A chain theorem for sequentially 3-rank-connected graphs with respect to vertex-minors

Duksang Lee^{*2,1,3} and Sang-il Oum^{$\dagger1,2$}

¹Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, South Korea ²Department of Mathematical Sciences, KAIST, Daejeon, South Korea ³Department of Industrial and Systems Engineering, KAIST, Daejeon, South Korea Email: duksang@kaist.ac.kr, sangil@ibs.re.kr

May 22, 2023

Abstract

Tutte (1961) proved the chain theorem for simple 3-connected graphs with respect to minors, which states that every simple 3-connected graph G has a simple 3-connected minor with one edge fewer than G, unless G is a wheel graph. Bouchet (1987) proved an analog for prime graphs with respect to vertex-minors. We present a chain theorem for higher connectivity with respect to vertex-minors, showing that every sequentially 3-rank-connected graph G has a sequentially 3-rank-connected vertex-minor with one vertex fewer than G, unless $|V(G)| \leq 12$.

1 Introduction

Tutte [11] proved the chain theorem for simple 3-connected graphs with respect to minors, which states that every simple 3-connected graph G has a simple 3-connected minor with one edge fewer than G, unless G is a wheel graph. We will present a chain theorem for vertex-minors.

For a vertex v of a graph G, the *local complementation* at v is an operation obtaining a new graph G * v from G by replacing the subgraph induced by the neighbors of v with its complement graph. A graph H is a *vertex-minor* of G if H can be obtained from G by a sequence of local complementations and vertex deletions.

For a graph G, the *cut-rank* function ρ_G is a function which maps a set X of vertices of G to the rank of a matrix over the binary field whose rows are labeled by X and columns are labeled by V(G) - X, where the (i, j)-entry is 1 if i and j are adjacent in G and 0 otherwise. A graph G is *prime* if there is no set X of vertices of G such that $|X| \ge 2$, $|V(G) - X| \ge 2$, and $\rho_G(X) \le 1$. Bouchet proved the following chain theorem for prime graphs with respect to vertex-minors. Later, Allys [1] proved a stronger theorem.

Theorem 1.1 (Bouchet [2, Theorem 3.2]). Every prime graph G has a prime vertex-minor H with |V(H)| = |V(G)| - 1, unless $|V(G)| \le 5$.

A set X of vertices of G is sequential in G if there is an ordering a_1, \ldots, a_k of the vertices in X such that $\rho_G(\{a_1, \ldots, a_i\}) \leq 2$ for each $1 \leq i \leq k$. A graph G is sequentially 3-rank-connected if it is prime and whenever $\rho_G(X) \leq 2$ for $X \subseteq V(G)$, either X or V(G) - X is sequential in G.

^{*}Supported by the Institute for Basic Science (IBS-R029-C1), the National Research Foundation of Korea(NRF) grant funded by the Korea government (NRF-2022M3J6A1063021), and the KAIST Starting Fund (KAIST-G04220016).

[†]Supported by the Institute for Basic Science (IBS-R029-C1).

Here is our chain theorem for sequentially 3-rank-connected graphs with respect to vertexminors.

Theorem 1.2. Every sequentially 3-rank-connected graph G has a sequentially 3-rank-connected vertex-minor H with |V(H)| = |V(G)| - 1, unless $|V(G)| \le 12$.

Our theorem is motivated by the following theorem for sequentially 4-connected matroids, proved by Geelen and Whittle.

Theorem 1.3 (Geelen and Whittle [5, Theorem 1.2]). Every sequentially 4-connected matroid M has a sequentially 4-connected minor N with |E(N)| = |E(M)| - 1, unless M is a wheel matroid or a whirl matroid.

Theorem 1.3 was motivated by the conjecture on the number of inequivalent representations over a fixed prime field. This conjecture was later proved by Geelen and Whittle [6] by using a stronger version of Theorem 1.3 due to Oxley, Semple, and Whittle [9]. It would be interesting to see if this stronger version also has a vertex-minor analog.

Let us briefly sketch the proof of Theorem 1.2. The proof consists of three parts. In the first part, we prove it for 3-rank-connected graphs that are prime graphs with no set X such that $\rho_G(X) \leq 2$, |X| > 2, and |V(G) - X| > 2. The second part discusses internally 3-rank-connected graphs that are not 3-rank-connected. The last part considers sequentially 3-rank-connected graphs that are not internally 3-rank-connected.

Essentially, the proof is based on the submodularity of the matrix rank function. We will also use Theorem 1.1. Proof ideas of some lemmas are from Geelen and Whittle [5]. We will also use triplets introduced by Oum [8].

Our paper is organized as follows. In Section 2, we review vertex-minors and several inequalities for cut-rank functions. In Section 3, we prove elementary lemmas on sequential sets and sequentially 3-rank-connected graphs. In Section 4, we prove the main theorem for 3-rankconnected graphs. In Section 5, we prove our theorem for internally 3-rank-connected graphs. In Section 6, we conclude the proof by dealing with sequentially 3-rank-connected graphs which are not internally 3-rank-connected.

2 Preliminaries

A graph is simple if it has no loops and parallel edges. In this paper, all graphs are finite and simple. For a graph G and a vertex v, let $N_G(v)$ be the set of vertices adjacent to v in G. For a graph G and a subset X of V(G), let G[X] be the subgraph of G induced on X.

Vertex-minors For a graph G and a vertex v of G, let G * v be the graph obtained by replacing $G[N_G(v)]$ with its complement. The operation obtaining G * v from G is called the *local complementation* at v. A graph H is *locally equivalent* to G if H can be obtained from G by a sequence of local complementations. A graph H is a *vertex-minor* of a graph G if H can be obtained from G by applying local complementations and deleting vertices.

For an edge uv of a graph G, let $G \wedge uv = G * u * v * u$. Then $G \wedge uv$ is obtained from G by pivoting uv. The graph $G \wedge uv$ is well defined since G * u * v * u = G * v * u * v [7, Corollary 2.2].

Lemma 2.1 (see Oum [7]). Let G be a graph and v be a vertex of G. If $x, y \in N_G(v)$, then $(G \wedge vx) \setminus v$ is locally equivalent to $(G \wedge vy) \setminus v$.

By Lemma 2.1, we write G/v to denote $G \wedge uv \setminus v$ for a neighbor u of v in G because we are only interested in graphs up to local equivalence.

Lemma 2.2 (Geelen and Oum [4, Lemma 3.1]). Let G be a graph and v and w be vertices of G. Then the following hold.

- (1) If $v \neq w$ and $vw \notin E(G)$, then $(G * w) \setminus v$, $(G * w * v) \setminus v$, and (G * w)/v are locally equivalent to $G \setminus v$, $G * v \setminus v$, and G/v respectively.
- (2) If $v \neq w$ and $vw \in E(G)$, then $(G * w) \setminus v$, $(G * w * v) \setminus v$, and (G * w)/v are locally equivalent to $G \setminus v$, G/v, and $(G * v) \setminus v$ respectively.
- (3) If v = w, then $(G * w) \setminus v$, $(G * w * v) \setminus v$, and (G * w)/v are locally equivalent to $G * v \setminus v$, $G \setminus v$, and G/v respectively.

Lemma 2.2 implies the following lemma, which was first proved by Bouchet.

Lemma 2.3 (Bouchet [3, Corollary 9.2]). Let H be a vertex-minor of a graph G such that $V(H) = V(G) - \{v\}$ for a vertex v of G. Then H is locally equivalent to one of $G \setminus v$, $G * v \setminus v$, and G/v.

Cut-rank function and rank-connectivity For an $X \times Y$ -matrix A and $I \subseteq X$, $J \subseteq Y$, let A[I, J] be an $I \times J$ -submatrix of A. Let A_G be the adjacency matrix of a graph G over the binary field GF(2). The *cut-rank* $\rho_G(X)$ of a subset X of V(G) is defined by

$$\rho_G(X) = \operatorname{rank}(A_G[X, V(G) - X]).$$

It is trivial to check that $\rho_G(X) = \rho_G(V(G) - X)$. For disjoint sets X, Y of a graph G, let $\rho_G(X, Y) = \operatorname{rank}(A_G[X, Y])$. A graph G is k-rank-connected if there is no partition (A, B) of V(G) such that $|A|, |B| > \rho_G(A)$ and $\rho_G(A) < k$. A graph is prime if it is 2-rank-connected. Observe that 1-rank-connected graphs are connected graphs.

Lemma 2.4. If G is a 3-rank-connected graph with at least 6 vertices, then $\deg_G(v) \ge 3$ for each $v \in V(G)$.

Proof. Suppose that $\deg_G(v) \leq 2$. Let X be the set of neighbors of v. Then $\rho_G(X \cup \{v\}) \leq |X| \leq 2$. However, $\rho_G(X \cup \{v\}) < |X \cup \{v\}|$ and $2 < |V(G) - (X \cup \{v\})|$, contradicting assumption that G is 3-rank-connected.

Lemma 2.5 (Oum [8, Proposition 2.4]). Let k be a positive integer. If a graph G is k-rank-connected and $|V(G)| \ge 2k$, then for each $v \in V(G)$, the graph $G \setminus v$ is (k-1)-rank-connected.

Lemma 2.6. Let k be a positive integer. A k-rank-connected graph with $|V(G)| \ge 2k$ is k-connected.

Proof. We use induction on k. Let G be a k-rank-connected graph with $|V(G)| \ge 2k$. We may assume that k > 1. Let X be a subset of V(G) with |X| < k. It is enough to prove that $G \setminus X$ is connected. Since G is 1-rank-connected, G is connected and therefore we may assume that X is nonempty. Let v be a vertex in X. By applying Lemma 2.5 and the induction hypothesis, $G \setminus v$ is (k-1)-connected and therefore $(G \setminus v) \setminus (X - \{v\}) = G \setminus X$ is connected. \Box

The following lemmas give properties of the matrix rank function and the cut-rank function.

Lemma 2.7 (see Oum [7, Proposition 2.6]). If a graph G' is locally equivalent to a graph G, then $\rho_G(X) = \rho_{G'}(X)$ for each $X \subseteq V(G)$.

Lemma 2.8. Let G be a graph and v be a vertex of G. For a subset X of $V(G) - \{v\}$, we have

(i)
$$\rho_{G\setminus v}(X) + 1 \ge \rho_G(X) \ge \rho_{G\setminus v}(X)$$
.

(ii) $\rho_{G\setminus v}(X) + 1 \ge \rho_G(X \cup \{v\}) \ge \rho_{G\setminus v}(X).$

Proof. Observe that removing a row or a column of a matrix decreases the rank by at most 1. \Box

Lemma 2.9 (see Truemper [10]). Let A be an $X \times Y$ -matrix. For sets $X_1, X_2 \subseteq X$ and $Y_1, Y_2 \subseteq Y$,

 $\operatorname{rank}(A[X_1, Y_1]) + \operatorname{rank}(A[X_2, Y_2]) \ge \operatorname{rank}(A[X_1 \cap X_2, Y_1 \cup Y_2]) + \operatorname{rank}(A[X_1 \cup X_2, Y_1 \cap Y_2]).$

Lemma 2.9 implies the following seven lemmas.

Lemma 2.10 (see Oum [7, Corollary 4.2]). Let G be a graph and let X, Y be subsets of V(G). Then,

$$\rho_G(X) + \rho_G(Y) \ge \rho_G(X \cap Y) + \rho_G(X \cup Y).$$

Lemma 2.11. Let G be a graph and X and Y be subsets of V(G). Then,

 $\rho_G(X) + \rho_G(Y) \ge \rho_G(Y - X) + \rho_G(X - Y).$

Proof. Apply Lemma 2.10 with X and V(G) - Y.

Lemma 2.12 (Oum [8, Lemma 2.3]). Let G be a graph and v be a vertex of G. Let X and Y be subsets of $V(G) - \{v\}$. Then, the following hold.

- (S1) $\rho_{G\setminus v}(X) + \rho_G(Y \cup \{v\}) \ge \rho_{G\setminus v}(X \cap Y) + \rho_G(X \cup Y \cup \{v\}).$
- (S2) $\rho_{G\setminus v}(X) + \rho_G(Y) \ge \rho_G(X \cap Y) + \rho_{G\setminus v}(X \cup Y).$

Lemma 2.13. Let G be a graph and v be a vertex of G. Let X, Y be subsets of $V(G \setminus v)$. If $X \subseteq Y$ and $\rho_{G \setminus v}(Y) \ge \rho_G(Y)$, then $\rho_{G \setminus v}(X) = \rho_G(X)$.

Proof. By (S2) of Lemma 2.12,

$$\rho_{G\setminus v}(X) + \rho_G(Y) \ge \rho_{G\setminus v}(Y) + \rho_G(X).$$

Therefore, by Lemma 2.8(i), $0 \le \rho_G(X) - \rho_{G\setminus v}(X) \le \rho_G(Y) - \rho_{G\setminus v}(Y) \le 0$. So we conclude that $\rho_{G\setminus v}(X) = \rho_G(X)$.

Lemma 2.14. Let G be a graph and v be a vertex of G. Let X, Y be subsets of V(G). If $v \in Y \subseteq X$ and $\rho_{G\setminus v}(Y - \{v\}) \ge \rho_G(Y)$, then $\rho_{G\setminus v}(X - \{v\}) = \rho_G(X)$.

Proof. We apply Lemma 2.13 for V(G) - X and V(G) - Y.

Lemma 2.15. Let G be a graph and v be a vertex of G. Let X and Y be subsets of $V(G) - \{v\}$. Then,

$$\rho_{G\setminus v}(X) + \rho_G(Y \cup \{v\}) \ge \rho_{G\setminus v}(Y - X) + \rho_G(X - Y).$$

Proof. Apply (S1) of Lemma 2.12 with $V(G) - (X \cup \{v\})$ and Y.

Lemma 2.16 (Oum [8, Lemma 2.2]). Let G be a graph and a, b be distinct vertices of G. Let $A \subseteq V(G) - \{a\}$ and $B \subseteq V(G) - \{b\}$. Then, the following hold.

(A1) If $b \notin A$ and $a \notin B$, then $\rho_G(A \cap B) + \rho_{G \setminus a \setminus b}(A \cup B) \leq \rho_{G \setminus a}(A) + \rho_{G \setminus b}(B)$.

(A2) If $b \in A$ and $a \notin B$, then $\rho_{G \setminus b}(A \cap B) + \rho_{G \setminus a}(A \cup B) \leq \rho_{G \setminus a}(A) + \rho_{G \setminus b}(B)$.

(A3) If $b \in A$ and $a \in B$, then $\rho_{G \setminus a \setminus b}(A \cap B) + \rho_G(A \cup B) \le \rho_{G \setminus a}(A) + \rho_{G \setminus b}(B)$.

Lemma 2.17 (Oum [7, Proposition 4.3]). Let G be a graph and x be a vertex of G. For a subset X of $V(G) - \{x\}$, the following hold.

(1)
$$\rho_{G*x\setminus x}(X) = \operatorname{rank} \begin{pmatrix} 1 & A_G[\{x\}, V(G) - (X \cup \{x\})] \\ A_G[X, \{x\}] & A_G[X, V(G) - (X \cup \{x\})] \end{pmatrix} - 1.$$

(2)
$$\rho_{G/x}(X) = \operatorname{rank} \begin{pmatrix} 0 & A_G[\{x\}, V(G) - (X \cup \{x\})] \\ A_G[X, \{x\}] & A_G[X, V(G) - (X \cup \{x\})] \end{pmatrix} - 1.$$

From Lemma 2.17, we deduce the following lemma.

Lemma 2.18. Let G be a graph and $x \in V(G)$. Let C be a subset of $V(G) - \{x\}$ such that $\rho_{G\setminus x}(C) = \rho_G(C)$. Then $\rho_{G*x\setminus x}(C) = \rho_G(C\cup\{x\}) - 1$ or $\rho_{G/x}(C) = \rho_G(C\cup\{x\}) - 1$.

Proof. Let $D = V(G) - (C \cup \{x\})$. Since $\rho_{G\setminus x}(C) = \rho_G(C)$, a column vector $A_G[C, \{x\}]$ is in the column space of $A_G[C, D]$. Then let A' and A'' be matrices over GF(2) such that

$$A' = \begin{pmatrix} 1 & A_G[\{x\}, D] \\ A_G[C, \{x\}] & A_G[C, D] \end{pmatrix} \text{ and } A'' = \begin{pmatrix} 0 & A_G[\{x\}, D] \\ A_G[C, \{x\}] & A_G[C, D] \end{pmatrix}.$$

Then $\operatorname{rank}(A') = \rho_G(C \cup \{x\})$ or $\operatorname{rank}(A'') = \rho_G(C \cup \{x\})$ and therefore, by Lemma 2.17, we have $\rho_{G*x \setminus x}(C) = \operatorname{rank}(A') - 1 = \rho_G(C \cup \{x\}) - 1$ or $\rho_{G/x}(C) = \operatorname{rank}(A'') - 1 = \rho_G(C \cup \{x\}) - 1$. \Box

Lemma 2.19 (Oum [7, Lemma 4.4]). Let G be a graph and x be a vertex of G. Let (X_1, Y_1) and (X_2, Y_2) be partitions of $V(G) - \{x\}$. Then the following hold:

- (P1) $\rho_{G\setminus x}(X_1) + \rho_{G*x\setminus x}(X_2) \ge \rho_G(X_1 \cap X_2) + \rho_G(Y_1 \cap Y_2) 1.$
- (P2) $\rho_{G\setminus x}(X_1) + \rho_{G/x}(X_2) \ge \rho_G(X_1 \cap X_2) + \rho_G(Y_1 \cap Y_2) 1.$

The following lemma is an easy consequence of Lemmas 2.7 and 2.19.

Lemma 2.20. Let G be a graph and x be a vertex of G. Let (X_1, Y_1) and (X_2, Y_2) be partitions of $V(G) - \{x\}$. Then,

$$\rho_{G*x\setminus x}(X_1) + \rho_{G/x}(X_2) \ge \rho_G(X_1 \cap X_2) + \rho_G(Y_1 \cap Y_2) - 1.$$

3 Sequentially 3-rank-connected graphs

Let us recall the definition of sequentially 3-rank-connected graphs introduced in Section 1. A subset A of V(G) is sequential in a graph G if there is an ordering $a_1, \ldots, a_{|A|}$ of the elements of A such that $\rho_G(\{a_1, \ldots, a_i\}) \leq 2$ for each $1 \leq i \leq |A|$. A graph G is sequentially 3-rank-connected if it is prime and for each subset X of V(G) with $\rho_G(X) \leq 2$, we have that X or V(G) - X is sequential in G.

We now present basic lemmas on sequential sets and sequentially 3-rank-connected graphs.

Lemma 3.1. Let G be a graph and A be a subset of V(G). Let t be a vertex of G such that $\rho_G(A \cup \{t\}) = \rho_G(A)$. Then $A \cup \{t\}$ is sequential in G if and only if A is sequential in G.

Proof. We may assume that $t \notin A$. The backward direction is obvious. So it is enough to show the forward direction.

Since $A \cup \{t\}$ is sequential in G, there is an ordering a_1, \ldots, a_m of the elements of $A \cup \{t\}$ such that $m = |A \cup \{t\}|$ and $\rho_G(\{a_1, \ldots, a_i\}) \leq 2$ for each $1 \leq i \leq m$. Let $1 \leq j \leq m$ be an index such that $a_j = t$. Then for each $j + 1 \leq i \leq m$, by Lemma 2.10, we have

$$\rho_G(\{a_1,\ldots,a_i\}) + \rho_G(A) \ge \rho_G(A \cup \{t\}) + \rho_G(\{a_1,\ldots,a_i\} - \{t\}),$$

and therefore $\rho_G(\{a_1, ..., a_i\} - \{t\}) \le \rho_G(\{a_1, ..., a_i\})$. For each $1 \le i \le m - 1$, let

$$a'_{i} = \begin{cases} a_{i} & \text{if } i < j, \\ a_{i+1} & \text{if } i \ge j. \end{cases}$$

Hence, by above inequality, A is sequential in G because a'_1, \ldots, a'_{m-1} is a desired ordering of the elements of A.

Lemma 3.2. Let G be a prime graph that is not sequentially 3-rank-connected and let T_1, \ldots, T_n be pairwise disjoint 3-element subsets of V(G) such that $\rho_G(T_i) = 2$ for each $1 \le i \le n$. Then there exists a subset A of V(G) such that $\rho_G(A) \le 2$, neither A nor V(G) - A is sequential in G, and for each $1 \le i \le n$, we have that $T_i \subseteq A$ or $T_i \subseteq V(G) - A$.

Proof. We proceed by induction on n. Since G is prime and not sequentially 3-rank-connected, there is a subset A of V(G) such that $\rho_G(A) \leq 2$, and neither A nor V(G) - A is sequential in G. So we can assume that $n \geq 1$. By the induction hypothesis, there is a subset A' of V(G) such that $\rho_G(A') \leq 2$, and neither A' nor V(G) - A' is sequential in G, and for each $1 \leq i \leq n - 1$, either $T_i \subseteq A'$ or $T_i \subseteq V(G) - A'$. Let B' = V(G) - A'. We may assume that $A' \cap T_n \neq \emptyset$ and $B' \cap T_n \neq \emptyset$. Then, by symmetry, we can assume that $|A' \cap T_n| = 2$ and let x be the element of $B' \cap T_n$. Since $|T_n - \{x\}| = 2$ and G is prime, we have $\rho_G(T_n - \{x\}) = 2 = \rho_G(T_n)$. Then, by Lemma 2.10,

$$\rho_G(A') + 2 = \rho_G(A') + \rho_G(T_n) \ge \rho_G(A' \cup \{x\}) + \rho_G(T_n - \{x\}) = \rho_G(A' \cup \{x\}) + 2.$$

Hence $\rho_G(A' \cup \{x\}) \leq \rho_G(A') \leq 2$. Since V(G) - A' is not sequential in G, $|V(G) - A'| \geq 4$ and so $|V(G) - (A' \cup \{x\})| \geq 3$. Hence $\rho_G(A') = \rho_G(A' \cup \{x\}) = 2$ because G is prime. Hence, by Lemma 3.1, neither $A' \cup \{x\}$ nor $V(G) - (A' \cup \{x\})$ is sequential in G.

For each $1 \leq i \leq n-1$, we have $x \notin T_i$ because T_n and T_i are disjoint. Therefore, $T_i \subseteq A' \cup \{x\}$ or $T_i \subseteq V(G) - (A' \cup \{x\})$ for each $1 \leq i \leq n$.

4 Treating 3-rank-connected graphs

In this section, we prove Theorem 1.2 for 3-rank-connected graphs.

The following lemma shows that every vertex-minor of a 3-rank-connected graph G with one vertex fewer than G is prime.

Lemma 4.1. Let G be a 3-rank-connected graph with $|V(G)| \ge 6$ and x be a vertex of G. Then all of $G \setminus x$, $G * x \setminus x$, and G/x are prime.

Proof. By Lemma 2.7, it is enough to show that $G \setminus x$ is prime. This is implied by Lemma 2.5. \Box

A graph G is weakly 3-rank-connected if G is prime and V(G) has no subset X such that $|X| \ge 5$, $|V(G) - X| \ge 5$, and $\rho_G(X) \le 2$. The following lemma can be deduced easily from [8, Proposition 2.6] and Lemma 2.2.

Lemma 4.2 (Oum [8]). Let G be a 3-rank-connected graph with $|V(G)| \ge 6$ and x be a vertex of G. Then at least two of $G \setminus x$, $G * x \setminus x$, and G/x are weakly 3-rank-connected.

Lemma 4.3. Let G be a 3-rank-connected graph with $|V(G)| \ge 6$ and let $S = \{v_1, \dots, v_t\}$ be the set of all vertices x of G such that $G \setminus x$ is not weakly 3-rank-connected. Let $G' = G * v_1 * \dots * v_t$. Then $G' \setminus v$ is weakly 3-rank-connected for every vertex v of G'.

Proof. If $v \notin S$, then $G' \setminus v = (G \setminus v) * v_1 * \cdots * v_t$ and so $G' \setminus v$ is weakly 3-rank-connected. If $v = v_i$ for some $1 \leq i \leq t$, then by Lemma 4.2, $G * v \setminus v$ is weakly 3-rank-connected. Since $G' \setminus v = (G * v \setminus v) * v_1 * \cdots * v_{i-1} * v_{i+1} * \cdots * v_t$ is locally equivalent to $G * v \setminus v$, we deduce that $G' \setminus v$ is weakly 3-rank-connected.

Lemma 4.4. Let G be a 3-rank-connected graph and x be a vertex of G. Let P be a 4-element subset of $V(G) - \{x\}$ such that $\rho_{G\setminus x}(P) \leq 2$ and (A, B) be a partition of $V(G) - \{x\}$ such that $|A|, |B| \geq 4$ and $\rho_H(A) \leq 2$ for some $H \in \{G * x \setminus x, G/x\}$. Then $|A \cap P| = |B \cap P| = 2$.

Proof. Suppose that $|A \cap P| \neq |B \cap P|$. We may assume that $|A \cap P| > |B \cap P|$. Since $\rho_{G\setminus x}(P) \leq 2$ and $\rho_H(A) \leq 2$, by (P1) and (P2) of Lemma 2.19, we have

$$4 \ge \rho_{G\setminus x}(P) + \rho_H(A) \ge \rho_G(A \cap P) + \rho_G(B - P) - 1.$$

Since $|A \cap P| > 2$ and G is 3-rank-connected, $\rho_G(A \cap P) > 2$. Hence $\rho_G(B - P) \le 2$. Since G is 3-rank-connected, $|B - P| \le 2$, which implies that $|B \cap P| \ge 2$, contradicting the fact that |P| = 4.

A 4-element subset P of V(G) is a quad of G if $\rho_G(P) = 2$ and $\rho_G(P - \{x\}) = 3$ for each $x \in P$.

Lemma 4.5. Let G be a prime graph and A be a subset of V(G) such that $\rho_G(A) = 2$ and $|A| \leq 4$. Then A is a quad of G or A is sequential in G.

Proof. Suppose that A is not sequential in G. Then |A| = 4 and $\rho_G(T) = 3$ for each 3-element subset T of A. Therefore, A is a quad of G.

Our key ingredient of this section is Proposition 4.6, which states that it is sufficient to identify a set $\{t_1, t_2, t_3\}$ of three vertices and a quad Q_i from $G \setminus t_i$ for each $i \in \{1, 2, 3\}$ that satisfy the following conditions:

- (1) $G \setminus t_i$ is weakly 3-rank-connected for each $i \in \{1, 2, 3\}$.
- (2) $Q_1 \cap Q_2 = \{t_3\}, Q_2 \cap Q_3 = \{t_1\}, \text{ and } Q_3 \cap Q_1 = \{t_2\}.$

The remainder of this section will focus on identifying these three vertices and quads.

Proposition 4.6. Let t_1 , t_2 , and t_3 be distinct vertices of a 3-rank-connected graph G such that $G \setminus t_1$, $G \setminus t_2$, and $G \setminus t_3$ are weakly 3-rank-connected. For each $i \in \{1, 2, 3\}$, let Q_i be a quad of $G \setminus t_i$. If $Q_1 \cap Q_2 = \{t_3\}$, $Q_2 \cap Q_3 = \{t_1\}$, and $Q_3 \cap Q_1 = \{t_2\}$, then for each $i \in \{1, 2, 3\}$, either $G * t_i \setminus t_i$ or G/t_i is sequentially 3-rank-connected.

Proof. Since $|V(G)| \ge |Q_1 \cup Q_2| = 7$, by Lemma 4.1, all of $G \setminus v$, $G * v \setminus v$, and G/v are prime for each vertex v of G. Observe that $\{t_2, t_3\} \subseteq Q_1$, $\{t_1, t_3\} \subseteq Q_2$, and $\{t_1, t_2\} \subseteq Q_3$. For each $i \in \{1, 2, 3\}$, let a_i and b_i be two distinct vertices of $Q_i - \{t_1, t_2, t_3\}$.

Suppose that neither $G * t_1 \setminus t_1$ nor G/t_1 is sequentially 3-rank-connected. Let us first show that $\rho_{G \setminus t_1}(\{t_2, a_3, b_3\}) = 3$. Since $G \setminus t_1$ is prime, $\rho_{G \setminus t_1}(\{a_3, b_3\}) = 2 = \rho_{G \setminus t_1}(\{t_2, t_3, a_1, b_1\})$. By Lemma 2.11,

$$\rho_{G\setminus t_1}(\{t_2, a_3, b_3\}) + \rho_{G\setminus t_1}(\{t_2, t_3, a_1, b_1\}) \ge \rho_{G\setminus t_1}(\{a_3, b_3\}) + \rho_{G\setminus t_1}(\{t_3, a_1, b_1\}),$$

and therefore $\rho_{G\setminus t_1}(\{t_2, a_3, b_3\}) \ge \rho_{G\setminus t_1}(\{t_3, a_1, b_1\})$. Since $Q_1 = \{t_2, t_3, a_1, b_1\}$ is a quad of $G\setminus t_1$, $\rho_{G\setminus t_1}(\{t_3, a_1, b_1\}) = 3$. Therefore $\rho_{G\setminus t_1}(\{t_2, a_3, b_3\}) = 3$ and, by symmetry, $\rho_{G\setminus t_1}(\{t_3, a_2, b_2\}) = 3$.

Since $3 = \rho_{G\setminus t_1}(\{t_2, a_3, b_3\}) \leq \rho_G(\{t_2, a_3, b_3\}) \leq 3$, we have $\rho_G(\{t_2, a_3, b_3\}) = 3$. Since $Q_3 = \{t_1, t_2, a_3, b_3\}$ is a quad of $G \setminus t_3$ and G is 3-rank-connected, we observe that $3 \leq \rho_G(\{t_1, t_2, a_3, b_3\}) \leq 1 + \rho_{G\setminus t_3}(\{t_1, t_2, a_3, b_3\}) = 3$ and therefore $\rho_G(\{t_1, t_2, a_3, b_3\}) = 3$. Similarly, $\rho_G(\{t_3, a_2, b_2\}) = \rho_G(\{t_1, t_3, a_2, b_2\}) = 3$. Therefore, by Lemma 2.18, the following hold.

(R1)
$$\rho_{G*t_1 \setminus t_1}(\{t_2, a_3, b_3\}) = 2 \text{ or } \rho_{G/t_1}(\{t_2, a_3, b_3\}) = 2.$$

(R2) $\rho_{G*t_1 \setminus t_1}(\{t_3, a_2, b_2\}) = 2 \text{ or } \rho_{G/t_1}(\{t_3, a_2, b_2\}) = 2.$

Since G is 3-rank connected, $\rho_G(\{t_2, a_3, b_3\}), \rho_G(\{t_3, a_2, b_2\}) \ge 3$. So by Lemma 2.20,

$$\begin{split} \rho_{G*t_1 \setminus t_1}(\{t_2, a_3, b_3\}) + \rho_{G/t_1}(V(G \setminus t_1) - \{t_3, a_2, b_2\}) &= \rho_G(\{t_2, a_3, b_3\}) + \rho_G(\{t_3, a_2, b_2\}) - 1 \geq 5. \\ \text{Hence, } \rho_{G*t_1 \setminus t_1}(\{t_2, a_3, b_3\}) + \rho_{G/t_1}(\{t_3, a_2, b_2\}) \geq 5 \text{ and similarly,} \end{split}$$

$$\rho_{G*t_1 \setminus t_1}(\{t_3, a_2, b_2\}) + \rho_{G/t_1}(\{t_2, a_3, b_3\}) \ge 5.$$

Therefore, by (R1) and (R2), either

- (a) $\rho_{G*t_1 \setminus t_1}(\{t_2, a_3, b_3\}) = \rho_{G*t_1 \setminus t_1}(\{t_3, a_2, b_2\}) = 2$, or
- (b) $\rho_{G/t_1}(\{t_2, a_3, b_3\}) = \rho_{G/t_1}(\{t_3, a_2, b_2\}) = 2.$

By Lemma 2.2, we may assume (a), because otherwise we can choose a neighbor $y \notin \{t_2, t_3\}$ of t_1 in G by Lemma 2.4 and replace G by G*y. By Lemma 3.2, there is a subset A of $V(G*t_1 \setminus t_1)$ such that

- $\rho_{G*t_1 \setminus t_1}(A) \leq 2$,
- neither A nor $V(G * t_1 \setminus t_1) A$ is sequential in $G * t_1 \setminus t_1$,
- $\{t_2, a_3, b_3\} \subseteq A$ or $\{t_2, a_3, b_3\} \subseteq V(G * t_1 \setminus t_1) A$, and
- $\{t_3, a_2, b_2\} \subseteq A \text{ or } \{t_3, a_2, b_2\} \subseteq V(G * t_1 \setminus t_1) A.$

We may assume that $\{t_2, a_3, b_3\} \subseteq A$ by replacing A with $V(G * t_1 \setminus t_1) - A$ if necessary. Let $B = V(G * t_1 \setminus t_1) - A$.

Suppose that $\{t_3, a_2, b_2\} \subseteq A$. Observe that $\rho_G(A) \leq \rho_{G*t_1 \setminus t_1}(A) + 1 \leq 3$. Since $\{t_1, t_2, a_3, b_3\}$ is a quad of $G \setminus t_3$, by (S1) of Lemma 2.12,

$$\begin{split} 3+2 &\geq \rho_G(A) + \rho_{G\backslash t_3}(\{t_1, t_2, a_3, b_3\}) = \rho_G((A - \{t_3\}) \cup \{t_3\}) + \rho_{G\backslash t_3}(\{t_1, t_2, a_3, b_3\}) \\ &\geq \rho_{G\backslash t_3}((A - \{t_3\}) \cap \{t_1, t_2, a_3, b_3\}) + \rho_G((A - \{t_3\}) \cup \{t_1, t_2, t_3, a_3, b_3\}) \\ &= \rho_{G\backslash t_3}(\{t_2, a_3, b_3\}) + \rho_G(A \cup \{t_1\}) \geq 3 + \rho_G(A \cup \{t_1\}). \end{split}$$

Therefore $\rho_G(A \cup \{t_1\}) \leq 2$, contradicting our assumption that G is 3-rank-connected. So we deduce that $\{t_3, a_2, b_2\} \subseteq B$.

By Lemma 4.4, $|A \cap \{t_2, t_3, a_1, b_1\}| = |B \cap \{t_2, t_3, a_1, b_1\}| = 2$. So $|A \cap \{a_1, b_1\}| = |B \cap \{a_1, b_1\}| = 1$ and we can assume that $\{a_1, t_2, a_3, b_3\} \subseteq A$ and $\{b_1, t_3, a_2, b_2\} \subseteq B$ by swapping a_1 and b_1 if necessary.

If |A| = 4, then A is sequential in $G * t_1 \setminus t_1$ because $\rho_{G * t_1 \setminus t_1}(\{t_2, a_3, b_3\}) \leq 2$ and $\{t_2, a_3, b_3\} \subseteq A$, contradicting our assumption on A. Hence $|A| \geq 5$.

If |B| = 4, then B is sequential in $G * t_1 \setminus t_1$ because $\rho_{G * t_1 \setminus t_1}(\{t_3, a_2, b_2\}) \leq 2$ and $\{t_3, a_2, b_2\} \subseteq B$, contradicting our assumption on B. So $|B| \geq 5$ and $|V(G)| = |A| + |B| + 1 \geq 11$.

For each $k \in \{1, 2, 3\}$, let $P_k = Q_k \cup \{t_k\} = \{t_1, t_2, t_3, a_k, b_k\}$. Observe that $\rho_G(P_k) \leq \rho_{G\setminus t_k}(Q_k) + 1 \leq 3$ for each $1 \leq k \leq 3$. Since G is 3-rank-connected and $|P_1 \cap P_3| = 3$, we have $\rho_G(P_1 \cap P_3) \geq 3$. By Lemma 2.10,

$$6 \ge \rho_G(P_1) + \rho_G(P_3) \ge \rho_G(P_1 \cup P_3) + \rho_G(P_1 \cap P_3) \ge \rho_G(P_1 \cup P_3) + 3,$$

which implies that $\rho_G(P_1 \cup P_3) \leq 3$. Observe that $|V(G) - (A \cup (P_1 \cup P_3))| \geq |B - \{b_1, t_3\}| \geq 3$. Since G is 3-rank-connected, $\rho_G(A \cup (P_1 \cup P_3)) \geq 3$. By Lemma 2.10,

Therefore $\rho_G(\{a_1, t_2, a_3, b_3\}) = \rho_G(A \cap (P_1 \cup P_3)) \le 3$. Hence by Lemma 2.10,

$$\begin{aligned} 3+2 &\geq \rho_{G\backslash t_3}(\{a_1,t_2,a_3,b_3\}) + \rho_{G\backslash t_3}(\{t_1,t_2,a_3,b_3\}) \\ &\geq \rho_{G\backslash t_3}(\{a_1,t_1,t_2,a_3,b_3\}) + \rho_{G\backslash t_3}(\{t_2,a_3,b_3\}) = \rho_{G\backslash t_3}(\{a_1,t_1,t_2,a_3,b_3\}) + 3. \end{aligned}$$

Hence $\rho_{G\setminus t_3}(\{a_1, t_1, t_2, a_3, b_3\}) \leq 2$, contradicting our assumption that $G \setminus t_3$ is weakly 3-rank-connected.

An *independent* set of a graph is a set of pairwise nonadjacent vertices. For sets A and B, let $A \triangle B = (A - B) \cup (B - A)$.

Lemma 4.7. Let G be a 3-rank-connected graph with $|V(G)| \ge 6$ and x be a vertex of G such that $G \setminus x$ is weakly 3-rank-connected. Let P be a quad of $G \setminus x$. Then there is a graph G' locally equivalent to G such that the following hold.

- (1) $G' \setminus v$ is weakly 3-rank-connected for each vertex $v \in P \cup \{x\}$.
- (2) $N_{G'}(t) P \neq \emptyset$ for each $t \in P$.
- (3) P is a quad of $G' \setminus x$.

Proof. Let $P = \{p, q, r, s\}$. By Lemma 4.3, there is a graph locally equivalent to G satisfying (1) and (3). We may assume that among all graphs locally equivalent to G satisfying (1) and (3), G maximizes the number of edges between vertices in P.

We may assume that $N_G(p) \subseteq \{q, r, s\}$ because otherwise (1), (2), and (3) hold for G' = G. Since P is a quad of $G \setminus x$, we have $\rho_{G \setminus x}(P) = 2$, which implies that $|V(G \setminus x) - P| \ge 2$. So $|V(G)| \ge 7$. Since G is 3-rank-connected, by Lemma 2.4, we have $N_G(p) = \{q, r, s\}$.

Suppose that $\{q, r, s\}$ is independent in G. Since G is 3-rank-connected, by Lemma 2.6, G is 3-connected and so $G \setminus x \setminus p$ is connected. Let X be a shortest path joining two vertices of $\{q, r, s\}$ in $G \setminus x \setminus p$. By symmetry, we may assume that $X = qv_1 \cdots v_m r$ and $v_i \neq s$ for each $1 \leq i \leq m$. Since $\{q, r, s\}$ is independent in G, we deduce that $m \geq 1$ and $\{v_1, \ldots, v_m\} \subseteq V(G) - (P \cup \{x\})$. Then let $G' = G * v_1 * \cdots * v_m$. Then G' satisfies (1) and (3). Moreover, $N_{G'}(p) = \{q, r, s\}$ and $qr \in E(G')$. Hence |E(G'[P])| > |E(G[P])|, contradicting the choice of G. Therefore, $\{q, r, s\}$ is not independent in G.

Since G is 3-rank-connected, we have $3 \leq \rho_G(P) \leq \rho_{G\setminus x}(P) + 1 = 3$. Hence $\rho_G(P) = 3$ and so $N_G(q) - P$, $N_G(r) - P$, and $N_G(s) - P$ are nonempty, pairwise distinct, and $(N_G(s) - P) \triangle (N_G(q) - P) \triangle (N_G(r) - P) \neq \emptyset$.

If $G * q \setminus q$ is weakly 3-rank-connected, then let G' = G * q. Obviously, (1) and (3) hold. We have $N_{G'}(p) - P = N_G(q) - P = N_{G'}(q) - P \neq \emptyset$. For each vertex $v \in \{r, s\}$,

$$N_{G'}(v) - P = \begin{cases} N_G(v) - P \neq \emptyset & \text{if } v \text{ is not adjacent to } q \text{ in } G, \\ (N_G(q) - P) \triangle (N_G(v) - P) \neq \emptyset & \text{if } v \text{ is adjacent to } q \text{ in } G, \end{cases}$$

and therefore G' satisfies (2). So we can assume that none of $G*q \setminus q$, $G*r \setminus r$, and $G*s \setminus s$ is weakly 3-rank-connected. Then by Lemma 4.2, all of G/q, G/r, and G/s are weakly 3-rank-connected.

Since $\{q, r, s\}$ is not independent in G, by symmetry, we may assume that q and r are adjacent in G. Let $G' = G \land qr$. For each vertex $v \in P \cup \{x\}$, if $v \in \{p, s, x\}$, then $G' \setminus v = (G \setminus v) \land qr$ and if $v \in \{q, r\}$, then $G' \setminus v = G/v$, which implies that (1) and (3) hold. Then $N_{G'}(q) - P = N_G(r) - P$ and $N_{G'}(r) - P = N_G(q) - P$. Since $p \in N_G(q) \cap N_G(r)$ and $N_G(q) - P \neq N_G(r) - P$, we have $N_{G'}(p) - P = (N_G(q) - P) \triangle (N_G(r) - P) \neq \emptyset$. Furthermore,

$$N_{G'}(s) - P = \begin{cases} N_G(s) - P \neq \emptyset & \text{if } s \notin N_G(q) \cup N_G(r), \\ (N_G(s) - P) \triangle (N_G(q) - P) \neq \emptyset & \text{if } s \in N_G(r) - N_G(q), \\ (N_G(s) - P) \triangle (N_G(r) - P) \neq \emptyset & \text{if } s \in N_G(q) - N_G(r), \\ (N_G(s) - P) \triangle (N_G(q) - P) \triangle (N_G(r) - P) \neq \emptyset & \text{if } s \in N_G(q) \cap N_G(r). \end{cases}$$

Hence, (2) holds.

Lemma 4.8. Let G be a 3-rank-connected graph with $|V(G)| \ge 6$ and x be a vertex of G. Let P be a quad of $G \setminus x$ and t be a vertex in P. If $G \setminus t$ is weakly 3-rank-connected, then one of the following holds.

- (Q1) $G \setminus t$ is sequentially 3-rank-connected.
- (Q2) There is a subset X of $V(G \setminus t)$ such that $\rho_{G \setminus t}(X) \leq 2$, $X \cap P \neq \emptyset$, $(V(G \setminus t) X) \cap P \neq \emptyset$, and neither X nor $V(G \setminus t) - X$ is sequential in $G \setminus t$.
- (Q3) $\rho_{G\setminus t}(P \{t\}) = 2$ and $G \setminus t$ has a quad Y containing x such that $Y \cap P = \emptyset$.

Proof. Suppose that $G \setminus t$ is not sequentially 3-rank-connected. Then there is a subset X of $V(G \setminus t)$ such that $\rho_{G \setminus t}(X) \leq 2$ and neither X nor $V(G \setminus t) - X$ is sequential in $G \setminus t$. Let $Y = V(G \setminus t) - X$ and $(Z_1, Z_2) = (X - \{x\}, Y - \{x\})$. Since both X and Y are non-sequential in $G \setminus t$, we have $|X|, |Y| \geq 4$ and so $|Z_1|, |Z_2| \geq 3$. If $X \cap P \neq \emptyset$ and $Y \cap P \neq \emptyset$, then (Q2) holds. So by symmetry, we may assume that $P - \{t\} \subseteq X$. Then $P - \{t\} \subseteq Z_1$. Since P is a quad of $G \setminus x$, we know that $\rho_{G \setminus x}(P) = 2 = \rho_{G \setminus x}(P - \{t\}) - 1 \leq \rho_{G \setminus x \setminus t}(P - \{t\})$. Then by Lemma 2.14, $\rho_{G \setminus x}(Z_1 \cup \{t\}) = \rho_{G \setminus x \setminus t}(Z_1)$.

By Lemma 4.1, $G \setminus x$ is prime and so $2 \leq \rho_{G \setminus x}(Z_1 \cup \{t\}) = \rho_{G \setminus x \setminus t}(Z_1) \leq \rho_{G \setminus t}(X) \leq 2$, which implies that

$$\rho_{G\setminus x}(Z_1\cup\{t\})=\rho_{G\setminus x\setminus t}(Z_1)=2.$$

Since G is 3-rank-connected and $|V(G) - (Z_1 \cup \{x,t\})| \ge |Z_2| \ge 3$, we have $\rho_G(Z_1 \cup \{x,t\}) \ge 3$. So by Lemma (A3) of Lemma 2.16,

$$2 + \rho_{G\setminus t}(Z_1 \cup \{x\}) \ge \rho_{G\setminus x}(Z_1 \cup \{t\}) + \rho_{G\setminus t}(Z_1 \cup \{x\}) \ge \rho_G(Z_1 \cup \{x,t\}) + \rho_{G\setminus x\setminus t}(Z_1) \ge 3 + 2.$$

Hence $\rho_{G\setminus t}(Z_1 \cup \{x\}) > 2$ and $x \in Y$. So $(Z_1, Z_2) = (X, Y - \{x\})$ and $\rho_{G\setminus x}(X \cup \{t\}) = \rho_{G\setminus x}(Z_1 \cup \{t\}) = 2$. Since $t \in P$ and $x \notin Z_1$, by (A2) of Lemma 2.16,

$$2 + 2 \ge \rho_{G\setminus x}(P) + \rho_{G\setminus t}(Z_1) \ge \rho_{G\setminus x}(Z_1 \cup \{t\}) + \rho_{G\setminus t}(P - \{t\}) \ge 2 + \rho_{G\setminus t}(P - \{t\}).$$

Therefore, $\rho_{G\setminus t}(P - \{t\}) = 2$ because $G \setminus t$ is prime. Since X is non-sequential in $G \setminus t$ and $\rho_{G\setminus t}(P - \{t\}) \leq 2$, we have $|X| \geq 5$. Hence |Y| = 4 because $G \setminus t$ is weakly 3-rank-connected. Since Y is non-sequential in $G \setminus t$, by Lemma 4.5, $Y = Z_2 \cup \{x\}$ is a quad of $G \setminus t$. Hence (Q3) holds.

Lemma 4.9. Let G be a 3-rank-connected graph such that $|V(G)| \ge 12$ and x be a vertex of G. Let P be a quad of $G \setminus x$ and t be a vertex of P. Let (X,Y) be a partition of $V(G) - \{t\}$ such that $\rho_{G\setminus t}(X) \le 2$ and neither X nor Y is sequential in $G \setminus t$. If $G \setminus x$ and $G \setminus t$ are weakly 3-rank-connected and $|X \cap P| = 1$, then the following hold.

- (K1) $\rho_{G\setminus x\setminus t}(X-\{x\})=\rho_{G\setminus t}(X)=2.$
- (K2) X is a quad of $G \setminus t$ containing x.

Proof. Since neither X nor Y is sequential in $G \setminus t$, we have $|X|, |Y| \ge 4$ and so $|X - \{x\}|, |Y - \{x\}| \ge 3$. Clearly, $\rho_{G \setminus x \setminus t}(X - \{x\}) \le \rho_{G \setminus t}(X) \le 2$. Let q be the element of $X \cap P$ and r, s be the elements of $Y \cap P$. Let $C = X - \{q, x\}$ and $D = Y - \{r, s, x\}$. Then we have $|D| \ge 1$ because $|Y| \ge 4$.

Let us show that $\rho_{G\setminus x\setminus t}(C) \leq 2$. Since P is a quad of $G\setminus x$, by (ii) of Lemma 2.8, $\rho_{G\setminus x\setminus t}(P-\{t\}) \leq \rho_{G\setminus x}(P) = 2 = \rho_{G\setminus x}(P-\{q\}) - 1 \leq \rho_{G\setminus x\setminus t}(\{r,s\})$. Hence, by Lemma 2.11,

$$\rho_{G\backslash x\backslash t}(X - \{x\}) + 2 \ge \rho_{G\backslash x\backslash t}(X - \{x\}) + \rho_{G\backslash x\backslash t}(P - \{t\})$$
$$\ge \rho_{G\backslash x\backslash t}(C) + \rho_{G\backslash x\backslash t}(\{r, s\}) \ge \rho_{G\backslash x\backslash t}(C) + 2$$

and therefore $\rho_{G\setminus x\setminus t}(C) \leq \rho_{G\setminus x\setminus t}(X-\{x\}) \leq 2$. Since P is a quad of $G\setminus x$, by (i) of Lemma 2.8, $\rho_{G\setminus x}(P) = 2 = \rho_{G\setminus x}(P-\{t\}) - 1 \leq \rho_{G\setminus x\setminus t}(P-\{t\})$. By Lemma 2.15,

$$2 + \rho_{G\backslash x\backslash t}(C) \ge \rho_{G\backslash x}((P - \{t\}) \cup \{t\}) + \rho_{G\backslash x\backslash t}(C) \ge \rho_{G\backslash x}(C) + \rho_{G\backslash x\backslash t}(P - \{t\}) \ge \rho_{G\backslash x}(C) + 2,$$

which implies that $\rho_{G\setminus x}(C) \leq \rho_{G\setminus x\setminus t}(C) \leq 2$. Hence $\rho_{G\setminus x}(C) = \rho_{G\setminus x\setminus t}(C) = \rho_{G\setminus x\setminus t}(X - \{x\}) = \rho_{G\setminus t}(X) = 2$ because $G\setminus x$ is prime and $|V(G\setminus x) - C| \geq 2$. Hence (K1) holds.

Since $G \setminus x$ is weakly 3-rank-connected and $|V(G \setminus x) - C| \ge |P| + |D| \ge 5$, we deduce that $|C| \le 4$ and $|X| \le 6$. So $|Y| \ge 11 - |X| \ge 5$.

Suppose that $x \notin X$. Then $X = X - \{x\}$ and $\rho_{G\setminus x\setminus t}(X) = \rho_{G\setminus t}(X) = 2$. Since $C \subseteq X$, by Lemma 2.13, we have $\rho_{G\setminus t}(C) = \rho_{G\setminus x\setminus t}(C) = 2$. By (A1) of Lemma 2.16,

$$\rho_{G\setminus x}(C) + \rho_{G\setminus t}(C) \ge \rho_G(C) + \rho_{G\setminus x\setminus t}(C),$$

which implies that $\rho_G(C) \leq 2$. So $|C| \leq 2$ because G is 3-rank-connected. Then $|X| = |C \cup \{q\}| \leq 3$, contradicting our assumption on X. Hence $x \in X$.

Since $G \setminus t$ is weakly 3-rank-connected, $\rho_{G \setminus t}(X) = 2$, and $|Y| \ge 5$, we have |X| = 4. Therefore, by Lemma 4.5, X is a quad of $G \setminus t$ and (K2) holds.

Lemma 4.10. Let G be a 3-rank-connected graph with $|V(G)| \ge 12$ and no sequentially 3-rankconnected vertex-minor on |V(G)| - 1 vertices. Let x be a vertex of G such that $G \setminus x$ is weakly 3-rank-connected and P be a quad of $G \setminus x$. Then there is a graph G' locally equivalent to G such that the following hold.

- (1) $G' \setminus v$ is weakly 3-rank-connected for each vertex v of $P \cup \{x\}$.
- (2) P is a quad of $G' \setminus x$.
- (3) There exist a 2-element subset S of P and a quad X_u of $G' \setminus u$ for each u in S such that $x \in X_u$, $|X_u \cap P| = 1$, and $V(G' \setminus u) X_u$ is not sequential in $G' \setminus u$.

Proof. By Lemma 4.1, $G \setminus v$ is prime for each vertex v of G. By Lemma 4.7, we can assume that $G \setminus v$ is weakly 3-rank-connected for each vertex v of $P \cup \{x\}$, the set P is a quad of $G \setminus x$, and $N_G(t) - P$ is nonempty for each $t \in P$.

By Lemma 4.8, each vertex t in P satisfies (Q2) or (Q3). Suppose that at most 1 vertex of P satisfies (Q2). Then by Lemma 4.8, there exist 3 vertices q, r, s of P such that $\rho_{G\setminus q}(P-\{q\})=2$, $\rho_{G\setminus r}(P-\{r\})=2$, and $\rho_{G\setminus s}(P-\{s\})=2$. Since P is a quad of $G\setminus x$, by (i) of Lemma 2.8, we have $\rho_G(P) \leq \rho_{G\setminus x}(P) + 1 \leq 3$. Since G is 3-rank-connected, $3 \leq \rho_G(P)$ and therefore, $\rho_G(P)=3$. By (A3) of Lemma 2.16,

$$2+2 = \rho_{G\backslash q}(P-\{q\}) + \rho_{G\backslash r}(P-\{r\})$$

$$\geq \rho_G(P) + \rho_{G\backslash q\backslash r}(P-\{q,r\}) = 3 + \rho_{G\backslash q\backslash r}(P-\{q,r\}).$$

Therefore, $\rho_{G\backslash q\backslash r}(P - \{q, r\}) \leq 1$ and by symmetry, $\rho_{G\backslash q\backslash s}(P - \{q, s\}) \leq 1$ and $\rho_{G\backslash r\backslash s}(P - \{r, s\}) \leq 1$. Let p be the element of $P - \{q, r, s\}$. Since $N_G(t) - P \neq \emptyset$ for each $t \in P$, we have $N_G(p) - P = N_G(q) - P = N_G(r) - P = N_G(s) - P$ and therefore $\rho_G(P) = 1$, contradicting our assumption.

Therefore, there exist a subset $S = \{p,q\}$ of P and a subset X_u of $V(G \setminus u)$ for each $u \in S$ such that $\rho_{G \setminus u}(X_u) \leq 2$, both $X_u \cap P$ and $(V(G \setminus u) - X_u) \cap P$ are nonempty, and neither X_u nor $V(G \setminus u) - X_u$ is sequential in $G \setminus u$.

Let $Y_p = V(G \setminus p) - X_p$ and $Y_q = V(G \setminus q) - X_q$. By symmetry, we may assume that $|X_p \cap P| = 1$ and $|X_q \cap P| = 1$. Then by (K2) of Lemma 4.9, X_p is a quad of $G \setminus p$, X_q is a quad of $G \setminus q$, and $x \in X_p \cap X_q$.

Lemma 4.11. Let G be a 3-rank-connected graph with $|V(G)| \ge 12$ and x, y be distinct vertices of G such that both $G \setminus x$ and $G \setminus y$ are weakly 3-rank-connected. Let A be a quad of $G \setminus x$ and B be a quad of $G \setminus y$. Then $|A \cap B| \le 2$.

Proof. Suppose that $|A \cap B| \ge 3$. First let us consider the case when $y \notin A$ and $x \notin B$. Since G is 3-rank-connected and $|V(G) - (A \cap B)| \ge 3$, we have $\rho_G(A \cap B) \ge 3$. So by (A1) of Lemma 2.16,

$$2+2 \ge \rho_{G\setminus x}(A) + \rho_{G\setminus y}(B) \ge \rho_G(A \cap B) + \rho_{G\setminus x\setminus y}(A \cup B) \ge 3 + \rho_{G\setminus x\setminus y}(A \cup B).$$

Hence $\rho_{G\setminus x\setminus y}(A\cup B) \leq 1$. Then by (ii) of Lemma 2.8, we have $\rho_{G\setminus x}(A\cup B\cup \{y\}) \leq 2$. Since $G\setminus x$ is weakly 3-rank-connected and $|A\cup B\cup \{y\}| \in \{5,6\}$, we deduce that $|V(G\setminus x) - (A\cup B\cup \{y\})| \leq 4$ and so $|V(G)| \leq 11$, contradicting our assumption.

Now we consider the case when either

- $y \in A$ and $x \notin B$, or
- $y \notin A$ and $x \in B$.

By symmetry, we may assume that $y \in A$ and $x \notin B$. Then $|A \cap B| = 3$ because $x \notin B$. Since $G \setminus x$ is weakly 3-rank-connected, $|A \cup B| = 5$, and $|V(G \setminus x) - (A \cup B)| \ge 6$, we have $\rho_{G \setminus x}(A \cup B) \ge 3$. By (A2) of Lemma 2.16,

$$2+2 \ge \rho_{G\setminus x}(A) + \rho_{G\setminus y}(B) \ge \rho_{G\setminus x}(A\cup B) + \rho_{G\setminus y}(A\cap B) \ge 3 + \rho_{G\setminus y}(A\cap B)$$

Hence $\rho_{G\setminus y}(A \cap B) \leq 1$, contradicting the fact that $G \setminus y$ is prime.

Now it remains to consider the case when $y \in A$ and $x \in B$. Since $x \notin A$ and $y \notin B$, we have $|A \cap B| = 3$. Since G is 3-rank-connected and $|V(G) - (A \cup B)| \ge 7$, we have $\rho_G(A \cup B) \ge 3$. By (A3) of Lemma 2.16,

$$2+2 \ge \rho_{G\setminus x}(A) + \rho_{G\setminus y}(B) \ge \rho_G(A \cup B) + \rho_{G\setminus x\setminus y}(A \cap B) \ge 3 + \rho_{G\setminus x\setminus y}(A \cap B).$$

So $\rho_{G\setminus x\setminus y}(A\cap B) \leq 1$ and $\rho_{G\setminus x}(A\cap B) \leq 2$, contradicting the assumption that A is a quad of $G\setminus x$.

Lemma 4.12. Let G be a 3-rank-connected graph with $|V(G)| \ge 12$ and x be a vertex of G. Let P be a quad of $G \setminus x$ and y be a vertex of P. Let Q be a quad of $G \setminus y$. If $G \setminus x$ is weakly 3-rank-connected and $|P \cap Q| = 2$, then $x \in Q$.

Proof. Suppose that $x \notin Q$. Since G is 3-rank-connected, by Lemma 4.1, $G \setminus y$ is prime. Therefore, $\rho_{G \setminus y}(P \cap Q) = 2$ because $|P \cap Q| = 2$. Since $y \in P$ and $x \notin Q$, by (A2) of Lemma 2.16,

$$2+2 \ge \rho_{G\setminus x}(P) + \rho_{G\setminus y}(Q) \ge \rho_{G\setminus x}(P \cup Q) + \rho_{G\setminus y}(P \cap Q) \ge \rho_{G\setminus x}(P \cup Q) + 2.$$

Hence $\rho_{G\setminus x}(P \cup Q) \leq 2$. Since $G \setminus x$ is weakly 3-rank-connected and $|P \cup Q| = 6$, we have $|V(G \setminus x) - (P \cup Q)| \leq 4$ and so $|V(G)| \leq 11$, contradicting our assumption.

Lemma 4.13. Let G be a 3-rank-connected graph with $|V(G)| \ge 13$ and x be a vertex of G. Let P be a quad of $G \setminus x$ and p, q be distinct vertices of P. For each $u \in \{p,q\}$, let A_u be a quad of $G \setminus u$ such that $x \in A_u$, $|A_u \cap P| = 1$, and $V(G \setminus u) - A_u$ is not sequential in $G \setminus u$. If $G \setminus x$, $G \setminus p$, and $G \setminus q$ are weakly 3-rank-connected, then $A_p \cap A_q \subseteq P \cup \{x\}$.

Proof. For each $u \in \{p,q\}$, let $B_u = A_u - (P \cup \{x\})$. Then $|B_u| = 2$ and $|A_u \cup P| = 7$ for each $u \in \{p,q\}$. Let t be the unique element of $A_p \cap P$.

Now we claim that $\rho_G(A_p \cup P) = 3$. By Lemma 2.4, $N_{G\setminus x\setminus p}(t) \neq \emptyset$ and so $\rho_{G\setminus x\setminus p}(\{t\}) = 1$. Since P is a quad of $G \setminus x$, we have $\rho_{G\setminus x\setminus p}(P - \{p\}) \leq \rho_{G\setminus x}(P) = 2$. By (K1) of Lemma 4.9, $\rho_{G\setminus x\setminus p}(A_p - \{x\}) = \rho_{G\setminus p}(A_p) = 2$. By Lemma 2.14, $\rho_{G\setminus p}(A_p \cup (P - \{p\})) = \rho_{G\setminus x\setminus p}((A_p - \{x\}) \cup (P - \{p\}))$. By Lemma 2.10,

$$2 + 2 \ge \rho_{G \setminus x \setminus p}(A_p - \{x\}) + \rho_{G \setminus x \setminus p}(P - \{p\}) \\ \ge \rho_{G \setminus x \setminus p}((A_p - \{x\}) \cup (P - \{p\})) + \rho_{G \setminus x \setminus p}(\{t\}) \ge \rho_{G \setminus x \setminus p}((A_p - \{x\}) \cup (P - \{p\})) + 1.$$

Hence $\rho_{G\setminus x\setminus p}((A_p - \{x\}) \cup (P - \{p\})) \leq 3.$

Since P is a quad of $G \setminus x$, we have $\rho_{G \setminus x}(P) = 2 = \rho_{G \setminus x}(P - \{p\}) - 1 \le \rho_{G \setminus x \setminus p}(P - \{p\})$. So by Lemma 2.14, $\rho_{G \setminus x}((A_p - \{x\}) \cup P) = \rho_{G \setminus x \setminus p}((A_p - \{x\}) \cup (P - \{p\}))$. By (A3) of Lemma 2.16,

$$\rho_{G\setminus x}((A_p - \{x\}) \cup P) + \rho_{G\setminus p}(A_p \cup (P - \{p\})) \ge \rho_G(A_p \cup P) + \rho_{G\setminus x\setminus p}((A_p - \{x\}) \cup (P - \{p\})).$$

It follows that $\rho_G(A_p \cup P) = \rho_{G \setminus x \setminus p}((A_p - \{x\}) \cup (P - \{p\})) \leq 3$. Since G is 3-rank-connected and $|A_p \cup P|, |V(G) - (A_p \cup P)| \geq 3$, we have $\rho_G(A_p \cup P) = 3$.

By Lemma 4.11, $|A_p \cap A_q| \le 2$. Since $x \in A_p \cap A_q$, we have $|B_p \cap B_q| \le 1$.

Suppose that $|B_p \cap B_q| = 1$. Then $|A_q \cap (A_p \cup P)| = |A_q| - |A_q - (A_p \cup P)| = |A_q| - |B_q - B_p| = |A_q| - (|B_q| - |B_p \cap B_q|) = 3$. So $\rho_{G \setminus q}(A_q \cap (A_p \cup P)) = 3$ because A_q is a quad of $G \setminus q$. Since $\rho_{G \setminus q}(A_q) = 2$ and $\rho_{G \setminus q}((A_p \cup P) - \{q\}) \le \rho_G(A_p \cup P) = 3$, by Lemma 2.11,

$$5 \ge \rho_{G \setminus q}(A_q) + \rho_{G \setminus q}((A_p \cup P) - \{q\}) \ge \rho_{G \setminus q}((A_q \cup (A_p \cup P)) - \{q\}) + \rho_{G \setminus q}(A_q \cap (A_p \cup P)) = \rho_{G \setminus q}((A_q \cup (A_p \cup P)) - \{q\}) + 3.$$

Hence $\rho_{G\backslash q}((A_q \cup (A_p \cup P)) - \{q\}) \leq 2$. Since $G\backslash q$ is weakly 3-rank-connected and $|(A_q \cup (A_p \cup P)) - \{q\}| = |A_q| + |A_p \cup P| - |A_q \cap (A_p \cup P)| - 1 = 7$, we deduce that $|V(G\backslash q) - ((A_q \cup (A_p \cup P)) - \{q\})| \leq 4$. Therefore, $|V(G)| \leq 12$, contradicting our assumption. Therefore, $B_p \cap B_q = \emptyset$ and so $A_p \cap A_q \subseteq P \cup \{x\}$.

Lemma 4.14. Let G be a 3-rank-connected graph with $|V(G)| \ge 6$ and a, b be distinct vertices of G. Let A be a quad of $G \setminus a$ and B be a quad of $G \setminus b$. If $|A \cap B| = 1$, then $b \in A$ and $a \in B$.

Proof. Suppose not. Then by symmetry, we may assume that $b \notin A$. Since B is a quad of $G \setminus b$, we know that $\rho_{G \setminus b}(B) < \rho_{G \setminus b}(B - A)$. Then by Lemma 2.11,

$$\rho_{G \setminus b}(B) + \rho_{G \setminus b}(A) \ge \rho_{G \setminus b}(A - B) + \rho_{G \setminus b}(B - A)$$

and therefore $\rho_{G\setminus b}(A - B) < \rho_{G\setminus b}(A)$. Since A is a quad of $G \setminus a$, we have that $\rho_G(A) \le \rho_{G\setminus a}(A) + 1 \le 3$. By Lemma 4.1, $G \setminus b$ is prime and so

$$2 \le \rho_{G \setminus b}(A - B) < \rho_{G \setminus b}(A) \le \rho_G(A) \le 3,$$

which implies that $\rho_{G\setminus b}(A-B) = 2$ and $\rho_{G\setminus b}(A) = 3$. Since $2 = \rho_{G\setminus b}(A) - 1 \le \rho_{G\setminus a\setminus b}(A) \le \rho_{G\setminus a}(A) = 2$, we have $\rho_{G\setminus a\setminus b}(A) = 2$. Since $a \notin A - B$ and $b \notin A$, by (A1) of Lemma 2.16,

$$2+2=\rho_{G\backslash a}(A)+\rho_{G\backslash b}(A-B)\geq\rho_G(A-B)+\rho_{G\backslash a\backslash b}(A)=\rho_G(A-B)+2.$$

Hence $\rho_G(A - B) \leq 2$, contradicting the condition that G is 3-rank-connected.

Proposition 4.15. Let G be a 3-rank-connected graph such that $|V(G)| \ge 13$. Then there exists a sequentially 3-rank-connected vertex-minor H of G such that |V(H)| = |V(G)| - 1.

Proof. Suppose that no vertex-minor of G on |V(G)|-1 vertices is sequentially 3-rank-connected. Let x be a vertex of G. By Lemma 4.3, we can assume that $G \setminus x$ is weakly 3-rank-connected. By Lemma 4.1, $G \setminus x$ is prime. Since $G \setminus x$ is not sequentially 3-rank-connected, there exists a subset P of $V(G \setminus x)$ such that $\rho_{G \setminus x}(P) \leq 2$ and neither P nor $V(G \setminus x) - P$ is sequential in $G \setminus x$. Since $G \setminus x$ is weakly 3-rank-connected, we may assume that |P| = 4. Since $|V(G \setminus x) - P| \geq 4$ and $G \setminus x$ is prime, $\rho_{G \setminus x}(P) = 2$. So by Lemma 4.5, P is a quad of $G \setminus x$. Then by Lemma 4.10, we can assume the following.

- (1) $G \setminus v$ is weakly 3-rank-connected for each vertex v of $P \cup \{x\}$.
- (2) P is a quad of $G \setminus x$.
- (3) There exist a 2-element subset S of P and a quad X_u of $G \setminus u$ for each u in S such that $x \in X_u$, $|X_u \cap P| = 1$, and $V(G \setminus u) X_u$ is not sequential in $G \setminus u$.

Let p and q be distinct vertices of S. By Lemma 4.13, $x \in X_p \cap X_q \subseteq P \cup \{x\}$. By Lemma 4.11, $|X_p \cap X_q| \leq 2$.

If $|X_p \cap X_q| = 1$, then, by Lemma 4.14, $q \in X_p$ and $p \in X_q$. Then, since $X_p \cap X_q = \{x\}$, $X_p \cap P = \{q\}$, and $X_q \cap P = \{p\}$, by Proposition 4.6, $G * x \setminus x$ or G/x is sequentially 3-rank-connected, contradicting the assumption.

So $|X_p \cap X_q| = 2$. Let $r \in X_p \cap X_q - \{x\}$. Since r does not satisfy (Q1), by Lemma 4.8, (Q2) or (Q3) holds for r.

If (Q2) holds, there is a subset R of $V(G \setminus r)$ such that $\rho_{G \setminus r}(R) \leq 2$, $R \cap P \neq \emptyset$, $(V(G \setminus r) - R) \cap P \neq \emptyset$, and neither R nor $V(G \setminus r) - R$ is sequential in $G \setminus r$. By symmetry, we may assume that $|P \cap R| = 1$ by replacing R by $V(G \setminus r) - R$. Then by (K2) of Lemma 4.9, R is a quad of $G \setminus r$ containing x. By Lemma 4.11, $|R \cap X_p|, |R \cap X_q| \leq 2$.

Suppose that $|R \cap X_p| = 2$ and $|R \cap X_q| = 2$. Then by applying Lemma 4.12 twice, we deduce that R contains both p and q, contradicting our assumption that $|P \cap R| = 1$. So by symmetry, we can assume that $|R \cap X_p| = 1$. Then by Lemma 4.14, $p \in R$. Since $R \cap X_p = \{x\}$, $P \cap R = \{p\}$, and $X_p \cap P = \{r\}$, by Lemma 4.6, we deduce that $G * x \setminus x$ or G/x is sequentially 3-rank-connected, contradicting our assumption.

If (Q3) holds, then there is a quad of R of $G \setminus r$ containing x such that $R \cap P = \emptyset$. By Lemma 4.11, $|R \cap X_p| \leq 2$. Since $p \notin R$, by Lemma 4.12, $|R \cap X_p| = 1$. Then Lemma 4.14 implies that $p \in R$, contradicting the assumption.

5 Treating internally 3-rank-connected graphs

In this section, we prove Theorem 1.2 for internally 3-rank-connected graphs.

A graph G is internally 3-rank-connected if G is prime and for each subset X of V(G), either $|X| \leq 3$ or $|V(G) - X| \leq 3$ whenever $\rho_G(X) \leq 2$. A 3-element set T of vertices of a graph G is a triplet of G if $\rho_G(T) = 2$ and $\rho_{G\setminus X}(T-x) = 2$ for each $x \in T$.

Here is a rough overview of our approach in this section. If G is an internally 3-rankconnected counterexample of Theorem 1.2 and $|V(G)| \ge 13$, then by pivoting, we may assume that G has a triplet $T = \{a, b, c\}$. Next we find a partition (A_b, A_c) of $V(G \setminus a)$, a partition (B_a, B_c) of $V(G \setminus b)$, and a partition (C_a, C_b) of $V(G \setminus c)$ satisfying the following conditions:

- (1) $b \in A_b, c \in A_c$, and neither A_b nor A_c is sequential in $G \setminus a$.
- (2) $a \in B_a, c \in B_c$, and neither B_a nor B_c is sequential in $G \setminus b$.
- (3) $a \in C_a, b \in C_b$, and neither C_a nor C_b is sequential in $G \setminus c$.

We then prove that all of $A_b, A_c, B_a, B_c, C_a, C_b$ must be small, contradicting the assumption that $|V(G)| \ge 13$.

The following lemma shows that if a graph is internally 3-rank-connected but not 3-rank-connected, then we can apply pivoting to obtain a graph with a triplet.

Lemma 5.1 (Oum [8, Lemma 5.1]). Let G be a prime graph and A be a 3-element subset of V(G) such that $\rho_G(A) = 2$. Then there is a graph G' pivot-equivalent to G such that A is a triplet of G'.

Lemma 5.2 (Oum [8, Lemma 5.2]). Let G be an internally 3-rank-connected graph and $T = \{a, b, c\}$ be a triplet of G. Then $G \setminus a$, $G \setminus b$, and $G \setminus c$ are prime.

Lemma 5.3. Let T be a triplet of an internally 3-rank-connected graph G and $a \in T$. Let (X, Y) be a partition of $V(G) - \{a\}$ such that $\rho_{G \setminus a}(X) \leq 2$ and neither X nor Y is sequential in $G \setminus a$. Then there exist $b \in X \cap T$ and $c \in Y \cap T$ such that $\rho_{G \setminus b}(X - \{b\}) = \rho_{G \setminus c}(Y - \{c\}) = 3$.

Proof. Since neither X nor Y is sequential in $G \setminus a$, $|X| \ge 4$ and $|Y| \ge 4$. So $\rho_{G\setminus a}(X) = 2$ because $G \setminus a$ is prime by Lemma 5.2. Since T is a triplet of G, we have $\rho_{G\setminus a}(T - \{a\}) = \rho_G(T)$. If $T \subseteq X \cup \{a\}$, then by Lemma 2.14, $\rho_G(X \cup \{a\}) = \rho_{G\setminus a}(X) = 2$, contradicting the assumption that G is internally 3-rank-connected.

Hence $T - \{a\} \nsubseteq X$ and similarly $T - \{a\} \nsubseteq Y$. Therefore, there exist $b \in X \cap T$ and $c \in Y \cap T$. Then $T = \{a, b, c\}$.

By (i) of Lemma 2.8, $\rho_G(X) \leq \rho_{G\setminus a}(X) + 1 \leq 3$. So by (ii) of Lemma 2.8, we have $\rho_{G\setminus b}(X - \{b\}) \leq 3$ and similarly, $\rho_{G\setminus c}(Y - \{c\}) \leq 3$.

Suppose that $\rho_{G\setminus c}(Y - \{c\}) < 3$. Since T is a triplet of G, by Lemma 2.9,

$$\rho_G(\{a,b\}, Y - \{c\}) + 2 = \rho_G(\{a,b\}, Y - \{c\}) + \rho_G(\{a,b,c\}, V(G) - \{a,b,c\})$$

$$\geq \rho_G(\{a,b,c\}, Y - \{c\}) + \rho_G(\{a,b\}, V(G) - \{a,b,c\})$$

$$= \rho_G(\{a,b,c\}, Y - \{c\}) + 2,$$

and therefore $\rho_G(\{a, b, c\}, Y - \{c\}) \leq \rho_G(\{a, b\}, Y - \{c\})$. Then by Lemma 2.9, we have

$$\rho_G(X \cup \{a\}, Y - \{c\}) + \rho_G(\{a, b, c\}, Y - \{c\}) \ge \rho_G(X \cup \{a, c\}, Y - \{c\}) + \rho_G(\{a, b\}, Y - \{c\}).$$

Hence $\rho_{G\setminus a}(X \cup \{c\}) \leq \rho_G(X \cup \{a,c\}, Y - \{c\}) \leq \rho_G(X \cup \{a\}, Y - \{c\}) = \rho_{G\setminus c}(Y - \{c\}) < 3$. Therefore, $\rho_{G\setminus a}(X \cup \{c\}) \leq 2 = \rho_{G\setminus a}(X)$. Since $|Y - \{c\}| \geq 3$ and $G \setminus a$ is prime, we have $\rho_{G\setminus a}(X \cup \{c\}) = 2$. Since Y is not sequential in $G \setminus a$, by Lemma 3.1, $Y - \{c\}$ is not sequential in $G \setminus a$ and therefore $|Y - \{c\}| \geq 4$. Since $T \subseteq X \cup \{a,c\}$, by Lemma 2.14, $\rho_G(X \cup \{a,c\}) = \rho_{G\setminus a}(X \cup \{c\}) = 2$, contradicting the assumption that G is internally 3-rank-connected. Therefore $\rho_{G\setminus c}(Y - \{c\}) = 3$. By symmetry, we deduce that $\rho_{G\setminus b}(X - \{b\}) = 3$.

Lemma 5.4. Let G be an internally 3-rank-connected graph with $|V(G)| \ge 12$ and $T = \{a, b, c\}$ be a triplet of G such that $G \setminus c$ is not sequentially 3-rank-connected. Let X be a subset of $V(G \setminus a \setminus b)$ such that $|X| \ge 3$, $|V(G \setminus a \setminus b) - X| \ge 2$, and $c \notin X$. Then $\rho_{G \setminus a \setminus b}(X) \ge 2$.

Proof. Suppose that $\rho_{G\backslash a\backslash b}(X) \leq 1$. Let $Y = V(G \setminus a \setminus b) - X$. Since $\{a, b, c\}$ is a triplet of G, we have $\rho_{G\backslash a}(\{b, c\}) = \rho_G(\{a, b, c\})$. By Lemma 2.14, $\rho_G(Y \cup \{a, b\}) = \rho_{G\backslash a}(Y \cup \{b\})$. Hence $\rho_G(Y \cup \{a, b\}) = \rho_{G\backslash a}(Y \cup \{b\}) \leq \rho_{G\backslash a\backslash b}(Y) + 1 = \rho_{G\backslash a\backslash b}(X) + 1 \leq 2$. So $|X| \leq 3$ because G is internally 3-rank-connected and $|Y \cup \{a, b\}| \geq 4$.

Since $G \setminus c$ is not sequentially 3-rank-connected, there exists a partition (C_a, C_b) of $V(G \setminus c)$ such that $\rho_{G \setminus c}(C_a) \leq 2$ and neither C_a nor C_b is sequential in $G \setminus c$.

Suppose that |X| = 3. By symmetry, we may assume that $|C_a \cap X| \ge 2$ by swapping C_a and C_b if necessary. If $|C_a \cap X| = 2$, then let x be the element in $C_b \cap X$. By Lemma 5.2, $G \setminus c$ is prime. Since $|(Y \cup \{a, b\}) - \{c\}| \ge 2$, we have $2 \le \rho_{G \setminus c}(X) \le \rho_G(X) = 2$. Since $|X - \{x\}| = 2$ and $G \setminus c$ is prime, we also have $\rho_{G \setminus c}(X - \{x\}) = 2$. So by Lemma 2.10,

$$\rho_{G\backslash c}(C_a) + \rho_{G\backslash c}(X) \ge \rho_{G\backslash c}(C_a \cup \{x\}) + \rho_{G\backslash c}(X - \{x\}).$$

Therefore, $\rho_{G\setminus c}(C_a \cup \{x\}) \leq \rho_{G\setminus c}(C_a) \leq 2$. Since $|C_b - \{x\}| \geq 3$ and by Lemma 5.2, $G \setminus c$ is prime, we have $\rho_{G\setminus c}(C_a \cup \{x\}) = \rho_{G\setminus c}(C_a) = 2$. So by Lemma 3.1, neither $C_a \cup \{x\}$ nor $C_b - \{x\}$ is sequential in $G \setminus c$. By replacing (C_a, C_b) with $(C_a \cup \{x\}, C_b - \{x\})$, we may assume that $|C_a \cap X| = 3$.

By Lemma 5.3, there is a unique element $t \in \{a, b\}$ of $C_b \cap T$. Then $X \subseteq C_a$ and $C_b - \{t\} \subseteq Y - \{c\} \subseteq Y$. Since $|V(G)| \ge 12$ and G is internally 3-rank-connected, we have $\rho_G(Y \cup \{t\}) \ge 3$. Since $\rho_{G \setminus t}(Y) \le \rho_{G \setminus a \setminus b}(Y) + 1 \le 2 < \rho_G(Y \cup \{t\})$ and $t \in C_b \subseteq Y \cup \{t\}$, by Lemma 2.14, $\rho_{G \setminus t}(C_b - \{t\}) < \rho_G(C_b) \le 3$, contradicting Lemma 5.3. **Lemma 5.5.** Let G be an internally 3-rank-connected graph with $|V(G)| \ge 12$ and $T = \{a, b, c\}$ be a triplet of G. Let (A_b, A_c) be a partition of $V(G \setminus a)$ such that $b \in A_b$, $c \in A_c$, $\rho_{G \setminus a}(A_b) \le 2$, and neither A_b nor A_c is sequential in $G \setminus a$ and let (B_a, B_c) be a partition of $V(G \setminus b)$ such that $a \in B_a$, $c \in B_c$, $\rho_{G \setminus b}(B_a) \le 2$, and neither B_a nor B_c is sequential in $G \setminus b$. If $G \setminus c$ is not sequentially 3-rank-connected, then the following hold.

- (1) $\rho_{G\setminus a\setminus b}(A_b\cap B_c) = \rho_G(A_b\cap B_c).$
- (2) $\rho_{G\setminus a\setminus b}(A_c\cap B_a) = \rho_G(A_c\cap B_a).$
- (3) $\rho_{G\setminus a\setminus b}(A_c\cap B_c) = \rho_G(A_c\cap B_c).$

Proof. Since none of A_b , A_c is sequential in $G \setminus a$ and none of B_a , B_c is sequential in $G \setminus b$, we have $|A_b|, |A_c|, |B_a|, |B_c| \ge 4$. By Lemma 5.2, $G \setminus a$ is prime and so $\rho_{G\setminus a}(A_c) = 2$. Since $c \notin A_b - \{b\}$ and $|A_b - \{b\}| \ge 3$, by Lemma 5.4, we have $\rho_{G\setminus a\setminus b}(A_c) = \rho_{G\setminus a\setminus b}(A_b - \{b\}) \ge 2$. So by Lemma 2.8(i), we have $\rho_{G\setminus a\setminus b}(A_c) = \rho_{G\setminus a\setminus b}(A_c) = 2$. Similarly, $\rho_{G\setminus a\setminus b}(B_c) = \rho_{G\setminus b}(B_c) = 2$.

Since $\rho_{G \setminus a \setminus b}(B_c) = \rho_{G \setminus b}(B_c) = 2$ and $A_b \cap B_c \subseteq B_c$, by Lemma 2.13, we have $\rho_{G \setminus b}(A_b \cap B_c) = \rho_{G \setminus a \setminus b}(A_b \cap B_c)$.

Since $\{a, b, c\}$ is a triplet of G, we have $\rho_G(\{a, b, c\}) = \rho_{G \setminus b}(\{a, c\})$. Observe that $\rho_{G \setminus b}(A_b \cap B_c) = \rho_{G \setminus b}(A_c \cup B_a)$ and $\rho_G(A_b \cap B_c) = \rho_G(A_c \cup B_a \cup \{b\})$. Since $\{a, b, c\} \subseteq A_c \cup B_a \cup \{b\}$, by Lemma 2.14, $\rho_{G \setminus b}(A_b \cap B_c) = \rho_{G \setminus b}(A_c \cup B_a) = \rho_G(A_c \cup B_a \cup \{b\}) = \rho_G(A_b \cap B_c)$.

Hence $\rho_{G\setminus a\setminus b}(A_b\cap B_c) = \rho_G(A_b\cap B_c)$ and (1) holds. By symmetry, (2) also holds.

Now let us prove (3). Since $\rho_{G \setminus a \setminus b}(B_c) = \rho_{G \setminus b}(B_c) = 2$ and $A_c \cap B_c \subseteq B_c$, by Lemma 2.13, we have $\rho_{G \setminus a \setminus b}(A_c \cap B_c) = \rho_{G \setminus b}(A_c \cap B_c)$.

By (A1) of Lemma 2.16,

4

$$2 + \rho_{G \setminus b}(A_c \cap B_c) = \rho_{G \setminus a}(A_c) + \rho_{G \setminus b}(A_c \cap B_c)$$

$$\geq \rho_{G \setminus a \setminus b}(A_c) + \rho_G(A_c \cap B_c) = 2 + \rho_G(A_c \cap B_c),$$

which implies that $\rho_{G\setminus b}(A_c\cap B_c) \ge \rho_G(A_c\cap B_c)$. By (i) of Lemma 2.8, $\rho_{G\setminus b}(A_c\cap B_c) \le \rho_G(A_c\cap B_c)$ and so $\rho_{G\setminus b}(A_c\cap B_c) = \rho_G(A_c\cap B_c)$. Hence $\rho_G(A_c\cap B_c) = \rho_{G\setminus a\setminus b}(A_c\cap B_c)$.

Lemma 5.6. Let G be an internally 3-rank-connected graph with $|V(G)| \ge 12$ and $T = \{a, b, c\}$ be a triplet of G. Let (A_b, A_c) be a partition of $V(G \setminus a)$ such that $b \in A_b$, $c \in A_c$, $\rho_{G \setminus a}(A_b) \le 2$, and neither A_b nor A_c is sequential in $G \setminus a$ and let (B_a, B_c) be a partition of $V(G \setminus b)$ such that $a \in B_a$, $c \in B_c$, $\rho_{G \setminus b}(B_a) \le 2$, and neither B_a nor B_c is sequential in $G \setminus b$. If $G \setminus c$ is not sequentially 3-rank-connected, then the following hold.

- (i) $\rho_G(A_c \cap B_a) \le 2 \text{ and } 2 \le |A_c \cap B_a| \le 3.$
- (ii) $\rho_G(A_b \cap B_c) \le 2 \text{ and } 2 \le |A_b \cap B_c| \le 3.$
- (iii) $\rho_{G \setminus a \setminus b}(A_b \cap B_a) \leq 2.$
- (iv) $|A_c \cap B_c| \ge 2$.
- (v) If $\rho_{G\setminus a\setminus b}(A_b\cap B_a) \geq 2$, then $\rho_G(A_c\cap B_c) \leq 2$ and $|A_c\cap B_c| \leq 3$.

Proof. Since none of A_b , A_c is sequential in $G \setminus a$ and none of B_a , B_c is sequential in $G \setminus b$, we have $|A_b|, |A_c|, |B_a|, |B_c| \ge 4$. Let us prove the following, which prove the lemma.

- (1) If $|A_b \cap B_c| \ge 2$, then $\rho_G(A_c \cap B_a) \le 2$ and $|A_c \cap B_a| \le 3$.
- (2) If $|A_c \cap B_a| \ge 2$, then $\rho_G(A_b \cap B_c) \le 2$ and $|A_b \cap B_c| \le 3$.
- (3) If $|A_c \cap B_c| \ge 2$, then $\rho_{G \setminus a \setminus b}(A_b \cap B_a) \le 2$.

- (4) If $\rho_{G\setminus a\setminus b}(A_b\cap B_a) \ge 2$, then $\rho_G(A_c\cap B_c) \le 2$ and $|A_c\cap B_c| \le 3$.
- $(5) |A_b \cap B_c| \ge 2.$
- $(6) |A_c \cap B_a| \ge 2.$
- $(7) |A_c \cap B_c| \ge 2.$

To prove (1), suppose that $|A_b \cap B_c| \ge 2$. Since G is prime and $|V(G) - (A_b \cap B_c)| \ge |A_c| \ge 4$, by (1) of Lemma 5.5, $\rho_{G \setminus a \setminus b}(A_b \cap B_c) = \rho_G(A_b \cap B_c) \ge 2$. Since $G \setminus b$ is prime and $|A_b \cap B_c| \ge 2$, we have $\rho_{G \setminus b}(A_c \cup B_a) = \rho_{G \setminus b}(A_b \cap B_c) \ge 2$. Since $\rho_{G \setminus a \setminus b}(A_c) = 2$, by (S1) of Lemma 2.12,

$$2 + 2 = \rho_{G \setminus a \setminus b}(A_c) + \rho_{G \setminus b}(B_a)$$

$$\geq \rho_{G \setminus a \setminus b}(A_c \cap B_a) + \rho_{G \setminus b}(A_c \cup B_a) \geq \rho_{G \setminus a \setminus b}(A_c \cap B_a) + 2.$$

Therefore, by (2) of Lemma 5.5, $\rho_G(A_c \cap B_a) = \rho_{G \setminus a \setminus b}(A_c \cap B_a) \leq 2$. Since G is internally 3-rank-connected and $|V(G) - (A_c \cap B_a)| \geq |A_b| \geq 4$, we deduce that $|A_c \cap B_a| \leq 3$. So this proves (1). By symmetry between a and b, (2) also holds.

Now we show (3). Suppose that $|A_c \cap B_c| \ge 2$. Since G is prime and $|V(G) - (A_b \cup B_a)| \ge |A_c| \ge 4$, we have $\rho_G(A_b \cup B_a) \ge 2$. By (A3) of Lemma 2.16,

$$4 \ge \rho_{G \setminus a}(A_b) + \rho_{G \setminus b}(B_a) \ge \rho_G(A_b \cup B_a) + \rho_{G \setminus a \setminus b}(A_b \cap B_a)$$

and therefore $\rho_{G \setminus a \setminus b}(A_b \cap B_a) \leq 2$.

Now let us prove (4). Suppose that $\rho_{G\setminus a\setminus b}(A_b\cap B_a) \geq 2$. By (A3) of Lemma 2.16,

$$4 \ge \rho_{G\backslash a}(A_b) + \rho_{G\backslash b}(B_a) \ge \rho_G(A_b \cup B_a) + \rho_{G\backslash a\backslash b}(A_b \cap B_a) \ge \rho_G(A_b \cup B_a) + 2.$$

Hence $\rho_G(A_b \cup B_a) = \rho_G(A_c \cap B_c) \leq 2$. Since G is internally 3-rank-connected and $|V(G) - (A_c \cap B_c)| \geq 4$, we conclude that $|A_c \cap B_c| \leq 3$.

To prove (5), suppose that $|A_b \cap B_c| \le 1$. Then $4 \le |A_b| = |\{b\}| + |A_b \cap B_c| + |A_b \cap B_a| \le 2 + |A_b \cap B_a|$ and so $|A_b \cap B_a| \ge 2$.

If $|A_b \cap B_a| \ge 3$, then, since $c \in A_c \cap B_c$, by Lemma 5.4, $\rho_{G \setminus a \setminus b}(A_b \cap B_a) \ge 2$. If $|A_b \cap B_a| = 2$, then $|A_b| = 4$ and by Lemma 4.5, A_b is a quad of $G \setminus a$. Then by (ii) of Lemma 2.8, $\rho_{G \setminus a \setminus b}(A_b \cap B_a) \ge \rho_{G \setminus a}((A_b \cap B_a) \cup \{b\}) - 1 = 2$. So, in both cases, we deduce that $\rho_{G \setminus a \setminus b}(A_b \cap B_a) \ge 2$.

Hence, by (4), $\rho_G(A_c \cap B_c) \leq 2$ and $|A_c \cap B_c| \leq 3$. Since $4 \leq |B_c| = |A_b \cap B_c| + |A_c \cap B_c| \leq 1 + |A_c \cap B_c| \leq 4$, we have $|A_c \cap B_c| = 3$ and $|B_c| = 4$. By (i) of Lemma 2.8, $\rho_{G\setminus b}(A_c \cap B_c) \leq \rho_G(A_c \cap B_c) \leq 2$. So B_c is sequential in $G \setminus b$, contradicting our assumption. So this proves that $|A_b \cap B_c| \geq 2$ and by symmetry between a and b, $|A_c \cap B_a| \geq 2$ and (6) holds.

Now let us prove (7). Suppose that $|A_c \cap B_c| \leq 1$. Then $4 \leq |A_c| = 1 + |A_c \cap B_c| + |A_c \cap B_a| \leq 2 + |A_c \cap B_a|$ and so $2 \leq |A_c \cap B_a|$. Then by (2), we have $\rho_G(A_b \cap B_c) \leq 2$ and $|A_b \cap B_c| \leq 3$. Since $4 \leq |B_c| = |A_b \cap B_c| + |A_c \cap B_c| \leq |A_b \cap B_c| + 1 \leq 4$, we have $|A_b \cap B_c| = 3$ and $|B_c| = 4$. By (i) of Lemma 2.8, $\rho_{G \setminus b}(A_b \cap B_c) \leq \rho_G(A_b \cap B_c) \leq 2$. So B_c is sequential in $G \setminus b$, contradicting our assumption.

Lemma 5.7. Let G be an internally 3-rank-connected graph with $|V(G)| \ge 12$ and $T = \{a, b, c\}$ be a triplet of G. Let (A_b, A_c) be a partition of $V(G \setminus a)$ such that $b \in A_b$, $c \in A_c$, $\rho_{G \setminus a}(A_b) \le 2$, and neither A_b nor A_c is sequential in $G \setminus a$, let (B_a, B_c) be a partition of $V(G \setminus b)$ such that $a \in B_a$, $c \in B_c$, $\rho_{G \setminus b}(B_a) \le 2$, and neither B_a nor B_c is sequential in $G \setminus b$, and let (C_a, C_b) be a partition of $V(G \setminus c)$ such that $a \in C_a$, $b \in C_b$, $\rho_{G \setminus c}(C_a) \le 2$, and neither C_a nor C_b is sequential in $G \setminus c$. Then the following hold.

(1) If $|A_c \cap B_c| \ge 3$ and $\rho_{G \setminus a \setminus b}(A_b \cap B_a) > 1$, then $|A_c \cap B_c| = 3$, $\rho_G(A_c \cap B_c) = 2$, and $|A_c \cap B_c \cap C_a| = |A_c \cap B_c \cap C_b| = 1$.

- (2) If $|A_c \cap B_c| \geq 3$ and $\rho_{G \setminus a \setminus b}(A_b \cap B_a) \leq 1$, then either
 - $A_b \cap B_a = \emptyset$, or
 - $1 \le |A_b \cap B_a| \le 2$ and $\rho_{G \setminus c}((A_b \cap B_a) \cup \{a, b\}) = 3$.

Proof. (1) Since $\rho_{G\setminus a\setminus b}(A_b\cap B_a) > 1$, we have $|A_b\cap B_a| \geq 2$ and by (v) of Lemma 5.6, $\rho_G(A_c\cap B_c) \leq 2$ and $|A_c\cap B_c| \leq 3$. Hence $|A_c\cap B_c| = 3$. Since G is prime and $|V(G) - (A_c\cap B_c)| \geq 3$, we have $\rho_G(A_c\cap B_c) = 2$. Now we prove that $|A_c\cap B_c\cap C_a| = |A_c\cap B_c\cap C_b| = 1$. Suppose not. Then, by symmetry, we may assume that $|A_c\cap B_c\cap C_a| = 2$ and $|A_c\cap B_c\cap C_b| = 0$. Since $|(A_c\cap B_c) - \{c\}| = 2$ and $G\setminus c$ is prime, $\rho_{G\setminus c}((A_c\cap B_c) - \{c\}) \geq 2$. By (ii) of Lemma 2.8, $\rho_{G\setminus c}((A_c\cap B_c) - \{c\}) = \rho_G(A_c\cap B_c) = 2$. Since $A_c\cap B_c \subseteq C_a\cup\{c\}$, by Lemma 2.14, $\rho_G(C_a\cup\{c\}) = \rho_{G\setminus c}(C_a) \leq 2$. Since G is internally 3-rank-connected and $|C_a\cup\{c\}| \geq 5$, we have $|C_b| \leq 3$, contradicting our assumption.

(2) By Lemma 5.4, $|A_b \cap B_a| \leq 2$. Suppose that $|A_b \cap B_a| \geq 1$. We can observe that $\rho_G((A_b \cap B_a) \cup \{a, b, c\}) \geq 3$ because G is internally 3-rank-connected, $|(A_b \cap B_a) \cup \{a, b, c\}| \geq 4$, and $|V(G) - ((A_b \cap B_a) \cup \{a, b, c\})| \geq 12 - 5 = 7$. Since $\{a, b, c\}$ is a triplet of G, we have $\rho_G(\{a, b, c\}) = \rho_{G\setminus C}(\{a, b\}) = 2$. Since $\{a, b, c\} \subseteq (A_b \cap B_a) \cup \{a, b, c\}$, by Lemma 2.14, $\rho_{G\setminus C}((A_b \cap B_a) \cup \{a, b\}) = \rho_G((A_b \cap B_a) \cup \{a, b, c\}) \geq 3$. By (i) and (ii) of Lemma 2.8, $\rho_{G\setminus C}((A_b \cap B_a) \cup \{a, b\}) \leq \rho_G((A_b \cap B_a) \cup \{a, b\}) \leq 2 + \rho_{G\setminus a\setminus b}(A_b \cap B_a) \leq 3$ and we conclude that $\rho_{G\setminus C}((A_b \cap B_a) \cup \{a, b\}) = 3$. \Box

Proposition 5.8. Let T be a triplet of an internally 3-rank-connected graph G. If $|V(G)| \ge 12$, then there exists $t \in T$ such that $G \setminus t$ is sequentially 3-rank-connected.

Proof. Let $T = \{a, b, c\}$. Suppose that none of $G \setminus a$, $G \setminus b$, and $G \setminus c$ is sequentially 3-rankconnected. Then there exist partitions (A_b, A_c) of $V(G) - \{a\}$, (B_a, B_c) of $V(G) - \{b\}$, and (C_a, C_b) of $V(G) - \{c\}$ such that $\rho_{G \setminus a}(A_b) \leq 2$, $\rho_{G \setminus b}(B_a) \leq 2$, $\rho_{G \setminus c}(C_a) \leq 2$, neither A_b nor A_c is sequential in $G \setminus a$, neither B_a nor B_c is sequential in $G \setminus b$, and neither C_a nor C_b is sequential in $G \setminus c$. Then $|A_b|, |A_c| \geq 4, |B_a|, |B_c| \geq 4$, and $|C_a|, |C_b| \geq 4$.

By Lemma 5.3, we may assume that $b \in A_b$, $c \in A_c$, $a \in B_a$, $c \in B_c$, $a \in C_a$, and $b \in C_b$.

By Lemma 5.6, we have $|A_b \cap B_c| \leq 3$, $|A_c \cap B_a| \leq 3$, and $\rho_G(A_b \cap B_c) \leq 2$. By symmetry between b and c, we have that $|A_c \cap C_b| \leq 3$ and $|A_b \cap C_a| \leq 3$. By symmetry between a and c, we have that $|B_c \cap C_a| \leq 3$ and $|B_a \cap C_b| \leq 3$. Now we show that we can assume the following.

- (B1) If $|A_b \cap B_c| = 3$, then $A_b \cap B_c \subseteq C_a$ or $A_b \cap B_c \subseteq C_b$.
- (B2) If $|A_c \cap B_a| = 3$, then $A_c \cap B_a \subseteq C_a$ or $A_c \cap B_a \subseteq C_b$.
- (B3) If $|A_c \cap C_b| = 3$, then $A_c \cap C_b \subseteq B_a$ or $A_c \cap C_b \subseteq B_c$.
- (B4) If $|A_b \cap C_a| = 3$, then $A_b \cap C_a \subseteq B_a$ or $A_b \cap C_a \subseteq B_c$.
- (B5) If $|B_c \cap C_a| = 3$, then $B_c \cap C_a \subseteq A_b$ or $B_c \cap C_a \subseteq A_c$.
- (B6) If $|B_a \cap C_b| = 3$, then $B_a \cap C_b \subseteq A_b$ or $B_a \cap C_b \subseteq A_c$.

We choose $(A_b, A_c, B_a, B_c, C_a, C_b)$ such that $b \in A_b, c \in A_c, a \in B_a, c \in B_c, a \in C_a, b \in C_b$, and it satisfies the maximum number of (B1)–(B6). Then we claim that all of (B1)–(B6) hold. Suppose not. Then by symmetry, we can assume that (B1) does not hold. Then $|A_b \cap B_c| = 3$, $A_b \cap B_c \notin C_a$, and $A_b \cap B_c \notin C_b$. Then either $|A_b \cap B_c \cap C_a| = 2$ and $|A_b \cap B_c \cap C_b| = 1$ or $|A_b \cap B_c \cap C_a| = 1$ and $|A_b \cap B_c \cap C_b| = 2$.

(i) Suppose that $|A_b \cap B_c \cap C_a| = 2$ and $|A_b \cap B_c \cap C_b| = 1$. Let x be the element of $A_b \cap B_c \cap C_b$. We have $\rho_{G \setminus c}(A_b \cap B_c) \leq \rho_G(A_b \cap B_c) \leq 2$. Since $|(A_b \cap B_c) - \{x\}| = 2$ and $G \setminus c$ is prime, $\rho_{G \setminus c}((A_b \cap B_c) - \{x\}) \geq 2$. So by Lemma 2.10,

$$2+2 \ge \rho_{G\backslash c}(C_a) + \rho_{G\backslash c}(A_b \cap B_c)$$
$$\ge \rho_{G\backslash c}((A_b \cap B_c) - \{x\}) + \rho_{G\backslash c}(C_a \cup \{x\}) \ge 2 + \rho_{G\backslash c}(C_a \cup \{x\})$$

Therefore, $\rho_{G\backslash c}(C_a \cup \{x\}) \leq \rho_{G\backslash c}(C_a) \leq 2$. Since $G \setminus c$ is prime and $|V(G \setminus c) - (C_a \cup \{x\})| = |C_b| - 1 \geq 3$, we have $\rho_{G\backslash c}(C_a \cup \{x\}) = \rho_{G\backslash c}(C_a) = 2$. Hence by Lemma 3.1, neither $C_a \cup \{x\}$ nor $C_b - \{x\}$ is sequential in $G \setminus c$. We deduce that $(A_b, A_c, B_a, B_c, C_a \cup \{x\}, C_b - \{x\})$ satisfies (B1). Since $x \notin A_c \cap B_a$, if $(A_b, A_c, B_a, B_c, C_a, C_b)$ satisfies (B2), then $(A_b, A_c, B_a, B_c, C_a \cup \{x\}, C_b - \{x\})$ satisfies (B2). Since $A_c \cap C_b = A_c \cap (C_b - \{x\})$, if $(A_b, A_c, B_a, B_c, C_a, C_b)$ satisfies (B3), then $(A_b, A_c, B_a, B_c, C_a \cup \{x\}, C_b - \{x\})$ satisfies (B6), then $(A_b, A_c, B_a, B_c, C_a, C_b)$ satisfies (B3). Since $B_a \cap (C_b - \{x\}) = B_a \cap C_b$, if $(A_b, A_c, B_a, B_c, C_a, C_b)$ satisfies (B6), then $(A_b, A_c, B_a, B_c, C_a \cup \{x\}, C_b - \{x\})$ satisfies (B6). Since $x \in A_b$, we have $|A_b \cap C_a| + 1 = |A_b \cap (C_a \cup \{x\})| \leq 3$ by applying Lemma 5.6(i) with (A_c, A_b) and $(C_a \cup \{x\}, C_b - \{x\})$. So $|A_b \cap C_a| \leq 2$. Since $|A_b \cap B_c \cap C_a| = 2$ we have $A_b \cap C_a \subseteq B_c$. So $A_b \cap (C_a \cup \{x\}) \subseteq B_c$ because $x \in B_c$. Hence $(A_b, A_c, B_a, B_c, C_a \cup \{x\}, C_b - \{x\})$ satisfies (B4).

Since $x \in B_c$, we have $|B_c \cap C_a| + 1 = |B_c \cap (C_a \cup \{x\})| \le 3$ by applying Lemma 5.6(ii) with (B_c, B_a) and $(C_b - \{x\}, C_a \cup \{x\})$. So $|B_c \cap C_a| \le 2$. Since $|A_b \cap B_c \cap C_a| = 2$ we have $B_c \cap C_a \subseteq A_b$. So $B_c \cap (C_a \cup \{x\}) \subseteq A_b$ because $x \in A_b$. Hence $(A_b, A_c, B_a, B_c, C_a \cup \{x\}, C_b - \{x\})$ satisfies (B5). Therefore, the number of (B1)–(B6) which $(A_b, A_c, B_a, B_c, C_a \cup \{x\}, C_b - \{x\})$ satisfies is larger than the number of (B1)–(B6) which $(A_b, A_c, B_a, B_c, C_a, C_b)$ satisfies, contradicting our assumption.

(*ii*) Suppose that $|A_b \cap B_c \cap C_a| = 1$ and $|A_b \cap B_c \cap C_b| = 2$. Let y be the element of $A_b \cap B_c \cap C_a$. Since $|(A_b \cap B_c) - \{y\}| = 2$ and $G \setminus c$ is prime, $\rho_{G \setminus c}((A_b \cap B_c) - \{y\}) \ge 2$. So by Lemma 2.10,

$$2+2 \ge \rho_{G\backslash c}(C_b) + \rho_{G\backslash c}(A_b \cap B_c)$$

$$\ge \rho_{G\backslash c}((A_b \cap B_c) - \{y\}) + \rho_{G\backslash c}(C_b \cup \{y\}) \ge 2 + \rho_{G\backslash c}(C_b \cup \{y\}).$$

Therefore, $\rho_{G\backslash c}(C_b \cup \{y\}) \leq \rho_{G\backslash c}(C_b) \leq 2$. Since $G \setminus c$ is prime and $|V(G \setminus c) - (C_b \cup \{y\})| = |C_a| - 1 \geq 3$, we have $\rho_{G\backslash c}(C_b \cup \{y\}) = \rho_{G\backslash c}(C_b) = 2$. Hence by Lemma 3.1, neither $C_a - \{y\}$ nor $C_b \cup \{y\}$ is sequential in $G \setminus c$. We deduce that $(A_b, A_c, B_a, B_c, C_a - \{y\}, C_b \cup \{y\})$ satisfies (B1). Since $y \notin A_c \cap B_a$, if $(A_b, A_c, B_a, B_c, C_a, C_b)$ satisfies (B2), then $(A_b, A_c, B_a, B_c, C_a - \{y\}, C_b \cup \{y\})$ satisfies (B2). Since $y \in A_b \cap C_a$, by applying Lemma 5.6(i) with (A_c, A_b) and (C_a, C_b) , we have $|A_b \cap (C_a - \{y\})| = |A_b \cap C_a| - 1 \leq 3 - 1 = 2$. Hence $(A_b, A_c, B_a, B_c, C_a - \{y\}, C_b \cup \{y\})$ satisfies (B4). Since $y \in B_c \cap C_a$, by applying Lemma 5.6(ii) with (B_c, B_a) and (C_b, C_a) , we have $|B_c \cap (C_a - \{y\})| = |B_c \cap C_a| - 1 \leq 3 - 1 = 2$. Hence $(A_b, A_c, B_a, B_c, C_a - \{y\}, C_b \cup \{y\})$ satisfies (B5).

Since $A_c \cap (C_b \cup \{y\}) = A_c \cap C_b$, if $(A_b, A_c, B_a, B_c, C_a, C_b)$ satisfies (B3), then $(A_b, A_c, B_a, B_c, C_a - \{y\}, C_b \cup \{y\})$ satisfies (B3). Since $B_a \cap (C_b \cup \{y\}) = B_a \cap C_b$, if $(A_b, A_c, B_a, B_c, C_a, C_b)$ satisfies (B6), then $(A_b, A_c, B_a, B_c, C_a - \{y\}, C_b \cup \{y\})$ satisfies (B6). Therefore, the number of (B1)–(B6) which $(A_b, A_c, B_a, B_c, C_a - \{y\}, C_b \cup \{y\})$ satisfies is larger than the number of (B1)–(B6) which $(A_b, A_c, B_a, B_c, C_a, C_b)$ satisfies, contradicting our assumption.

Therefore, the claim is proved and $(A_b, A_c, B_a, B_c, C_a, C_b)$ satisfies (B1)–(B6).

Claim 5.9. $|A_b \cap B_a \cap C_a| \le 1$.

Proof. Suppose that $|A_b \cap B_a \cap C_a| \ge 2$. If $|A_b \cap C_a| = 2$, then $A_b \cap C_a \subseteq B_a$ and so $A_b \cap B_c \cap C_a = \emptyset$. If $|A_b \cap C_a| = 3$, then by (B4), $A_b \cap B_c \cap C_a = \emptyset$. Since $2 \le |A_b \cap C_a| \le 3$, we deduce that $A_b \cap B_c \cap C_a = \emptyset$.

By applying Lemma 5.6(ii) with (B_c, B_a) and (C_b, C_a) , we have that $|B_c \cap C_a| \ge 2$. Since $A_b \cap B_c \cap C_a = \emptyset$, we have $|A_c \cap B_c \cap C_a| = |B_c \cap C_a| \ge 2$ and so $|A_c \cap B_c| \ge |\{c\}| + |A_c \cap B_c \cap C_a| \ge 3$. Since $|A_c \cap B_c \cap C_a| \ge 2$, by Lemma 5.7(1), $\rho_{G\backslash a\backslash b}(A_b \cap B_a) \le 1$. So by Lemma 5.7(2),

$$|A_b \cap B_a| = 2$$
 and $\rho_{G \setminus c}((A_b \cap B_a) \cup \{a, b\}) = 3$,

because $|A_b \cap B_a| \ge |A_b \cap B_a \cap C_a| \ge 2$. Hence $A_b \cap B_a \subseteq C_a$.

By Lemma 5.2, $G \setminus a$ is prime and so $\rho_{G \setminus a}(A_b \cap B_a) = 2$. By (ii) of Lemma 2.8, we have $\rho_{G \setminus a}((A_b \cap B_a) \cup \{b\}) \leq \rho_{G \setminus a \setminus b}(A_b \cap B_a) + 1 \leq 2$. So by (A2) of Lemma 2.16,

$$\rho_{G\backslash c}((A_b \cap B_a) \cup \{a\}) + 2 \ge \rho_{G\backslash c}((A_b \cap B_a) \cup \{a\}) + \rho_{G\backslash a}((A_b \cap B_a) \cup \{b\})$$
$$\ge \rho_{G\backslash c}((A_b \cap B_a) \cup \{a,b\}) + \rho_{G\backslash a}(A_b \cap B_a) = 3 + 2,$$

which implies that $\rho_{G\setminus c}((A_b \cap B_a) \cup \{a\}) \geq 3$. Therefore, by Lemma 2.11,

$$3+2 \ge \rho_{G\backslash c}((A_b \cap B_a) \cup \{a,b\}) + \rho_{G\backslash c}(C_b)$$
$$\ge \rho_{G\backslash c}((A_b \cap B_a) \cup \{a\}) + \rho_{G\backslash c}(C_b - \{b\}) \ge 3 + \rho_{G\backslash c}(C_b - \{b\}).$$

Therefore, $\rho_{G\setminus c}(C_b - \{b\}) \leq 2$. By Lemma 5.2, $G \setminus c$ is prime. Since $|C_b - \{b\}| \geq 3$, we have $\rho_{G\setminus c}(C_b - \{b\}) = 2$. So by Lemma 3.1, neither $C_a \cup \{b\}$ nor $C_b - \{b\}$ is sequential in $G \setminus c$, contradicting Lemma 5.3 because $\{a, b\} \subseteq C_a \cup \{b\}$.

Hence, by symmetry, we have $|A_b \cap B_a \cap C_a| \leq 1$, $|A_c \cap B_a \cap C_a| \leq 1$, $|A_b \cap B_a \cap C_b| \leq 1$, $|A_b \cap B_c \cap C_b| \leq 1$, $|A_c \cap B_c \cap C_a| \leq 1$, and $|A_c \cap B_c \cap C_b| \leq 1$.

Claim 5.10.
$$|A_b \cap B_c \cap C_a| \le 1$$
.

Proof. Suppose that $|A_b \cap B_c \cap C_a| \ge 2$. If $|A_b \cap B_c| = 2$, then $A_b \cap B_c \subseteq C_a$ and $A_b \cap B_c \cap C_b = \emptyset$. If $|A_b \cap B_c| = 3$, then by (B1), $A_b \cap B_c \cap C_b = \emptyset$. By Lemma 5.6(i), we have $2 \le |A_b \cap B_c| \le 3$. So we deduce that $A_b \cap B_c \cap C_b = \emptyset$.

By symmetry between (a, b, c) and (c, a, b), we deduce that $C_a \cap A_b \cap B_a = \emptyset$. By symmetry between (a, b, c) and (b, c, a), we deduce that $B_c \cap C_a \cap A_c = \emptyset$. By Lemma 5.6(iv), $|A_c \cap B_c| \ge 2$. So we deduce that

$$1 \le |A_c \cap B_c| - |\{c\}| - |A_c \cap B_c \cap C_a| = |A_c \cap B_c \cap C_b| \le 1,$$

and therefore $|A_c \cap B_c \cap C_b| = 1$.

If $|A_c \cap C_b| = 3$, then by (B3), $|A_c \cap B_a \cap C_b| = 0$. If $|A_c \cap C_b| \le 2$, then $|A_c \cap B_a \cap C_b| = |A_c \cap C_b| - |A_c \cap B_c \cap C_b| \le 2 - 1 = 1$. Since $|A_c \cap C_b| \le 3$, in both cases, we deduce that $|A_c \cap B_a \cap C_b| \le 1$. Then we have

$$\begin{aligned} |V(G)| &= |A_b \cap B_a \cap C_a| + |A_b \cap B_a \cap C_b| + |A_b \cap B_c \cap C_a| + |A_b \cap B_c \cap C_b| \\ &+ |A_c \cap B_a \cap C_a| + |A_c \cap B_a \cap C_b| + |A_c \cap B_c \cap C_a| + |A_c \cap B_c \cap C_b| + |\{a, b, c\}| \\ &= 0 + |A_b \cap B_a \cap C_b| + |A_b \cap B_c \cap C_a| + 0 \\ &+ |A_c \cap B_a \cap C_a| + |A_c \cap B_a \cap C_b| + 0 + |A_c \cap B_c \cap C_b| + |\{a, b, c\}| \\ &\leq 0 + 1 + |A_b \cap B_c| + 0 + 1 + 1 + 0 + 1 + 3 \leq 10, \end{aligned}$$

contradicting our assumption.

By symmetry, we have $|A_c \cap B_a \cap C_b| \leq 1$. Therefore, we have

$$|V(G)| = |A_b \cap B_a \cap C_a| + |A_b \cap B_a \cap C_b| + |A_b \cap B_c \cap C_a| + |A_b \cap B_c \cap C_b| + |A_c \cap B_a \cap C_a| + |A_c \cap B_a \cap C_b| + |A_c \cap B_c \cap C_a| + |A_c \cap B_c \cap C_b| + |\{a, b, c\}| \le 11,$$

contradicting our assumption.

6 Completing the proof

A set X of vertices of a graph G is fully closed if $\rho_G(X \cup \{v\}) > \rho_G(X)$ for all $v \in V(G) - X$.

Lemma 6.1 (Oum [8, Proposition 3.1]). Let G be a prime graph with $|V(G)| \ge 8$. Suppose that G has a fully closed set A such that $\rho_G(A) \ge 2$. Then there is a vertex v of A such that $G \setminus v$ or G/v is prime.

Lemma 6.2. Let G be a sequentially 3-rank-connected graph and a_1, a_2, \ldots, a_k be distinct vertices of G such that $k \ge 4$ and $\rho_G(\{a_1, \ldots, a_i\}) \le 2$ for each $i \le k$. For each $1 \le i \le k$, if $G \setminus a_i$ is prime, then $G \setminus a_i$ is sequentially 3-rank-connected.

Proof. Since G is prime, we know that $\rho_G(\{a_1, \ldots, a_j\}) = \min\{2, |V(G)| - j\}$ for each $2 \le j \le k$. So $\rho_G(\{a_1, \ldots, a_{j-1}\}) \ge \rho_G(\{a_1, \ldots, a_j\})$ for each $2 \le j \le k$. For each $3 \le j \le i - 1$, by (S2) of Lemma 2.12, we have

 $\rho_G(\{a_1, \dots, a_j\}) + \rho_{G \setminus a_i}(\{a_1, \dots, a_{j-1}\}) \ge \rho_G(\{a_1, \dots, a_{j-1}\}) + \rho_{G \setminus a_i}(\{a_1, \dots, a_j\})$

and therefore $\rho_{G\setminus a_i}(\{a_1,\ldots,a_{j-1}\}) \ge \rho_{G\setminus a_i}(\{a_1,\ldots,a_j\}).$

Suppose that $G \setminus a_i$ is prime and not sequentially 3-rank-connected.

Let us first consider the case when i > 3. By Lemma 3.2, there is a subset X of $V(G \setminus a_i)$ such that $\rho_{G \setminus a_i}(X) \leq 2$, neither X nor $V(G \setminus a_i) - X$ is sequential in $G \setminus a_i$, and $\{a_1, a_2, a_3\} \subseteq X$. We may assume that X is maximal among all such sets.

We claim that $\{a_1, \ldots, a_{i-1}\} \subseteq X$. Suppose not. Let $j \leq i-1$ be the minimum index such that $a_j \notin X$. Then $\{a_1, \ldots, a_{j-1}\} \subseteq X$. Note that $j \geq 4$. Let $Y = V(G \setminus a_i) - X$. Since neither X nor Y is sequential in $G \setminus a_i$, we have $|X|, |Y| \geq 4$. Since $\rho_{G \setminus a_i}(\{a_1, \ldots, a_{j-1}\}) \geq \rho_{G \setminus a_i}(\{a_1, \ldots, a_j\})$, by Lemma 2.10,

$$\rho_{G\setminus a_i}(X) + \rho_{G\setminus a_i}(\{a_1, \dots, a_j\}) \ge \rho_{G\setminus a_i}(X \cup \{a_j\}) + \rho_{G\setminus a_i}(\{a_1, \dots, a_{j-1}\}),$$

and therefore $\rho_{G\setminus a_i}(X \cup \{a_j\}) \leq \rho_{G\setminus a_i}(X) \leq 2$. Since $G \setminus a_i$ is prime and $|Y - \{a_i\}| \geq 3$, we have $\rho_{G\setminus a_i}(X \cup \{a_j\}) = \rho_{G\setminus a_i}(X) = 2$. Hence by Lemma 3.1, neither $X \cup \{a_j\}$ nor $Y - \{a_j\}$ is sequential in $G \setminus a_i$, contradicting the maximality of X. Hence $\{a_1, \ldots, a_{i-1}\} \subseteq X$.

Then by (S1) of Lemma 2.12,

$$\rho_{G\setminus a_i}(X) + \rho_G(\{a_1, \dots, a_i\}) \ge \rho_G(X \cup \{a_i\}) + \rho_{G\setminus a_i}(\{a_1, \dots, a_{i-1}\}).$$

Since $G \setminus a_i$ is prime and i > 3, we have $\rho_{G \setminus a_i}(\{a_1, \ldots, a_{i-1}\}) \ge \min\{2, |V(G)| - i\} = \rho_G(\{a_1, \ldots, a_i\})$. So $\rho_G(X \cup \{a_i\}) \le \rho_{G \setminus a_i}(X) \le 2$. Since G is sequentially 3-rank-connected, $X \cup \{a_i\}$ or Y is sequential in G. Then by (i), (ii) of Lemma 2.8, X or Y is sequential in $G \setminus a_i$, contradicting our assumption.

Now we consider the case when $i \leq 3$. By permuting a_1, a_2, a_3 , we can assume that i = 3. Suppose that $G \setminus a_3$ is prime. By Lemma 2.8(ii), we have $\rho_{G \setminus a_3}(\{a_1, a_2, a_4\}) \leq \rho_G(\{a_1, a_2, a_3, a_4\}) \leq 2$. Since a_1, a_2, a_4, a_3 is another sequence satisfying all the requirements, we conclude that $G \setminus a_3$ is sequentially 3-rank-connected because we proved the statement for i > 3.

Lemma 6.3. Let G be a sequentially 3-rank-connected graph with $|V(G)| \ge 8$ and a_1, a_2, \ldots, a_k be distinct vertices of G such that $k \ge 4$, $k \ne |V(G)| - 1$, and $\rho_G(\{a_1, \ldots, a_i\}) \le 2$ for each $i \le k$. If $\{a_1, \ldots, a_k\}$ is a fully closed set of G, then there exists $i \in \{1, \ldots, k\}$ such that $G \setminus a_i$ or G/a_i is sequentially 3-rank-connected.

Proof. By Theorem 1.1 and Lemma 6.2, we may assume that $k \neq |V(G)|$ and therefore $k \leq |V(G)| - 2$. Since G is prime, we have $\rho_G(\{a_1, \ldots, a_k\}) = 2$ and so, by Lemma 6.1, there is a vertex a_i of G such that $G \setminus a_i$ or G/a_i is prime. By pivoting, we may assume that $G \setminus a_i$ is prime. Then, by Lemma 6.2, $G \setminus a_i$ is sequentially 3-rank-connected.

Proof of Theorem 1.2. By Proposition 4.15, we may assume that G is not 3-rank-connected. So there is a subset A of V(G) such that $\rho_G(A) \leq 2$, $|A| \geq 3$, and $|V(G) - A| \geq 3$. If G is internally 3-rank-connected, then we may assume that |A| = 3. By Lemma 5.1, we can assume that A is a triplet of G by pivoting. By Proposition 5.8, there is a vertex $a \in A$ such that $G \setminus a$ is sequentially 3-rank-connected. Hence we may assume that G is not internally 3-rank-connected.

Therefore, we may assume that $|A| \ge 4$ and $|V(G) - A| \ge 4$. Since G is sequentially 3-rankconnected, A or V(G) - A is sequential in G. Therefore there exists a sequential set with at least 4 elements.

Let X be a maximum sequential set of G. Then X is a fully closed set of G. Furthermore, $|X| \neq |V(G)| - 1$ because otherwise V(G) is sequential in G. Since $|X| \geq 4$, we conclude the proof by Lemma 6.3.

Acknowledgements The authors would like to thank the anonymous reviewers for their careful reading and useful comments. In particular, the paragraph following Theorem 1.3 was suggested by one of the anonymous reviewers.

References

- Loïc Allys, Minimally 3-connected isotropic systems, Combinatorica 14 (1994), no. 3, 247– 262. MR 1305894
- [2] André Bouchet, Reducing prime graphs and recognizing circle graphs, Combinatorica 7 (1987), no. 3, 243–254. MR 918395
- [3] _____, Graphic presentations of isotropic systems, J. Combin. Theory Ser. B **45** (1988), no. 1, 58–76. MR 953895
- [4] Jim Geelen and Sang-il Oum, Circle graph obstructions under pivoting, J. Graph Theory 61 (2009), no. 1, 1–11. MR 2514095
- [5] Jim Geelen and Geoff Whittle, Matroid 4-connectivity: a deletion-contraction theorem, J. Combin. Theory Ser. B 83 (2001), no. 1, 15–37. MR 1855794
- [6] _____, Inequivalent representations of matroids over prime fields, Adv. in Appl. Math. 51 (2013), no. 1, 1–175. MR 3056744
- [7] Sang-il Oum, Rank-width and vertex-minors, J. Combin. Theory Ser. B 95 (2005), no. 1, 79–100. MR 2156341
- [8] Sang-il Oum, Rank connectivity and pivot-minors of graphs, European J. Combin. 108 (2023), 103634.
- [9] James Oxley, Charles Semple, and Geoff Whittle, An upgraded wheels-and-whirls theorem for 3-connected matroids, J. Combin. Theory Ser. B 102 (2012), no. 3, 610–637. MR 2900807
- [10] Klaus Truemper, A decomposition theory for matroids. I. General results, J. Combin. Theory Ser. B 39 (1985), no. 1, 43–76. MR 805456
- [11] William T. Tutte, A theory of 3-connected graphs, Nederl. Akad. Wetensch. Proc. Ser. A 64 Indag. Math. 23 (1961), 441–455. MR 0140094