now in a position to prove our main result. By Lemmas 2.5 and 2.6, we obtain that, if n is odd then

$$\sum_{\ell \in \mathsf{L}(P_0)} m_\ell^2 \equiv x(\theta_{n-1} - 1 + x)$$
$$\equiv x(\theta_{n-1} - 1 + x) - 2w(x - w) \mod 2(q+1),$$

i.e.,

$$w(w-x) \equiv 0 \mod (q+1),$$

and if n is even then

$$\begin{split} \sum_{\ell \in \mathsf{L}(P_0)} m_{\ell}^2 &\equiv x(\theta_{n-1} - 1 + x) \\ &\equiv x\theta_{n-1} + 2w(x - w) \mod 2(q+1), \end{split}$$

i.e.,

$$\binom{x}{2} + w(w - x) \equiv 0 \mod (q+1),$$

which completes the proof of Theorem 1.2.

Acknowledgements

The author is supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (grant number NRF-2018R1D1A1B07047427).

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