# A chain theorem for sequentially 3-rank-connected graphs with respect to vertex-minors 

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#### Abstract

Tutte (1961) proved the chain theorem for simple 3-connected graphs with respect to minors, which states that every simple 3-connected graph $G$ has a simple 3-connected minor with one edge fewer than $G$, unless $G$ is a wheel graph. Bouchet (1987) proved an analog for prime graphs with respect to vertex-minors. We present a chain theorem for higher connectivity with respect to vertex-minors, showing that every sequentially 3 -rank-connected graph $G$ has a sequentially 3 -rank-connected vertex-minor with one vertex fewer than $G$, unless $|V(G)| \leq 12$.


## 1 Introduction

Tutte [11] proved the chain theorem for simple 3-connected graphs with respect to minors, which states that every simple 3 -connected graph $G$ has a simple 3 -connected minor with one edge fewer than $G$, unless $G$ is a wheel graph. We will present a chain theorem for vertex-minors.

For a vertex $v$ of a graph $G$, the local complementation at $v$ is an operation obtaining a new graph $G * v$ from $G$ by replacing the subgraph induced by the neighbors of $v$ with its complement graph. A graph $H$ is a vertex-minor of $G$ if $H$ can be obtained from $G$ by a sequence of local complementations and vertex deletions.

For a graph $G$, the cut-rank function $\rho_{G}$ is a function which maps a set $X$ of vertices of $G$ to the rank of a matrix over the binary field whose rows are labeled by $X$ and columns are labeled by $V(G)-X$, where the $(i, j)$-entry is 1 if $i$ and $j$ are adjacent in $G$ and 0 otherwise. A graph $G$ is prime if there is no set $X$ of vertices of $G$ such that $|X| \geq 2,|V(G)-X| \geq 2$, and $\rho_{G}(X) \leq 1$. Bouchet proved the following chain theorem for prime graphs with respect to vertex-minors. Later, Allys [1] proved a stronger theorem.

Theorem 1.1 (Bouchet [2, Theorem 3.2]). Every prime graph $G$ has a prime vertex-minor $H$ with $|V(H)|=|V(G)|-1$, unless $|V(G)| \leq 5$.

A set $X$ of vertices of $G$ is sequential in $G$ if there is an ordering $a_{1}, \ldots, a_{k}$ of the vertices in $X$ such that $\rho_{G}\left(\left\{a_{1}, \ldots, a_{i}\right\}\right) \leq 2$ for each $1 \leq i \leq k$. A graph $G$ is sequentially 3 -rank-connected if it is prime and whenever $\rho_{G}(X) \leq 2$ for $X \subseteq V(G)$, either $X$ or $V(G)-X$ is sequential in $G$.

[^0]Here is our chain theorem for sequentially 3 -rank-connected graphs with respect to vertexminors.

Theorem 1.2. Every sequentially 3-rank-connected graph $G$ has a sequentially 3-rank-connected vertex-minor $H$ with $|V(H)|=|V(G)|-1$, unless $|V(G)| \leq 12$.

Our theorem is motivated by the following theorem for sequentially 4 -connected matroids, proved by Geelen and Whittle.
Theorem 1.3 (Geelen and Whittle [5, Theorem 1.2]). Every sequentially 4-connected matroid $M$ has a sequentially 4-connected minor $N$ with $|E(N)|=|E(M)|-1$, unless $M$ is a wheel matroid or a whirl matroid.

Theorem 1.3 was motivated by the conjecture on the number of inequivalent representations over a fixed prime field. This conjecture was later proved by Geelen and Whittle [6] by using a stronger version of Theorem 1.3 due to Oxley, Semple, and Whittle [9]. It would be interesting to see if this stronger version also has a vertex-minor analog.

Let us briefly sketch the proof of Theorem 1.2. The proof consists of three parts. In the first part, we prove it for 3 -rank-connected graphs that are prime graphs with no set $X$ such that $\rho_{G}(X) \leq 2,|X|>2$, and $|V(G)-X|>2$. The second part discusses internally 3-rank-connected graphs that are not 3 -rank-connected. The last part considers sequentially 3 -rank-connected graphs that are not internally 3 -rank-connected.

Essentially, the proof is based on the submodularity of the matrix rank function. We will also use Theorem 1.1. Proof ideas of some lemmas are from Geelen and Whittle [5]. We will also use triplets introduced by Oum [8].

Our paper is organized as follows. In Section 2, we review vertex-minors and several inequalities for cut-rank functions. In Section 3, we prove elementary lemmas on sequential sets and sequentially 3 -rank-connected graphs. In Section 4, we prove the main theorem for 3 -rankconnected graphs. In Section 5, we prove our theorem for internally 3 -rank-connected graphs. In Section 6, we conclude the proof by dealing with sequentially 3 -rank-connected graphs which are not internally 3 -rank-connected.

## 2 Preliminaries

A graph is simple if it has no loops and parallel edges. In this paper, all graphs are finite and simple. For a graph $G$ and a vertex $v$, let $N_{G}(v)$ be the set of vertices adjacent to $v$ in $G$. For a graph $G$ and a subset $X$ of $V(G)$, let $G[X]$ be the subgraph of $G$ induced on $X$.

Vertex-minors For a graph $G$ and a vertex $v$ of $G$, let $G * v$ be the graph obtained by replacing $G\left[N_{G}(v)\right]$ with its complement. The operation obtaining $G * v$ from $G$ is called the local complementation at $v$. A graph $H$ is locally equivalent to $G$ if $H$ can be obtained from $G$ by a sequence of local complementations. A graph $H$ is a vertex-minor of a graph $G$ if $H$ can be obtained from $G$ by applying local complementations and deleting vertices.

For an edge $u v$ of a graph $G$, let $G \wedge u v=G * u * v * u$. Then $G \wedge u v$ is obtained from $G$ by pivoting $u v$. The graph $G \wedge u v$ is well defined since $G * u * v * u=G * v * u * v$ [7, Corollary 2.2].

Lemma 2.1 (see Oum [7]). Let $G$ be a graph and $v$ be a vertex of $G$. If $x, y \in N_{G}(v)$, then $(G \wedge v x) \backslash v$ is locally equivalent to $(G \wedge v y) \backslash v$.

By Lemma 2.1, we write $G / v$ to denote $G \wedge u v \backslash v$ for a neighbor $u$ of $v$ in $G$ because we are only interested in graphs up to local equivalence.

Lemma 2.2 (Geelen and Oum [4, Lemma 3.1]). Let $G$ be a graph and $v$ and $w$ be vertices of $G$. Then the following hold.
(1) If $v \neq w$ and $v w \notin E(G)$, then $(G * w) \backslash v,(G * w * v) \backslash v$, and $(G * w) / v$ are locally equivalent to $G \backslash v, G * v \backslash v$, and $G / v$ respectively.
(2) If $v \neq w$ and $v w \in E(G)$, then $(G * w) \backslash v,(G * w * v) \backslash v$, and $(G * w) / v$ are locally equivalent to $G \backslash v, G / v$, and $(G * v) \backslash v$ respectively.
(3) If $v=w$, then $(G * w) \backslash v,(G * w * v) \backslash v$, and $(G * w) / v$ are locally equivalent to $G * v \backslash v$, $G \backslash v$, and $G / v$ respectively.

Lemma 2.2 implies the following lemma, which was first proved by Bouchet.
Lemma 2.3 (Bouchet [3, Corollary 9.2]). Let $H$ be a vertex-minor of a graph $G$ such that $V(H)=V(G)-\{v\}$ for a vertex $v$ of $G$. Then $H$ is locally equivalent to one of $G \backslash v, G * v \backslash v$, and $G / v$.

Cut-rank function and rank-connectivity For an $X \times Y$-matrix $A$ and $I \subseteq X, J \subseteq Y$, let $A[I, J]$ be an $I \times J$-submatrix of $A$. Let $A_{G}$ be the adjacency matrix of a graph $G$ over the binary field GF(2). The cut-rank $\rho_{G}(X)$ of a subset $X$ of $V(G)$ is defined by

$$
\rho_{G}(X)=\operatorname{rank}\left(A_{G}[X, V(G)-X]\right) .
$$

It is trivial to check that $\rho_{G}(X)=\rho_{G}(V(G)-X)$. For disjoint sets $X, Y$ of a graph $G$, let $\rho_{G}(X, Y)=\operatorname{rank}\left(A_{G}[X, Y]\right)$. A graph $G$ is $k$-rank-connected if there is no partition $(A, B)$ of $V(G)$ such that $|A|,|B|>\rho_{G}(A)$ and $\rho_{G}(A)<k$. A graph is prime if it is 2-rank-connected. Observe that 1-rank-connected graphs are connected graphs.

Lemma 2.4. If $G$ is a 3 -rank-connected graph with at least 6 vertices, then $\operatorname{deg}_{G}(v) \geq 3$ for each $v \in V(G)$.

Proof. Suppose that $\operatorname{deg}_{G}(v) \leq 2$. Let $X$ be the set of neighbors of $v$. Then $\rho_{G}(X \cup\{v\}) \leq|X| \leq$ 2. However, $\rho_{G}(X \cup\{v\})<|X \cup\{v\}|$ and $2<|V(G)-(X \cup\{v\})|$, contradicting assumption that $G$ is 3 -rank-connected.

Lemma 2.5 (Oum [8, Proposition 2.4]). Let $k$ be a positive integer. If a graph $G$ is $k$-rankconnected and $|V(G)| \geq 2 k$, then for each $v \in V(G)$, the graph $G \backslash v$ is $(k-1)$-rank-connected.

Lemma 2.6. Let $k$ be a positive integer. A $k$-rank-connected graph with $|V(G)| \geq 2 k$ is $k$ connected.

Proof. We use induction on $k$. Let $G$ be a $k$-rank-connected graph with $|V(G)| \geq 2 k$. We may assume that $k>1$. Let $X$ be a subset of $V(G)$ with $|X|<k$. It is enough to prove that $G \backslash X$ is connected. Since $G$ is 1-rank-connected, $G$ is connected and therefore we may assume that $X$ is nonempty. Let $v$ be a vertex in $X$. By applying Lemma 2.5 and the induction hypothesis, $G \backslash v$ is $(k-1)$-connected and therefore $(G \backslash v) \backslash(X-\{v\})=G \backslash X$ is connected.

The following lemmas give properties of the matrix rank function and the cut-rank function.
Lemma 2.7 (see Oum [7, Proposition 2.6]). If a graph $G^{\prime}$ is locally equivalent to a graph $G$, then $\rho_{G}(X)=\rho_{G^{\prime}}(X)$ for each $X \subseteq V(G)$.

Lemma 2.8. Let $G$ be a graph and $v$ be a vertex of $G$. For a subset $X$ of $V(G)-\{v\}$, we have
(i) $\rho_{G \backslash v}(X)+1 \geq \rho_{G}(X) \geq \rho_{G \backslash v}(X)$.
(ii) $\rho_{G \backslash v}(X)+1 \geq \rho_{G}(X \cup\{v\}) \geq \rho_{G \backslash v}(X)$.

Proof. Observe that removing a row or a column of a matrix decreases the rank by at most 1.

Lemma 2.9 (see Truemper [10]). Let $A$ be an $X \times Y$-matrix. For sets $X_{1}, X_{2} \subseteq X$ and $Y_{1}, Y_{2} \subseteq Y$,

$$
\operatorname{rank}\left(A\left[X_{1}, Y_{1}\right]\right)+\operatorname{rank}\left(A\left[X_{2}, Y_{2}\right]\right) \geq \operatorname{rank}\left(A\left[X_{1} \cap X_{2}, Y_{1} \cup Y_{2}\right]\right)+\operatorname{rank}\left(A\left[X_{1} \cup X_{2}, Y_{1} \cap Y_{2}\right]\right)
$$

Lemma 2.9 implies the following seven lemmas.
Lemma 2.10 (see Oum [7, Corollary 4.2]). Let $G$ be a graph and let $X, Y$ be subsets of $V(G)$. Then,

$$
\rho_{G}(X)+\rho_{G}(Y) \geq \rho_{G}(X \cap Y)+\rho_{G}(X \cup Y)
$$

Lemma 2.11. Let $G$ be a graph and $X$ and $Y$ be subsets of $V(G)$. Then,

$$
\rho_{G}(X)+\rho_{G}(Y) \geq \rho_{G}(Y-X)+\rho_{G}(X-Y)
$$

Proof. Apply Lemma 2.10 with $X$ and $V(G)-Y$.
Lemma 2.12 (Oum [8, Lemma 2.3]). Let $G$ be a graph and $v$ be a vertex of $G$. Let $X$ and $Y$ be subsets of $V(G)-\{v\}$. Then, the following hold.
(S1) $\rho_{G \backslash v}(X)+\rho_{G}(Y \cup\{v\}) \geq \rho_{G \backslash v}(X \cap Y)+\rho_{G}(X \cup Y \cup\{v\})$.
(S2) $\rho_{G \backslash v}(X)+\rho_{G}(Y) \geq \rho_{G}(X \cap Y)+\rho_{G \backslash v}(X \cup Y)$.
Lemma 2.13. Let $G$ be a graph and $v$ be a vertex of $G$. Let $X, Y$ be subsets of $V(G \backslash v)$. If $X \subseteq Y$ and $\rho_{G \backslash v}(Y) \geq \rho_{G}(Y)$, then $\rho_{G \backslash v}(X)=\rho_{G}(X)$.

Proof. By (S2) of Lemma 2.12,

$$
\rho_{G \backslash v}(X)+\rho_{G}(Y) \geq \rho_{G \backslash v}(Y)+\rho_{G}(X)
$$

Therefore, by Lemma 2.8(i), $0 \leq \rho_{G}(X)-\rho_{G \backslash v}(X) \leq \rho_{G}(Y)-\rho_{G \backslash v}(Y) \leq 0$. So we conclude that $\rho_{G \backslash v}(X)=\rho_{G}(X)$.

Lemma 2.14. Let $G$ be a graph and $v$ be a vertex of $G$. Let $X, Y$ be subsets of $V(G)$. If $v \in Y \subseteq X$ and $\rho_{G \backslash v}(Y-\{v\}) \geq \rho_{G}(Y)$, then $\rho_{G \backslash v}(X-\{v\})=\rho_{G}(X)$.

Proof. We apply Lemma 2.13 for $V(G)-X$ and $V(G)-Y$.
Lemma 2.15. Let $G$ be a graph and $v$ be a vertex of $G$. Let $X$ and $Y$ be subsets of $V(G)-\{v\}$. Then,

$$
\rho_{G \backslash v}(X)+\rho_{G}(Y \cup\{v\}) \geq \rho_{G \backslash v}(Y-X)+\rho_{G}(X-Y)
$$

Proof. Apply (S1) of Lemma 2.12 with $V(G)-(X \cup\{v\})$ and $Y$.
Lemma 2.16 (Oum [8, Lemma 2.2]). Let $G$ be a graph and $a, b$ be distinct vertices of $G$. Let $A \subseteq V(G)-\{a\}$ and $B \subseteq V(G)-\{b\}$. Then, the following hold.
(A1) If $b \notin A$ and $a \notin B$, then $\rho_{G}(A \cap B)+\rho_{G \backslash a \backslash b}(A \cup B) \leq \rho_{G \backslash a}(A)+\rho_{G \backslash b}(B)$.
(A2) If $b \in A$ and $a \notin B$, then $\rho_{G \backslash b}(A \cap B)+\rho_{G \backslash a}(A \cup B) \leq \rho_{G \backslash a}(A)+\rho_{G \backslash b}(B)$.
(A3) If $b \in A$ and $a \in B$, then $\rho_{G \backslash a \backslash b}(A \cap B)+\rho_{G}(A \cup B) \leq \rho_{G \backslash a}(A)+\rho_{G \backslash b}(B)$.
Lemma 2.17 (Oum [7, Proposition 4.3]). Let $G$ be a graph and $x$ be a vertex of $G$. For a subset $X$ of $V(G)-\{x\}$, the following hold.
(1) $\rho_{G * x \backslash x}(X)=\operatorname{rank}\left(\begin{array}{cc}1 & A_{G}[\{x\}, V(G)-(X \cup\{x\})] \\ A_{G}[X,\{x\}] & A_{G}[X, V(G)-(X \cup\{x\})]\end{array}\right)-1$.

$$
\rho_{G / x}(X)=\operatorname{rank}\left(\begin{array}{cc}
0 & A_{G}[\{x\}, V(G)-(X \cup\{x\})]  \tag{2}\\
A_{G}[X,\{x\}] & A_{G}[X, V(G)-(X \cup\{x\})]
\end{array}\right)-1 .
$$

From Lemma 2.17, we deduce the following lemma.
Lemma 2.18. Let $G$ be a graph and $x \in V(G)$. Let $C$ be a subset of $V(G)-\{x\}$ such that $\rho_{G \backslash x}(C)=\rho_{G}(C)$. Then $\rho_{G * x \backslash x}(C)=\rho_{G}(C \cup\{x\})-1$ or $\rho_{G / x}(C)=\rho_{G}(C \cup\{x\})-1$.

Proof. Let $D=V(G)-(C \cup\{x\})$. Since $\rho_{G \backslash x}(C)=\rho_{G}(C)$, a column vector $A_{G}[C,\{x\}]$ is in the column space of $A_{G}[C, D]$. Then let $A^{\prime}$ and $A^{\prime \prime}$ be matrices over GF(2) such that

$$
A^{\prime}=\left(\begin{array}{cc}
1 & A_{G}[\{x\}, D] \\
A_{G}[C,\{x\}] & A_{G}[C, D]
\end{array}\right) \text { and } A^{\prime \prime}=\left(\begin{array}{cc}
0 & A_{G}[\{x\}, D] \\
A_{G}[C,\{x\}] & A_{G}[C, D]
\end{array}\right) .
$$

Then $\operatorname{rank}\left(A^{\prime}\right)=\rho_{G}(C \cup\{x\})$ or $\operatorname{rank}\left(A^{\prime \prime}\right)=\rho_{G}(C \cup\{x\})$ and therefore, by Lemma 2.17, we have $\rho_{G * x \backslash x}(C)=\operatorname{rank}\left(A^{\prime}\right)-1=\rho_{G}(C \cup\{x\})-1$ or $\rho_{G / x}(C)=\operatorname{rank}\left(A^{\prime \prime}\right)-1=\rho_{G}(C \cup\{x\})-1$.

Lemma 2.19 (Oum [7, Lemma 4.4]). Let $G$ be a graph and $x$ be a vertex of $G$. Let ( $X_{1}, Y_{1}$ ) and $\left(X_{2}, Y_{2}\right)$ be partitions of $V(G)-\{x\}$. Then the following hold:
(P1) $\rho_{G \backslash x}\left(X_{1}\right)+\rho_{G * x \backslash x}\left(X_{2}\right) \geq \rho_{G}\left(X_{1} \cap X_{2}\right)+\rho_{G}\left(Y_{1} \cap Y_{2}\right)-1$.
(P2) $\rho_{G \backslash x}\left(X_{1}\right)+\rho_{G / x}\left(X_{2}\right) \geq \rho_{G}\left(X_{1} \cap X_{2}\right)+\rho_{G}\left(Y_{1} \cap Y_{2}\right)-1$.
The following lemma is an easy consequence of Lemmas 2.7 and 2.19.
Lemma 2.20. Let $G$ be a graph and $x$ be a vertex of $G$. Let $\left(X_{1}, Y_{1}\right)$ and $\left(X_{2}, Y_{2}\right)$ be partitions of $V(G)-\{x\}$. Then,

$$
\rho_{G * x \backslash x}\left(X_{1}\right)+\rho_{G / x}\left(X_{2}\right) \geq \rho_{G}\left(X_{1} \cap X_{2}\right)+\rho_{G}\left(Y_{1} \cap Y_{2}\right)-1 .
$$

## 3 Sequentially 3-rank-connected graphs

Let us recall the definition of sequentially 3 -rank-connected graphs introduced in Section 1. A subset $A$ of $V(G)$ is sequential in a graph $G$ if there is an ordering $a_{1}, \ldots, a_{|A|}$ of the elements of $A$ such that $\rho_{G}\left(\left\{a_{1}, \ldots, a_{i}\right\}\right) \leq 2$ for each $1 \leq i \leq|A|$. A graph $G$ is sequentially 3 -rankconnected if it is prime and for each subset $X$ of $V(G)$ with $\rho_{G}(X) \leq 2$, we have that $X$ or $V(G)-X$ is sequential in $G$.

We now present basic lemmas on sequential sets and sequentially 3 -rank-connected graphs.
Lemma 3.1. Let $G$ be a graph and $A$ be a subset of $V(G)$. Let $t$ be a vertex of $G$ such that $\rho_{G}(A \cup\{t\})=\rho_{G}(A)$. Then $A \cup\{t\}$ is sequential in $G$ if and only if $A$ is sequential in $G$.
Proof. We may assume that $t \notin A$. The backward direction is obvious. So it is enough to show the forward direction.

Since $A \cup\{t\}$ is sequential in $G$, there is an ordering $a_{1}, \ldots, a_{m}$ of the elements of $A \cup\{t\}$ such that $m=|A \cup\{t\}|$ and $\rho_{G}\left(\left\{a_{1}, \ldots, a_{i}\right\}\right) \leq 2$ for each $1 \leq i \leq m$. Let $1 \leq j \leq m$ be an index such that $a_{j}=t$. Then for each $j+1 \leq i \leq m$, by Lemma 2.10, we have

$$
\rho_{G}\left(\left\{a_{1}, \ldots, a_{i}\right\}\right)+\rho_{G}(A) \geq \rho_{G}(A \cup\{t\})+\rho_{G}\left(\left\{a_{1}, \ldots, a_{i}\right\}-\{t\}\right),
$$

and therefore $\rho_{G}\left(\left\{a_{1}, \ldots, a_{i}\right\}-\{t\}\right) \leq \rho_{G}\left(\left\{a_{1}, \ldots, a_{i}\right\}\right)$. For each $1 \leq i \leq m-1$, let

$$
a_{i}^{\prime}= \begin{cases}a_{i} & \text { if } i<j, \\ a_{i+1} & \text { if } i \geq j .\end{cases}
$$

Hence, by above inequality, $A$ is sequential in $G$ because $a_{1}^{\prime}, \ldots, a_{m-1}^{\prime}$ is a desired ordering of the elements of $A$.

Lemma 3.2. Let $G$ be a prime graph that is not sequentially 3 -rank-connected and let $T_{1}, \ldots, T_{n}$ be pairwise disjoint 3 -element subsets of $V(G)$ such that $\rho_{G}\left(T_{i}\right)=2$ for each $1 \leq i \leq n$. Then there exists a subset $A$ of $V(G)$ such that $\rho_{G}(A) \leq 2$, neither $A$ nor $V(G)-A$ is sequential in $G$, and for each $1 \leq i \leq n$, we have that $T_{i} \subseteq A$ or $T_{i} \subseteq V(G)-A$.

Proof. We proceed by induction on $n$. Since $G$ is prime and not sequentially 3 -rank-connected, there is a subset $A$ of $V(G)$ such that $\rho_{G}(A) \leq 2$, and neither $A$ nor $V(G)-A$ is sequential in $G$. So we can assume that $n \geq 1$. By the induction hypothesis, there is a subset $A^{\prime}$ of $V(G)$ such that $\rho_{G}\left(A^{\prime}\right) \leq 2$, and neither $A^{\prime}$ nor $V(G)-A^{\prime}$ is sequential in $G$, and for each $1 \leq i \leq n-1$, either $T_{i} \subseteq A^{\prime}$ or $T_{i} \subseteq V(G)-A^{\prime}$. Let $B^{\prime}=V(G)-A^{\prime}$. We may assume that $A^{\prime} \cap T_{n} \neq \emptyset$ and $B^{\prime} \cap T_{n} \neq \emptyset$. Then, by symmetry, we can assume that $\left|A^{\prime} \cap T_{n}\right|=2$ and let $x$ be the element of $B^{\prime} \cap T_{n}$. Since $\left|T_{n}-\{x\}\right|=2$ and $G$ is prime, we have $\rho_{G}\left(T_{n}-\{x\}\right)=2=\rho_{G}\left(T_{n}\right)$. Then, by Lemma 2.10,

$$
\rho_{G}\left(A^{\prime}\right)+2=\rho_{G}\left(A^{\prime}\right)+\rho_{G}\left(T_{n}\right) \geq \rho_{G}\left(A^{\prime} \cup\{x\}\right)+\rho_{G}\left(T_{n}-\{x\}\right)=\rho_{G}\left(A^{\prime} \cup\{x\}\right)+2 .
$$

Hence $\rho_{G}\left(A^{\prime} \cup\{x\}\right) \leq \rho_{G}\left(A^{\prime}\right) \leq 2$. Since $V(G)-A^{\prime}$ is not sequential in $G,\left|V(G)-A^{\prime}\right| \geq 4$ and so $\left|V(G)-\left(A^{\prime} \cup\{x\}\right)\right| \geq 3$. Hence $\rho_{G}\left(A^{\prime}\right)=\rho_{G}\left(A^{\prime} \cup\{x\}\right)=2$ because $G$ is prime. Hence, by Lemma 3.1, neither $A^{\prime} \cup\{x\}$ nor $V(G)-\left(A^{\prime} \cup\{x\}\right)$ is sequential in $G$.

For each $1 \leq i \leq n-1$, we have $x \notin T_{i}$ because $T_{n}$ and $T_{i}$ are disjoint. Therefore, $T_{i} \subseteq A^{\prime} \cup\{x\}$ or $T_{i} \subseteq V(G)-\left(A^{\prime} \cup\{x\}\right)$ for each $1 \leq i \leq n$.

## 4 Treating 3-rank-connected graphs

In this section, we prove Theorem 1.2 for 3 -rank-connected graphs.
The following lemma shows that every vertex-minor of a 3 -rank-connected graph $G$ with one vertex fewer than $G$ is prime.

Lemma 4.1. Let $G$ be a 3 -rank-connected graph with $|V(G)| \geq 6$ and $x$ be a vertex of $G$. Then all of $G \backslash x, G * x \backslash x$, and $G / x$ are prime.

Proof. By Lemma 2.7, it is enough to show that $G \backslash x$ is prime. This is implied by Lemma 2.5.
A graph $G$ is weakly 3-rank-connected if $G$ is prime and $V(G)$ has no subset $X$ such that $|X| \geq 5,|V(G)-X| \geq 5$, and $\rho_{G}(X) \leq 2$. The following lemma can be deduced easily from [8, Proposition 2.6] and Lemma 2.2.

Lemma 4.2 (Oum [8]). Let $G$ be a 3 -rank-connected graph with $|V(G)| \geq 6$ and $x$ be a vertex of $G$. Then at least two of $G \backslash x, G * x \backslash x$, and $G / x$ are weakly 3-rank-connected.

Lemma 4.3. Let $G$ be a 3 -rank-connected graph with $|V(G)| \geq 6$ and let $S=\left\{v_{1}, \cdots, v_{t}\right\}$ be the set of all vertices $x$ of $G$ such that $G \backslash x$ is not weakly 3-rank-connected. Let $G^{\prime}=G * v_{1} * \cdots * v_{t}$. Then $G^{\prime} \backslash v$ is weakly 3 -rank-connected for every vertex $v$ of $G^{\prime}$.

Proof. If $v \notin S$, then $G^{\prime} \backslash v=(G \backslash v) * v_{1} * \cdots * v_{t}$ and so $G^{\prime} \backslash v$ is weakly 3 -rank-connected. If $v=v_{i}$ for some $1 \leq i \leq t$, then by Lemma 4.2, $G * v \backslash v$ is weakly 3-rank-connected. Since $G^{\prime} \backslash v=(G * v \backslash v) * v_{1} * \cdots * v_{i-1} * v_{i+1} * \cdots * v_{t}$ is locally equivalent to $G * v \backslash v$, we deduce that $G^{\prime} \backslash v$ is weakly 3 -rank-connected.

Lemma 4.4. Let $G$ be a 3-rank-connected graph and $x$ be a vertex of $G$. Let $P$ be a 4 -element subset of $V(G)-\{x\}$ such that $\rho_{G \backslash x}(P) \leq 2$ and $(A, B)$ be a partition of $V(G)-\{x\}$ such that $|A|,|B| \geq 4$ and $\rho_{H}(A) \leq 2$ for some $H \in\{G * x \backslash x, G / x\}$. Then $|A \cap P|=|B \cap P|=2$.

Proof. Suppose that $|A \cap P| \neq|B \cap P|$. We may assume that $|A \cap P|>|B \cap P|$. Since $\rho_{G \backslash x}(P) \leq 2$ and $\rho_{H}(A) \leq 2$, by (P1) and (P2) of Lemma 2.19, we have

$$
4 \geq \rho_{G \backslash x}(P)+\rho_{H}(A) \geq \rho_{G}(A \cap P)+\rho_{G}(B-P)-1
$$

Since $|A \cap P|>2$ and $G$ is 3-rank-connected, $\rho_{G}(A \cap P)>2$. Hence $\rho_{G}(B-P) \leq 2$. Since $G$ is 3-rank-connected, $|B-P| \leq 2$, which implies that $|B \cap P| \geq 2$, contradicting the fact that $|P|=4$.

A 4-element subset $P$ of $V(G)$ is a quad of $G$ if $\rho_{G}(P)=2$ and $\rho_{G}(P-\{x\})=3$ for each $x \in P$.

Lemma 4.5. Let $G$ be a prime graph and $A$ be a subset of $V(G)$ such that $\rho_{G}(A)=2$ and $|A| \leq 4$. Then $A$ is a quad of $G$ or $A$ is sequential in $G$.

Proof. Suppose that $A$ is not sequential in $G$. Then $|A|=4$ and $\rho_{G}(T)=3$ for each 3-element subset $T$ of $A$. Therefore, $A$ is a quad of $G$.

Our key ingredient of this section is Proposition 4.6, which states that it is sufficient to identify a set $\left\{t_{1}, t_{2}, t_{3}\right\}$ of three vertices and a quad $Q_{i}$ from $G \backslash t_{i}$ for each $i \in\{1,2,3\}$ that satisfy the following conditions:
(1) $G \backslash t_{i}$ is weakly 3 -rank-connected for each $i \in\{1,2,3\}$.
(2) $Q_{1} \cap Q_{2}=\left\{t_{3}\right\}, Q_{2} \cap Q_{3}=\left\{t_{1}\right\}$, and $Q_{3} \cap Q_{1}=\left\{t_{2}\right\}$.

The remainder of this section will focus on identifying these three vertices and quads.
Proposition 4.6. Let $t_{1}, t_{2}$, and $t_{3}$ be distinct vertices of a 3 -rank-connected graph $G$ such that $G \backslash t_{1}, G \backslash t_{2}$, and $G \backslash t_{3}$ are weakly 3-rank-connected. For each $i \in\{1,2,3\}$, let $Q_{i}$ be a quad of $G \backslash t_{i}$. If $Q_{1} \cap Q_{2}=\left\{t_{3}\right\}, Q_{2} \cap Q_{3}=\left\{t_{1}\right\}$, and $Q_{3} \cap Q_{1}=\left\{t_{2}\right\}$, then for each $i \in\{1,2,3\}$, either $G * t_{i} \backslash t_{i}$ or $G / t_{i}$ is sequentially 3-rank-connected.
Proof. Since $|V(G)| \geq\left|Q_{1} \cup Q_{2}\right|=7$, by Lemma 4.1, all of $G \backslash v, G * v \backslash v$, and $G / v$ are prime for each vertex $v$ of $G$. Observe that $\left\{t_{2}, t_{3}\right\} \subseteq Q_{1},\left\{t_{1}, t_{3}\right\} \subseteq Q_{2}$, and $\left\{t_{1}, t_{2}\right\} \subseteq Q_{3}$. For each $i \in\{1,2,3\}$, let $a_{i}$ and $b_{i}$ be two distinct vertices of $Q_{i}-\left\{t_{1}, t_{2}, t_{3}\right\}$.

Suppose that neither $G * t_{1} \backslash t_{1}$ nor $G / t_{1}$ is sequentially 3-rank-connected. Let us first show that $\rho_{G \backslash t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)=3$. Since $G \backslash t_{1}$ is prime, $\rho_{G \backslash t_{1}}\left(\left\{a_{3}, b_{3}\right\}\right)=2=\rho_{G \backslash t_{1}}\left(\left\{t_{2}, t_{3}, a_{1}, b_{1}\right\}\right)$. By Lemma 2.11,

$$
\rho_{G \backslash t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)+\rho_{G \backslash t_{1}}\left(\left\{t_{2}, t_{3}, a_{1}, b_{1}\right\}\right) \geq \rho_{G \backslash t_{1}}\left(\left\{a_{3}, b_{3}\right\}\right)+\rho_{G \backslash t_{1}}\left(\left\{t_{3}, a_{1}, b_{1}\right\}\right),
$$

and therefore $\rho_{G \backslash t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right) \geq \rho_{G \backslash t_{1}}\left(\left\{t_{3}, a_{1}, b_{1}\right\}\right)$. Since $Q_{1}=\left\{t_{2}, t_{3}, a_{1}, b_{1}\right\}$ is a quad of $G \backslash t_{1}$, $\rho_{G \backslash t_{1}}\left(\left\{t_{3}, a_{1}, b_{1}\right\}\right)=3$. Therefore $\rho_{G \backslash t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)=3$ and, by symmetry, $\rho_{G \backslash t_{1}}\left(\left\{t_{3}, a_{2}, b_{2}\right\}\right)=$ 3.

Since $3=\rho_{G \backslash t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right) \leq \rho_{G}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right) \leq 3$, we have $\rho_{G}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)=3$. Since $Q_{3}=\left\{t_{1}, t_{2}, a_{3}, b_{3}\right\}$ is a quad of $G \backslash t_{3}$ and $G$ is 3 -rank-connected, we observe that $3 \leq$ $\rho_{G}\left(\left\{t_{1}, t_{2}, a_{3}, b_{3}\right\}\right) \leq 1+\rho_{G \backslash t_{3}}\left(\left\{t_{1}, t_{2}, a_{3}, b_{3}\right\}\right)=3$ and therefore $\rho_{G}\left(\left\{t_{1}, t_{2}, a_{3}, b_{3}\right\}\right)=3$. Similarly, $\rho_{G}\left(\left\{t_{3}, a_{2}, b_{2}\right\}\right)=\rho_{G}\left(\left\{t_{1}, t_{3}, a_{2}, b_{2}\right\}\right)=3$. Therefore, by Lemma 2.18, the following hold.
(R1) $\rho_{G * t_{1} \backslash t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)=2$ or $\rho_{G / t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)=2$.
(R2) $\rho_{G * t_{1} \backslash t_{1}}\left(\left\{t_{3}, a_{2}, b_{2}\right\}\right)=2$ or $\rho_{G / t_{1}}\left(\left\{t_{3}, a_{2}, b_{2}\right\}\right)=2$.

Since $G$ is 3-rank connected, $\rho_{G}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right), \rho_{G}\left(\left\{t_{3}, a_{2}, b_{2}\right\}\right) \geq 3$. So by Lemma 2.20, $\rho_{G * t_{1} \backslash t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)+\rho_{G / t_{1}}\left(V\left(G \backslash t_{1}\right)-\left\{t_{3}, a_{2}, b_{2}\right\}\right)=\rho_{G}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)+\rho_{G}\left(\left\{t_{3}, a_{2}, b_{2}\right\}\right)-1 \geq 5$. Hence, $\rho_{G * t_{1} \backslash t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)+\rho_{G / t_{1}}\left(\left\{t_{3}, a_{2}, b_{2}\right\}\right) \geq 5$ and similarly,

$$
\rho_{G * t_{1} \backslash t_{1}}\left(\left\{t_{3}, a_{2}, b_{2}\right\}\right)+\rho_{G / t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right) \geq 5
$$

Therefore, by (R1) and (R2), either
(a) $\rho_{G * t_{1} \backslash t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)=\rho_{G * t_{1} \backslash t_{1}}\left(\left\{t_{3}, a_{2}, b_{2}\right\}\right)=2$, or
(b) $\rho_{G / t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)=\rho_{G / t_{1}}\left(\left\{t_{3}, a_{2}, b_{2}\right\}\right)=2$.

By Lemma 2.2, we may assume (a), because otherwise we can choose a neighbor $y \notin\left\{t_{2}, t_{3}\right\}$ of $t_{1}$ in $G$ by Lemma 2.4 and replace $G$ by $G * y$. By Lemma 3.2, there is a subset $A$ of $V\left(G * t_{1} \backslash t_{1}\right)$ such that

- $\rho_{G * t_{1} \backslash t_{1}}(A) \leq 2$,
- neither $A$ nor $V\left(G * t_{1} \backslash t_{1}\right)-A$ is sequential in $G * t_{1} \backslash t_{1}$,
- $\left\{t_{2}, a_{3}, b_{3}\right\} \subseteq A$ or $\left\{t_{2}, a_{3}, b_{3}\right\} \subseteq V\left(G * t_{1} \backslash t_{1}\right)-A$, and
- $\left\{t_{3}, a_{2}, b_{2}\right\} \subseteq A$ or $\left\{t_{3}, a_{2}, b_{2}\right\} \subseteq V\left(G * t_{1} \backslash t_{1}\right)-A$.

We may assume that $\left\{t_{2}, a_{3}, b_{3}\right\} \subseteq A$ by replacing $A$ with $V\left(G * t_{1} \backslash t_{1}\right)-A$ if necessary. Let $B=V\left(G * t_{1} \backslash t_{1}\right)-A$.

Suppose that $\left\{t_{3}, a_{2}, b_{2}\right\} \subseteq A$. Observe that $\rho_{G}(A) \leq \rho_{G * t_{1} \backslash t_{1}}(A)+1 \leq 3$. Since $\left\{t_{1}, t_{2}, a_{3}, b_{3}\right\}$ is a quad of $G \backslash t_{3}$, by (S1) of Lemma 2.12,

$$
\begin{aligned}
3+2 & \geq \rho_{G}(A)+\rho_{G \backslash t_{3}}\left(\left\{t_{1}, t_{2}, a_{3}, b_{3}\right\}\right)=\rho_{G}\left(\left(A-\left\{t_{3}\right\}\right) \cup\left\{t_{3}\right\}\right)+\rho_{G \backslash t_{3}}\left(\left\{t_{1}, t_{2}, a_{3}, b_{3}\right\}\right) \\
& \geq \rho_{G \backslash t_{3}}\left(\left(A-\left\{t_{3}\right\}\right) \cap\left\{t_{1}, t_{2}, a_{3}, b_{3}\right\}\right)+\rho_{G}\left(\left(A-\left\{t_{3}\right\}\right) \cup\left\{t_{1}, t_{2}, t_{3}, a_{3}, b_{3}\right\}\right) \\
& =\rho_{G \backslash t_{3}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)+\rho_{G}\left(A \cup\left\{t_{1}\right\}\right) \geq 3+\rho_{G}\left(A \cup\left\{t_{1}\right\}\right) .
\end{aligned}
$$

Therefore $\rho_{G}\left(A \cup\left\{t_{1}\right\}\right) \leq 2$, contradicting our assumption that $G$ is 3 -rank-connected. So we deduce that $\left\{t_{3}, a_{2}, b_{2}\right\} \subseteq B$.

By Lemma 4.4, $\left|A \cap\left\{t_{2}, t_{3}, a_{1}, b_{1}\right\}\right|=\left|B \cap\left\{t_{2}, t_{3}, a_{1}, b_{1}\right\}\right|=2$. So $\left|A \cap\left\{a_{1}, b_{1}\right\}\right|=\mid B \cap$ $\left\{a_{1}, b_{1}\right\} \mid=1$ and we can assume that $\left\{a_{1}, t_{2}, a_{3}, b_{3}\right\} \subseteq A$ and $\left\{b_{1}, t_{3}, a_{2}, b_{2}\right\} \subseteq B$ by swapping $a_{1}$ and $b_{1}$ if necessary.

If $|A|=4$, then $A$ is sequential in $G * t_{1} \backslash t_{1}$ because $\rho_{G * t_{1} \backslash t_{1}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right) \leq 2$ and $\left\{t_{2}, a_{3}, b_{3}\right\} \subseteq$ $A$, contradicting our assumption on $A$. Hence $|A| \geq 5$.

If $|B|=4$, then $B$ is sequential in $G * t_{1} \backslash t_{1}$ because $\rho_{G * t_{1} \backslash t_{1}}\left(\left\{t_{3}, a_{2}, b_{2}\right\}\right) \leq 2$ and $\left\{t_{3}, a_{2}, b_{2}\right\} \subseteq$ $B$, contradicting our assumption on $B$. So $|B| \geq 5$ and $|V(G)|=|A|+|B|+1 \geq 11$.

For each $k \in\{1,2,3\}$, let $P_{k}=Q_{k} \cup\left\{t_{k}\right\}=\left\{t_{1}, t_{2}, t_{3}, a_{k}, b_{k}\right\}$. Observe that $\rho_{G}\left(P_{k}\right) \leq$ $\rho_{G \backslash t_{k}}\left(Q_{k}\right)+1 \leq 3$ for each $1 \leq k \leq 3$. Since $G$ is 3-rank-connected and $\left|P_{1} \cap P_{3}\right|=3$, we have $\rho_{G}\left(P_{1} \cap P_{3}\right) \geq 3$. By Lemma 2.10,

$$
6 \geq \rho_{G}\left(P_{1}\right)+\rho_{G}\left(P_{3}\right) \geq \rho_{G}\left(P_{1} \cup P_{3}\right)+\rho_{G}\left(P_{1} \cap P_{3}\right) \geq \rho_{G}\left(P_{1} \cup P_{3}\right)+3
$$

which implies that $\rho_{G}\left(P_{1} \cup P_{3}\right) \leq 3$. Observe that $\left|V(G)-\left(A \cup\left(P_{1} \cup P_{3}\right)\right)\right| \geq\left|B-\left\{b_{1}, t_{3}\right\}\right| \geq 3$. Since $G$ is 3-rank-connected, $\rho_{G}\left(A \cup\left(P_{1} \cup P_{3}\right)\right) \geq 3$. By Lemma 2.10,
$3+3 \geq \rho_{G}(A)+\rho_{G}\left(P_{1} \cup P_{3}\right) \geq \rho_{G}\left(A \cap\left(P_{1} \cup P_{3}\right)\right)+\rho_{G}\left(A \cup\left(P_{1} \cup P_{3}\right)\right) \geq \rho_{G}\left(A \cap\left(P_{1} \cup P_{3}\right)\right)+3$. Therefore $\rho_{G}\left(\left\{a_{1}, t_{2}, a_{3}, b_{3}\right\}\right)=\rho_{G}\left(A \cap\left(P_{1} \cup P_{3}\right)\right) \leq 3$. Hence by Lemma 2.10,

$$
\begin{aligned}
3+2 & \geq \rho_{G \backslash t_{3}}\left(\left\{a_{1}, t_{2}, a_{3}, b_{3}\right\}\right)+\rho_{G \backslash t_{3}}\left(\left\{t_{1}, t_{2}, a_{3}, b_{3}\right\}\right) \\
& \geq \rho_{G \backslash t_{3}}\left(\left\{a_{1}, t_{1}, t_{2}, a_{3}, b_{3}\right\}\right)+\rho_{G \backslash t_{3}}\left(\left\{t_{2}, a_{3}, b_{3}\right\}\right)=\rho_{G \backslash t_{3}}\left(\left\{a_{1}, t_{1}, t_{2}, a_{3}, b_{3}\right\}\right)+3 .
\end{aligned}
$$

Hence $\rho_{G \backslash t_{3}}\left(\left\{a_{1}, t_{1}, t_{2}, a_{3}, b_{3}\right\}\right) \leq 2$, contradicting our assumption that $G \backslash t_{3}$ is weakly 3-rankconnected.

An independent set of a graph is a set of pairwise nonadjacent vertices. For sets $A$ and $B$, let $A \triangle B=(A-B) \cup(B-A)$.

Lemma 4.7. Let $G$ be a 3 -rank-connected graph with $|V(G)| \geq 6$ and $x$ be a vertex of $G$ such that $G \backslash x$ is weakly 3-rank-connected. Let $P$ be a quad of $G \backslash x$. Then there is a graph $G^{\prime}$ locally equivalent to $G$ such that the following hold.
(1) $G^{\prime} \backslash v$ is weakly 3-rank-connected for each vertex $v \in P \cup\{x\}$.
(2) $N_{G^{\prime}}(t)-P \neq \emptyset$ for each $t \in P$.
(3) $P$ is a quad of $G^{\prime} \backslash x$.

Proof. Let $P=\{p, q, r, s\}$. By Lemma 4.3, there is a graph locally equivalent to $G$ satisfying (1) and (3). We may assume that among all graphs locally equivalent to $G$ satisfying (1) and (3), $G$ maximizes the number of edges between vertices in $P$.

We may assume that $N_{G}(p) \subseteq\{q, r, s\}$ because otherwise (1), (2), and (3) hold for $G^{\prime}=G$. Since $P$ is a quad of $G \backslash x$, we have $\rho_{G \backslash x}(P)=2$, which implies that $|V(G \backslash x)-P| \geq 2$. So $|V(G)| \geq 7$. Since $G$ is 3-rank-connected, by Lemma 2.4, we have $N_{G}(p)=\{q, r, s\}$.

Suppose that $\{q, r, s\}$ is independent in $G$. Since $G$ is 3-rank-connected, by Lemma 2.6, $G$ is 3 -connected and so $G \backslash x \backslash p$ is connected. Let $X$ be a shortest path joining two vertices of $\{q, r, s\}$ in $G \backslash x \backslash p$. By symmetry, we may assume that $X=q v_{1} \cdots v_{m} r$ and $v_{i} \neq s$ for each $1 \leq i \leq m$. Since $\{q, r, s\}$ is independent in $G$, we deduce that $m \geq 1$ and $\left\{v_{1}, \ldots, v_{m}\right\} \subseteq V(G)-(P \cup\{x\})$. Then let $G^{\prime}=G * v_{1} * \cdots * v_{m}$. Then $G^{\prime}$ satisfies (1) and (3). Moreover, $N_{G^{\prime}}(p)=\{q, r, s\}$ and $q r \in E\left(G^{\prime}\right)$. Hence $\left|E\left(G^{\prime}[P]\right)\right|>|E(G[P])|$, contradicting the choice of $G$. Therefore, $\{q, r, s\}$ is not independent in $G$.

Since $G$ is 3 -rank-connected, we have $3 \leq \rho_{G}(P) \leq \rho_{G \backslash x}(P)+1=3$. Hence $\rho_{G}(P)=3$ and so $N_{G}(q)-P, N_{G}(r)-P$, and $N_{G}(s)-P$ are nonempty, pairwise distinct, and ( $N_{G}(s)-$ P) $\triangle\left(N_{G}(q)-P\right) \triangle\left(N_{G}(r)-P\right) \neq \emptyset$.

If $G * q \backslash q$ is weakly 3-rank-connected, then let $G^{\prime}=G * q$. Obviously, (1) and (3) hold. We have $N_{G^{\prime}}(p)-P=N_{G}(q)-P=N_{G^{\prime}}(q)-P \neq \emptyset$. For each vertex $v \in\{r, s\}$,

$$
N_{G^{\prime}}(v)-P= \begin{cases}N_{G}(v)-P \neq \emptyset & \text { if } v \text { is not adjacent to } q \text { in } G, \\ \left(N_{G}(q)-P\right) \triangle\left(N_{G}(v)-P\right) \neq \emptyset & \text { if } v \text { is adjacent to } q \text { in } G,\end{cases}
$$

and therefore $G^{\prime}$ satisfies (2). So we can assume that none of $G * q \backslash q, G * r \backslash r$, and $G * s \backslash s$ is weakly 3 -rank-connected. Then by Lemma 4.2, all of $G / q, G / r$, and $G / s$ are weakly 3 -rank-connected.

Since $\{q, r, s\}$ is not independent in $G$, by symmetry, we may assume that $q$ and $r$ are adjacent in $G$. Let $G^{\prime}=G \wedge q r$. For each vertex $v \in P \cup\{x\}$, if $v \in\{p, s, x\}$, then $G^{\prime} \backslash v=(G \backslash v) \wedge q r$ and if $v \in\{q, r\}$, then $G^{\prime} \backslash v=G / v$, which implies that (1) and (3) hold. Then $N_{G^{\prime}}(q)-P=N_{G}(r)-P$ and $N_{G^{\prime}}(r)-P=N_{G}(q)-P$. Since $p \in N_{G}(q) \cap N_{G}(r)$ and $N_{G}(q)-P \neq N_{G}(r)-P$, we have $N_{G^{\prime}}(p)-P=\left(N_{G}(q)-P\right) \triangle\left(N_{G}(r)-P\right) \neq \emptyset$. Furthermore,

$$
N_{G^{\prime}}(s)-P= \begin{cases}N_{G}(s)-P \neq \emptyset & \text { if } s \notin N_{G}(q) \cup N_{G}(r), \\ \left(N_{G}(s)-P\right) \triangle\left(N_{G}(q)-P\right) \neq \emptyset & \text { if } s \in N_{G}(r)-N_{G}(q), \\ \left(N_{G}(s)-P\right) \triangle\left(N_{G}(r)-P\right) \neq \emptyset & \text { if } s \in N_{G}(q)-N_{G}(r), \\ \left(N_{G}(s)-P\right) \triangle\left(N_{G}(q)-P\right) \triangle\left(N_{G}(r)-P\right) \neq \emptyset & \text { if } s \in N_{G}(q) \cap N_{G}(r)\end{cases}
$$

Hence, (2) holds.
Lemma 4.8. Let $G$ be a 3-rank-connected graph with $|V(G)| \geq 6$ and $x$ be a vertex of $G$. Let $P$ be a quad of $G \backslash x$ and $t$ be a vertex in $P$. If $G \backslash t$ is weakly 3 -rank-connected, then one of the following holds.
(Q1) $G \backslash t$ is sequentially 3-rank-connected.
(Q2) There is a subset $X$ of $V(G \backslash t)$ such that $\rho_{G \backslash t}(X) \leq 2, X \cap P \neq \emptyset,(V(G \backslash t)-X) \cap P \neq \emptyset$, and neither $X$ nor $V(G \backslash t)-X$ is sequential in $G \backslash t$.
(Q3) $\rho_{G \backslash t}(P-\{t\})=2$ and $G \backslash t$ has a quad $Y$ containing $x$ such that $Y \cap P=\emptyset$.
Proof. Suppose that $G \backslash t$ is not sequentially 3 -rank-connected. Then there is a subset $X$ of $V(G \backslash t)$ such that $\rho_{G \backslash t}(X) \leq 2$ and neither $X$ nor $V(G \backslash t)-X$ is sequential in $G \backslash t$. Let $Y=V(G \backslash t)-X$ and $\left(Z_{1}, Z_{2}\right)=(X-\{x\}, Y-\{x\})$. Since both $X$ and $Y$ are non-sequential in $G \backslash t$, we have $|X|,|Y| \geq 4$ and so $\left|Z_{1}\right|,\left|Z_{2}\right| \geq 3$. If $X \cap P \neq \emptyset$ and $Y \cap P \neq \emptyset$, then (Q2) holds. So by symmetry, we may assume that $P-\{t\} \subseteq X$. Then $P-\{t\} \subseteq Z_{1}$. Since $P$ is a quad of $G \backslash x$, we know that $\rho_{G \backslash x}(P)=2=\rho_{G \backslash x}(P-\{t\})-1 \leq \rho_{G \backslash x \backslash t}(P-\{t\})$. Then by Lemma 2.14, $\rho_{G \backslash x}\left(Z_{1} \cup\{t\}\right)=\rho_{G \backslash x \backslash t}\left(Z_{1}\right)$.

By Lemma 4.1, $G \backslash x$ is prime and so $2 \leq \rho_{G \backslash x}\left(Z_{1} \cup\{t\}\right)=\rho_{G \backslash x \backslash t}\left(Z_{1}\right) \leq \rho_{G \backslash t}(X) \leq 2$, which implies that

$$
\rho_{G \backslash x}\left(Z_{1} \cup\{t\}\right)=\rho_{G \backslash x \backslash t}\left(Z_{1}\right)=2 .
$$

Since $G$ is 3-rank-connected and $\left|V(G)-\left(Z_{1} \cup\{x, t\}\right)\right| \geq\left|Z_{2}\right| \geq 3$, we have $\rho_{G}\left(Z_{1} \cup\{x, t\}\right) \geq 3$. So by Lemma (A3) of Lemma 2.16,

$$
2+\rho_{G \backslash t}\left(Z_{1} \cup\{x\}\right) \geq \rho_{G \backslash x}\left(Z_{1} \cup\{t\}\right)+\rho_{G \backslash t}\left(Z_{1} \cup\{x\}\right) \geq \rho_{G}\left(Z_{1} \cup\{x, t\}\right)+\rho_{G \backslash x \backslash t}\left(Z_{1}\right) \geq 3+2 .
$$

Hence $\rho_{G \backslash t}\left(Z_{1} \cup\{x\}\right)>2$ and $x \in Y$. So $\left(Z_{1}, Z_{2}\right)=(X, Y-\{x\})$ and $\rho_{G \backslash x}(X \cup\{t\})=$ $\rho_{G \backslash x}\left(Z_{1} \cup\{t\}\right)=2$. Since $t \in P$ and $x \notin Z_{1}$, by (A2) of Lemma 2.16,

$$
2+2 \geq \rho_{G \backslash x}(P)+\rho_{G \backslash t}\left(Z_{1}\right) \geq \rho_{G \backslash x}\left(Z_{1} \cup\{t\}\right)+\rho_{G \backslash t}(P-\{t\}) \geq 2+\rho_{G \backslash t}(P-\{t\}) .
$$

Therefore, $\rho_{G \backslash t}(P-\{t\})=2$ because $G \backslash t$ is prime. Since $X$ is non-sequential in $G \backslash t$ and $\rho_{G \backslash t}(P-\{t\}) \leq 2$, we have $|X| \geq 5$. Hence $|Y|=4$ because $G \backslash t$ is weakly 3-rank-connected. Since $Y$ is non-sequential in $G \backslash t$, by Lemma 4.5, $Y=Z_{2} \cup\{x\}$ is a quad of $G \backslash t$. Hence (Q3) holds.

Lemma 4.9. Let $G$ be a 3-rank-connected graph such that $|V(G)| \geq 12$ and $x$ be a vertex of $G$. Let $P$ be a quad of $G \backslash x$ and $t$ be a vertex of $P$. Let $(X, Y)$ be a partition of $V(G)-\{t\}$ such that $\rho_{G \backslash t}(X) \leq 2$ and neither $X$ nor $Y$ is sequential in $G \backslash t$. If $G \backslash x$ and $G \backslash t$ are weakly 3-rank-connected and $|X \cap P|=1$, then the following hold.
(K1) $\rho_{G \backslash x \backslash t}(X-\{x\})=\rho_{G \backslash t}(X)=2$.
(K2) $X$ is a quad of $G \backslash t$ containing $x$.
Proof. Since neither $X$ nor $Y$ is sequential in $G \backslash t$, we have $|X|,|Y| \geq 4$ and so $|X-\{x\}|, \mid Y-$ $\{x\} \mid \geq 3$. Clearly, $\rho_{G \backslash x \backslash t}(X-\{x\}) \leq \rho_{G \backslash t}(X) \leq 2$. Let $q$ be the element of $X \cap P$ and $r, s$ be the elements of $Y \cap P$. Let $C=X-\{q, x\}$ and $D=Y-\{r, s, x\}$. Then we have $|D| \geq 1$ because $|Y| \geq 4$.

Let us show that $\rho_{G \backslash x \backslash t}(C) \leq 2$. Since $P$ is a quad of $G \backslash x$, by (ii) of Lemma 2.8, $\rho_{G \backslash x \backslash t}(P-$ $\{t\}) \leq \rho_{G \backslash x}(P)=2=\rho_{G \backslash x}(P-\{q\})-1 \leq \rho_{G \backslash x \backslash t}(\{r, s\})$. Hence, by Lemma 2.11,

$$
\begin{aligned}
\rho_{G \backslash x \backslash t}(X-\{x\})+2 & \geq \rho_{G \backslash x \backslash t}(X-\{x\})+\rho_{G \backslash x \backslash t}(P-\{t\}) \\
& \geq \rho_{G \backslash x \backslash t}(C)+\rho_{G \backslash x \backslash t}(\{r, s\}) \geq \rho_{G \backslash x \backslash t}(C)+2
\end{aligned}
$$

and therefore $\rho_{G \backslash x \backslash t}(C) \leq \rho_{G \backslash x \backslash t}(X-\{x\}) \leq 2$. Since $P$ is a quad of $G \backslash x$, by (i) of Lemma 2.8, $\rho_{G \backslash x}(P)=2=\rho_{G \backslash x}(P-\{t\})-1 \leq \rho_{G \backslash x \backslash t}(P-\{t\})$. By Lemma 2.15,
$2+\rho_{G \backslash x \backslash t}(C) \geq \rho_{G \backslash x}((P-\{t\}) \cup\{t\})+\rho_{G \backslash x \backslash t}(C) \geq \rho_{G \backslash x}(C)+\rho_{G \backslash x \backslash t}(P-\{t\}) \geq \rho_{G \backslash x}(C)+2$,
which implies that $\rho_{G \backslash x}(C) \leq \rho_{G \backslash x \backslash t}(C) \leq 2$. Hence $\rho_{G \backslash x}(C)=\rho_{G \backslash x \backslash t}(C)=\rho_{G \backslash x \backslash t}(X-\{x\})=$ $\rho_{G \backslash t}(X)=2$ because $G \backslash x$ is prime and $|V(G \backslash x)-C| \geq 2$. Hence (K1) holds.

Since $G \backslash x$ is weakly 3-rank-connected and $|V(G \backslash x)-C| \geq|P|+|D| \geq 5$, we deduce that $|C| \leq 4$ and $|X| \leq 6$. So $|Y| \geq 11-|X| \geq 5$.

Suppose that $x \notin X$. Then $X=X-\{x\}$ and $\rho_{G \backslash x \backslash t}(X)=\rho_{G \backslash t}(X)=2$. Since $C \subseteq X$, by Lemma 2.13, we have $\rho_{G \backslash t}(C)=\rho_{G \backslash x \backslash t}(C)=2$. By (A1) of Lemma 2.16,

$$
\rho_{G \backslash x}(C)+\rho_{G \backslash t}(C) \geq \rho_{G}(C)+\rho_{G \backslash x \backslash t}(C),
$$

which implies that $\rho_{G}(C) \leq 2$. So $|C| \leq 2$ because $G$ is 3-rank-connected. Then $|X|=|C \cup\{q\}| \leq$ 3 , contradicting our assumption on $X$. Hence $x \in X$.

Since $G \backslash t$ is weakly 3-rank-connected, $\rho_{G \backslash t}(X)=2$, and $|Y| \geq 5$, we have $|X|=4$. Therefore, by Lemma 4.5, $X$ is a quad of $G \backslash t$ and (K2) holds.

Lemma 4.10. Let $G$ be a 3-rank-connected graph with $|V(G)| \geq 12$ and no sequentially 3-rankconnected vertex-minor on $|V(G)|-1$ vertices. Let $x$ be a vertex of $G$ such that $G \backslash x$ is weakly 3 -rank-connected and $P$ be a quad of $G \backslash x$. Then there is a graph $G^{\prime}$ locally equivalent to $G$ such that the following hold.
(1) $G^{\prime} \backslash v$ is weakly 3-rank-connected for each vertex $v$ of $P \cup\{x\}$.
(2) $P$ is a quad of $G^{\prime} \backslash x$.
(3) There exist a 2-element subset $S$ of $P$ and a quad $X_{u}$ of $G^{\prime} \backslash u$ for each $u$ in $S$ such that $x \in X_{u},\left|X_{u} \cap P\right|=1$, and $V\left(G^{\prime} \backslash u\right)-X_{u}$ is not sequential in $G^{\prime} \backslash u$.

Proof. By Lemma 4.1, $G \backslash v$ is prime for each vertex $v$ of $G$. By Lemma 4.7, we can assume that $G \backslash v$ is weakly 3-rank-connected for each vertex $v$ of $P \cup\{x\}$, the set $P$ is a quad of $G \backslash x$, and $N_{G}(t)-P$ is nonempty for each $t \in P$.

By Lemma 4.8, each vertex $t$ in $P$ satisfies (Q2) or (Q3). Suppose that at most 1 vertex of $P$ satisfies (Q2). Then by Lemma 4.8, there exist 3 vertices $q, r, s$ of $P$ such that $\rho_{G \backslash q}(P-\{q\})=2$, $\rho_{G \backslash r}(P-\{r\})=2$, and $\rho_{G \backslash s}(P-\{s\})=2$. Since $P$ is a quad of $G \backslash x$, by (i) of Lemma 2.8, we have $\rho_{G}(P) \leq \rho_{G \backslash x}(P)+1 \leq 3$. Since $G$ is 3 -rank-connected, $3 \leq \rho_{G}(P)$ and therefore, $\rho_{G}(P)=3$. By (A3) of Lemma 2.16,

$$
\begin{aligned}
2+2 & =\rho_{G \backslash q}(P-\{q\})+\rho_{G \backslash r}(P-\{r\}) \\
& \geq \rho_{G}(P)+\rho_{G \backslash q \backslash r}(P-\{q, r\})=3+\rho_{G \backslash q \backslash r}(P-\{q, r\})
\end{aligned}
$$

Therefore, $\rho_{G \backslash q \backslash r}(P-\{q, r\}) \leq 1$ and by symmetry, $\rho_{G \backslash q \backslash s}(P-\{q, s\}) \leq 1$ and $\rho_{G \backslash r \backslash s}(P-$ $\{r, s\}) \leq 1$. Let $p$ be the element of $P-\{q, r, s\}$. Since $N_{G}(t)-P \neq \emptyset$ for each $t \in P$, we have $N_{G}(p)-P=N_{G}(q)-P=N_{G}(r)-P=N_{G}(s)-P$ and therefore $\rho_{G}(P)=1$, contradicting our assumption.

Therefore, there exist a subset $S=\{p, q\}$ of $P$ and a subset $X_{u}$ of $V(G \backslash u)$ for each $u \in S$ such that $\rho_{G \backslash u}\left(X_{u}\right) \leq 2$, both $X_{u} \cap P$ and $\left(V(G \backslash u)-X_{u}\right) \cap P$ are nonempty, and neither $X_{u}$ nor $V(G \backslash u)-X_{u}$ is sequential in $G \backslash u$.

Let $Y_{p}=V(G \backslash p)-X_{p}$ and $Y_{q}=V(G \backslash q)-X_{q}$. By symmetry, we may assume that $\left|X_{p} \cap P\right|=1$ and $\left|X_{q} \cap P\right|=1$. Then by (K2) of Lemma 4.9, $X_{p}$ is a quad of $G \backslash p, X_{q}$ is a quad of $G \backslash q$, and $x \in X_{p} \cap X_{q}$.

Lemma 4.11. Let $G$ be a 3-rank-connected graph with $|V(G)| \geq 12$ and $x, y$ be distinct vertices of $G$ such that both $G \backslash x$ and $G \backslash y$ are weakly 3-rank-connected. Let $A$ be a quad of $G \backslash x$ and $B$ be a quad of $G \backslash y$. Then $|A \cap B| \leq 2$.

Proof. Suppose that $|A \cap B| \geq 3$. First let us consider the case when $y \notin A$ and $x \notin B$. Since $G$ is 3 -rank-connected and $|V(G)-(A \cap B)| \geq 3$, we have $\rho_{G}(A \cap B) \geq 3$. So by (A1) of Lemma 2.16,

$$
2+2 \geq \rho_{G \backslash x}(A)+\rho_{G \backslash y}(B) \geq \rho_{G}(A \cap B)+\rho_{G \backslash x \backslash y}(A \cup B) \geq 3+\rho_{G \backslash x \backslash y}(A \cup B)
$$

Hence $\rho_{G \backslash x \backslash y}(A \cup B) \leq 1$. Then by (ii) of Lemma 2.8, we have $\rho_{G \backslash x}(A \cup B \cup\{y\}) \leq 2$. Since $G \backslash x$ is weakly 3 -rank-connected and $|A \cup B \cup\{y\}| \in\{5,6\}$, we deduce that $|V(G \backslash x)-(A \cup B \cup\{y\})| \leq 4$ and so $|V(G)| \leq 11$, contradicting our assumption.

Now we consider the case when either

- $y \in A$ and $x \notin B$, or
- $y \notin A$ and $x \in B$.

By symmetry, we may assume that $y \in A$ and $x \notin B$. Then $|A \cap B|=3$ because $x \notin B$. Since $G \backslash x$ is weakly 3-rank-connected, $|A \cup B|=5$, and $|V(G \backslash x)-(A \cup B)| \geq 6$, we have $\rho_{G \backslash x}(A \cup B) \geq 3$. By (A2) of Lemma 2.16,

$$
2+2 \geq \rho_{G \backslash x}(A)+\rho_{G \backslash y}(B) \geq \rho_{G \backslash x}(A \cup B)+\rho_{G \backslash y}(A \cap B) \geq 3+\rho_{G \backslash y}(A \cap B)
$$

Hence $\rho_{G \backslash y}(A \cap B) \leq 1$, contradicting the fact that $G \backslash y$ is prime.
Now it remains to consider the case when $y \in A$ and $x \in B$. Since $x \notin A$ and $y \notin B$, we have $|A \cap B|=3$. Since $G$ is 3-rank-connected and $|V(G)-(A \cup B)| \geq 7$, we have $\rho_{G}(A \cup B) \geq 3$. By (A3) of Lemma 2.16,

$$
2+2 \geq \rho_{G \backslash x}(A)+\rho_{G \backslash y}(B) \geq \rho_{G}(A \cup B)+\rho_{G \backslash x \backslash y}(A \cap B) \geq 3+\rho_{G \backslash x \backslash y}(A \cap B)
$$

So $\rho_{G \backslash x \backslash y}(A \cap B) \leq 1$ and $\rho_{G \backslash x}(A \cap B) \leq 2$, contradicting the assumption that $A$ is a quad of $G \backslash x$.

Lemma 4.12. Let $G$ be a 3-rank-connected graph with $|V(G)| \geq 12$ and $x$ be a vertex of $G$. Let $P$ be a quad of $G \backslash x$ and $y$ be a vertex of $P$. Let $Q$ be a quad of $G \backslash y$. If $G \backslash x$ is weakly 3-rank-connected and $|P \cap Q|=2$, then $x \in Q$.

Proof. Suppose that $x \notin Q$. Since $G$ is 3-rank-connected, by Lemma 4.1, $G \backslash y$ is prime. Therefore, $\rho_{G \backslash y}(P \cap Q)=2$ because $|P \cap Q|=2$. Since $y \in P$ and $x \notin Q$, by (A2) of Lemma 2.16,

$$
2+2 \geq \rho_{G \backslash x}(P)+\rho_{G \backslash y}(Q) \geq \rho_{G \backslash x}(P \cup Q)+\rho_{G \backslash y}(P \cap Q) \geq \rho_{G \backslash x}(P \cup Q)+2
$$

Hence $\rho_{G \backslash x}(P \cup Q) \leq 2$. Since $G \backslash x$ is weakly 3 -rank-connected and $|P \cup Q|=6$, we have $|V(G \backslash x)-(P \cup Q)| \leq 4$ and so $|V(G)| \leq 11$, contradicting our assumption.

Lemma 4.13. Let $G$ be a 3-rank-connected graph with $|V(G)| \geq 13$ and $x$ be a vertex of $G$. Let $P$ be a quad of $G \backslash x$ and $p, q$ be distinct vertices of $P$. For each $u \in\{p, q\}$, let $A_{u}$ be a quad of $G \backslash u$ such that $x \in A_{u},\left|A_{u} \cap P\right|=1$, and $V(G \backslash u)-A_{u}$ is not sequential in $G \backslash u$. If $G \backslash x$, $G \backslash p$, and $G \backslash q$ are weakly 3-rank-connected, then $A_{p} \cap A_{q} \subseteq P \cup\{x\}$.

Proof. For each $u \in\{p, q\}$, let $B_{u}=A_{u}-(P \cup\{x\})$. Then $\left|B_{u}\right|=2$ and $\left|A_{u} \cup P\right|=7$ for each $u \in\{p, q\}$. Let $t$ be the unique element of $A_{p} \cap P$.

Now we claim that $\rho_{G}\left(A_{p} \cup P\right)=3$. By Lemma $2.4, N_{G \backslash x \backslash p}(t) \neq \emptyset$ and so $\rho_{G \backslash x \backslash p}(\{t\})=1$. Since $P$ is a quad of $G \backslash x$, we have $\rho_{G \backslash x \backslash p}(P-\{p\}) \leq \rho_{G \backslash x}(P)=2$. By (K1) of Lemma 4.9, $\rho_{G \backslash x \backslash p}\left(A_{p}-\{x\}\right)=\rho_{G \backslash p}\left(A_{p}\right)=2$. By Lemma 2.14, $\rho_{G \backslash p}\left(A_{p} \cup(P-\{p\})\right)=\rho_{G \backslash x \backslash p}\left(\left(A_{p}-\{x\}\right) \cup\right.$ $(P-\{p\}))$.

By Lemma 2.10,

$$
\begin{aligned}
2+2 & \geq \rho_{G \backslash x \backslash p}\left(A_{p}-\{x\}\right)+\rho_{G \backslash x \backslash p}(P-\{p\}) \\
& \geq \rho_{G \backslash x \backslash p}\left(\left(A_{p}-\{x\}\right) \cup(P-\{p\})\right)+\rho_{G \backslash x \backslash p}(\{t\}) \geq \rho_{G \backslash x \backslash p}\left(\left(A_{p}-\{x\}\right) \cup(P-\{p\})\right)+1 .
\end{aligned}
$$

Hence $\rho_{G \backslash x \backslash p}\left(\left(A_{p}-\{x\}\right) \cup(P-\{p\})\right) \leq 3$.
Since $P$ is a quad of $G \backslash x$, we have $\rho_{G \backslash x}(P)=2=\rho_{G \backslash x}(P-\{p\})-1 \leq \rho_{G \backslash x \backslash p}(P-\{p\})$. So by Lemma 2.14, $\rho_{G \backslash x}\left(\left(A_{p}-\{x\}\right) \cup P\right)=\rho_{G \backslash x \backslash p}\left(\left(A_{p}-\{x\}\right) \cup(P-\{p\})\right)$. By (A3) of Lemma 2.16, $\rho_{G \backslash x}\left(\left(A_{p}-\{x\}\right) \cup P\right)+\rho_{G \backslash p}\left(A_{p} \cup(P-\{p\})\right) \geq \rho_{G}\left(A_{p} \cup P\right)+\rho_{G \backslash x \backslash p}\left(\left(A_{p}-\{x\}\right) \cup(P-\{p\})\right)$.

It follows that $\rho_{G}\left(A_{p} \cup P\right)=\rho_{G \backslash x \backslash p}\left(\left(A_{p}-\{x\}\right) \cup(P-\{p\})\right) \leq 3$. Since $G$ is 3-rank-connected and $\left|A_{p} \cup P\right|,\left|V(G)-\left(A_{p} \cup P\right)\right| \geq 3$, we have $\rho_{G}\left(A_{p} \cup P\right)=3$.

By Lemma 4.11, $\left|A_{p} \cap A_{q}\right| \leq 2$. Since $x \in A_{p} \cap A_{q}$, we have $\left|B_{p} \cap B_{q}\right| \leq 1$.
Suppose that $\left|B_{p} \cap B_{q}\right|=1$. Then $\left|A_{q} \cap\left(A_{p} \cup P\right)\right|=\left|A_{q}\right|-\left|A_{q}-\left(A_{p} \cup P\right)\right|=\left|A_{q}\right|-\left|B_{q}-B_{p}\right|=$ $\left|A_{q}\right|-\left(\left|B_{q}\right|-\left|B_{p} \cap B_{q}\right|\right)=3$. So $\rho_{G \backslash q}\left(A_{q} \cap\left(A_{p} \cup P\right)\right)=3$ because $A_{q}$ is a quad of $G \backslash q$. Since $\rho_{G \backslash q}\left(A_{q}\right)=2$ and $\rho_{G \backslash q}\left(\left(A_{p} \cup P\right)-\{q\}\right) \leq \rho_{G}\left(A_{p} \cup P\right)=3$, by Lemma 2.11,

$$
\begin{aligned}
5 \geq \rho_{G \backslash q}\left(A_{q}\right)+\rho_{G \backslash q}\left(\left(A_{p} \cup P\right)-\{q\}\right) & \geq \rho_{G \backslash q}\left(\left(A_{q} \cup\left(A_{p} \cup P\right)\right)-\{q\}\right)+\rho_{G \backslash q}\left(A_{q} \cap\left(A_{p} \cup P\right)\right) \\
& =\rho_{G \backslash q}\left(\left(A_{q} \cup\left(A_{p} \cup P\right)\right)-\{q\}\right)+3 .
\end{aligned}
$$

Hence $\rho_{G \backslash q}\left(\left(A_{q} \cup\left(A_{p} \cup P\right)\right)-\{q\}\right) \leq 2$. Since $G \backslash q$ is weakly 3-rank-connected and $\mid\left(A_{q} \cup\left(A_{p} \cup P\right)\right)-$ $\{q\}\left|=\left|A_{q}\right|+\left|A_{p} \cup P\right|-\left|A_{q} \cap\left(A_{p} \cup P\right)\right|-1=7\right.$, we deduce that $| V(G \backslash q)-\left(\left(A_{q} \cup\left(A_{p} \cup P\right)\right)-\{q\}\right) \mid \leq 4$. Therefore, $|V(G)| \leq 12$, contradicting our assumption. Therefore, $B_{p} \cap B_{q}=\emptyset$ and so $A_{p} \cap A_{q} \subseteq$ $P \cup\{x\}$.

Lemma 4.14. Let $G$ be a 3-rank-connected graph with $|V(G)| \geq 6$ and a, b be distinct vertices of $G$. Let $A$ be a quad of $G \backslash a$ and $B$ be a quad of $G \backslash b$. If $|A \cap B|=1$, then $b \in A$ and $a \in B$.

Proof. Suppose not. Then by symmetry, we may assume that $b \notin A$. Since $B$ is a quad of $G \backslash b$, we know that $\rho_{G \backslash b}(B)<\rho_{G \backslash b}(B-A)$. Then by Lemma 2.11,

$$
\rho_{G \backslash b}(B)+\rho_{G \backslash b}(A) \geq \rho_{G \backslash b}(A-B)+\rho_{G \backslash b}(B-A)
$$

and therefore $\rho_{G \backslash b}(A-B)<\rho_{G \backslash b}(A)$. Since $A$ is a quad of $G \backslash a$, we have that $\rho_{G}(A) \leq$ $\rho_{G \backslash a}(A)+1 \leq 3$. By Lemma 4.1, $G \backslash b$ is prime and so

$$
2 \leq \rho_{G \backslash b}(A-B)<\rho_{G \backslash b}(A) \leq \rho_{G}(A) \leq 3,
$$

which implies that $\rho_{G \backslash b}(A-B)=2$ and $\rho_{G \backslash b}(A)=3$. Since $2=\rho_{G \backslash b}(A)-1 \leq \rho_{G \backslash a \backslash b}(A) \leq$ $\rho_{G \backslash a}(A)=2$, we have $\rho_{G \backslash a \backslash b}(A)=2$. Since $a \notin A-B$ and $b \notin A$, by (A1) of Lemma 2.16,

$$
2+2=\rho_{G \backslash a}(A)+\rho_{G \backslash b}(A-B) \geq \rho_{G}(A-B)+\rho_{G \backslash a \backslash b}(A)=\rho_{G}(A-B)+2 .
$$

Hence $\rho_{G}(A-B) \leq 2$, contradicting the condition that $G$ is 3 -rank-connected.
Proposition 4.15. Let $G$ be a 3-rank-connected graph such that $|V(G)| \geq 13$. Then there exists a sequentially 3-rank-connected vertex-minor $H$ of $G$ such that $|V(H)|=|V(G)|-1$.

Proof. Suppose that no vertex-minor of $G$ on $|V(G)|-1$ vertices is sequentially 3-rank-connected. Let $x$ be a vertex of $G$. By Lemma 4.3, we can assume that $G \backslash x$ is weakly 3-rank-connected. By Lemma 4.1, $G \backslash x$ is prime. Since $G \backslash x$ is not sequentially 3 -rank-connected, there exists a subset $P$ of $V(G \backslash x)$ such that $\rho_{G \backslash x}(P) \leq 2$ and neither $P$ nor $V(G \backslash x)-P$ is sequential in $G \backslash x$. Since $G \backslash x$ is weakly 3 -rank-connected, we may assume that $|P|=4$. Since $|V(G \backslash x)-P| \geq 4$ and $G \backslash x$ is prime, $\rho_{G \backslash x}(P)=2$. So by Lemma 4.5, $P$ is a quad of $G \backslash x$. Then by Lemma 4.10, we can assume the following.
(1) $G \backslash v$ is weakly 3-rank-connected for each vertex $v$ of $P \cup\{x\}$.
(2) $P$ is a quad of $G \backslash x$.
(3) There exist a 2-element subset $S$ of $P$ and a quad $X_{u}$ of $G \backslash u$ for each $u$ in $S$ such that $x \in X_{u},\left|X_{u} \cap P\right|=1$, and $V(G \backslash u)-X_{u}$ is not sequential in $G \backslash u$.

Let $p$ and $q$ be distinct vertices of $S$. By Lemma 4.13, $x \in X_{p} \cap X_{q} \subseteq P \cup\{x\}$. By Lemma 4.11, $\left|X_{p} \cap X_{q}\right| \leq 2$.

If $\left|X_{p} \cap X_{q}\right|=1$, then, by Lemma 4.14, $q \in X_{p}$ and $p \in X_{q}$. Then, since $X_{p} \cap X_{q}=\{x\}$, $X_{p} \cap P=\{q\}$, and $X_{q} \cap P=\{p\}$, by Proposition 4.6, $G * x \backslash x$ or $G / x$ is sequentially 3-rankconnected, contradicting the assumption.

So $\left|X_{p} \cap X_{q}\right|=2$. Let $r \in X_{p} \cap X_{q}-\{x\}$. Since $r$ does not satisfy (Q1), by Lemma 4.8, (Q2) or (Q3) holds for $r$.

If (Q2) holds, there is a subset $R$ of $V(G \backslash r)$ such that $\rho_{G \backslash r}(R) \leq 2, R \cap P \neq \emptyset,(V(G \backslash$ $r)-R) \cap P \neq \emptyset$, and neither $R$ nor $V(G \backslash r)-R$ is sequential in $G \backslash r$. By symmetry, we may assume that $|P \cap R|=1$ by replacing $R$ by $V(G \backslash r)-R$. Then by (K2) of Lemma 4.9, $R$ is a quad of $G \backslash r$ containing $x$. By Lemma 4.11, $\left|R \cap X_{p}\right|,\left|R \cap X_{q}\right| \leq 2$.

Suppose that $\left|R \cap X_{p}\right|=2$ and $\left|R \cap X_{q}\right|=2$. Then by applying Lemma 4.12 twice, we deduce that $R$ contains both $p$ and $q$, contradicting our assumption that $|P \cap R|=1$. So by symmetry, we can assume that $\left|R \cap X_{p}\right|=1$. Then by Lemma 4.14, $p \in R$. Since $R \cap X_{p}=\{x\}$, $P \cap R=\{p\}$, and $X_{p} \cap P=\{r\}$, by Lemma 4.6, we deduce that $G * x \backslash x$ or $G / x$ is sequentially 3 -rank-connected, contradicting our assumption.

If (Q3) holds, then there is a quad of $R$ of $G \backslash r$ containing $x$ such that $R \cap P=\emptyset$. By Lemma 4.11, $\left|R \cap X_{p}\right| \leq 2$. Since $p \notin R$, by Lemma 4.12, $\left|R \cap X_{p}\right|=1$. Then Lemma 4.14 implies that $p \in R$, contradicting the assumption.

## 5 Treating internally 3-rank-connected graphs

In this section, we prove Theorem 1.2 for internally 3 -rank-connected graphs.
A graph $G$ is internally 3-rank-connected if $G$ is prime and for each subset $X$ of $V(G)$, either $|X| \leq 3$ or $|V(G)-X| \leq 3$ whenever $\rho_{G}(X) \leq 2$. A 3 -element set $T$ of vertices of a graph $G$ is a triplet of $G$ if $\rho_{G}(T)=2$ and $\rho_{G \backslash x}(T-x)=2$ for each $x \in T$.

Here is a rough overview of our approach in this section. If $G$ is an internally 3 -rankconnected counterexample of Theorem 1.2 and $|V(G)| \geq 13$, then by pivoting, we may assume that $G$ has a triplet $T=\{a, b, c\}$. Next we find a partition $\left(A_{b}, A_{c}\right)$ of $V(G \backslash a)$, a partition $\left(B_{a}, B_{c}\right)$ of $V(G \backslash b)$, and a partition $\left(C_{a}, C_{b}\right)$ of $V(G \backslash c)$ satisfying the following conditions:
(1) $b \in A_{b}, c \in A_{c}$, and neither $A_{b}$ nor $A_{c}$ is sequential in $G \backslash a$.
(2) $a \in B_{a}, c \in B_{c}$, and neither $B_{a}$ nor $B_{c}$ is sequential in $G \backslash b$.
(3) $a \in C_{a}, b \in C_{b}$, and neither $C_{a}$ nor $C_{b}$ is sequential in $G \backslash c$.

We then prove that all of $A_{b}, A_{c}, B_{a}, B_{c}, C_{a}, C_{b}$ must be small, contradicting the assumption that $|V(G)| \geq 13$.

The following lemma shows that if a graph is internally 3 -rank-connected but not 3 -rankconnected, then we can apply pivoting to obtain a graph with a triplet.

Lemma 5.1 (Oum [8, Lemma 5.1]). Let $G$ be a prime graph and $A$ be a 3 -element subset of $V(G)$ such that $\rho_{G}(A)=2$. Then there is a graph $G^{\prime}$ pivot-equivalent to $G$ such that $A$ is a triplet of $G^{\prime}$.
Lemma 5.2 (Oum [8, Lemma 5.2]). Let $G$ be an internally 3-rank-connected graph and $T=$ $\{a, b, c\}$ be a triplet of $G$. Then $G \backslash a, G \backslash b$, and $G \backslash c$ are prime.

Lemma 5.3. Let $T$ be a triplet of an internally 3-rank-connected graph $G$ and $a \in T$. Let ( $X, Y$ ) be a partition of $V(G)-\{a\}$ such that $\rho_{G \backslash a}(X) \leq 2$ and neither $X$ nor $Y$ is sequential in $G \backslash a$. Then there exist $b \in X \cap T$ and $c \in Y \cap T$ such that $\rho_{G \backslash b}(X-\{b\})=\rho_{G \backslash c}(Y-\{c\})=3$.
Proof. Since neither $X$ nor $Y$ is sequential in $G \backslash a,|X| \geq 4$ and $|Y| \geq 4$. So $\rho_{G \backslash a}(X)=2$ because $G \backslash a$ is prime by Lemma 5.2. Since $T$ is a triplet of $G$, we have $\rho_{G \backslash a}(T-\{a\})=\rho_{G}(T)$. If $T \subseteq X \cup\{a\}$, then by Lemma 2.14, $\rho_{G}(X \cup\{a\})=\rho_{G \backslash a}(X)=2$, contradicting the assumption that $G$ is internally 3 -rank-connected.

Hence $T-\{a\} \nsubseteq X$ and similarly $T-\{a\} \nsubseteq Y$. Therefore, there exist $b \in X \cap T$ and $c \in Y \cap T$. Then $T=\{a, b, c\}$.

By (i) of Lemma 2.8, $\rho_{G}(X) \leq \rho_{G \backslash a}(X)+1 \leq 3$. So by (ii) of Lemma 2.8, we have $\rho_{G \backslash b}(X-\{b\}) \leq 3$ and similarly, $\rho_{G \backslash c}(Y-\{c\}) \leq 3$.

Suppose that $\rho_{G \backslash c}(Y-\{c\})<3$. Since $T$ is a triplet of $G$, by Lemma 2.9,

$$
\begin{aligned}
\rho_{G}(\{a, b\}, Y-\{c\})+2 & =\rho_{G}(\{a, b\}, Y-\{c\})+\rho_{G}(\{a, b, c\}, V(G)-\{a, b, c\}) \\
& \geq \rho_{G}(\{a, b, c\}, Y-\{c\})+\rho_{G}(\{a, b\}, V(G)-\{a, b, c\}) \\
& =\rho_{G}(\{a, b, c\}, Y-\{c\})+2,
\end{aligned}
$$

and therefore $\rho_{G}(\{a, b, c\}, Y-\{c\}) \leq \rho_{G}(\{a, b\}, Y-\{c\})$. Then by Lemma 2.9, we have

$$
\rho_{G}(X \cup\{a\}, Y-\{c\})+\rho_{G}(\{a, b, c\}, Y-\{c\}) \geq \rho_{G}(X \cup\{a, c\}, Y-\{c\})+\rho_{G}(\{a, b\}, Y-\{c\}) .
$$

Hence $\rho_{G \backslash a}(X \cup\{c\}) \leq \rho_{G}(X \cup\{a, c\}, Y-\{c\}) \leq \rho_{G}(X \cup\{a\}, Y-\{c\})=\rho_{G \backslash c}(Y-\{c\})<$ 3. Therefore, $\rho_{G \backslash a}(X \cup\{c\}) \leq 2=\rho_{G \backslash a}(X)$. Since $|Y-\{c\}| \geq 3$ and $G \backslash a$ is prime, we have $\rho_{G \backslash a}(X \cup\{c\})=2$. Since $Y$ is not sequential in $G \backslash a$, by Lemma 3.1, $Y-\{c\}$ is not sequential in $G \backslash a$ and therefore $|Y-\{c\}| \geq 4$. Since $T \subseteq X \cup\{a, c\}$, by Lemma 2.14, $\rho_{G}(X \cup\{a, c\})=\rho_{G \backslash a}(X \cup\{c\})=2$, contradicting the assumption that $G$ is internally 3-rankconnected. Therefore $\rho_{G \backslash c}(Y-\{c\})=3$. By symmetry, we deduce that $\rho_{G \backslash b}(X-\{b\})=3$.

Lemma 5.4. Let $G$ be an internally 3-rank-connected graph with $|V(G)| \geq 12$ and $T=\{a, b, c\}$ be a triplet of $G$ such that $G \backslash c$ is not sequentially 3-rank-connected. Let $X$ be a subset of $V(G \backslash a \backslash b)$ such that $|X| \geq 3,|V(G \backslash a \backslash b)-X| \geq 2$, and $c \notin X$. Then $\rho_{G \backslash a \backslash b}(X) \geq 2$.
Proof. Suppose that $\rho_{G \backslash a \backslash b}(X) \leq 1$. Let $Y=V(G \backslash a \backslash b)-X$. Since $\{a, b, c\}$ is a triplet of $G$, we have $\rho_{G \backslash a}(\{b, c\})=\rho_{G}(\{a, b, c\})$. By Lemma 2.14, $\rho_{G}(Y \cup\{a, b\})=\rho_{G \backslash a}(Y \cup\{b\})$. Hence $\rho_{G}(Y \cup\{a, b\})=\rho_{G \backslash a}(Y \cup\{b\}) \leq \rho_{G \backslash a \backslash b}(Y)+1=\rho_{G \backslash a \backslash b}(X)+1 \leq 2$. So $|X| \leq 3$ because $G$ is internally 3-rank-connected and $|Y \cup\{a, b\}| \geq 4$.

Since $G \backslash c$ is not sequentially 3-rank-connected, there exists a partition ( $C_{a}, C_{b}$ ) of $V(G \backslash c)$ such that $\rho_{G \backslash c}\left(C_{a}\right) \leq 2$ and neither $C_{a}$ nor $C_{b}$ is sequential in $G \backslash c$.

Suppose that $|X|=3$. By symmetry, we may assume that $\left|C_{a} \cap X\right| \geq 2$ by swapping $C_{a}$ and $C_{b}$ if necessary. If $\left|C_{a} \cap X\right|=2$, then let $x$ be the element in $C_{b} \cap X$. By Lemma 5.2, $G \backslash c$ is prime. Since $|(Y \cup\{a, b\})-\{c\}| \geq 2$, we have $2 \leq \rho_{G \backslash c}(X) \leq \rho_{G}(X)=2$. Since $|X-\{x\}|=2$ and $G \backslash c$ is prime, we also have $\rho_{G \backslash c}(X-\{x\})=2$. So by Lemma 2.10,

$$
\rho_{G \backslash c}\left(C_{a}\right)+\rho_{G \backslash c}(X) \geq \rho_{G \backslash c}\left(C_{a} \cup\{x\}\right)+\rho_{G \backslash c}(X-\{x\}) .
$$

Therefore, $\rho_{G \backslash c}\left(C_{a} \cup\{x\}\right) \leq \rho_{G \backslash c}\left(C_{a}\right) \leq 2$. Since $\left|C_{b}-\{x\}\right| \geq 3$ and by Lemma 5.2, $G \backslash c$ is prime, we have $\rho_{G \backslash c}\left(C_{a} \cup\{x\}\right)=\rho_{G \backslash c}\left(C_{a}\right)=2$. So by Lemma 3.1, neither $C_{a} \cup\{x\}$ nor $C_{b}-\{x\}$ is sequential in $G \backslash c$. By replacing $\left(C_{a}, C_{b}\right)$ with $\left(C_{a} \cup\{x\}, C_{b}-\{x\}\right)$, we may assume that $\left|C_{a} \cap X\right|=3$.

By Lemma 5.3, there is a unique element $t \in\{a, b\}$ of $C_{b} \cap T$. Then $X \subseteq C_{a}$ and $C_{b}-\{t\} \subseteq$ $Y-\{c\} \subseteq Y$. Since $|V(G)| \geq 12$ and $G$ is internally 3-rank-connected, we have $\rho_{G}(Y \cup\{t\}) \geq 3$. Since $\rho_{G \backslash t}(Y) \leq \rho_{G \backslash a \backslash b}(Y)+1 \leq 2<\rho_{G}(Y \cup\{t\})$ and $t \in C_{b} \subseteq Y \cup\{t\}$, by Lemma 2.14, $\rho_{G \backslash t}\left(C_{b}-\{t\}\right)<\rho_{G}\left(C_{b}\right) \leq 3$, contradicting Lemma 5.3.

Lemma 5.5. Let $G$ be an internally 3-rank-connected graph with $|V(G)| \geq 12$ and $T=\{a, b, c\}$ be a triplet of $G$. Let $\left(A_{b}, A_{c}\right)$ be a partition of $V(G \backslash a)$ such that $b \in A_{b}, c \in A_{c}, \rho_{G \backslash a}\left(A_{b}\right) \leq 2$, and neither $A_{b}$ nor $A_{c}$ is sequential in $G \backslash a$ and let $\left(B_{a}, B_{c}\right)$ be a partition of $V(G \backslash b)$ such that $a \in B_{a}, c \in B_{c}, \rho_{G \backslash b}\left(B_{a}\right) \leq 2$, and neither $B_{a}$ nor $B_{c}$ is sequential in $G \backslash b$. If $G \backslash c$ is not sequentially 3-rank-connected, then the following hold.

$$
\begin{equation*}
\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{c}\right)=\rho_{G}\left(A_{b} \cap B_{c}\right) . \tag{1}
\end{equation*}
$$

$$
\begin{equation*}
\rho_{G \backslash a \backslash b}\left(A_{c} \cap B_{a}\right)=\rho_{G}\left(A_{c} \cap B_{a}\right) . \tag{2}
\end{equation*}
$$

3) $\rho_{G \backslash a \backslash b}\left(A_{c} \cap B_{c}\right)=\rho_{G}\left(A_{c} \cap B_{c}\right)$.

Proof. Since none of $A_{b}, A_{c}$ is sequential in $G \backslash a$ and none of $B_{a}, B_{c}$ is sequential in $G \backslash b$, we have $\left|A_{b}\right|,\left|A_{c}\right|,\left|B_{a}\right|,\left|B_{c}\right| \geq 4$. By Lemma 5.2, $G \backslash a$ is prime and so $\rho_{G \backslash a}\left(A_{c}\right)=2$. Since $c \notin A_{b}-\{b\}$ and $\left|A_{b}-\{b\}\right| \geq 3$, by Lemma 5.4, we have $\rho_{G \backslash a \backslash b}\left(A_{c}\right)=\rho_{G \backslash a \backslash b}\left(A_{b}-\{b\}\right) \geq 2$. So by Lemma 2.8(i), we have $\rho_{G \backslash \backslash \backslash b}\left(A_{c}\right)=\rho_{G \backslash a}\left(A_{c}\right)=2$. Similarly, $\rho_{G \backslash a \backslash b}\left(B_{c}\right)=\rho_{G \backslash b}\left(B_{c}\right)=2$.

Since $\rho_{G \backslash \backslash \backslash b}\left(B_{c}\right)=\rho_{G \backslash b}\left(B_{c}\right)=2$ and $A_{b} \cap B_{c} \subseteq B_{c}$, by Lemma 2.13, we have $\rho_{G \backslash b}\left(A_{b} \cap B_{c}\right)=$ $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{c}\right)$.

Since $\{a, b, c\}$ is a triplet of $G$, we have $\rho_{G}(\{a, b, c\})=\rho_{G \backslash b}(\{a, c\})$. Observe that $\rho_{G \backslash b}\left(A_{b} \cap\right.$ $\left.B_{c}\right)=\rho_{G \backslash b}\left(A_{c} \cup B_{a}\right)$ and $\rho_{G}\left(A_{b} \cap B_{c}\right)=\rho_{G}\left(A_{c} \cup B_{a} \cup\{b\}\right)$. Since $\{a, b, c\} \subseteq A_{c} \cup B_{a} \cup\{b\}$, by Lemma 2.14, $\rho_{G \backslash b}\left(A_{b} \cap B_{c}\right)=\rho_{G \backslash b}\left(A_{c} \cup B_{a}\right)=\rho_{G}\left(A_{c} \cup B_{a} \cup\{b\}\right)=\rho_{G}\left(A_{b} \cap B_{c}\right)$.

Hence $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{c}\right)=\rho_{G}\left(A_{b} \cap B_{c}\right)$ and (1) holds. By symmetry, (2) also holds.
Now let us prove (3). Since $\rho_{G \backslash a \backslash b}\left(B_{c}\right)=\rho_{G \backslash b}\left(B_{c}\right)=2$ and $A_{c} \cap B_{c} \subseteq B_{c}$, by Lemma 2.13, we have $\rho_{G \backslash a \backslash b}\left(A_{c} \cap B_{c}\right)=\rho_{G \backslash b}\left(A_{c} \cap B_{c}\right)$.

By (A1) of Lemma 2.16,

$$
\begin{aligned}
2+\rho_{G \backslash b}\left(A_{c} \cap B_{c}\right) & =\rho_{G \backslash a}\left(A_{c}\right)+\rho_{G \backslash b}\left(A_{c} \cap B_{c}\right) \\
& \geq \rho_{G \backslash a \backslash b}\left(A_{c}\right)+\rho_{G}\left(A_{c} \cap B_{c}\right)=2+\rho_{G}\left(A_{c} \cap B_{c}\right),
\end{aligned}
$$

which implies that $\rho_{G \backslash b}\left(A_{c} \cap B_{c}\right) \geq \rho_{G}\left(A_{c} \cap B_{c}\right)$. By (i) of Lemma 2.8, $\rho_{G \backslash b}\left(A_{c} \cap B_{c}\right) \leq \rho_{G}\left(A_{c} \cap B_{c}\right)$ and so $\rho_{G \backslash b}\left(A_{c} \cap B_{c}\right)=\rho_{G}\left(A_{c} \cap B_{c}\right)$. Hence $\rho_{G}\left(A_{c} \cap B_{c}\right)=\rho_{G \backslash \backslash \backslash b}\left(A_{c} \cap B_{c}\right)$.
Lemma 5.6. Let $G$ be an internally 3-rank-connected graph with $|V(G)| \geq 12$ and $T=\{a, b, c\}$ be a triplet of $G$. Let $\left(A_{b}, A_{c}\right)$ be a partition of $V(G \backslash a)$ such that $b \in A_{b}, c \in A_{c}, \rho_{G \backslash a}\left(A_{b}\right) \leq 2$, and neither $A_{b}$ nor $A_{c}$ is sequential in $G \backslash a$ and let $\left(B_{a}, B_{c}\right)$ be a partition of $V(G \backslash b)$ such that $a \in B_{a}, c \in B_{c}, \rho_{G \backslash b}\left(B_{a}\right) \leq 2$, and neither $B_{a}$ nor $B_{c}$ is sequential in $G \backslash b$. If $G \backslash c$ is not sequentially 3 -rank-connected, then the following hold.
(i) $\rho_{G}\left(A_{c} \cap B_{a}\right) \leq 2$ and $2 \leq\left|A_{c} \cap B_{a}\right| \leq 3$.
(ii) $\rho_{G}\left(A_{b} \cap B_{c}\right) \leq 2$ and $2 \leq\left|A_{b} \cap B_{c}\right| \leq 3$.
(iii) $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \leq 2$.
(iv) $\left|A_{c} \cap B_{c}\right| \geq 2$.
(v) If $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \geq 2$, then $\rho_{G}\left(A_{c} \cap B_{c}\right) \leq 2$ and $\left|A_{c} \cap B_{c}\right| \leq 3$.

Proof. Since none of $A_{b}, A_{c}$ is sequential in $G \backslash a$ and none of $B_{a}, B_{c}$ is sequential in $G \backslash b$, we have $\left|A_{b}\right|,\left|A_{c}\right|,\left|B_{a}\right|,\left|B_{c}\right| \geq 4$. Let us prove the following, which prove the lemma.
(1) If $\left|A_{b} \cap B_{c}\right| \geq 2$, then $\rho_{G}\left(A_{c} \cap B_{a}\right) \leq 2$ and $\left|A_{c} \cap B_{a}\right| \leq 3$.
(2) If $\left|A_{c} \cap B_{a}\right| \geq 2$, then $\rho_{G}\left(A_{b} \cap B_{c}\right) \leq 2$ and $\left|A_{b} \cap B_{c}\right| \leq 3$.
(3) If $\left|A_{c} \cap B_{c}\right| \geq 2$, then $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \leq 2$.
(4) If $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \geq 2$, then $\rho_{G}\left(A_{c} \cap B_{c}\right) \leq 2$ and $\left|A_{c} \cap B_{c}\right| \leq 3$.
(5) $\left|A_{b} \cap B_{c}\right| \geq 2$.
(6) $\left|A_{c} \cap B_{a}\right| \geq 2$.
(7) $\left|A_{c} \cap B_{c}\right| \geq 2$.

To prove (1), suppose that $\left|A_{b} \cap B_{c}\right| \geq 2$. Since $G$ is prime and $\left|V(G)-\left(A_{b} \cap B_{c}\right)\right| \geq\left|A_{c}\right| \geq 4$, by (1) of Lemma 5.5, $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{c}\right)=\rho_{G}\left(A_{b} \cap B_{c}\right) \geq 2$. Since $G \backslash b$ is prime and $\left|A_{b} \cap B_{c}\right| \geq 2$, we have $\rho_{G \backslash b}\left(A_{c} \cup B_{a}\right)=\rho_{G \backslash b}\left(A_{b} \cap B_{c}\right) \geq 2$. Since $\rho_{G \backslash \backslash \backslash b}\left(A_{c}\right)=2$, by (S1) of Lemma 2.12,

$$
\begin{aligned}
2+2 & =\rho_{G \backslash a \backslash b}\left(A_{c}\right)+\rho_{G \backslash b}\left(B_{a}\right) \\
& \geq \rho_{G \backslash a \backslash b}\left(A_{c} \cap B_{a}\right)+\rho_{G \backslash b}\left(A_{c} \cup B_{a}\right) \geq \rho_{G \backslash a \backslash b}\left(A_{c} \cap B_{a}\right)+2 .
\end{aligned}
$$

Therefore, by (2) of Lemma 5.5, $\rho_{G}\left(A_{c} \cap B_{a}\right)=\rho_{G \backslash a \backslash b}\left(A_{c} \cap B_{a}\right) \leq 2$. Since $G$ is internally 3-rank-connected and $\left|V(G)-\left(A_{c} \cap B_{a}\right)\right| \geq\left|A_{b}\right| \geq 4$, we deduce that $\left|A_{c} \cap B_{a}\right| \leq 3$. So this proves (1). By symmetry between $a$ and $b,(2)$ also holds.

Now we show (3). Suppose that $\left|A_{c} \cap B_{c}\right| \geq 2$. Since $G$ is prime and $\left|V(G)-\left(A_{b} \cup B_{a}\right)\right| \geq$ $\left|A_{c}\right| \geq 4$, we have $\rho_{G}\left(A_{b} \cup B_{a}\right) \geq 2$. By (A3) of Lemma 2.16,

$$
4 \geq \rho_{G \backslash a}\left(A_{b}\right)+\rho_{G \backslash b}\left(B_{a}\right) \geq \rho_{G}\left(A_{b} \cup B_{a}\right)+\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right)
$$

and therefore $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \leq 2$.
Now let us prove (4). Suppose that $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \geq 2$. By (A3) of Lemma 2.16,

$$
4 \geq \rho_{G \backslash a}\left(A_{b}\right)+\rho_{G \backslash b}\left(B_{a}\right) \geq \rho_{G}\left(A_{b} \cup B_{a}\right)+\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \geq \rho_{G}\left(A_{b} \cup B_{a}\right)+2 .
$$

Hence $\rho_{G}\left(A_{b} \cup B_{a}\right)=\rho_{G}\left(A_{c} \cap B_{c}\right) \leq 2$. Since $G$ is internally 3-rank-connected and $\mid V(G)-$ $\left(A_{c} \cap B_{c}\right) \mid \geq 4$, we conclude that $\left|A_{c} \cap B_{c}\right| \leq 3$.

To prove (5), suppose that $\left|A_{b} \cap B_{c}\right| \leq 1$. Then $4 \leq\left|A_{b}\right|=|\{b\}|+\left|A_{b} \cap B_{c}\right|+\left|A_{b} \cap B_{a}\right| \leq$ $2+\left|A_{b} \cap B_{a}\right|$ and so $\left|A_{b} \cap B_{a}\right| \geq 2$.

If $\left|A_{b} \cap B_{a}\right| \geq 3$, then, since $c \in A_{c} \cap B_{c}$, by Lemma 5.4, $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \geq 2$. If $\left|A_{b} \cap B_{a}\right|=2$, then $\left|A_{b}\right|=4$ and by Lemma 4.5, $A_{b}$ is a quad of $G \backslash a$. Then by (ii) of Lemma 2.8, $\rho_{G \backslash a \backslash b}\left(A_{b} \cap\right.$ $\left.B_{a}\right) \geq \rho_{G \backslash a}\left(\left(A_{b} \cap B_{a}\right) \cup\{b\}\right)-1=2$. So, in both cases, we deduce that $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \geq 2$.

Hence, by (4), $\rho_{G}\left(A_{c} \cap B_{c}\right) \leq 2$ and $\left|A_{c} \cap B_{c}\right| \leq 3$. Since $4 \leq\left|B_{c}\right|=\left|A_{b} \cap B_{c}\right|+\left|A_{c} \cap B_{c}\right| \leq$ $1+\left|A_{c} \cap B_{c}\right| \leq 4$, we have $\left|A_{c} \cap B_{c}\right|=3$ and $\left|B_{c}\right|=4$. By (i) of Lemma 2.8, $\rho_{G \backslash b}\left(A_{c} \cap B_{c}\right) \leq$ $\rho_{G}\left(A_{c} \cap B_{c}\right) \leq 2$. So $B_{c}$ is sequential in $G \backslash b$, contradicting our assumption. So this proves that $\left|A_{b} \cap B_{c}\right| \geq 2$ and by symmetry between $a$ and $b,\left|A_{c} \cap B_{a}\right| \geq 2$ and (6) holds.

Now let us prove (7). Suppose that $\left|A_{c} \cap B_{c}\right| \leq 1$. Then $4 \leq\left|A_{c}\right|=1+\left|A_{c} \cap B_{c}\right|+\left|A_{c} \cap B_{a}\right| \leq$ $2+\left|A_{c} \cap B_{a}\right|$ and so $2 \leq\left|A_{c} \cap B_{a}\right|$. Then by (2), we have $\rho_{G}\left(A_{b} \cap B_{c}\right) \leq 2$ and $\left|A_{b} \cap B_{c}\right| \leq 3$. Since $4 \leq\left|B_{c}\right|=\left|A_{b} \cap B_{c}\right|+\left|A_{c} \cap B_{c}\right| \leq\left|A_{b} \cap B_{c}\right|+1 \leq 4$, we have $\left|A_{b} \cap B_{c}\right|=3$ and $\left|B_{c}\right|=4$. By (i) of Lemma 2.8, $\rho_{G \backslash b}\left(A_{b} \cap B_{c}\right) \leq \rho_{G}\left(A_{b} \cap B_{c}\right) \leq 2$. So $B_{c}$ is sequential in $G \backslash b$, contradicting our assumption.

Lemma 5.7. Let $G$ be an internally 3 -rank-connected graph with $|V(G)| \geq 12$ and $T=\{a, b, c\}$ be a triplet of $G$. Let $\left(A_{b}, A_{c}\right)$ be a partition of $V(G \backslash a)$ such that $b \in A_{b}, c \in A_{c}, \rho_{G \backslash a}\left(A_{b}\right) \leq 2$, and neither $A_{b}$ nor $A_{c}$ is sequential in $G \backslash a$, let $\left(B_{a}, B_{c}\right)$ be a partition of $V(G \backslash b)$ such that $a \in B_{a}, c \in B_{c}, \rho_{G \backslash b}\left(B_{a}\right) \leq 2$, and neither $B_{a}$ nor $B_{c}$ is sequential in $G \backslash b$, and let ( $C_{a}, C_{b}$ ) be a partition of $V(G \backslash c)$ such that $a \in C_{a}, b \in C_{b}, \rho_{G \backslash c}\left(C_{a}\right) \leq 2$, and neither $C_{a}$ nor $C_{b}$ is sequential in $G \backslash c$. Then the following hold.
(1) If $\left|A_{c} \cap B_{c}\right| \geq 3$ and $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right)>1$, then $\left|A_{c} \cap B_{c}\right|=3$, $\rho_{G}\left(A_{c} \cap B_{c}\right)=2$, and $\left|A_{c} \cap B_{c} \cap C_{a}\right|=\left|A_{c} \cap B_{c} \cap C_{b}\right|=1$.
(2) If $\left|A_{c} \cap B_{c}\right| \geq 3$ and $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \leq 1$, then either

- $A_{b} \cap B_{a}=\emptyset$, or
- $1 \leq\left|A_{b} \cap B_{a}\right| \leq 2$ and $\rho_{G \backslash c}\left(\left(A_{b} \cap B_{a}\right) \cup\{a, b\}\right)=3$.

Proof. (1) Since $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right)>1$, we have $\left|A_{b} \cap B_{a}\right| \geq 2$ and by (v) of Lemma 5.6, $\rho_{G}\left(A_{c} \cap\right.$ $\left.B_{c}\right) \leq 2$ and $\left|A_{c} \cap B_{c}\right| \leq 3$. Hence $\left|A_{c} \cap B_{c}\right|=3$. Since $G$ is prime and $\left|V(G)-\left(A_{c} \cap B_{c}\right)\right| \geq 3$, we have $\rho_{G}\left(A_{c} \cap B_{c}\right)=2$. Now we prove that $\left|A_{c} \cap B_{c} \cap C_{a}\right|=\left|A_{c} \cap B_{c} \cap C_{b}\right|=1$. Suppose not. Then, by symmetry, we may assume that $\left|A_{c} \cap B_{c} \cap C_{a}\right|=2$ and $\left|A_{c} \cap B_{c} \cap C_{b}\right|=0$. Since $\left|\left(A_{c} \cap B_{c}\right)-\{c\}\right|=2$ and $G \backslash c$ is prime, $\rho_{G \backslash c}\left(\left(A_{c} \cap B_{c}\right)-\{c\}\right) \geq 2$. By (ii) of Lemma 2.8, $\rho_{G \backslash c}\left(\left(A_{c} \cap B_{c}\right)-\{c\}\right)=\rho_{G}\left(A_{c} \cap B_{c}\right)=2$. Since $A_{c} \cap B_{c} \subseteq C_{a} \cup\{c\}$, by Lemma 2.14, $\rho_{G}\left(C_{a} \cup\{c\}\right)=$ $\rho_{G \backslash c}\left(C_{a}\right) \leq 2$. Since $G$ is internally 3-rank-connected and $\left|C_{a} \cup\{c\}\right| \geq 5$, we have $\left|C_{b}\right| \leq 3$, contradicting our assumption.
(2) By Lemma 5.4, $\left|A_{b} \cap B_{a}\right| \leq 2$. Suppose that $\left|A_{b} \cap B_{a}\right| \geq 1$. We can observe that $\rho_{G}\left(\left(A_{b} \cap\right.\right.$ $\left.\left.B_{a}\right) \cup\{a, b, c\}\right) \geq 3$ because $G$ is internally 3-rank-connected, $\left|\left(A_{b} \cap B_{a}\right) \cup\{a, b, c\}\right| \geq 4$, and $\left|V(G)-\left(\left(A_{b} \cap B_{a}\right) \cup\{a, b, c\}\right)\right| \geq 12-5=7$. Since $\{a, b, c\}$ is a triplet of $G$, we have $\rho_{G}(\{a, b, c\})=$ $\rho_{G \backslash c}(\{a, b\})=2$. Since $\{a, b, c\} \subseteq\left(A_{b} \cap B_{a}\right) \cup\{a, b, c\}$, by Lemma 2.14, $\rho_{G \backslash c}\left(\left(A_{b} \cap B_{a}\right) \cup\{a, b\}\right)=$ $\rho_{G}\left(\left(A_{b} \cap B_{a}\right) \cup\{a, b, c\}\right) \geq 3$. By (i) and (ii) of Lemma 2.8, $\rho_{G \backslash c}\left(\left(A_{b} \cap B_{a}\right) \cup\{a, b\}\right) \leq \rho_{G}\left(\left(A_{b} \cap\right.\right.$ $\left.\left.B_{a}\right) \cup\{a, b\}\right) \leq 2+\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \leq 3$ and we conclude that $\rho_{G \backslash c}\left(\left(A_{b} \cap B_{a}\right) \cup\{a, b\}\right)=3$.

Proposition 5.8. Let $T$ be a triplet of an internally 3-rank-connected graph $G$. If $|V(G)| \geq 12$, then there exists $t \in T$ such that $G \backslash t$ is sequentially 3-rank-connected.
Proof. Let $T=\{a, b, c\}$. Suppose that none of $G \backslash a, G \backslash b$, and $G \backslash c$ is sequentially 3-rankconnected. Then there exist partitions $\left(A_{b}, A_{c}\right)$ of $V(G)-\{a\},\left(B_{a}, B_{c}\right)$ of $V(G)-\{b\}$, and $\left(C_{a}, C_{b}\right)$ of $V(G)-\{c\}$ such that $\rho_{G \backslash a}\left(A_{b}\right) \leq 2, \rho_{G \backslash b}\left(B_{a}\right) \leq 2, \rho_{G \backslash c}\left(C_{a}\right) \leq 2$, neither $A_{b}$ nor $A_{c}$ is sequential in $G \backslash a$, neither $B_{a}$ nor $B_{c}$ is sequential in $G \backslash b$, and neither $C_{a}$ nor $C_{b}$ is sequential in $G \backslash c$. Then $\left|A_{b}\right|,\left|A_{c}\right| \geq 4,\left|B_{a}\right|,\left|B_{c}\right| \geq 4$, and $\left|C_{a}\right|,\left|C_{b}\right| \geq 4$.

By Lemma 5.3, we may assume that $b \in A_{b}, c \in A_{c}, a \in B_{a}, c \in B_{c}, a \in C_{a}$, and $b \in C_{b}$.
By Lemma 5.6, we have $\left|A_{b} \cap B_{c}\right| \leq 3,\left|A_{c} \cap B_{a}\right| \leq 3$, and $\rho_{G}\left(A_{b} \cap B_{c}\right) \leq 2$. By symmetry between $b$ and $c$, we have that $\left|A_{c} \cap C_{b}\right| \leq 3$ and $\left|A_{b} \cap C_{a}\right| \leq 3$. By symmetry between $a$ and $c$, we have that $\left|B_{c} \cap C_{a}\right| \leq 3$ and $\left|B_{a} \cap C_{b}\right| \leq 3$. Now we show that we can assume the following.
(B1) If $\left|A_{b} \cap B_{c}\right|=3$, then $A_{b} \cap B_{c} \subseteq C_{a}$ or $A_{b} \cap B_{c} \subseteq C_{b}$.
(B2) If $\left|A_{c} \cap B_{a}\right|=3$, then $A_{c} \cap B_{a} \subseteq C_{a}$ or $A_{c} \cap B_{a} \subseteq C_{b}$.
(B3) If $\left|A_{c} \cap C_{b}\right|=3$, then $A_{c} \cap C_{b} \subseteq B_{a}$ or $A_{c} \cap C_{b} \subseteq B_{c}$.
(B4) If $\left|A_{b} \cap C_{a}\right|=3$, then $A_{b} \cap C_{a} \subseteq B_{a}$ or $A_{b} \cap C_{a} \subseteq B_{c}$.
(B5) If $\left|B_{c} \cap C_{a}\right|=3$, then $B_{c} \cap C_{a} \subseteq A_{b}$ or $B_{c} \cap C_{a} \subseteq A_{c}$.
(B6) If $\left|B_{a} \cap C_{b}\right|=3$, then $B_{a} \cap C_{b} \subseteq A_{b}$ or $B_{a} \cap C_{b} \subseteq A_{c}$.
We choose ( $A_{b}, A_{c}, B_{a}, B_{c}, C_{a}, C_{b}$ ) such that $b \in A_{b}, c \in A_{c}, a \in B_{a}, c \in B_{c}, a \in C_{a}, b \in C_{b}$, and it satisfies the maximum number of (B1)-(B6). Then we claim that all of (B1)-(B6) hold. Suppose not. Then by symmetry, we can assume that (B1) does not hold. Then $\left|A_{b} \cap B_{c}\right|=3$, $A_{b} \cap B_{c} \nsubseteq C_{a}$, and $A_{b} \cap B_{c} \nsubseteq C_{b}$. Then either $\left|A_{b} \cap B_{c} \cap C_{a}\right|=2$ and $\left|A_{b} \cap B_{c} \cap C_{b}\right|=1$ or $\left|A_{b} \cap B_{c} \cap C_{a}\right|=1$ and $\left|A_{b} \cap B_{c} \cap C_{b}\right|=2$.
(i) Suppose that $\left|A_{b} \cap B_{c} \cap C_{a}\right|=2$ and $\left|A_{b} \cap B_{c} \cap C_{b}\right|=1$. Let $x$ be the element of $A_{b} \cap B_{c} \cap C_{b}$. We have $\rho_{G \backslash c}\left(A_{b} \cap B_{c}\right) \leq \rho_{G}\left(A_{b} \cap B_{c}\right) \leq 2$. Since $\left|\left(A_{b} \cap B_{c}\right)-\{x\}\right|=2$ and $G \backslash c$ is prime, $\rho_{G \backslash c}\left(\left(A_{b} \cap B_{c}\right)-\{x\}\right) \geq 2$. So by Lemma 2.10,

$$
\begin{aligned}
2+2 & \geq \rho_{G \backslash c}\left(C_{a}\right)+\rho_{G \backslash c}\left(A_{b} \cap B_{c}\right) \\
& \geq \rho_{G \backslash c}\left(\left(A_{b} \cap B_{c}\right)-\{x\}\right)+\rho_{G \backslash c}\left(C_{a} \cup\{x\}\right) \geq 2+\rho_{G \backslash c}\left(C_{a} \cup\{x\}\right) .
\end{aligned}
$$

Therefore, $\rho_{G \backslash c}\left(C_{a} \cup\{x\}\right) \leq \rho_{G \backslash c}\left(C_{a}\right) \leq 2$. Since $G \backslash c$ is prime and $\left|V(G \backslash c)-\left(C_{a} \cup\{x\}\right)\right|=$ $\left|C_{b}\right|-1 \geq 3$, we have $\rho_{G \backslash c}\left(C_{a} \cup\{x\}\right)=\rho_{G \backslash c}\left(C_{a}\right)=2$. Hence by Lemma 3.1, neither $C_{a} \cup\{x\}$ nor $C_{b}-\{x\}$ is sequential in $G \backslash c$. We deduce that $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a} \cup\{x\}, C_{b}-\{x\}\right)$ satisfies (B1). Since $x \notin A_{c} \cap B_{a}$, if $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}, C_{b}\right)$ satisfies (B2), then $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a} \cup\right.$ $\left.\{x\}, C_{b}-\{x\}\right)$ satisfies (B2). Since $A_{c} \cap C_{b}=A_{c} \cap\left(C_{b}-\{x\}\right)$, if $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}, C_{b}\right)$ satisfies (B3), then $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a} \cup\{x\}, C_{b}-\{x\}\right)$ satisfies (B3). Since $B_{a} \cap\left(C_{b}-\{x\}\right)=B_{a} \cap C_{b}$, if $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}, C_{b}\right)$ satisfies (B6), then $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a} \cup\{x\}, C_{b}-\{x\}\right)$ satisfies (B6). Since $x \in A_{b}$, we have $\left|A_{b} \cap C_{a}\right|+1=\left|A_{b} \cap\left(C_{a} \cup\{x\}\right)\right| \leq 3$ by applying Lemma 5.6(i) with $\left(A_{c}, A_{b}\right)$ and $\left(C_{a} \cup\{x\}, C_{b}-\{x\}\right)$. So $\left|A_{b} \cap C_{a}\right| \leq 2$. Since $\left|A_{b} \cap B_{c} \cap C_{a}\right|=2$ we have $A_{b} \cap C_{a} \subseteq B_{c}$. So $A_{b} \cap\left(C_{a} \cup\{x\}\right) \subseteq B_{c}$ because $x \in B_{c}$. Hence $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a} \cup\{x\}, C_{b}-\{x\}\right)$ satisfies (B4).

Since $x \in B_{c}$, we have $\left|B_{c} \cap C_{a}\right|+1=\left|B_{c} \cap\left(C_{a} \cup\{x\}\right)\right| \leq 3$ by applying Lemma 5.6(ii) with $\left(B_{c}, B_{a}\right)$ and $\left(C_{b}-\{x\}, C_{a} \cup\{x\}\right)$. So $\left|B_{c} \cap C_{a}\right| \leq 2$. Since $\left|A_{b} \cap B_{c} \cap C_{a}\right|=2$ we have $B_{c} \cap C_{a} \subseteq A_{b}$. So $B_{c} \cap\left(C_{a} \cup\{x\}\right) \subseteq A_{b}$ because $x \in A_{b}$. Hence $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a} \cup\{x\}, C_{b}-\{x\}\right)$ satisfies (B5). Therefore, the number of (B1)-(B6) which $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a} \cup\{x\}, C_{b}-\{x\}\right)$ satisfies is larger than the number of (B1)-(B6) which $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}, C_{b}\right)$ satisfies, contradicting our assumption.
(ii) Suppose that $\left|A_{b} \cap B_{c} \cap C_{a}\right|=1$ and $\left|A_{b} \cap B_{c} \cap C_{b}\right|=2$. Let $y$ be the element of $A_{b} \cap B_{c} \cap C_{a}$. Since $\left|\left(A_{b} \cap B_{c}\right)-\{y\}\right|=2$ and $G \backslash c$ is prime, $\rho_{G \backslash c}\left(\left(A_{b} \cap B_{c}\right)-\{y\}\right) \geq 2$. So by Lemma 2.10,

$$
\begin{aligned}
2+2 & \geq \rho_{G \backslash c}\left(C_{b}\right)+\rho_{G \backslash c}\left(A_{b} \cap B_{c}\right) \\
& \geq \rho_{G \backslash c}\left(\left(A_{b} \cap B_{c}\right)-\{y\}\right)+\rho_{G \backslash c}\left(C_{b} \cup\{y\}\right) \geq 2+\rho_{G \backslash c}\left(C_{b} \cup\{y\}\right)
\end{aligned}
$$

Therefore, $\rho_{G \backslash c}\left(C_{b} \cup\{y\}\right) \leq \rho_{G \backslash c}\left(C_{b}\right) \leq 2$. Since $G \backslash c$ is prime and $\left|V(G \backslash c)-\left(C_{b} \cup\{y\}\right)\right|=$ $\left|C_{a}\right|-1 \geq 3$, we have $\rho_{G \backslash c}\left(C_{b} \cup\{y\}\right)=\rho_{G \backslash c}\left(C_{b}\right)=2$. Hence by Lemma 3.1, neither $C_{a}-\{y\}$ nor $C_{b} \cup\{y\}$ is sequential in $G \backslash c$. We deduce that $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}-\{y\}, C_{b} \cup\{y\}\right)$ satisfies (B1). Since $y \notin A_{c} \cap B_{a}$, if ( $A_{b}, A_{c}, B_{a}, B_{c}, C_{a}, C_{b}$ ) satisfies (B2), then $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}-\{y\}, C_{b} \cup\{y\}\right)$ satisfies (B2). Since $y \in A_{b} \cap C_{a}$, by applying Lemma 5.6(i) with $\left(A_{c}, A_{b}\right)$ and $\left(C_{a}, C_{b}\right)$, we have $\left|A_{b} \cap\left(C_{a}-\{y\}\right)\right|=\left|A_{b} \cap C_{a}\right|-1 \leq 3-1=2$. Hence $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}-\{y\}, C_{b} \cup\{y\}\right)$ satisfies (B4). Since $y \in B_{c} \cap C_{a}$, by applying Lemma 5.6(ii) with ( $B_{c}, B_{a}$ ) and ( $C_{b}, C_{a}$ ), we have $\left|B_{c} \cap\left(C_{a}-\{y\}\right)\right|=\left|B_{c} \cap C_{a}\right|-1 \leq 3-1=2$. Hence ( $\left.A_{b}, A_{c}, B_{a}, B_{c}, C_{a}-\{y\}, C_{b} \cup\{y\}\right)$ satisfies (B5).

Since $A_{c} \cap\left(C_{b} \cup\{y\}\right)=A_{c} \cap C_{b}$, if $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}, C_{b}\right)$ satisfies (B3), then $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}-\right.$ $\left.\{y\}, C_{b} \cup\{y\}\right)$ satisfies (B3). Since $B_{a} \cap\left(C_{b} \cup\{y\}\right)=B_{a} \cap C_{b}$, if $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}, C_{b}\right)$ satisfies (B6), then $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}-\{y\}, C_{b} \cup\{y\}\right)$ satisfies (B6). Therefore, the number of (B1)-(B6) which $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}-\{y\}, C_{b} \cup\{y\}\right)$ satisfies is larger than the number of (B1)-(B6) which $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}, C_{b}\right)$ satisfies, contradicting our assumption.

Therefore, the claim is proved and $\left(A_{b}, A_{c}, B_{a}, B_{c}, C_{a}, C_{b}\right)$ satisfies (B1)-(B6).
Claim 5.9. $\left|A_{b} \cap B_{a} \cap C_{a}\right| \leq 1$.
Proof. Suppose that $\left|A_{b} \cap B_{a} \cap C_{a}\right| \geq 2$. If $\left|A_{b} \cap C_{a}\right|=2$, then $A_{b} \cap C_{a} \subseteq B_{a}$ and so $A_{b} \cap B_{c} \cap C_{a}=\emptyset$. If $\left|A_{b} \cap C_{a}\right|=3$, then by $(\mathrm{B} 4), A_{b} \cap B_{c} \cap C_{a}=\emptyset$. Since $2 \leq\left|A_{b} \cap C_{a}\right| \leq 3$, we deduce that $A_{b} \cap B_{c} \cap C_{a}=\emptyset$.

By applying Lemma 5.6(ii) with $\left(B_{c}, B_{a}\right)$ and ( $C_{b}, C_{a}$ ), we have that $\left|B_{c} \cap C_{a}\right| \geq 2$. Since $A_{b} \cap B_{c} \cap C_{a}=\emptyset$, we have $\left|A_{c} \cap B_{c} \cap C_{a}\right|=\left|B_{c} \cap C_{a}\right| \geq 2$ and so $\left|A_{c} \cap B_{c}\right| \geq|\{c\}|+\left|A_{c} \cap B_{c} \cap C_{a}\right| \geq 3$. Since $\left|A_{c} \cap B_{c} \cap C_{a}\right| \geq 2$, by Lemma 5.7(1), $\rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right) \leq 1$. So by Lemma 5.7(2),

$$
\left|A_{b} \cap B_{a}\right|=2 \text { and } \rho_{G \backslash c}\left(\left(A_{b} \cap B_{a}\right) \cup\{a, b\}\right)=3
$$

because $\left|A_{b} \cap B_{a}\right| \geq\left|A_{b} \cap B_{a} \cap C_{a}\right| \geq 2$. Hence $A_{b} \cap B_{a} \subseteq C_{a}$.

By Lemma 5.2, $G \backslash a$ is prime and so $\rho_{G \backslash a}\left(A_{b} \cap B_{a}\right)=2$. By (ii) of Lemma 2.8, we have $\rho_{G \backslash a}\left(\left(A_{b} \cap B_{a}\right) \cup\{b\}\right) \leq \rho_{G \backslash a \backslash b}\left(A_{b} \cap B_{a}\right)+1 \leq 2$. So by (A2) of Lemma 2.16,

$$
\begin{aligned}
\rho_{G \backslash c}\left(\left(A_{b} \cap B_{a}\right) \cup\{a\}\right)+2 & \geq \rho_{G \backslash c}\left(\left(A_{b} \cap B_{a}\right) \cup\{a\}\right)+\rho_{G \backslash a}\left(\left(A_{b} \cap B_{a}\right) \cup\{b\}\right) \\
& \geq \rho_{G \backslash c}\left(\left(A_{b} \cap B_{a}\right) \cup\{a, b\}\right)+\rho_{G \backslash a}\left(A_{b} \cap B_{a}\right)=3+2,
\end{aligned}
$$

which implies that $\rho_{G \backslash c}\left(\left(A_{b} \cap B_{a}\right) \cup\{a\}\right) \geq 3$. Therefore, by Lemma 2.11,

$$
\begin{aligned}
3+2 & \geq \rho_{G \backslash c}\left(\left(A_{b} \cap B_{a}\right) \cup\{a, b\}\right)+\rho_{G \backslash c}\left(C_{b}\right) \\
& \geq \rho_{G \backslash c}\left(\left(A_{b} \cap B_{a}\right) \cup\{a\}\right)+\rho_{G \backslash c}\left(C_{b}-\{b\}\right) \geq 3+\rho_{G \backslash c}\left(C_{b}-\{b\}\right) .
\end{aligned}
$$

Therefore, $\rho_{G \backslash c}\left(C_{b}-\{b\}\right) \leq 2$. By Lemma 5.2, $G \backslash c$ is prime. Since $\left|C_{b}-\{b\}\right| \geq 3$, we have $\rho_{G \backslash c}\left(C_{b}-\{b\}\right)=2$. So by Lemma 3.1, neither $C_{a} \cup\{b\}$ nor $C_{b}-\{b\}$ is sequential in $G \backslash c$, contradicting Lemma 5.3 because $\{a, b\} \subseteq C_{a} \cup\{b\}$.

Hence, by symmetry, we have $\left|A_{b} \cap B_{a} \cap C_{a}\right| \leq 1,\left|A_{c} \cap B_{a} \cap C_{a}\right| \leq 1,\left|A_{b} \cap B_{a} \cap C_{b}\right| \leq 1$, $\left|A_{b} \cap B_{c} \cap C_{b}\right| \leq 1,\left|A_{c} \cap B_{c} \cap C_{a}\right| \leq 1$, and $\left|A_{c} \cap B_{c} \cap C_{b}\right| \leq 1$.
Claim 5.10. $\left|A_{b} \cap B_{c} \cap C_{a}\right| \leq 1$.
Proof. Suppose that $\left|A_{b} \cap B_{c} \cap C_{a}\right| \geq 2$. If $\left|A_{b} \cap B_{c}\right|=2$, then $A_{b} \cap B_{c} \subseteq C_{a}$ and $A_{b} \cap B_{c} \cap C_{b}=\emptyset$. If $\left|A_{b} \cap B_{c}\right|=3$, then by (B1), $A_{b} \cap B_{c} \cap C_{b}=\emptyset$. By Lemma 5.6(i), we have $2 \leq\left|A_{b} \cap B_{c}\right| \leq 3$. So we deduce that $A_{b} \cap B_{c} \cap C_{b}=\emptyset$.

By symmetry between ( $a, b, c$ ) and ( $c, a, b$ ), we deduce that $C_{a} \cap A_{b} \cap B_{a}=\emptyset$. By symmetry between ( $a, b, c$ ) and ( $b, c, a$ ), we deduce that $B_{c} \cap C_{a} \cap A_{c}=\emptyset$. By Lemma 5.6(iv), $\left|A_{c} \cap B_{c}\right| \geq 2$. So we deduce that

$$
1 \leq\left|A_{c} \cap B_{c}\right|-|\{c\}|-\left|A_{c} \cap B_{c} \cap C_{a}\right|=\left|A_{c} \cap B_{c} \cap C_{b}\right| \leq 1,
$$

and therefore $\left|A_{c} \cap B_{c} \cap C_{b}\right|=1$.
If $\left|A_{c} \cap C_{b}\right|=3$, then by (B3), $\left|A_{c} \cap B_{a} \cap C_{b}\right|=0$. If $\left|A_{c} \cap C_{b}\right| \leq 2$, then $\left|A_{c} \cap B_{a} \cap C_{b}\right|=$ $\left|A_{c} \cap C_{b}\right|-\left|A_{c} \cap B_{c} \cap C_{b}\right| \leq 2-1=1$. Since $\left|A_{c} \cap C_{b}\right| \leq 3$, in both cases, we deduce that $\left|A_{c} \cap B_{a} \cap C_{b}\right| \leq 1$. Then we have

$$
\begin{aligned}
|V(G)|= & \left|A_{b} \cap B_{a} \cap C_{a}\right|+\left|A_{b} \cap B_{a} \cap C_{b}\right|+\left|A_{b} \cap B_{c} \cap C_{a}\right|+\left|A_{b} \cap B_{c} \cap C_{b}\right| \\
& +\left|A_{c} \cap B_{a} \cap C_{a}\right|+\left|A_{c} \cap B_{a} \cap C_{b}\right|+\left|A_{c} \cap B_{c} \cap C_{a}\right|+\left|A_{c} \cap B_{c} \cap C_{b}\right|+|\{a, b, c\}| \\
= & 0+\left|A_{b} \cap B_{a} \cap C_{b}\right|+\left|A_{b} \cap B_{c} \cap C_{a}\right|+0 \\
& +\left|A_{c} \cap B_{a} \cap C_{a}\right|+\left|A_{c} \cap B_{a} \cap C_{b}\right|+0+\left|A_{c} \cap B_{c} \cap C_{b}\right|+|\{a, b, c\}| \\
\leq & 0+1+\left|A_{b} \cap B_{c}\right|+0+1+1+0+1+3 \leq 10,
\end{aligned}
$$

contradicting our assumption.
By symmetry, we have $\left|A_{c} \cap B_{a} \cap C_{b}\right| \leq 1$. Therefore, we have

$$
\begin{aligned}
|V(G)|= & \left|A_{b} \cap B_{a} \cap C_{a}\right|+\left|A_{b} \cap B_{a} \cap C_{b}\right|+\left|A_{b} \cap B_{c} \cap C_{a}\right|+\left|A_{b} \cap B_{c} \cap C_{b}\right| \\
& +\left|A_{c} \cap B_{a} \cap C_{a}\right|+\left|A_{c} \cap B_{a} \cap C_{b}\right|+\left|A_{c} \cap B_{c} \cap C_{a}\right|+\left|A_{c} \cap B_{c} \cap C_{b}\right|+|\{a, b, c\}| \leq 11,
\end{aligned}
$$

contradicting our assumption.

## 6 Completing the proof

A set $X$ of vertices of a graph $G$ is fully closed if $\rho_{G}(X \cup\{v\})>\rho_{G}(X)$ for all $v \in V(G)-X$.
Lemma 6.1 (Oum [8, Proposition 3.1]). Let $G$ be a prime graph with $|V(G)| \geq 8$. Suppose that $G$ has a fully closed set $A$ such that $\rho_{G}(A) \geq 2$. Then there is a vertex $v$ of $A$ such that $G \backslash v$ or $G / v$ is prime.

Lemma 6.2. Let $G$ be a sequentially 3 -rank-connected graph and $a_{1}, a_{2}, \ldots, a_{k}$ be distinct vertices of $G$ such that $k \geq 4$ and $\rho_{G}\left(\left\{a_{1}, \ldots, a_{i}\right\}\right) \leq 2$ for each $i \leq k$. For each $1 \leq i \leq k$, if $G \backslash a_{i}$ is prime, then $G \backslash a_{i}$ is sequentially 3-rank-connected.
Proof. Since $G$ is prime, we know that $\rho_{G}\left(\left\{a_{1}, \ldots, a_{j}\right\}\right)=\min \{2,|V(G)|-j\}$ for each $2 \leq j \leq k$. So $\rho_{G}\left(\left\{a_{1}, \ldots, a_{j-1}\right\}\right) \geq \rho_{G}\left(\left\{a_{1}, \ldots, a_{j}\right\}\right)$ for each $2 \leq j \leq k$. For each $3 \leq j \leq i-1$, by (S2) of Lemma 2.12, we have

$$
\rho_{G}\left(\left\{a_{1}, \ldots, a_{j}\right\}\right)+\rho_{G \backslash a_{i}}\left(\left\{a_{1}, \ldots, a_{j-1}\right\}\right) \geq \rho_{G}\left(\left\{a_{1}, \ldots, a_{j-1}\right\}\right)+\rho_{G \backslash a_{i}}\left(\left\{a_{1}, \ldots, a_{j}\right\}\right)
$$

and therefore $\rho_{G \backslash a_{i}}\left(\left\{a_{1}, \ldots, a_{j-1}\right\}\right) \geq \rho_{G \backslash a_{i}}\left(\left\{a_{1}, \ldots, a_{j}\right\}\right)$.
Suppose that $G \backslash a_{i}$ is prime and not sequentially 3 -rank-connected.
Let us first consider the case when $i>3$. By Lemma 3.2, there is a subset $X$ of $V\left(G \backslash a_{i}\right)$ such that $\rho_{G \backslash a_{i}}(X) \leq 2$, neither $X$ nor $V\left(G \backslash a_{i}\right)-X$ is sequential in $G \backslash a_{i}$, and $\left\{a_{1}, a_{2}, a_{3}\right\} \subseteq X$. We may assume that $X$ is maximal among all such sets.

We claim that $\left\{a_{1}, \ldots, a_{i-1}\right\} \subseteq X$. Suppose not. Let $j \leq i-1$ be the minimum index such that $a_{j} \notin X$. Then $\left\{a_{1}, \ldots, a_{j-1}\right\} \subseteq X$. Note that $j \geq 4$. Let $Y=V\left(G \backslash a_{i}\right)-X$. Since neither $X$ nor $Y$ is sequential in $G \backslash a_{i}$, we have $|X|,|Y| \geq 4$. Since $\rho_{G \backslash a_{i}}\left(\left\{a_{1}, \ldots, a_{j-1}\right\}\right) \geq$ $\rho_{G \backslash a_{i}}\left(\left\{a_{1}, \ldots, a_{j}\right\}\right)$, by Lemma 2.10,

$$
\rho_{G \backslash a_{i}}(X)+\rho_{G \backslash a_{i}}\left(\left\{a_{1}, \ldots, a_{j}\right\}\right) \geq \rho_{G \backslash a_{i}}\left(X \cup\left\{a_{j}\right\}\right)+\rho_{G \backslash a_{i}}\left(\left\{a_{1}, \ldots, a_{j-1}\right\}\right),
$$

and therefore $\rho_{G \backslash a_{i}}\left(X \cup\left\{a_{j}\right\}\right) \leq \rho_{G \backslash a_{i}}(X) \leq 2$. Since $G \backslash a_{i}$ is prime and $\left|Y-\left\{a_{i}\right\}\right| \geq 3$, we have $\rho_{G \backslash a_{i}}\left(X \cup\left\{a_{j}\right\}\right)=\rho_{G \backslash a_{i}}(X)=2$. Hence by Lemma 3.1, neither $X \cup\left\{a_{j}\right\}$ nor $Y-\left\{a_{j}\right\}$ is sequential in $G \backslash a_{i}$, contradicting the maximality of $X$. Hence $\left\{a_{1}, \ldots, a_{i-1}\right\} \subseteq X$.

Then by (S1) of Lemma 2.12,

$$
\rho_{G \backslash a_{i}}(X)+\rho_{G}\left(\left\{a_{1}, \ldots, a_{i}\right\}\right) \geq \rho_{G}\left(X \cup\left\{a_{i}\right\}\right)+\rho_{G \backslash a_{i}}\left(\left\{a_{1}, \ldots, a_{i-1}\right\}\right) .
$$

Since $G \backslash a_{i}$ is prime and $i>3$, we have $\rho_{G \backslash a_{i}}\left(\left\{a_{1}, \ldots, a_{i-1}\right\}\right) \geq \min \{2,|V(G)|-i\}=\rho_{G}\left(\left\{a_{1}, \ldots, a_{i}\right\}\right)$. So $\rho_{G}\left(X \cup\left\{a_{i}\right\}\right) \leq \rho_{G \backslash a_{i}}(X) \leq 2$. Since $G$ is sequentially 3-rank-connected, $X \cup\left\{a_{i}\right\}$ or $Y$ is sequential in $G$. Then by (i), (ii) of Lemma 2.8, $X$ or $Y$ is sequential in $G \backslash a_{i}$, contradicting our assumption.

Now we consider the case when $i \leq 3$. By permuting $a_{1}, a_{2}, a_{3}$, we can assume that $i=3$. Suppose that $G \backslash a_{3}$ is prime. By Lemma 2.8(ii), we have $\rho_{G \backslash a_{3}}\left(\left\{a_{1}, a_{2}, a_{4}\right\}\right) \leq$ $\rho_{G}\left(\left\{a_{1}, a_{2}, a_{3}, a_{4}\right\}\right) \leq 2$. Since $a_{1}, a_{2}, a_{4}, a_{3}$ is another sequence satisfying all the requirements, we conclude that $G \backslash a_{3}$ is sequentially 3 -rank-connected because we proved the statement for $i>3$.

Lemma 6.3. Let $G$ be a sequentially 3-rank-connected graph with $|V(G)| \geq 8$ and $a_{1}, a_{2}, \ldots, a_{k}$ be distinct vertices of $G$ such that $k \geq 4, k \neq|V(G)|-1$, and $\rho_{G}\left(\left\{a_{1}, \ldots, a_{i}\right\}\right) \leq 2$ for each $i \leq k$. If $\left\{a_{1}, \ldots, a_{k}\right\}$ is a fully closed set of $G$, then there exists $i \in\{1, \ldots, k\}$ such that $G \backslash a_{i}$ or $G / a_{i}$ is sequentially 3-rank-connected.
Proof. By Theorem 1.1 and Lemma 6.2, we may assume that $k \neq|V(G)|$ and therefore $k \leq$ $|V(G)|-2$. Since $G$ is prime, we have $\rho_{G}\left(\left\{a_{1}, \ldots, a_{k}\right\}\right)=2$ and so, by Lemma 6.1, there is a vertex $a_{i}$ of $G$ such that $G \backslash a_{i}$ or $G / a_{i}$ is prime. By pivoting, we may assume that $G \backslash a_{i}$ is prime. Then, by Lemma 6.2, $G \backslash a_{i}$ is sequentially 3-rank-connected.

Proof of Theorem 1.2. By Proposition 4.15, we may assume that $G$ is not 3-rank-connected. So there is a subset $A$ of $V(G)$ such that $\rho_{G}(A) \leq 2,|A| \geq 3$, and $|V(G)-A| \geq 3$. If $G$ is internally 3 -rank-connected, then we may assume that $|A|=3$. By Lemma 5.1, we can assume that $A$ is a triplet of $G$ by pivoting. By Proposition 5.8, there is a vertex $a \in A$ such that $G \backslash a$ is sequentially 3 -rank-connected. Hence we may assume that $G$ is not internally 3 -rank-connected.

Therefore, we may assume that $|A| \geq 4$ and $|V(G)-A| \geq 4$. Since $G$ is sequentially 3-rankconnected, $A$ or $V(G)-A$ is sequential in $G$. Therefore there exists a sequential set with at least 4 elements.

Let $X$ be a maximum sequential set of $G$. Then $X$ is a fully closed set of $G$. Furthermore, $|X| \neq|V(G)|-1$ because otherwise $V(G)$ is sequential in $G$. Since $|X| \geq 4$, we conclude the proof by Lemma 6.3.

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