# Stability of intersecting families* 

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#### Abstract

The celebrated Erdős-Ko-Rado theorem [1] states that the maximum intersecting $k$-uniform family on [ $n$ ] is a full star if $n \geq 2 k+1$. Furthermore, Hilton-Milner [9] showed that if an intersecting $k$-uniform family on $[n]$ is not a subfamily of a full star, then its maximum size achieves only on a family isomorphic to $H M(n, k):=\left\{G \in\binom{[n]}{k}: 1 \in G, G \cap[2, k+1] \neq \emptyset\right\} \cup\{[2, k+1]\}$ if $n>2 k$ and $k \geq 4$, and there is one more possibility in the case of $k=3$. Han and Kohayakawa [8] determined the maximum intersecting $k$-uniform family on $[n]$ which is neither a subfamily of a full star nor a subfamily of the extremal family in Hilton-Milner theorm, and they asked what is the next maximum intersecting $k$-uniform family on [ $n$ ]. Kostochka and Mubayi [11] gave the answer for large enough $n$. In this paper, we are going to get rid of the requirement that $n$ is large enough in the result by Kostochka and Mubayi [11] and answer the question of Han and Kohayakawa [8].


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## 1 Introduction

For a positive interge $n$, let $[n]=\{1,2, \ldots, n\}$ and $2^{[n]}$ be the family of all subsets of $[n]$. An $i$-element subset $A \subseteq[n]$ is called an $i$-set. For $0 \leq k \leq n$, let $\binom{[n]}{k}$ denote the collection of all $k$-sets of [n]. A family $\mathcal{F} \subseteq\binom{[n]}{k}$ is called $k$-uniform. For a family $\mathcal{F} \subseteq 2^{[n]}$, we say $\mathcal{F}$ is intersecting if for any two distinct sets $F$ and $F^{\prime}$ in $\mathcal{F}$ we have $\left|F \cap F^{\prime}\right| \geq 1$. In this paper, we always consider a $k$-uniform intersecting family on

[^0]$[n]$. The following celebrated theorem of Erdős-Ko-Rado determines the maximum intersecting family.

For $x \in[n]$ denote $\mathcal{F}_{x}:=\left\{F \in\binom{[n]}{k}: x \in F\right\}$ by the full star centered at $x$. We say $\mathcal{F}$ is $E K R$ if $\mathcal{F}$ is contained in a full star.

Theorem 1.1 (Erdős-Ko-Rado [1]). Let $n \geq 2 k$ be integer and $\mathcal{F}$ be a $k$-uniform intersecting family of subsets of $[n]$. Then

$$
|\mathcal{F}| \leq\binom{ n-1}{k-1}
$$

Moreover, when $n>2 k$, equality holds if and only if $\mathcal{F}$ is a full star.
The theorem of Hilton-Milner determines the maximum size of non-EKR families.
Theorem 1.2 (Hilton-Milner [9]). Let $k \geq 2$ and $n \geq 2 k$ be integers and $\mathcal{F} \subseteq\binom{[n]}{k}$ be an intersecting family. If $\mathcal{F}$ is not EKR, then

$$
|\mathcal{F}| \leq\binom{ n-1}{k-1}-\binom{n-k-1}{k-1}+1
$$

Moreover, for $n>2 k$ and $k \geq 4$, equality holds if and only if $\mathcal{F}$ is isomorphic to

$$
H M(n, k):=\left\{G \in\binom{[n]}{k}: 1 \in G, G \cap[2, k+1] \neq \emptyset\right\} \cup\{[2, k+1]\} .
$$

For the case $k=3$, there is one more possibility, namely

$$
\mathcal{T}(n, 3):=\left\{F \in\binom{[n]}{3}:|F \cap[3]| \geq 2\right\} .
$$

We say a family $\mathcal{F}$ is $H M$ if it is isomorphic to a subfamily of $\operatorname{HM}(n, k)$. We say that 1 is the center of $\operatorname{HM}(n, k)$.

Let $E \subseteq[n]$ be an $i$-set and $x \in[n]$. We define

$$
\mathcal{G}_{i}:=\left\{G \in\binom{[n]}{k}: E \subseteq G\right\} \cup\left\{G \in\binom{[n]}{k}: x \in G \text { and } G \cap E \neq \emptyset\right\} .
$$

We call $x$ the center, and $E$ the core of $\mathcal{G}_{i}$ for $i \geq 3$. With a slight tweaking, we call $\{x\} \cup E$ the core of $\mathcal{G}_{2}$. Note that $\mathcal{G}_{k}=H M(n, k)$.

For a $(k-1)$-set $E$, a point $x \in[n] \backslash E$, and an $i$-set $J \subset[n] \backslash(E \cup\{x\})$, we denote

$$
\begin{aligned}
\mathcal{J}_{i}:=\left\{G \in\binom{[n]}{k}: E \subseteq G \text { and } G \cap J \neq \emptyset\right\} & \cup\left\{G \in\binom{[n]}{k}: J \cup\{x\} \subseteq G\right\} \\
& \cup\left\{G \in\binom{[n]}{k}: x \in G, G \cap E \neq \emptyset\right\} .
\end{aligned}
$$

We call $x$ the center, $E$ the kernel, and $J$ the set of pages.

For two $k$-sets $E_{1}$ and $E_{2} \subseteq[n]$ with $\left|E_{1} \cap E_{2}\right|=k-2$, and $x \in[n] \backslash\left(E_{1} \cup E_{2}\right)$, we define

$$
\mathcal{K}_{2}:=\left\{G \in\binom{[n]}{k}: x \in G, G \cap E_{1} \neq \emptyset \text { and } G \cap E_{2} \neq \emptyset\right\} \cup\left\{E_{1}, E_{2}\right\}
$$

and call $x$ the center of $\mathcal{K}_{2}$.
In [8], Han and Kohayakawa obtained the size of a maximum non-EKR, non-HM intersecting family.

Theorem 1.3 (Han-Kohayakawa [8]). Suppose $k \geq 3$ and $n \geq 2 k+1$ and let $\mathcal{H}$ be an intersecting $k$-uniform family on $[n]$. Furthermore, assume that $\mathcal{H}$ is neither $E K R$ nor $H M$, if $k=3, \mathcal{H} \nsubseteq \mathcal{G}_{2}$. Then

$$
|\mathcal{H}| \leq\binom{ n-1}{k-1}-\binom{n-k-1}{k-1}-\binom{n-k-2}{k-2}+2
$$

For $k=4$, equality holds if and only if $\mathcal{H}=\mathcal{J}_{2}, \mathcal{G}_{2}$ or $\mathcal{G}_{3}$. For every other $k$, equality holds if and only if $\mathcal{H}=\mathcal{J}_{2}$.

Han and Kohayakawa [8] proposed the following question.
Question 1.4. Let $n \geq 2 k+1$. What is the maximum size of an intersecting family $\mathcal{H}$ that is neither $E K R$ nor $H M$, and $\mathcal{H} \nsubseteq \mathcal{J}_{2}$ (in addition $\mathcal{H} \nsubseteq \mathcal{G}_{2}$ and $\mathcal{H} \nsubseteq \mathcal{G}_{3}$ if $k=4)$ ?

Regarding this question, Kostochka and Mubayi [11] showed that the answer is $\left|\mathcal{J}_{3}\right|$ for sufficiently large $n$. In fact they proved that the maximum size of an intersecting family that is neither EKR, nor HM, nor contained in $\mathcal{J}_{i}$ for each $i, 2 \leq i \leq k-1$ (nor in $\mathcal{G}_{2}, \mathcal{G}_{3}$ for $k=4$ ) is $\left|\mathcal{K}_{2}\right|$ for all large enough $n$. In paper [11], they also established the structure of almost all intersecting 3 -uniform families. Sometimes, it is relatively easier to get extremal families under the assumption that $n$ is large enough. For example, Erdős matching conjecture [2] states that for a $k$-uniform family $\mathcal{F}$ on finite set $[n],|\mathcal{F}| \leq \max \left\{\binom{k(s+1)-1}{k},\binom{n}{k}-\binom{n-s}{k}\right\}$ if there is no $s+1$ pairwise disjoint members of $\mathcal{F}$ and $n \geq(s+1) k$, and it was proved to be true for large enough $n$ in [2]. There has been a lot of recent studies for small $n$ (see [5, 7, 10, 12]). However, the conjecture is not completely verified for small $n$. Up to now, the best condition on $n$ was given by Frankl in [5, 6] that $n \geq k(2 s+1)-s$, for $(s+1) k \leq n \leq k(2 s+1)-s-1$.

As mentioned by Han and Kohayakawa in [8], for $k \geq 4$, the bound in Theorem 1.3 can be deduced from Theorem 3 in [9] which was established by Hilton and Milner in 1967. However, family $\mathcal{H}$ in Question 1.4 does not satisfy the hypothesis of Theorem 3 in 9 for $k \geq 4$. This makes Question 1.4 more interesting. In this paper, we answer Question 1.4. We are going to get rid of the requirement that $n$ is large enough in the result by Kostochka and Mubayi [11]. As in the proofs of Theorem 1.1. Theorem 1.2 and Theorem 1.3 , we will apply the shifting method. The main difficulty in our proof is to guarantee that we can get a stable family which is not EKR, not HM, $\nsubseteq \mathcal{J}_{2}$ (in
addition $\nsubseteq \mathcal{G}_{2}, \nsubseteq \mathcal{G}_{3}$ if $\left.k=4\right)$ after performing a series of shifts to a family which is not EKR , not $\mathrm{HM}, \nsubseteq \mathcal{J}_{2}$ (in addition $\nsubseteq \mathcal{G}_{2}, \nsubseteq \mathcal{G}_{3}$ if $k=4$ ). Our main result is as follows.

Theorem 1.5. Let $k \geq 4$ and $\mathcal{H} \subseteq\binom{[n]}{k}$ be an intersecting family which is neither EKR nor HM. Furthermore, $\mathcal{H} \nsubseteq \mathcal{J}_{2}$ (in addition $\mathcal{H} \nsubseteq \mathcal{G}_{2}$ and $\mathcal{H} \nsubseteq \mathcal{G}_{3}$ if $k=4$ ).
(i) If $2 k+1 \leq n \leq 3 k-3$, then

$$
|\mathcal{H}| \leq\binom{ n-1}{k-1}-2\binom{n-k-1}{k-1}+\binom{n-k-3}{k-1}+2
$$

Moreover, the equality holds only for $\mathcal{H}=\mathcal{K}_{2}$ if $k \geq 5$, and $\mathcal{H}=\mathcal{K}_{2}$ or $\mathcal{J}_{3}$ if $k=4$. (ii) If $n \geq 3 k-2$, then

$$
|\mathcal{H}| \leq\binom{ n-1}{k-1}-\binom{n-k-1}{k-1}-\binom{n-k-2}{k-2}-\binom{n-k-3}{k-3}+3
$$

Moreover, for $k=5$, the equality holds only for $\mathcal{H}=\mathcal{J}_{3}$ or $\mathcal{G}_{4}$. For every other $k$, equality holds only for $\mathcal{H}=\mathcal{J}_{3}$.

In Section 2, we will give the proof of Theorem 1.5. The proofs of some crucial lemmas for the proof of Theorem 1.5 are given in Section 3 .

## 2 Proof of Theorem 1.5

In this section, we always assume that $\mathcal{H}$ is a maximum intersecting family which satisfies the conditions of Theorem 1.5, that is, $\mathcal{H}$ is not EKR, not HM, $\mathcal{H} \nsubseteq \mathcal{J}_{2}$ (in addition $\mathcal{H} \nsubseteq \mathcal{G}_{2}, \mathcal{H} \nsubseteq \mathcal{G}_{3}$ if $k=4$ ). By direct calculation, we have the following fact.

Fact 2.1. (i) Suppose that there is $x \in[n]$ such that there are only 2 sets, say, $E_{1}$ and $E_{2} \in \mathcal{H}$ missing $x$. If $\left|E_{1} \cap E_{2}\right|=k-i$ and $i \geq 2$, then

$$
\begin{align*}
|\mathcal{H}| & \leq\binom{ n-1}{k-1}-2\binom{n-k-1}{k-1}+\binom{n-k-i-1}{k-1}+2 \\
& \leq\binom{ n-1}{k-1}-2\binom{n-k-1}{k-1}+\binom{n-k-3}{k-1}+2 . \tag{1}
\end{align*}
$$

The equality in (1) holds if and only if $\left|E_{1} \cap E_{2}\right|=k-2$, that is $\mathcal{H}=\mathcal{K}_{2}$.
(ii) By the definiton of $\mathcal{J}_{i}$, we have

$$
\begin{equation*}
\left|\mathcal{J}_{3}\right|=\binom{n-1}{k-1}-\binom{n-k-1}{k-1}-\binom{n-k-2}{k-2}-\binom{n-k-3}{k-3}+3 \tag{2}
\end{equation*}
$$

(iii) Comparing the right hand sides of (1) and (2), we can see that if $2 k+1 \leq n \leq$ $3 k-3$, then $\left|\mathcal{K}_{2}\right| \geq\left|\mathcal{J}_{3}\right|$, the equality holds only for $k=4$; and if $n \geq 3 k-2$, then $\left|\mathcal{K}_{2}\right|<\left|\mathcal{J}_{3}\right|$.

By Fact 2.1, we may assume that for any $x$, at least 3 sets in $\mathcal{H}$ do not contain $x$. To show Theorem 1.5, it is sufficient to show the following result.

Theorem 2.2. Let $k \geq 4, n \geq 2 k+1$ and $\mathcal{H} \subseteq\binom{[n]}{k}$ be an intersecting family which is not $E K R$, not $H M$ and $\mathcal{H} \nsubseteq \mathcal{J}_{2}$ (in addition $\mathcal{H} \nsubseteq \mathcal{G}_{2}, \mathcal{H} \nsubseteq \mathcal{G}_{3}$ if $k=4$ ). Moreover, for any $x \in[n]$, there are at least 3 sets in $\mathcal{H}$ not containing $x$. Then

$$
|\mathcal{H}| \leq\binom{ n-1}{k-1}-\binom{n-k-1}{k-1}-\binom{n-k-2}{k-2}-\binom{n-k-3}{k-3}+3
$$

Moreover if $k \neq 5$, the equality holds only for $\mathcal{H}=\mathcal{J}_{3}$; if $k=5$, the equality holds for $\mathcal{H}=\mathcal{J}_{3}$ or $\mathcal{G}_{4}$.

From now on, we always assume that $\mathcal{H}$ is a maximum intersecting family which satisfies the conditions of Theorem 2.2, that is $\mathcal{H}$ is not EKR, not HM, $\mathcal{H} \nsubseteq \mathcal{J}_{2}$ (in addition $\mathcal{H} \nsubseteq \mathcal{G}_{2}, \mathcal{H} \nsubseteq \mathcal{G}_{3}$ if $\left.k=4\right)$ and for any $x \in[n]$, there are at least 3 sets in $\mathcal{H}$ not containing $x$.

We first give some definition related to the shifting method. For $x$ and $y \in[n], x<$ $y$, and $F \in \mathcal{F}$, we call the following operation a shift:

$$
S_{x y}(F)= \begin{cases}(F \backslash\{y\}) \cup\{x\}, & \text { if } x \notin F, y \in F \text { and }(F \backslash\{y\}) \cup\{x\} \notin \mathcal{F} ; \\ F, & \text { otherwise. }\end{cases}
$$

We say that $F$ is stable under the shift $S_{x y}$ if $S_{x y}(F)=F$. If $z \in F$ and $z \in S_{x y}(F)$ still, we say that $F$ is stable at $z$ after the shift $S_{x y}$. For a family $\mathcal{F}$, we define

$$
S_{x y}(\mathcal{F})=\left\{S_{x y}(F): F \in \mathcal{F}\right\}
$$

Clearly, $\left|S_{x y}(\mathcal{F})\right|=|\mathcal{F}|$. We say that $\mathcal{F}$ is stable if $S_{x y}(\mathcal{F})=\mathcal{F}$ for all $x, y \in[n]$ with $x<y$.

An important property shown in [4] is that if $\mathcal{F}$ is intersecting, then $S_{x y}(\mathcal{F})$ is still intersecting. Let us rewrite is as a remark.

Remark 2.3. [4] If $\mathcal{F}$ is a maximum intersecting family, then $S_{x y}(\mathcal{F})$ is still a maximum intersecting family.

This property guarantees that performing shifts repeatedly to a maximum intersecting family will yield a stable maximum intersecting family. The main difficulty we need to overcome is to guarantee that we can get a stable maximum intersecting family with further properties: not EKR, not $\mathrm{HM}, \nsubseteq \mathcal{J}_{2}$ (in addition $\nsubseteq \mathcal{G}_{2}, \nsubseteq \mathcal{G}_{3}$ if $k=4$ ). The following facts and lemmas are for this purpose.

Fact 2.4. The following properties hold.
(i) If $S_{x y}(\mathcal{H})$ is EKR (or HM), then $x$ must be the center.
(ii) If $S_{x y}(\mathcal{H}) \subseteq \mathcal{G}_{2}$, then the core is $\left\{x, x_{1}, x_{2}\right\}$ for some $x_{1}, x_{2} \in[n] \backslash\{x, y\}$.
(iii) If $S_{x y}(\mathcal{H}) \subseteq \mathcal{J}_{2}$, then $x$ is the center.
(iv) If $S_{x y}(\mathcal{H}) \subseteq \mathcal{G}_{3}$, then $x$ is the center or $x$ is in the core.

Proof. For (i) and (ii), Han and Kohaykawa proved them in [8]. We prove (iii) and (iv) only.

For (iii), suppose that $S_{x y}(\mathcal{H}) \subseteq \mathcal{J}_{2}$ at center $z \in[n] \backslash\{x\}$. Since $\mathcal{H} \nsubseteq \mathcal{J}_{2}$ at $z$, there are at least three sets $E_{1}, E_{2}$ and $E_{3}$ in $\mathcal{H}$ missing $z$, after doing the shift $S_{x y}$, these 3 sets still miss $z$, so $S_{x y}(\mathcal{H})$ is not contained in $\mathcal{J}_{2}$ center at $z$.

For (iv), let $S_{x y}(\mathcal{H}) \subseteq \mathcal{G}_{3}$ at center $x_{0}$ and core $E=\left\{x_{1}, x_{2}, x_{3}\right\}$, and let $B=$ $\left\{x_{0}, x_{1}, x_{2}, x_{3}\right\}$. Since $\mathcal{H} \nsubseteq \mathcal{G}_{3}$, there is a set $G \in \mathcal{H}$ that satisfies one of the following two cases: (a) $\left\{y, x_{0}\right\} \subseteq G, G \cap E=\emptyset$; (b) $y \in G, x_{0} \notin G,|G \cap E| \in\{1,2\}$. If (a) holds, then $x \neq x_{0}$ and $x$ must be in the core, $y \notin B$. If (b) holds, then either $x=x_{0}$ is the center or $x$ is in the core and $y \notin B$.
Remark 2.5. By Fact 2.4, if applying $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}\right)$ repeatedly to $\mathcal{H}$, we may reach a family which belong to one of the following cases.
Case 1: a family $\mathcal{H}_{1}$ such that $S_{x y}\left(\mathcal{H}_{1}\right)$ is EKR with center $x$;
Case 2: a family $\mathcal{H}_{2}$ such that $S_{x y}\left(\mathcal{H}_{2}\right)$ is $H M$ with center $x$;
Case 3: a family $\mathcal{H}_{3}$ such that $S_{x y}\left(\mathcal{H}_{3}\right) \subseteq \mathcal{J}_{2}$ with center $x$;
Case 4: a family $\mathcal{H}_{4}$ such that $S_{x y}\left(\mathcal{H}_{4}\right) \subseteq \mathcal{G}_{2}$ with core $\left\{x, x_{1}, x_{2}\right\}$ for some $\left\{x_{1}, x_{2}\right\} \in$ $X \backslash\{x, y\}(k=4$ only $)$;
Case 5: a family $\mathcal{H}_{5}$ such that $S_{x y}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}$ with center $x$ or $x$ being in the core ( $k=4$ only);
Case 6: a stable family $\mathcal{H}_{6}$ satisfies the conditions of Theorem 2.2, that is we will not meet Cases 1-5 after doing all shifts.

By Remark 2.3 , we know that for any shift $S_{x y}$ on $[n]$ we have $\left|S_{x y}(\mathcal{H})\right|=|\mathcal{H}|$ and $S_{x y}(\mathcal{H})$ is also intersecting. We hope to get a stable family satisfying the conditions of Theorem 2.2 after some shifts, that is neither EKR, nor HM, nor contained in $\mathcal{J}_{2}$ (nor in $\mathcal{G}_{2}, \mathcal{G}_{3}$ if $k=4$ ). By Fact 2.1, we can assume that a family $\mathcal{G}$ obtained by performing shifts to $\mathcal{H}$ has the property that for any $x$, at least 3 sets in $\mathcal{G}$ do not contain $x$. What we are going to do is: If any case of Cases $1-5$ happens, we will not perform $S_{x y}$. Instead we will adjust the shifts as shown in Lemma 2.6 to guarantee that the terminating family is a stable family satisfying the conditions of Theorem 2.2. We will prove the following two crucial lemmas in Section 3 .

Lemma 2.6. Let $i \in$ [5]. If we reach $\mathcal{H}_{i}$ in Case $i$ in Remark 2.5, then there is a set $X_{i} \subseteq[n]$ with $\left|X_{i}\right| \leq 5$ (when $k \geq 5,\left|X_{i}\right| \leq 3$ for $i \in[3]$ ), such that after a series of shifts $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}\right.$ and $\left.x^{\prime}, y^{\prime} \in[n] \backslash X_{i}\right)$ to $\mathcal{H}_{i}$, we can reach a stable family satisfying the conditions of Theorem [2.2. Moreover, for any set $G$ in the final family $\mathcal{G}$, we have $G \cap X_{i} \neq \emptyset$.

From now on, let $X_{i}$ be the corresponding sets in Lemma 2.6 for $1 \leq i \leq 5$ and $X_{6}=\emptyset$. For $k \geq 5$ and $i \in\{1,2,3,6\}$, let $Y_{i}$ be the set of the first $2 k-\left|X_{i}\right|$ elements of $[n] \backslash X_{i}$, and for $k=4$ and $i \in\{1,2,3,4,5,6\}$, let $Y_{i}$ be the first $9-\left|X_{i}\right|$ elements of $[n] \backslash X_{i}$. Let $Y=Y_{i} \cup X_{i}$, then $\left|Y_{i}\right| \geq 2 k-4$ and $|Y|=2 k$ if $k \geq 5$. If $k=4$ then $|Y|=9$. Let

$$
\mathcal{A}_{i}:=\{G \cap Y: G \in \mathcal{G},|G \cap Y|=i\}
$$

$$
\widetilde{\mathcal{A}_{i}}:=\{G: G \in \mathcal{G},|G \cap Y|=i\} .
$$

Lemma 2.7. Let $\mathcal{G}$ be the final stable family guaranteed by Lemma 2.6 satisfying the conditions of Theorem 2.2, and let $X_{i}$ be inherit from Lemma 2.6. In other words, $\mathcal{G}$ is stable; $\mathcal{G}$ is neither $E K R$, nor $H M$, nor contained in $\mathcal{J}_{2}$ (nor in $\mathcal{G}_{2}, \mathcal{G}_{3}$ if $k=4$ ); for any $x \in[n]$, there are at least 3 sets in $\mathcal{G}$ not containing $x$; and $G \cap X_{i} \neq \emptyset$ for any $G \in \mathcal{G}$. Then
(i) $\mathcal{A}_{1}=\emptyset$.
(ii) For all $G$ and $G^{\prime} \in \mathcal{G}$, we have $G \cap G^{\prime} \cap Y \neq \emptyset$, or equivalently, $\cup_{i=2}^{k} \mathcal{A}_{i} \cup \mathcal{G}$ is intersecting.

### 2.1 Quantitative Part of Theorem 2.2

Lemma 2.8. For $k=4$, we have $\left|\mathcal{A}_{1}\right|=0,\left|\mathcal{A}_{2}\right| \leq 3,\left|\mathcal{A}_{3}\right| \leq 18$ and $\left|\mathcal{A}_{4}\right| \leq 50$. For $k \geq 5$, we have

$$
\begin{aligned}
& \left|\mathcal{A}_{i}\right| \leq\binom{ 2 k-1}{i-1}-\binom{k-1}{i-1}-\binom{k-2}{i-2}-\binom{k-3}{i-3}, 1 \leq i \leq k-1, \\
& \left|\mathcal{A}_{k}\right| \leq \frac{1}{2}\binom{2 k}{k}=\binom{2 k-1}{k-1}-\binom{k-1}{k-1}-\binom{k-2}{k-2}-\binom{k-3}{k-3}+3 .
\end{aligned}
$$

Proof. By Lemma 2.7 (i), we have $\left|\mathcal{A}_{1}\right|=0$.
First consider $k=4$. If $\left|\mathcal{A}_{2}\right| \geq 4$, since $\mathcal{A}_{2}$ is intersecting, it must be a star. Let its center be $x$. Since $\mathcal{A}_{2} \cup \mathcal{A}_{3} \cup \mathcal{A}_{4}$ is intersecting, $\mathcal{A}_{3}$ must be a star with center $x$ and there is at most one set in $\mathcal{A}_{4}$ missing $x$, this implies that $\mathcal{G}$ is EKR or HM, which contradicts the fact that $\mathcal{G}$ is neither EKR nor HM.

Suppose that $\left|\mathcal{A}_{3}\right| \geq 19$. By Theorem $1.3, \mathcal{A}_{3}$ must be EKR, HM or $\mathcal{G}_{2}$.
If $\mathcal{A}_{3}$ is EKR with center $x$, then since $\mathcal{G}$ is not EKR and $\mathcal{A}_{1}=\emptyset$, there must exist $G \in \mathcal{G}$, such that either $x \notin G$ and $G \cap Y \in \mathcal{A}_{2}$, or $x \notin G$ and $G \cap Y \in \mathcal{A}_{4}$. If the former holds, by the intersecting property of $\mathcal{A}_{2} \cup \mathcal{A}_{3}$, every set in $\mathcal{A}_{3}$ must contain at least one of the elements in $G \cap Y$, so $\left|\mathcal{A}_{3}\right| \leq 13$, a contradiction. Otherwise, the latter holds and $\mathcal{A}_{2}$ is a star with center $x$, and all sets of $\mathcal{G}$ missing $x$ lie in $Y$ completely. Recall that the number of these sets is at leat 3 , say $x \notin G_{1}, G_{2}, G_{3} \in \mathcal{G}$. Since $\mathcal{G}$ is not $\mathcal{G}_{3}$, it's impossible that $G_{1}, G_{2}, G_{3}$ form a 3 -star (each member contains a fixed 3 -set). If any two sets in $G_{1}, G_{2}, G_{3}$ intersect at 3 vertices, then $G_{1}, G_{2}, G_{3}$ must be a 2 -star. Since $\mathcal{A}_{3} \cup \mathcal{A}_{4}$ is intersecting, calculating directly the number of triples of $Y$ containing $x$ and intersecting with $G_{1}, G_{2}$ and $G_{3}$, we have $\left|\mathcal{A}_{3}\right| \leq 16$, a contradiction. Otherwise, there are two members, w.l.o.g., say, $G_{1}, G_{2}$, such that $\left|G_{1} \cap G_{2}\right|=2$. Since $\mathcal{A}_{3} \cup \mathcal{A}_{4}$ is intersecting, calculating directly the number of triples of $Y$ containing $x$ and intersecting with $G_{1}$ and $G_{2}$, we have $\left|\mathcal{A}_{3}\right| \leq 17$, also a contradiction.

If $\mathcal{A}_{3}$ is HM with center $x$, let $\left\{z_{1}, z_{2}, z_{3}\right\} \in \mathcal{A}_{3}$. By Theorem 1.2, we have $\left|\mathcal{A}_{3}\right| \leq 19$, so we may assume $\left|\mathcal{A}_{3}\right|=19$ and $\mathcal{A}_{3}$ is isomorphic to $\operatorname{HM}(9,3)$. Suppose
that there is a set $G$ such that $x \notin G, G \cap Y \in \mathcal{A}_{2}$, w.l.o.g., assume $z_{1} \notin G$. Since $\mid Y \backslash$ $\left(\left\{x, z_{1}, z_{2}, z_{3}\right\} \cup G\right) \mid \geq 3$, there is $a \in Y \backslash\left(\left\{x, z_{1}, z_{2}, z_{3}\right\} \cup G\right)$ such that $\left\{x, z_{1}, a\right\} \cap G=\emptyset$. By the intersecting property of $\mathcal{A}_{3} \cup \mathcal{A}_{4}$, we have $\left\{x, z_{1}, a\right\} \notin \mathcal{A}_{3}$, so $\left|\mathcal{A}_{3}\right|<19$, a contradiction. Now we may assume that $\mathcal{A}_{2}$ is a star with center $x$. Since $\mathcal{G}$ is neither HM nor contained in $\mathcal{G}_{3}$, there must be a 4 -set $G$ in $\mathcal{A}_{4}$ such that either $x \notin G$ and $1 \leq\left|G \cap\left\{z_{1}, z_{2}, z_{3}\right\}\right| \leq 2$, w.l.o.g., assume $z_{1} \notin G$ or $x \in G$ and $\left|G \cap\left\{z_{1}, z_{2}, z_{3}\right\}\right|=0$. But since $\mathcal{A}_{3} \cup \mathcal{A}_{4}$ is intersecting, the latter case will not happen. Assume the former holds. Since $\left|Y \backslash\left(\left\{x, z_{1}, z_{2}, z_{3}\right\} \cup G\right)\right| \geq 2$, there is $a \in Y \backslash\left(\left\{x, z_{1}, z_{2}, z_{3}\right\} \cup G\right)$ such that $\left\{x, z_{1}, a\right\} \cap G=\emptyset$. By the intersecting property of $\mathcal{A}_{3} \cup \mathcal{A}_{4}$, we have $\left\{x, z_{1}, a\right\} \notin \mathcal{A}_{3}$, so $\left|\mathcal{A}_{3}\right|<19$.

At last, assume that $\mathcal{A}_{3} \subseteq \mathcal{G}_{2}$ with core $\left\{x_{1}, x_{2}, x_{3}\right\}$. Since $\mathcal{A}_{3}$ is intersecting, by calculating the number of triples in $Y$ containing at least 2 vertices in core $\left\{x_{1}, x_{2}, x_{3}\right\}$, we have $\left|\mathcal{A}_{3}\right| \leq 19$, so we may assume that $\left|\mathcal{A}_{3}\right|=19$. Since $\mathcal{G} \nsubseteq \mathcal{G}_{2}$, there exists a set $G \in \mathcal{G}$ such that $\left|G \cap\left\{x_{1}, x_{2}, x_{3}\right\}\right| \leq 1$. w.l.o.g., let $G \cap\left\{x_{1}, x_{2}\right\}=\emptyset$. Since $\left|Y \backslash\left(\left\{x_{1}, x_{2}, x_{3}\right\} \cup G\right)\right| \geq 2$, we can pick $a \in Y \backslash\left(\left\{x_{1}, x_{2}, x_{3}\right\} \cup G\right)$ such that $G \cap$ $\left\{x_{1}, x_{2}, a\right\}=\emptyset$. By the intersecting property of $\mathcal{A}_{3} \cup \mathcal{G}$, we have $\left\{x_{1}, x_{2}, a\right\} \notin \mathcal{A}_{3}$, hence $\left|\mathcal{A}_{3}\right| \leq 18$, as desired.

So we have proved that $\left|\mathcal{A}_{3}\right| \leq 18$ for $k=4$.
Next, we prove $\left|\mathcal{A}_{4}\right| \leq 50$. On the contrary, suppose that $\left|\mathcal{A}_{4}\right| \geq 51$. By Theorem 1.3, $\mathcal{A}_{4}$ must be EKR, HM, or contained in $\mathcal{J}_{2}, \mathcal{G}_{2}$ or $\mathcal{G}_{3}$.

Suppose that $\mathcal{A}_{4}$ is EKR at $x$. Since $\mathcal{G}$ is not EKR and $\mathcal{A}_{1}=\emptyset$, there must exist $G \in \mathcal{G}$ such that either $x \notin G$ and $G \cap Y \in \mathcal{A}_{2}$ or $x \notin G$ and $G \cap Y \in \mathcal{A}_{3}$. If the former holds, since $\mathcal{A}_{2} \cup \mathcal{A}_{4}$ is intersecting, by calculating the number of 4 -sets in $Y$ containing $x$ and intersecting with $G \cap Y$ directly, we have $\left|\mathcal{A}_{4}\right| \leq 36$. If the latter holds, since $\mathcal{A}_{3} \cup \mathcal{A}_{4}$ is intersecting, by calculating the number of 4 -sets in $Y$ containing $x$ and intersecting with $G \cap Y$ directly, we have $\left|\mathcal{A}_{4}\right| \leq 46$.

Suppose that $\mathcal{A}_{4}$ is HM at $x$. Since $\mathcal{G}$ is not HM at $x$, there exists $G \in \mathcal{G}$ such that either $x \notin G$ and $G \cap Y \in \mathcal{A}_{2}$ or $x \notin G$ and $G \cap Y \in \mathcal{A}_{3}$, since $\mathcal{A}_{4}$ is HM at $x$ and $\mathcal{A}_{2} \cup \mathcal{A}_{4}$ (or $\mathcal{A}_{3} \cup \mathcal{A}_{4}$ ) is intersecting, by calculating the number of 4 -subsets containing $x$ and intersecting with $G \cap Y$, and adding 1 set not containing $x$, we have $\left|\mathcal{A}_{4}\right| \leq 37$ (or $\left|\mathcal{A}_{4}\right| \leq 47$ ).

Suppose that $\mathcal{A}_{4} \subseteq \mathcal{G}_{2}$ with core $\left\{x_{1}, x_{2}, x_{3}\right\}=A$. By calculating the number of 4subsets in $Y$ containing at least 2 of $\left\{x_{1}, x_{2}, x_{3}\right\}$, we have $\left|\mathcal{A}_{4}\right| \leq 51$, so we may assume $\left|\mathcal{A}_{4}\right|=51$. Since $\mathcal{G} \nsubseteq \mathcal{G}_{2}$, there exists a set $G$ in $\mathcal{G}$ such that $|G \cap A| \leq 1, G \cap Y \in \mathcal{A}_{2}$ or $\mathcal{A}_{3}$. w.l.o.g., let $G \cap\left\{x_{1}, x_{2}\right\}=\emptyset$. Since $|Y \backslash(A \cup G)| \geq 2$, we can pick $a, b \in Y \backslash(A \cup G)$ such that $(G \cap Y) \cap\left\{x_{1}, x_{2}, a, b\right\}=\emptyset$. By the intersecting property of $\mathcal{A}_{2} \cup \mathcal{A}_{3} \cup \mathcal{A}_{4}$, we have $\left\{x_{1}, x_{2}, a, b\right\} \notin \mathcal{A}_{4}$. Hence $\left|\mathcal{A}_{4}\right| \leq 50$, as desired.

Suppose that $\mathcal{A}_{4} \subseteq \mathcal{G}_{3}$ with core $\left\{x_{1}, x_{2}, x_{3}\right\}$ and center $x$. By direct calculation, $\left|\mathcal{A}_{4}\right| \leq 51$, so we may assume $\left|\mathcal{A}_{4}\right|=51$ and $\mathcal{A}_{4}=\mathcal{G}_{3}$. Since $\mathcal{G} \nsubseteq \mathcal{G}_{3}$, there must be $G \in \mathcal{G}$ and $G \cap Y \in \mathcal{A}_{2}$ or $\mathcal{A}_{3}$, such that either $x \notin G$ and $\left\{x_{1}, x_{2}, x_{3}\right\} \nsubseteq G \cap Y$ or $x \in G$ and $\left\{x_{1}, x_{2}, x_{3}\right\} \cap(G \cap Y)=\emptyset$. By the intersecting property of $\mathcal{A}_{2} \cup \mathcal{A}_{3} \cup \mathcal{A}_{4}$, in either case, we have $\mathcal{A}_{4} \neq \mathcal{G}_{3}$ and $\left|\mathcal{A}_{4}\right|<51$.

At last, suppose that $\mathcal{A}_{4} \subseteq \mathcal{J}_{2}$ with center $x$, kernel $\left\{x_{1}, x_{2}, x_{3}\right\}$ and the set of
pages $\left\{x_{4}, x_{5}\right\}$. By Theorem 1.4, we may assume $\left|\mathcal{A}_{4}\right|=51$ and $\mathcal{A}_{4}=\mathcal{J}_{2}$. Since $\mathcal{A}_{2} \cup \mathcal{A}_{3} \cup \mathcal{A}_{4}$ is intersecting, there is no member in $\mathcal{A}_{2}$ or $\mathcal{A}_{3}$ avoiding $x$. And each member in $\mathcal{A}_{2}$ must interset with $\left\{x_{1}, x_{2}, x_{3}\right\}$, each member in $\mathcal{A}_{3}$ must interset with $\left\{x_{1}, x_{2}, x_{3}\right\}$ or contain $\left\{x_{4}, x_{5}\right\}$, to satisfy these conditions, $G$ must be contained in $\mathcal{J}_{2}$, a contradiction.

So we have proved that $\mathcal{A}_{4} \leq 50$ for $k=4$.
Next consider $k \geq 5$. Suppose on the contrary that there exists $i \in\{2, \ldots, k-1\}$ such that

$$
\begin{equation*}
\left|\mathcal{A}_{i}\right|>\binom{2 k-1}{i-1}-\binom{k-1}{i-1}-\binom{k-2}{i-2}-\binom{k-3}{i-3} . \tag{3}
\end{equation*}
$$

Note that for $i=2$,

$$
\binom{2 k-1}{i-1}-\binom{k-1}{i-1}-\binom{k-2}{i-2}-\binom{k-3}{i-3}=k-1
$$

If $\left|\mathcal{A}_{2}\right| \geq k(k \geq 5)$, then $\mathcal{A}_{2}$ is EKR, moreover, since $\mathcal{A}_{2} \cup \mathcal{G}$ is intersecting, $\mathcal{G}$ must be EKR or HM, a contradiction. Hence $\left|\mathcal{A}_{2}\right| \leq k-1$, as desired.

Now consider $i \geq 3$. Under the assumption (3), we claim that

$$
\begin{equation*}
\left|\mathcal{A}_{i}\right|>\binom{2 k-1}{i-1}-\binom{2 k-i-1}{i-1}-\binom{2 k-i-2}{i-2}+2 \tag{4}
\end{equation*}
$$

Let us explain inequality (4). We write

$$
\begin{equation*}
\binom{2 k-i-2}{i-2}=\binom{2 k-i-3}{i-2}+\binom{2 k-i-3}{i-3} \tag{5}
\end{equation*}
$$

For $k \geq 5$ and $3 \leq i \leq k-1$, we have

$$
\begin{gather*}
\binom{2 k-1-i}{i-1}-\binom{k-1}{i-1}=\binom{k-1}{i-2}+\binom{k}{i-2}+\cdots+\binom{2 k-2-i}{i-2} \geq 4  \tag{6}\\
\binom{2 k-i-3}{i-2}-\binom{k-2}{i-2} \geq 0, \quad\binom{2 k-i-3}{i-3}-\binom{k-3}{i-3} \geq 0 \tag{7}
\end{gather*}
$$

Combining (3), (5), (6) and (7), we obtain (4). Since $\mathcal{A}_{i}$ is intersecting, we may assume, by Theorem 1.3 that $\mathcal{A}_{i}$ is EKR or HM or for $i=3, \mathcal{A}_{i} \subseteq \mathcal{G}_{2}$.

Case (i): $\mathcal{A}_{i}$ is EKR or HM at center $x$.
In this case $\mathcal{A}_{i}$ contains at most $1 i$-set missing $x$. Recall that there are at least three sets missing $x$ in $\mathcal{G}$. Pick three sets $G_{1}, G_{2}, G_{3} \in \mathcal{G}$ missing $x$. Denote

$$
\begin{aligned}
T & =G_{1} \cap G_{2} \cap G_{3} \cap Y, t=|T|, \\
T_{1} & =\left(G_{1} \cap Y\right) \backslash\left(G_{2} \cup G_{3}\right), t_{1}=\left|T_{1}\right|, \\
T_{2} & =\left(G_{2} \cap Y\right) \backslash\left(G_{1} \cup G_{3}\right), t_{2}=\left|T_{2}\right|, \\
T_{3} & =\left(G_{3} \cap Y\right) \backslash\left(G_{1} \cup G_{2}\right), t_{3}=\left|T_{3}\right|, \\
T_{4} & =\left(G_{1} \cap G_{2} \cap Y\right) \backslash G_{3}, t_{4}=\left|T_{4}\right|, \\
T_{5} & =\left(G_{1} \cap G_{3} \cap Y\right) \backslash G_{2}, t_{5}=\left|T_{5}\right|, \\
T_{6} & =\left(G_{2} \cap G_{3} \cap Y\right) \backslash G_{1}, t_{6}=\left|T_{6}\right| .
\end{aligned}
$$



Clearly, $t+t_{1}+t_{4}+t_{5} \leq k, t+t_{2}+t_{4}+t_{6} \leq k, t+t_{3}+t_{5}+t_{6} \leq k$. By Lemma 2.7 $\mathcal{A}_{i} \cup\left\{G_{1} \cap Y, G_{2} \cap Y, G_{3} \cap Y\right\}$ is intersecting. Applying Inclusion-Exclusion principle, we have

$$
\begin{array}{r}
\mathcal{A}_{i} \leq\binom{ 2 k-1}{i-1}-\binom{2 k-1-t-t_{1}-t_{4}-t_{5}}{i-1}-\binom{2 k-1-t-t_{2}-t_{4}-t_{6}}{i-1} \\
-\binom{2 k-1-t-t_{3}-t_{5}-t_{6}}{i-1}+\binom{2 k-1-t-t_{1}-t_{2}-t_{4}-t_{5}-t_{6}}{i-1} \\
+\binom{2 k-1-t-t_{1}-t_{3}-t_{4}-t_{5}-t_{6}}{i-1}+\binom{2 k-1-t-t_{2}-t_{3}-t_{4}-t_{5}-t_{6}}{i-1}  \tag{8}\\
\quad-\binom{2 k-1-t-t_{1}-t_{2}-t_{3}-t_{4}-t_{5}-t_{6}}{i-1}+c,
\end{array}
$$

where $c=0$ (if $\mathcal{A}_{i}$ is EKR) or 1 (if $\mathcal{A}_{i}$ is HM). Denote the right side of equality (8) by $f$. We rewrite it as

$$
\begin{align*}
f=\binom{2 k-1}{i-1} & -\binom{2 k-2-t-t_{1}-t_{4}-t_{5}}{i-2}-\cdots-\binom{2 k-1-t-t_{1}-t_{3}-t_{4}-t_{5}-t_{6}}{i-2} \\
& -\binom{2 k-2-t-t_{2}-t_{4}-t_{6}}{i-2}-\cdots-\binom{2 k-1-t-t_{1}-t_{2}-t_{4}-t_{5}-t_{6}}{i-2} \\
& -\binom{2 k-2-t-t_{3}-t_{5}-t_{6}}{i-2}-\cdots-\binom{2 k-1-t-t_{2}-t_{3}-t_{4}-t_{5}-t_{6}}{i-2} \\
& -\binom{2 k-1-t-t_{1}-t_{2}-t_{3}-t_{4}-t_{5}-t_{6}}{i-1}+c . \tag{9}
\end{align*}
$$

We can see that the right side of (9), consequently (8) does not decrease as $t+t_{1}+t_{4}+$ $t_{5}, t+t_{2}+t_{4}+t_{6}, t+t_{3}+t_{5}+t_{6}$ increase. Since $t+t_{1}+t_{4}+t_{5}, t+t_{2}+t_{4}+t_{6}, t+t_{3}+t_{5}+t_{6} \leq k$, we can substitute $t+t_{1}+t_{4}+t_{5}=k, t_{2}+t_{4}+t_{6}=k-t, t_{3}+t_{5}+t_{6}=k-t$ into
inequality (8), and this will not decrease $f$. So we have

$$
\begin{align*}
\left|\mathcal{A}_{i}\right| & \leq\binom{ 2 k-1}{i-1}-3\binom{k-1}{i-1}+\binom{t+t_{4}-1}{i-1}+\binom{t+t_{5}-1}{i-1}+\binom{t+t_{6}-1}{i-1} \\
& -\binom{t+t_{5}-t_{2}-1}{i-1}+c \\
& =\binom{2 k-1}{i-1}-3\binom{k-1}{i-1}+\binom{t+t_{4}-1}{i-1}+\binom{t+t_{6}-1}{i-1}+\binom{t+t_{5}-2}{i-2}  \tag{10}\\
& +\cdots+\binom{t+t_{5}-t_{2}-1}{i-2}+c \\
& \triangleq g .
\end{align*}
$$

Clearly, $g$ does not decrease as $t+t_{4}, t+t_{5}, t+t_{6}$ increase and $t+t_{4} \leq k-1, t+t_{5} \leq k-1$ $t+t_{6} \leq k-1$. If $t+t_{5}-t_{2}-1 \geq k-3$, then

$$
\begin{aligned}
\left|\mathcal{A}_{i}\right| & \leq\binom{ 2 k-1}{i-1}-3\binom{k-1}{i-1}+3\binom{k-2}{i-1}-\binom{k-3}{i-1}+c \\
& =\binom{2 k-1}{i-1}-\binom{k-1}{i-1}-\binom{k-2}{i-2}-\binom{k-3}{i-3}+c .
\end{aligned}
$$

The equality holds only if $t=k-1, t_{1}=t_{2}=t_{3}=1, t_{4}=t_{5}=t_{6}=0$. If $t+t_{5}-t_{2}-1 \leq k-4(*)$, then $t \leq k-2$ since $t=k-1$ implies $t_{5}=0$ and combining with $(*)$, we have $t_{2} \geq 2$, so $t+t_{2} \geq k+1$, a contradiction. Since $t+t_{4} \leq k-1, t+t_{5} \leq k-1$ and $t+t_{6} \leq k-1$, by (9) and (10), taking $t+t_{1}+t_{4}+t_{5}=$ $k, t+t_{2}+t_{4}+t_{6}=k, t+t_{3}+t_{5}+t_{6}=k$ and $t+t_{4}=k-1, t+t_{5}=k-1, t+t_{6}=k-1$ (this implies that $t=k-2, t_{4}=t_{5}=t_{6}=1$ and $t_{1}=t_{2}=t_{3}=0$ ) does not decrease $f$. So

$$
\begin{aligned}
g & \leq\binom{ 2 k-1}{i-1}-3\binom{k-1}{i-1}+3\binom{k-2}{i-1}-\binom{k-2}{i-1}+c \\
& =\binom{2 k-1}{i-1}-\binom{k-1}{i-1}-\binom{k-2}{i-2}-\binom{k-3}{i-3}-\binom{k-3}{i-2}+c \\
& \leq\binom{ 2 k-1}{i-1}-\binom{k-1}{i-1}-\binom{k-2}{i-2}-\binom{k-3}{i-3}-2+c .
\end{aligned}
$$

So

$$
\left|\mathcal{A}_{i}\right| \leq\binom{ 2 k-1}{i-1}-\binom{k-1}{i-1}-\binom{k-2}{i-2}-\binom{k-3}{i-3}+c .
$$

To reach $c=1$, there is a set $A$ in $\mathcal{A}_{i}$ not containing $x$. Let $G_{1}$ be such that $G_{1} \cap Y=A$. So $\left|G_{1} \cap Y\right|=i \leq k-1$. This implies that $t+t_{1}+t_{4}+t_{5} \leq k-1$. In view of (8) and (9), $\left|\mathcal{A}_{i}\right|$ strictly decreases as $t+t_{1}+t_{4}+t_{5}$ strictly decreases. So we have

$$
\left|\mathcal{A}_{i}\right| \leq\binom{ 2 k-1}{i-1}-\binom{k-1}{i-1}-\binom{k-2}{i-2}-\binom{k-3}{i-3}
$$

as desired.
Case (ii): For $i=3, \mathcal{A}_{i} \subseteq \mathcal{G}_{2}$ with core, say $\left\{x_{1}, x_{2}, x_{3}\right\}$.
By direct calculation, we have $\left|\mathcal{A}_{3}\right| \leq 3(2 k-3)+1=6 k-8$. When $k \geq 5$, we have

$$
6 k-8<\binom{2 k-1}{2}-\binom{k-1}{2}-\binom{k-2}{1}-\binom{k-3}{0}
$$

as desired.
Lemma 2.9. Let $\mathcal{G}$ be the final stable family as in Lemma 2.7. Then

$$
|\mathcal{G}| \leq\binom{ n-1}{k-1}-\binom{n-k-1}{k-1}-\binom{n-k-2}{k-2}-\binom{n-k-3}{k-3}+3
$$

Proof. Note that for any $A \in \mathcal{A}_{i}$, there are at most $\binom{n-|Y|}{k-i} k$-sets in $\mathcal{G}$ containing $A$. For $k=4$, we have

$$
|\mathcal{G}| \leq \sum_{i=1}^{4}\left|\mathcal{A}_{i}\right|\binom{n-9}{4-i}
$$

By Lemma 2.8,

$$
\begin{align*}
|\mathcal{G}| & \leq 3\binom{n-9}{2}+18\binom{n-9}{1}+50 \\
& =\frac{3}{2} n^{2}-\frac{21}{2} n+23 \\
& =\binom{n-1}{3}-\binom{n-5}{3}-\binom{n-6}{2}-\binom{n-7}{1}+3 . \tag{11}
\end{align*}
$$

For $k \geq 5$, we have

$$
\begin{align*}
|\mathcal{G}| & \leq \sum_{i=1}^{k}\left|\mathcal{A}_{i}\right|\binom{n-2 k}{k-i} \\
& \leq{ }^{\text {Lemma }} \text { [2.8] } \\
& 3+\sum_{i=1}^{k}\left(\binom{2 k-1}{i-1}-\binom{k-1}{i-1}-\binom{k-2}{i-2}-\binom{k-3}{i-3}\right)\binom{n-2 k}{k-i}  \tag{12}\\
& =\binom{n-1}{k-1}-\binom{n-k-1}{k-1}-\binom{n-k-2}{k-2}-\binom{n-k-3}{k-3}+3 .
\end{align*}
$$

By Lemma 2.9, we have obtained the quantitative part of Theorem 2.2.

### 2.2 Uniqueness Part of Theorem 2.2

Let $\mathcal{G}$ be a $k$-uniform family such that the equality holds in Lemma 2.9. We first show the structure of $\mathcal{G}$.

Theorem 2.10. Let $\mathcal{G}$ be a family as in Lemma 2.9 such that the equality holds. If $k=5$, then $\mathcal{G}=\mathcal{J}_{3}$ or $\mathcal{G}_{4} ;$ if $k \neq 5$, then $\mathcal{G}=\mathcal{J}_{3}$.

Proof. To make the equalities (11) and (12p hold, we must get all the equalities in Lemma 2.8. So $\left|\mathcal{A}_{2}\right|=k-1$. By Lemma 2.7, $\mathcal{A}_{2}$ is intersecting, so $\mathcal{A}_{2}$ is a star, say with center $x$ and leaves $\left\{x_{1}, x_{2}, \ldots, x_{k-1}\right\}$, or a triangle on $\{x, y, z\}$ (only for $k=4$ ). First consider $k=4$. If $\mathcal{A}_{2}$ is a triangle, then $\mathcal{G}=\mathcal{G}_{2}$, a contradiction. Otherwise, $\mathcal{A}_{2}$ is a star, this implies that all sets in $\mathcal{G}$ missing $x$ must contain $\left\{x_{1}, x_{2}, x_{3}\right\}$, and the number of such sets is at least 3 . Then either $\mathcal{G}=\mathcal{G}_{3}$ or $\mathcal{G}=\mathcal{J}_{i}, 3 \leq i \leq k-1$. By the assumption that $\mathcal{G} \nsubseteq \mathcal{G}_{3}$, the former is impossible, and the latter implies $\mathcal{G}=\mathcal{J}_{3}$. Hence, the equality in (21) holds only if $\mathcal{G}=\mathcal{J}_{3}$. For $k \geq 5, \mathcal{A}_{2}$ must be a star. Similarly, in this condition, we have either $\mathcal{G}=\mathcal{G}_{k-1}$ or $\mathcal{G}=\mathcal{J}_{i}, 3 \leq i \leq k-1$. In particular, for $k=5$, we can see that the extremal value of $|\mathcal{G}|$ can be achieved by $\left|\mathcal{G}_{4}\right|$ and $\left|\mathcal{J}_{3}\right|$, and for $k>5$, by $\left|\mathcal{J}_{3}\right|$ only.

We will use some results in [8]. We say two families $\mathcal{G}$ and $\mathcal{F}$ are cross-intersecting if for any $G \in \mathcal{G}$ and $F \in \mathcal{F}, G \cap F \neq \emptyset$. We say that a family $\mathcal{F}$ is non-separable if $\mathcal{F}$ cannot be partitioned into the union of two cross-intersecting non-empty subfamilies.

Proposition 2.11. ([8]) Let $r \geq 2$. Let $Z$ be a set of size $m \geq 2 r+1$ and let $A \subseteq Z$ such that $|A| \in\{r-1, r\}$. Let $\mathcal{B}$ be an r-uniform family on $Z$ such that $\mathcal{B}=\{B \subseteq Z: 0<|B \cap A|<|A|\}$. Then $\mathcal{B}$ is non-separable.

Lemma 2.12. ([8]) Let $\mathcal{F}$ be a $k$-uniform intersecting family. If $k \geq 3$ and $S_{x y}(\mathcal{F}) \in$ $\left\{\mathcal{J}_{2}, \mathcal{G}_{k-1}, \mathcal{G}_{2}\right\}$, then $\mathcal{F}$ is isomorphic to $S_{x y}(\mathcal{F})$.

Combining with Theorem 2.10 and Lemma 2.12, the uniqueness part of Theorem 2.2 will be completed by showing the following lemma.

Lemma 2.13. Let $\mathcal{F}$ be a $k$-uniform intersecting family. If $k \geq 4$ and $S_{x y}(\mathcal{F})=\mathcal{J}_{3}$, then $\mathcal{F}$ is isomorphic to $\mathcal{J}_{3}$.

Proof. Assume that $S_{x y}(\mathcal{F})=\mathcal{J}_{3}$ with center $x_{0}$, kernel $E$ and the set of pages $\left\{x_{1}, x_{2}, x_{3}\right\}$. That is
$\mathcal{J}_{3}=\left\{G:\left\{x_{0}, x_{1}, x_{2}, x_{3}\right\} \subseteq G\right\} \cup\left\{G: x_{0} \in G, G \cap E \neq \emptyset\right\} \cup\left\{E \cup\left\{x_{1}\right\}, E \cup\left\{x_{2}\right\}, E \cup\left\{x_{3}\right\}\right\}$.
Define

$$
\begin{aligned}
\mathcal{B}_{x} & :=\left\{G \in \mathcal{J}_{3}: x \in G, y \notin G,(G \backslash x) \cup y \notin \mathcal{J}_{3}\right\} \\
\mathcal{C}_{x} & :=\left\{G \in \mathcal{B}_{x}: G \in \mathcal{F}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_{x} & :=\left\{G \in \mathcal{B}_{x}: G \notin \mathcal{F}\right\}, \\
\mathcal{B}^{\prime} & :=\left\{G \backslash\{x\}: G \in \mathcal{B}_{x}\right\}, \\
\mathcal{C}^{\prime} & :=\left\{G \backslash\{x\}: G \in \mathcal{C}_{x}\right\}, \\
\mathcal{D}^{\prime} & :=\left\{G \backslash\{x\}: G \in \mathcal{D}_{x}\right\} .
\end{aligned}
$$

Then $\mathcal{B}_{x}=\mathcal{C}_{x} \sqcup \mathcal{D}_{x}$ and $\mathcal{B}^{\prime}=\mathcal{C}^{\prime} \sqcup \mathcal{D}^{\prime}$. The definition of $\mathcal{D}_{x}$ implies that for any $G \in \mathcal{D}_{x}, G \backslash\{x\} \cup\{y\} \in \mathcal{F}$, and the definition of $\mathcal{C}_{x}$ implies that for any $G \in \mathcal{C}_{x}$, $G \backslash\{x\} \cup\{y\} \notin \mathcal{F}$. Clearly, only the sets in $\mathcal{D}_{x}$ are in $S_{x y}(\mathcal{F}) \backslash \mathcal{F}$. If $\mathcal{D}_{x}=\emptyset$, then $S_{x y}(\mathcal{F})=\mathcal{F}=\mathcal{J}_{3}$, and if $\mathcal{C}_{x}=\emptyset$, then $\mathcal{F}$ is still $\mathcal{J}_{3}$ with center $y$. On the other hand, notice that $\mathcal{C}_{x}$ and $\left\{G \backslash\{x\} \cup\{y\}: G \in \mathcal{D}_{x}\right\}$ are cross intersecting, so $\mathcal{C}^{\prime}$ and $\mathcal{D}^{\prime}$ are cross intersecting. We are going to prove that $\mathcal{B}^{\prime}$ is non-separable, this means that $\mathcal{C}^{\prime}=\emptyset$ or $\mathcal{D}^{\prime}=\emptyset$, and hence $\mathcal{C}_{x}=\emptyset$ or $\mathcal{D}_{x}=\emptyset$, we can conclude the proof. So what remains is to show the following claim.

Claim 2.14. $\mathcal{B}^{\prime}$ is non-separable.
Proof. We say the shift $S_{x y}: \mathcal{F} \rightarrow \mathcal{J}_{3}$ is trivial if $\mathcal{B}_{x}=\emptyset$. Let $Z:=[n] \backslash\{x, y\}$. If $r=k-1$, then $|Z| \geq 2 k+1-2=2 r+1$.

Let $T_{1}:=\left\{x_{0}\right\}, T_{2}:=E, T_{3}:=\left\{x_{1}, x_{2}, x_{3}\right\}, T_{4}:=[n] \backslash\left(T_{1} \cup T_{2} \cup T_{3}\right)$.
Since for $x, y \in T_{i}$ or for $x \in T_{i}, y \in T_{j}, i>j$, the shift is trivial, we only need to consider the following three cases.

Case (i): $x=x_{0}$ and $y \in T_{2} \cup T_{3} \cup T_{4}$.
If $y \in T_{3}$, let $A=E$, then $\mathcal{B}^{\prime}=\{B \subseteq Z: 0<|B \cap A|<|A|\}$. By Proposition 2.11, $\mathcal{B}^{\prime}$ is non-separable. If $y \in T_{2} \cup T_{4}$, let $A:=E \backslash\{y\}$, then $|A| \in\{r-1, r\}$. Assume that $\mathcal{B}^{\prime}$ has a partition $\mathcal{B}^{\prime}{ }_{1} \cup \mathcal{B}^{\prime}{ }_{2}$ such that $\mathcal{B}^{\prime}{ }_{1}$ and $\mathcal{B}^{\prime}{ }_{2}$ are cross-intersecting. We now partition $\mathcal{B}^{\prime}$ into three parts $\mathcal{P}_{1} \sqcup \mathcal{P}_{2} \sqcup \mathcal{P}_{3}$, where

$$
\begin{gathered}
\mathcal{P}_{1}:=\{B \subseteq Z: 0<|B \cap A|<|A|\} \\
\mathcal{P}_{2}:=\left\{B \in \mathcal{B}^{\prime}: B \cap A=\emptyset\right\}=\left\{T_{3} \cup F: F \subseteq T_{4} \backslash\{y\},|F|=k-4\right\},
\end{gathered}
$$

and

$$
\mathcal{P}_{3}:=\left\{B \in \mathcal{B}^{\prime}: A \subseteq B\right\}= \begin{cases}\left\{A \cup\{z\}: z \in T_{4}\right\}, & y \in T_{2} \\ \{A\}, & y \in T_{4}\end{cases}
$$

Obviously, $\mathcal{P}_{1} \neq \emptyset$. By Proposition 2.11, $\mathcal{P}_{1}$ is non-separable. For any $P \in \mathcal{P}_{2}$, and any $a \in A$, we have $|Z \backslash\{a\}| \geq 2 r$, then in $\mathcal{P}_{1}$ we can always find $P^{\prime} \subseteq Z \backslash(\{a\} \cup P)$ such that $0<\left|P^{\prime} \cap A\right|<|A|$ and $P \cap P^{\prime}=\emptyset$. This implies that $P$ and $P^{\prime}$ must be in the same $\mathcal{B}^{\prime}{ }_{i}(i=1$ or 2$)$ (recall that we assumed that $\mathcal{B}^{\prime}$ has a partition $\mathcal{B}^{\prime}{ }_{1} \cup \mathcal{B}^{\prime}{ }_{2}$ such that $\mathcal{B}^{\prime}{ }_{1}$ and $\mathcal{B}^{\prime}{ }_{2}$ are cross-intersecting), hence $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ are in the same $\mathcal{B}^{\prime}{ }_{i}$. For any $P \in \mathcal{P}_{3}$, we have $\left|P \cap T_{4}\right| \leq 1$. Since $\left|T_{4}\right| \geq k-2$, there is a $(k-4)$-set $F \subseteq T_{4} \backslash\{y\}$, such that $P \cap F=\emptyset$. Note that $P^{\prime}:=F \cup T_{3} \in \mathcal{P}_{2}$ and $P^{\prime} \cap P=\emptyset$, so $\mathcal{P}_{2}$ and $\mathcal{P}_{3}$ are in the same $\mathcal{B}^{\prime}{ }_{i}$. Hence $\mathcal{B}^{\prime}=\mathcal{B}^{\prime}{ }_{1}$ or $\mathcal{B}^{\prime}{ }_{2}$, as desired.

Case (ii): $x \in T_{2}$ and $y \in T_{3} \cup T_{4}$.

Let $E_{i}:=\left(E \cup\left\{x_{i}\right\}\right) \backslash\{x\}, i=1,2,3$.
If $y \in T_{4}$, then

$$
\mathcal{B}^{\prime}=\left\{E_{1}, E_{2}, E_{3}\right\} \cup\left\{G \in\binom{[n] \backslash\{x\}}{k-1}: x_{0} \in G, G \cap E=\emptyset,\left|G \cap T_{3}\right| \leq 2, y \notin G\right\} .
$$

Since $\left|T_{4} \backslash\{y\}\right| \geq k-3$, there is $P \in \mathcal{B}^{\prime} \backslash\left\{E_{1}, E_{2}\right\}$, such that $P \cap E_{1}=P \cap E_{2}=\emptyset$. Hence, $E_{1}$ and $E_{2}$ belong to the same part $\mathcal{B}^{\prime}{ }_{i}$. Similarly, $E_{1}$ and $E_{3}$ belong to the same part. Thus $E_{1}, E_{2}$ and $E_{3}$ are in the same $\mathcal{B}^{\prime}$. Moreover, for any $P^{\prime} \in \mathcal{B}^{\prime} \backslash\left\{E_{1}, E_{2}, E_{3}\right\}$, because $\left|P^{\prime} \cap\left\{x_{1}, x_{2}, x_{3}\right\}\right| \leq 2$, we have $P^{\prime} \cap E_{1}=\emptyset$, or $P^{\prime} \cap E_{2}=\emptyset$ or $P^{\prime} \cap E_{3}=\emptyset$. Hence, $\mathcal{B}^{\prime}$ is non-separable, as desired.

If $y \in T_{3}$, w.l.o.g., let $y=x_{1}$. Then

$$
\mathcal{B}^{\prime}=\left\{E_{2}, E_{3}\right\} \cup\left\{G \in\binom{[n] \backslash\{x\}}{k-1}: x_{0} \in G, G \cap E=\emptyset,\left|G \cap T_{3}\right| \leq 1, y \notin G\right\} .
$$

Since $\left|T_{4}\right| \geq k-2$, there exists $P \in \mathcal{B}^{\prime} \backslash\left\{E_{2}, E_{3}\right\}$ such that $P \cap T_{3}=\emptyset$, then $P \cap E_{2}=\emptyset$, and $P \cap E_{3}=\emptyset$, this implies that $E_{2}$ and $E_{3}$ are in the same $\mathcal{B}^{\prime}{ }_{i}$. Because $\left|G \cap T_{3}\right| \leq 1$ and $G \cap E=\emptyset$, it's not hard to see that each $P \in \mathcal{B}^{\prime} \backslash\left\{E_{2}, E_{3}\right\}$ is disjoint from one of $E_{2}$ and $E_{3}$. Hence $\mathcal{B}^{\prime}$ is non-separable.

Case (iii): $x \in T_{3}$ and $y \in T_{4}$. w.l.o.g., let $x=x_{1}$.
Under this condition,

$$
\mathcal{B}^{\prime}=\{E\} \cup\left\{G \in\binom{[n] \backslash\{x\}}{k-1}:\left\{x_{0}, x_{2}, x_{3}\right\} \subseteq G, G \cap E=\emptyset, y \notin G\right\}
$$

Since $E$ is disjoint from every other set in $\mathcal{B}^{\prime} \backslash\{E\}, \mathcal{B}^{\prime}$ is non-separable.
The proof of Lemma 2.13 is complete.

## 3 Proofs of Lemma 2.6 and Lemma 2.7

### 3.1 Proof of Lemma 2.6

We first show the following preliminary results. For a family $\mathcal{F} \subseteq 2^{[n]}$ and $x_{1}, x_{2}, x_{3} \in$ [ $n$ ], let $d_{\left\{x_{1}, x_{2}\right\}}$ be the number of sets containing $\left\{x_{1}, x_{2}\right\}$ in $\mathcal{F}$, and $d_{\left\{x_{1}, x_{2}, x_{3}\right\}}$ be the number of sets containing $\left\{x_{1}, x_{2}, x_{3}\right\}$ in $\mathcal{F}$.

Claim 3.1. Let $\mathcal{F} \subseteq \mathcal{G}_{2}$ be a 4-uniform family with core $A$ satisfying $d_{\left\{x_{1}, x_{2}\right\}}>2 n-7$. Then $\left\{x_{1}, x_{2}\right\} \subseteq A$.

Proof. If $\left\{x_{1}, x_{2}\right\} \subseteq[n] \backslash A$, then a set in $\mathcal{F}$ containing $\left\{x_{1}, x_{2}\right\}$ must have two elements from $A$, so $d_{\left(x_{1}, x_{2}\right)} \leq 3$, a contraction. If $\left|\left\{x_{1}, x_{2}\right\} \cap A\right|=1$, then a set in $\mathcal{F}$ containing $\left\{x_{1}, x_{2}\right\}$ must have at least one element from $A$, so $d_{\left(x_{1}, x_{2}\right)} \leq 2 n-7$, a contraction again. So $\left\{x_{1}, x_{2}\right\} \subseteq A$, as desired.

Claim 3.2. Let $\mathcal{F} \subseteq \mathcal{G}_{3}$ be a 4-uniform family with center $x$ and core $E$ and let $B=\{x\} \cup E$.
(i) If $d_{\left\{x_{1}, x_{2}\right\}} \geq 3 n-12$, then $x \in\left\{x_{1}, x_{2}\right\}$.
(ii) If $d_{\left\{x_{1}, x_{2}\right\}}>3 n-12$, then $\left\{x_{1}, x_{2}\right\} \subseteq B$ and $x \in\left\{x_{1}, x_{2}\right\}$.

Proof. For (i), assume that $x \notin\left\{x_{1}, x_{2}\right\}$. If $\left\{x_{1}, x_{2}\right\} \cap B=\emptyset$, then the sets containing $\left\{x_{1}, x_{2}\right\}$ must contain the center $x$ and another vertex from core $E$, so $d_{\left(x_{1}, x_{2}\right)} \leq 3<$ $3 n-12$, a contradiction. So $\left\{x_{1}, x_{2}\right\} \subseteq E$ or $\left|\left\{x_{1}, x_{2}\right\} \cap E\right|=1$. If the former holds, then the sets containing $\left\{x_{1}, x_{2}\right\}$ must contain the center $x$ or contain the core $E$, so $d_{\left(x_{1}, x_{2}\right)} \leq(n-3)+(n-4)=2 n-7<3 n-12$, a contradiction. If the latter holds, w.l.o.g., let $\left\{x_{1}, x_{2}\right\} \cap E=\left\{x_{1}\right\}$, then the sets containing $\left\{x_{1}, x_{2}\right\}$ must contain the center $x$ or just the set $E \cup\left\{x_{2}\right\}$, so $d_{\left(x_{1}, x_{2}\right)} \leq(n-3)+1<3 n-12$, also a contradiction. Hence, $x \in\left\{x_{1}, x_{2}\right\}$, as desired.

For (ii), we have shown that $x \in\left\{x_{1}, x_{2}\right\}$ by (i), w.l.o.g, let $x_{1}=x$ be the center. If $x_{2} \notin E$, then the sets containing $\left\{x_{1}, x_{2}\right\}$ must intersect with $E$, so $d_{\left(x_{1}, x_{2}\right)} \leq$ $\binom{n-2}{2}-\binom{n-5}{2}=3 n-12$, a contradiction to that $d_{\left\{x_{1}, x_{2}\right\}}>3 n-12$, so $x_{2} \in E$, that is $\left\{x_{1}, x_{2}\right\} \subseteq B$, as desired.

Claim 3.3. Fix $n>6$. Let $\mathcal{F} \subseteq \mathcal{G}_{3}$ be a 4-uniform family with center $x$ and core $E$ and let $B=\{x\} \cup E$. If $d_{\left\{x_{1}, x_{2}, x_{3}\right\}} \geq n-3$, then either $\left\{x_{1}, x_{2}, x_{3}\right\} \subset B$ or $\left|\left\{x_{1}, x_{2}, x_{3}\right\} \cap B\right|=2$ with $x \in\left\{x_{1}, x_{2}, x_{3}\right\}$.

Proof. Suppose on the contrary that neither $\left\{x_{1}, x_{2}, x_{3}\right\} \subset B$ nor $\left|\left\{x_{1}, x_{2}, x_{3}\right\} \cap B\right|=2$ with $x \in\left\{x_{1}, x_{2}, x_{3}\right\}$. Since $\mathcal{F} \subseteq \mathcal{G}_{3}$, it's easy to see that if $\left\{x_{1}, x_{2}, x_{3}\right\} \subseteq[n] \backslash B$, then $d_{\left\{x_{1}, x_{2}, x_{3}\right\}}=0$, so $1 \leq\left|\left\{x_{1}, x_{2}, x_{3}\right\} \cap B\right| \leq 2$. First consider that $\left|\left\{x_{1}, x_{2}, x_{3}\right\} \cap B\right|=1$. If $\left\{x_{1}, x_{2}, x_{3}\right\} \cap B=\{x\}$, then the sets containing $\left\{x_{1}, x_{2}, x_{3}\right\}$ in $\mathcal{F}$ must intersect with $E$, so $d_{\left\{x_{1}, x_{2}, x_{3}\right\}} \leq 3<n-3$, a contradiction. If $\left|\left\{x_{1}, x_{2}, x_{3}\right\} \cap E\right|=1$, then the set containing $\left\{x_{1}, x_{2}, x_{3}\right\}$ in $\mathcal{F}$ must contain $x$, so $d_{\left\{x_{1}, x_{2}, x_{3}\right\}} \leq 1<n-3$, also a contradiction. Hence $\left|\left\{x_{1}, x_{2}, x_{3}\right\} \cap B\right|=2$. By hypothesis, $\left|\left\{x_{1}, x_{2}, x_{3}\right\} \cap E\right|=2$, w.l.o.g., let $\left\{x_{1}, x_{2}, x_{3}\right\} \cap E=\left\{x_{1}, x_{2}\right\}$, then $d_{\left\{x_{1}, x_{2}, x_{3}\right\}} \leq 2$ since the possible sets in $\mathcal{F}$ containing $\left\{x_{1}, x_{2}, x_{3}\right\}$ are $\left\{x_{1}, x_{2}, x_{3}\right\} \cup\{x\}$ and $E \cup\left\{x_{3}\right\}$, a contradiction.

Proof of Lemma 2.6. We first consider that $k \geq 5$.
In Case 1, i.e., $S_{x y}\left(\mathcal{H}_{1}\right)$ is EKR with center $x$, we take $X_{1}=\{x, y\}$. In Case 2, since $S_{x y}\left(\mathcal{H}_{2}\right)$ is HM at center $x$, let $E=\left\{z_{1}, z_{2}, \ldots, z_{k}\right\}$ be the only member missing $x$, and without loss of generality, we assume $z_{1} \neq y$, and take $X_{2}=\left\{x, y, z_{1}\right\}$. In Case 3, $S_{x y}\left(\mathcal{H}_{3}\right) \subseteq \mathcal{J}_{2}$ with center $x$, kernal $\left\{z_{1}, z_{2}, \ldots, z_{k-1}\right\}$. Without loss of generality, we assume $z_{1} \neq y$, and take $X_{3}=\left\{x, y, z_{1}\right\}$. We can see that for any set $G \in \mathcal{H}_{i}$, $G \cap X_{i} \neq \emptyset$, for $i=1,2,3$. After the shifts $S_{x^{\prime} y^{\prime}}$ for all $x^{\prime}<y^{\prime}, x^{\prime}, y^{\prime} \in[n] \backslash X_{i}$ to $\mathcal{H}_{i}$, the resulting family $\mathcal{H}_{i}^{\prime}$ satisfies that for every set $G^{\prime} \in \mathcal{H}^{\prime}{ }_{i}, G^{\prime} \cap X_{i} \neq \emptyset$. By the maximality of $|\mathcal{H}|$, we may assume that all $k$-sets containing $X_{i}(i=1,2,3)$ are in $\mathcal{H}$, so is in $\mathcal{H}_{i}$. These sets will keep stable after any shift $S_{x^{\prime} y^{\prime}}$, so there are at least $\binom{n-3}{k-2}\left(\right.$ or $\left.\binom{n-4}{k-3}\right)>2$ sets missing $x^{\prime}$ in $\mathcal{H}_{i}^{\prime}$. Fact 2.4 (i), (ii) and (iii) implies that $\mathcal{H}_{i}^{\prime}$ is neither EKR nor HM nor contained in $\mathcal{J}_{2}$. We are done for $k \geq 5$.

We now assume that $k=4$. We will complete the proof by showing the following Lemmas corresponding to Cases 1-5 in Remark 2.5

Lemma 3.4 (Case 1). If we each a 4-uniform family $\mathcal{H}_{1}$ such that $S_{x y}\left(\mathcal{H}_{1}\right)$ is EKR at $x$, then there is a set $X_{1}=\left\{x, y, y^{\prime}, z, w\right\}$ such that after a series of shifts $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}\right.$ and $\left.x^{\prime}, y^{\prime} \in[n] \backslash X_{1}\right)$ to $\mathcal{H}_{1}$, we will reach a stable family $\mathcal{G}$ satisfying the conditions of Theorem 2.2. Moreover, $\left\{y, y^{\prime}, z, w\right\}$ or $\left\{x, y^{\prime}, z, w\right\}$ is in $\mathcal{G}$. Furthermore, $G \cap\{x, y\} \neq$ $\emptyset$ for any $G \in \mathcal{G}$.

Proof. Since $S_{x y}\left(\mathcal{H}_{1}\right)$ is EKR, for any $F \in \mathcal{H}_{1}$, we have $F \cap\{x, y\} \neq \emptyset$. Any set obtained by performing shifts $[n] \backslash\{x, y\}$ to a set in $\mathcal{H}_{1}$ still contains $x$ or $y$. We will show Claims 3.5, 3.6 and 3.8 implying Lemma 3.4.
Claim 3.5. Performing shifts in $[n] \backslash\{x, y\}$ to $\mathcal{H}_{1}$ repeatedly will not reach Cases 1-3 in Remark 2.5.

Proof. Since $S_{x y}\left(\mathcal{H}_{1}\right)$ is EKR, for any $G \in \mathcal{H}_{1}$, we have $G \cap\{x, y\} \neq \emptyset$. By the maximality of $|\mathcal{H}|\left(\left|\mathcal{H}_{1}\right|\right.$ as well $)$, we have

$$
\begin{gather*}
\left\{G \in\binom{[n]}{k}:\{x, y\} \subseteq G\right\} \subseteq \mathcal{H}_{1}, \\
\left|\left\{G \in \mathcal{H}_{1}:\{x, y\} \subseteq G\right\}\right|=\binom{n-2}{2} . \tag{13}
\end{gather*}
$$

All these sets containing $\{x, y\}$ are stable after performing $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}, x^{\prime}, y^{\prime} \notin\right.$ $\{x, y\})$. So there are still at least $\binom{n-3}{2}>2$ sets missing $x^{\prime}$ after $S_{x^{\prime} y^{\prime}}$, so we will not reach Case 1-3.

Claim 3.6. If performing some shifts in $[n] \backslash\{x, y\}$ repeatedly to $\mathcal{H}_{1}$ reaches $\mathcal{H}_{4}$ in Case $4\left(S_{x^{\prime} y^{\prime}}\left(\mathcal{H}_{4}\right) \subseteq \mathcal{G}_{2}\right)$, then there exists $X_{1}=\left\{x, y, y^{\prime}, z, w\right\}$ such that performing shifts in $[n] \backslash X_{1}$ repeatedly to $\mathcal{H}_{4}$ will not reach Cases 1-5 as in Remark 2.5, and $\left\{y, y^{\prime}, z, w\right\}$ or $\left\{x, y^{\prime}, z, w\right\}$ is in the final stable family $\mathcal{G}$.

Proof. Assume that after some shifts in $[n] \backslash\{x, y\}$ to $\mathcal{H}_{1}$, we get $\mathcal{H}_{4}$ such that $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}_{4}\right) \subseteq \mathcal{G}_{2}$ with core $A$. Since there are $\binom{n-2}{2}$ sets containing $\{x, y\}$ in $\mathcal{H}_{1}$ and they are stable (so in $\mathcal{H}_{4}$ ), and $\binom{n-2}{2}>2 n-7(n \geq 6)$, by Fact 2.4 (ii) and Claim 3.1, $A=\left\{x^{\prime}, x, y\right\}$. Since $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}_{4}\right) \subseteq \mathcal{G}_{2}$ with core $\left\{x^{\prime}, x, y\right\}$, there exists $\left\{y, y^{\prime}, z_{1}, w_{1}\right\}$ (or $\left\{x, y^{\prime}, z_{2}, w_{2}\right\}$ ) in $\mathcal{H}_{4}$. Let $X_{1}:=\left\{x, y, y^{\prime}, z_{1}, w_{1}\right\}$ (or $X_{1}:=\left\{x, y, y^{\prime}, z_{2}, w_{2}\right\}$ ). Clearly, any set containing $\{x, y\}$ and missing $x^{\prime \prime} \in[n] \backslash X_{1}$ are stable after performing shifts in $[n] \backslash X_{1}$ repeatedly to $\mathcal{H}_{4}$, so performing shifts $S_{x^{\prime \prime} y^{\prime \prime}}, x^{\prime \prime}, y^{\prime \prime} \in[n] \backslash X_{1}$ to $\mathcal{H}_{4}$ will not reach Cases 1-3. If we reach Case 4 , that is we get a family $\mathcal{H}^{\prime}{ }_{4}$, such that $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}^{\prime}{ }_{4}\right) \subseteq \mathcal{G}_{2}$ with core $A^{\prime}$. By Fact 2.4 and Claim 3.1, we have $A^{\prime}=\left\{x^{\prime \prime}, x, y\right\}$. However, $\left\{y, y^{\prime}, z_{1}, w_{1}\right\}$ (or $\left\{x, y^{\prime}, z_{2}, w_{2}\right\}$ ) is stable under all the shifts in $[n] \backslash X_{1}$, so it is still in $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}^{\prime}{ }_{4}\right)$, contradicting that $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}^{\prime}{ }_{4}\right) \subseteq \mathcal{G}_{2}$ with core $\left\{x^{\prime \prime}, x, y\right\}$. Thus we can not reach Case 4.

Now assume that after some shifts in $[n] \backslash X_{1}$ to $\mathcal{H}_{4}$, we get $\mathcal{H}_{5}$ such that $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}$ with center and core forming a 4 -set $B$ for some $x^{\prime \prime}$ and $y^{\prime \prime} \in[n] \backslash X_{1}$. By Fact 2.4 (iv), (13) and Claim 3.2 (ii), we have $\left\{x, y, x^{\prime \prime}\right\} \subseteq B$. Since there are $\binom{n-2}{2}$ sets which contain $\{x, y\}$ in $\mathcal{H}_{1}$ (so in $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$ ), we have one of the following cases:
$(*) x$ is the center, and $y$ is in the core;
$(* *) y$ is the center, and $x$ is in the core.
Recall that there exists $\left\{y, y^{\prime}, z_{1}, w_{1}\right\}$ or $\left\{x, y^{\prime}, z_{2}, w_{2}\right\}$ in $\mathcal{H}_{4}$. We will meet one of the following three cases:
(a) There is no set $G \in \mathcal{H}_{4}$ such that $G \cap\left\{x, y, x^{\prime}\right\}=\{x\}$. So there exists $\left\{y, y^{\prime}, z_{1}, w_{1}\right\} \in \mathcal{H}_{4}$, and all sets containing $\left\{x^{\prime}, x\right\}$ in $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}_{4}\right)$ are originally in $\mathcal{H}_{4}$. Take $X_{1}:=\left\{x, y, y^{\prime}, z_{1}, w_{1}\right\}$. By the maximality of $|\mathcal{H}|$ (so is $\left.\left|\mathcal{H}_{4}\right|\right)$, there are $\binom{n-2}{2}$ sets containing $\left\{x^{\prime}, x\right\}$ in $\mathcal{H}_{4}$ (so in $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$ as well). This implies that $x^{\prime} \in E$, and $x$ is the center. However, $\left\{y, y^{\prime}, z_{1}, w_{1}\right\}$ is contained in $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$, a contraction to that $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}$ with center $x$ and core $\left\{y, x^{\prime}, x^{\prime \prime}\right\}$.
(b) There is no set $G \in \mathcal{H}_{4}$ such that $G \cap\left\{x, y, x^{\prime}\right\}=\{y\}$. So there exists $\left\{x, y^{\prime}, z_{2}, w_{2}\right\} \in \mathcal{H}_{4}$, and all sets containing $\left\{x^{\prime}, y\right\}$ in $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}_{4}\right)$ are originally in $\mathcal{H}_{4}$. Take $X_{1}:=\left\{x, y, y^{\prime}, z_{2}, w_{2}\right\}$. By the maximality of $|\mathcal{H}|$ (so is $\left.\left|\mathcal{H}_{4}\right|\right)$, there are $\binom{n-2}{2}$ sets containing $\left\{x^{\prime}, y\right\}$ in $\mathcal{H}_{4}$, so in $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$. This implies that $x^{\prime} \in E$ and $y$ is the center for $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$. However, $\left\{x, y^{\prime}, z_{2}, w_{2}\right\}$ is in $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$, contradicting to that $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}$ at center $y$ and core $\left\{x, x^{\prime}, x^{\prime \prime}\right\}$.
(c) There are both $\left\{y, y^{\prime}, z_{1}, w_{1}\right\}$ and $\left\{x, y^{\prime}, z_{2}, w_{2}\right\}$ in $\mathcal{H}_{4}$. We choose $X_{1}:=$ $\left\{x, y, y^{\prime}, z_{1}, w_{1}\right\}$ first. Assume that $(*)$ happens. Since $\left\{y, y^{\prime}, z_{1}, w_{1}\right\}$ is still in $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$, this contradicts that $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}$ with center $x$ and $\left\{y, x^{\prime \prime}\right\}$ contained in the core. So we assume that $(* *)$ happens. Let $B=\left\{x, y, x^{\prime \prime}, u\right\}$ for some $u$. If $u=x^{\prime}$, then the existence of $\left\{y, y^{\prime}, z_{1}, w_{1}\right\}$ makes a contradiction again. Now consider $u \neq x^{\prime}$.
Claim 3.7. If $u \neq x^{\prime}$, then $u=y^{\prime}$.
Proof. Assume on the contrary that $u \neq y^{\prime}$. We have shown that $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$ can not be contained in $\mathcal{J}_{2}$ at center $y$, then there are at least 3 sets containing $\left\{x, u, x^{\prime \prime}\right\}$. Although $\left\{x, x^{\prime}, x^{\prime \prime}, u\right\}$ and $\left\{x, y^{\prime}, x^{\prime \prime}, u\right\}$ may be two such sets, there must be $\left\{x, u, x^{\prime \prime}, v\right\} \in$ $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$ for some $v \in[n] \backslash\left\{x, y, u, x^{\prime}, y^{\prime}, x^{\prime \prime}\right\}$. However, every set in $\mathcal{H}_{4}$ contains $\{x, y\}$, or $\left\{x^{\prime}, x\right\}$, or $\left\{x^{\prime}, y\right\}$, or $\left\{x, y^{\prime}\right\}$, or $\left\{y, y^{\prime}\right\}$ by recalling that $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}_{4}\right) \subseteq \mathcal{G}_{2}$ with core $\left\{x, y, x^{\prime}\right\}$, so is every set in $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$ since $x^{\prime \prime}, y^{\prime \prime} \in[n] \backslash\left\{x, y, y^{\prime}, z_{1}, w_{1}\right\}$, a contradiction.

By Claim 3.7, we have that $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}$ at center $y$ and core $\left\{x, x^{\prime \prime}, y^{\prime}\right\}$. This time, we change $X_{1}$ to $X_{1}^{\prime}:=\left\{x, y, y^{\prime}, z_{2}, w_{2}\right\}$. Similar to the lines in the first paragraph of the proof of Claim 3.6, we will not reach Cases 1-4 after performing shifts $S_{x^{\prime} y^{\prime}}$ in $[n] \backslash X_{1}^{\prime}$. If we reach Case 5 , that is, after some shifts in $[n] \backslash X_{1}^{\prime}$ to $\mathcal{H}_{4}$, we get $\mathcal{H}^{\prime}{ }_{5}$ such that $S_{x^{\prime \prime \prime} y^{\prime \prime \prime}}\left(\mathcal{H}^{\prime}{ }_{5}\right) \subseteq \mathcal{G}_{3}$ with center and core forming a 4 -set $B^{\prime}$ for some $x^{\prime \prime \prime}, y^{\prime \prime \prime} \in[n] \backslash X_{1}^{\prime}$. By the previous analysis, $B^{\prime}=\left\{x, y, x^{\prime \prime \prime}, y^{\prime}\right\}$, and we only need to consider the case that $x$ is the center (If $y$ is the center, since $\left\{y, y^{\prime}, z_{2}, w_{2}\right\}$
is still in $S_{x^{\prime \prime \prime} y^{\prime \prime \prime}}\left(\mathcal{H}^{\prime}{ }_{5}\right)$, this contradicts that $S_{x^{\prime \prime \prime}} y^{\prime \prime \prime}\left(\mathcal{H}_{5}^{\prime}\right) \subseteq \mathcal{G}_{3}$ with center $y$ and core $\left.\left\{x, y^{\prime}, x^{\prime \prime \prime}\right\}\right)$. We have shown that $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$ can not be contained in $\mathcal{G}_{2}$ with core $\left\{x, y, y^{\prime}\right\}$, so there is $G \in S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$ such that $G \cap\{x, y\}=\emptyset$ or $G \cap\left\{x, y^{\prime}\right\}=\emptyset$ or $G \cap\left\{y, y^{\prime}\right\}=\emptyset$. Since $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}$ with core $\left\{x, x^{\prime \prime}, y^{\prime}\right\}$ and center $y, G$ must contain $x$ or $y$. If $G \cap\left\{y, y^{\prime}\right\}=\emptyset$, it contradicts that $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}$ with core $\left\{x, x^{\prime \prime}, y^{\prime}\right\}$ and center $y$. So there is $G \in S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}\right)$ such that $G \cap\left\{x, y^{\prime}\right\}=\emptyset$. After shifts in $[n] \backslash X_{1}^{\prime}$ to $G$, we get $G^{\prime}$ missing $x$ and $y^{\prime}$ still. This contradicts that $S_{x^{\prime \prime \prime} y^{\prime \prime \prime}}\left(\mathcal{H}_{5}^{\prime}\right) \subseteq \mathcal{G}_{3}$ with core $\left\{y, x^{\prime \prime \prime}, y^{\prime}\right\}$ and center $x$. Hence, we will not reach Case 5 .

In summary, we have shown that there exists $X_{1}$ in the form of $\left\{x, y, y^{\prime}, z, w\right\}$ such that performing shifts in $[n] \backslash X_{1}$ repeatedly to $\mathcal{H}_{4}$ will not reach Cases 1-5 as in Remark 2.5. Moreover, $\left\{y, y^{\prime}, z, w\right\}$ or $\left\{x, y^{\prime}, z, w\right\}$ is in the final stable family $\mathcal{G}$. This completes the proof of Claim 3.6.

Claim 3.8. If performing some shifts in $[n] \backslash\{x, y\}$ repeatedly to $\mathcal{H}_{1}$ does not reach Cases1-4, but reaches $\mathcal{H}_{5}$ in Case $5\left(S_{x^{\prime} y^{\prime}}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}\right)$, then there exists $X_{1}$ in the form of $\left\{x, y, y^{\prime}, z, w\right\}$ such that performing shifts in $[n] \backslash X_{1}$ repeatedly to $\mathcal{H}_{4}$ will not reach Cases 1-5 as in Remark 2.5. Moreover, $\left\{y, y^{\prime}, z, w\right\}$ or $\left\{x, y^{\prime}, z, w\right\}$ is in the final stable family $\mathcal{G}$.

Proof. Suppose that we get some $\mathcal{H}_{5}$ such that $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}$ with center and core forming a 4 -set $B$. By (13) and Claim 3.2, the center must be $x$ or $y$, and $\{x, y\} \subset B$. By Fact 2.4 (iv), $X^{\prime} \in B$ and $y^{\prime} \notin B$. Let $B=\left\{x, y, x^{\prime}, z\right\}$. We consider the case that $x$ is the center, the proof for $y$ being the center is similar.

Since $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}$, and recall that we are under Case 1, every set in $\mathcal{H}_{5}$ intersects $\{x, y\}$, there exists $\left\{y, y^{\prime}, z, w\right\}$ (or $\left.\left\{x, y^{\prime}, z_{1}, z_{2}\right\}\right) \in \mathcal{H}_{5}$. And by the maximality of $|\mathcal{H}|$ (so is $\left.\left|\mathcal{H}_{5}\right|\right)$, we may assume that all the sets containing $\{x, z\}$ in $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}_{5}\right)$ are originally in $\mathcal{H}_{5}$. Let $X_{1}:=\left\{x, y, y^{\prime}, z, w\right\}$ (or $\left\{x, y, y^{\prime}, z_{1}, z_{2}\right\}$ ). Similar to the analysis in the first paragraph of the proof of Claim 3.6 , for any shifts $S_{x^{\prime \prime} y^{\prime \prime}}$ to $\mathcal{H}_{5}$ in $[n] \backslash X_{1}$, we won't reach Cases 1-4. If we reach Case 5 again, then the resulting family $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}^{\prime}\right)$ $\left(x^{\prime \prime}\right.$ and $\left.y^{\prime \prime} \in[n] \backslash X_{1}\right)$ must be contained in $\mathcal{G}_{3}$ with core $\left\{y, x^{\prime \prime}, z\right\}$ and center $x$. However $\left\{y, y^{\prime}, z, w\right\}$ (or $\left\{x, y^{\prime}, z_{1}, z_{2}\right\}$ ) is still in $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}^{\prime}{ }_{5}\right)$, and misses $x^{\prime \prime}$ and $x$ (or $\left.\left\{x^{\prime \prime}, z, y\right\} \cap\left\{x, y^{\prime}, z_{1}, z_{2}\right\}=\emptyset\right)$, contradicting that the family $S_{x^{\prime \prime} y^{\prime \prime}}\left(\mathcal{H}_{5}^{\prime}\right) \subseteq \mathcal{G}_{3}$ with core $\left\{y, x^{\prime \prime}, z\right\}$ and center $x$. So we will not achieve Case 5, as desired.

By Claims 3.5, 3.6 and 3.8, we have shown that if we reach a 4-uniform family $\mathcal{H}_{1}$ such that $\mathcal{H}_{1}$ is EKR, then there exists a set $X_{1}$ with $\left|X_{1}\right| \leq 5$ and $\{x, y\} \subseteq X_{1}$ such that performing shifts $S_{x^{\prime} y^{\prime}}$ in $[n] \backslash X_{1}$ repeatedly to $\mathcal{H}_{1}$ will result in a stable family satisfying the conditions of Lemma 3.4. This completes the proof of Lemma 3.4.

Lemma 3.9 (Case 2). If we each a 4-uniform family $\mathcal{H}_{2}$ such that $S_{x y}\left(\mathcal{H}_{2}\right)$ is HM at $x$, then there is a set $X_{2}=\left\{x, y, z_{1}, z_{2}, z_{3}\right\}$ such that after a series of shifts $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<\right.$ $y^{\prime}$ and $\left.x^{\prime}, y^{\prime} \in[n] \backslash X_{2}\right)$ to $\mathcal{H}_{2}$, we will reach a stable family $\mathcal{G}$ satisfying the conditions of Theorem 2.2. Moreover, $\left\{z_{1}, z_{2}, z_{3}, y\right\}$ or $\left\{z_{1}, z_{2}, z_{3}, z_{4}^{\prime}\right\} \in \mathcal{G}$. Furthermore, if $\left\{z_{1}, z_{2}, z_{3}, y\right\} \in \mathcal{G}$, then every member in $\mathcal{G}$ contains $x$ or $y$. If $\left\{z_{1}, z_{2}, z_{3}, z_{4}^{\prime}\right\} \in \mathcal{G}$, then every other member in $\mathcal{G}$ contains $x$ or $y$.

Proof. Note that $S_{x y}\left(\mathcal{H}_{2}\right)$ contains exactly one set, say, $G_{0}=\left\{z_{1}, z_{2}, z_{3}, z_{4}\right\}$, that misses $x$. W.l.o.g., let $z_{1}, z_{2}, z_{3} \neq y$. Let $X_{2}:=\left\{x, y, z_{1}, z_{2}, z_{3}\right\}$. By the maximality of $\left|\mathcal{H}_{2}\right|$, we may assume

$$
\left\{G \in\binom{X}{4}:\{x, y\} \subseteq G, G \cap G_{0} \neq \emptyset\right\} \subseteq \mathcal{H}_{2}
$$

If $y \in G_{0}$, that is, $y=z_{4}$, then

$$
\begin{equation*}
\left|\left\{G \in \mathcal{H}_{2}:\{x, y\}\right\}\right|=\binom{n-2}{2} \tag{14}
\end{equation*}
$$

Otherwise, $y \notin G_{0}$. We have

$$
\begin{equation*}
\left|\left\{G \in \mathcal{H}_{2}:\{x, y\}\right\}\right|=4 n-18 \tag{15}
\end{equation*}
$$

In particular, $\left\{x, y, z_{1}, z_{2}\right\},\left\{x, y, z_{1}, z_{3}\right\}$ and $\left\{x, y, z_{2}, z_{3}\right\}$ are in $\mathcal{H}_{2}$. Assume that applying shifts in $[n] \backslash X_{2}$ to $\mathcal{H}_{2}$, we get $\mathcal{H}^{\prime}$, such that $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right)$ is EKR or HM or contained in $\mathcal{J}_{2}$ at center $x^{\prime}$. However, the three sets $\left\{x, y, z_{1}, z_{2}\right\},\left\{x, y, z_{1}, z_{3}\right\}$ and $\left\{x, y, z_{2}, z_{3}\right\}$ are still in $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right)$ and they miss $x^{\prime}$, a contradiction. Thus we will not reach Cases 1-3.

Assume we reach Case 4 as in Remark 2.5, i.e., $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right) \subseteq \mathcal{G}_{2}$ with core $A$. By (14), (15), Claim 3.1 and Fact 2.4 (ii), we have $A=\left\{x, y, x^{\prime}\right\}$. However $\left\{z_{1}, z_{2}, z_{3}\right\} \cap$ $\left\{x, y, x^{\prime}, y^{\prime}\right\}=\emptyset$, after a series of shifts of $[n] \backslash X_{2}$ to $G_{0}=\left\{z_{1}, z_{2}, z_{3}, z_{4}\right\}$, we get the resulting set $G_{0}^{\prime} \in \mathcal{H}^{\prime}$ satisfying that $\left|G_{0}^{\prime} \cap\right| \leq\left\{x, y, x^{\prime}, y^{\prime}\right\} 1$, a contradiction to that $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right) \subseteq \mathcal{G}_{2}$ with core $\left\{x, y, x^{\prime}\right\}$. Thus we will not reach Case 4 .

At last, assume $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right) \subseteq \mathcal{G}_{3}$ as in Remark 2.5 (Case 5) with center and core forming a 4 -set $B$. By Fact 2.4 (iv), $x^{\prime} \in B$. By Claim 3.2 (ii) and (14), (15), there are at least $4 n-18>3 n-12(n>6)$ sets containing $\{x, y\}$, so $\left\{x, y, x^{\prime}\right\} \subset B$. And if $\{x, y\} \subset E$, then the number of sets containing $\{x, y\}$ in $\mathcal{H}^{\prime}$ is at most $2 n-7$, which is smaller than $4 n-18$, this contradicts to (15). Thus the resulting family can only have center $x$ or center $y$. First assume $y \in G_{0}$, that is $y=z_{4}$ and $G_{0}=\left\{y, z_{1}, z_{2}, z_{3}\right\}$. This implies that $\left\{x, z_{1}, z_{2}, z_{3}\right\} \in \mathcal{H}_{2}$. Both $\left\{x, z_{1}, z_{2}, z_{3}\right\}$ and $G_{0}=\left\{z_{1}, z_{2}, z_{3}, z_{4}\right\}$ are stable under shifts $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}\right.$ and $\left.x^{\prime}, y^{\prime} \in[n] \backslash X_{2}\right)$, so both of them are in $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right)$. Since $x, x^{\prime} \notin G_{0}$ and $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right) \subseteq \mathcal{G}_{3}$ with $B \supset\left\{x, y, x^{\prime}\right\}, x$ can not be the center. But if $y$ is the center, since $x^{\prime}, y \notin\left\{x, z_{1}, z_{2}, z_{3}\right\}$, also a contradiction. Next assume $y \notin G_{0}$. Notice that $\left\{z_{1}, z_{2}, z_{3}\right\} \cap\left\{x, y, x^{\prime}, y^{\prime}\right\}=\emptyset$, after a series of shifts of $[n] \backslash X_{2}$ to $G_{0}$, the resulting set $G_{0}^{\prime} \in S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right)$ satisfies that $G_{0}^{\prime} \cap\{x, y\}=\emptyset$, also contradicts that $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right) \subseteq \mathcal{G}_{3}$ with $B \supset\left\{x, y, x^{\prime}\right\}$, hence we will not reach Case 5 .

Notice that if $y \in G_{0}$, we have $\left\{x, z_{1}, z_{2}, z_{3}\right\} \in \mathcal{H}_{2}$ and $G_{0}=\left\{y, z_{1}, z_{2}, z_{3}\right\} \in \mathcal{H}_{2}$. Note that $\left\{z_{1}, z_{2}, z_{3}, y\right\}$ is stable under shifts $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}\right.$ and $\left.x^{\prime}, y^{\prime} \in[n] \backslash X_{2}\right)$, so $G_{0}=\left\{z_{1}, z_{2}, z_{3}, y\right\} \in \mathcal{G}$. In this case, every member in $\mathcal{H}_{2}$ contains $x$ or $y$, Since every member in $\mathcal{H}_{2}$ is stable at $x$ and $y$, every member in $\mathcal{G}$ contains $x$ or $y$. If $y \notin G_{0}$, then $G_{0}^{\prime}=\left\{z_{1}, z_{2}, z_{3}, z_{4}^{\prime}\right\} \in \mathcal{G}$ for some $z_{4}^{\prime} \neq y$, and this is the only set in $\mathcal{G}$ that disjoint from set $\{x, y\}$.

Lemma 3.10 (Case 3). If we each a 4-uniform $\mathcal{H}_{3}$ such that $S_{x y}\left(\mathcal{H}_{3}\right) \subseteq \mathcal{J}_{2}$ at center $x$, kernel $E$ and the set of pages $J$, then there is a set $X_{3}=\left\{x, y, z_{1}, z_{2}, z_{3}\right\}$ such that after a series of shifts $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}\right.$ and $\left.x^{\prime}, y^{\prime} \in[n] \backslash X_{3}\right)$ to $\mathcal{H}_{3}$, we will reach a stable family $\mathcal{G}$ satisfying the conditions of Theorem 2.2 and $G \cap X_{3} \neq \emptyset$ for any $G \in \mathcal{G}$. Moreover, either $\left|G \cap X_{3}\right| \geq 2$ for any $G \in \mathcal{G}$, or $\left|G \cap G^{\prime}\right| \geq 2$ if $G \cap X_{3}=\{x\}$ and $G^{\prime} \cap X_{3}=\{y\}$.

Proof. We will meet one of the following three cases. Case (a): $y \in E$. In this case, let $E=\left\{y, z_{1}, z_{2}\right\}, J=\left\{z_{3}, z_{4}\right\}$ and $X_{3}:=\left\{x, y, z_{1}, z_{2}, z_{3}\right\}$. Case (b): $y \in J$. In this case, let $E=\left\{z_{1}, z_{2}, z_{3}\right\}, J=\left\{y, z_{4}\right\}$ and $X_{3}:=\left\{x, y, z_{1}, z_{2}, z_{3}\right\}$. Case (c): $y \in[n] \backslash(E \cup J \cup\{x\})$. In this case, let $E=\left\{z_{1}, z_{2}, z_{3}\right\}, J=\left\{z_{4}, z_{5}\right\}$ and $X_{3}:=$ $\left\{x, y, z_{1}, z_{2}, z_{3}\right\}$.

In each of the above three cases, by the maximality of $|\mathcal{H}|\left(\left|\mathcal{H}_{3}\right|\right.$ as well $),\left\{x, y, z_{1}, z_{2}\right\}$, $\left\{x, y, z_{1}, z_{3}\right\},\left\{x, y, z_{2}, z_{3}\right\}$ are in $\mathcal{H}_{3}$, and they are stable after a series of shifts in $[n] \backslash X_{3}$, so we will not reach Cases 1-3 after performing shifts in $[n] \backslash X_{3}$. Assume that applying shifts in $[n] \backslash X_{3}$ to $\mathcal{H}_{3}$, we get $\mathcal{H}^{\prime \prime}$, such that $\mathcal{H}^{\prime}:=S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime \prime}\right) \subseteq \mathcal{G}_{2}$ with core $A$. Similarly, by the maximality of $\left|\mathcal{H}_{3}\right|$ and direct computation, we have the following claim:
Claim 3.11. There are at least $\binom{n-2}{2}, 4 n-18,3 n-11$ members that contain $\{x, y\}$ in Cases (a), (b), (c) respectively.

Notice that $\binom{n-2}{2}, 4 n-18,3 n-11>2 n-7$. By Claim 3.1. Claim 3.11 and Fact 2.4 (ii), $A=\left\{x, y, x^{\prime}\right\}$. In Case (a) or (b), we can see that $\left\{y, z_{1}, z_{2}, z_{3}\right\} \in$ $\mathcal{H}^{\prime},\left|\left\{y, z_{1}, z_{2}, z_{3}\right\} \cap A\right|=1$, a contradiction. In Case (c), we have $\left\{z_{1}, z_{2}, z_{3}, z_{4}\right\} \in \mathcal{H}_{3}$, after some shifts in $[n] \backslash X_{3}$, it becomes $F$ in $\mathcal{H}^{\prime}$, and $|F \cap A| \leq 1$, a contradiction to that $\mathcal{H}^{\prime} \subseteq \mathcal{G}_{2}$ with core $\left\{x, y, x^{\prime}\right\}$. Thus we will not reach Case 4 after performing shifts in $[n] \backslash X_{3}$ repeatedly.

At last, we assume that $\mathcal{H}^{\prime}:=S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime \prime}\right) \subseteq \mathcal{G}_{3}$ with center and core forming a 4 -set $B$. By Claim 3.2. Claim 3.11 and Fact 2.4 (iv), we have $\left\{x, y, x^{\prime}\right\} \subseteq B$, and the center of $\mathcal{H}^{\prime}$ must be $x$ or $y$. In Cases (a) and (b), we have $\left\{y, z_{1}, z_{2}, z_{3}\right\} \in \mathcal{H}_{3}$, so in $\mathcal{H}^{\prime}$. Since $x, x^{\prime} \notin\left\{y, z_{1}, z_{2}, z_{3}\right\}, \mathcal{H}^{\prime}$ can not be contained in $\mathcal{G}_{3}$ with $B \supset\left\{x, y, x^{\prime}\right\}$ and center $x$. Since $\left\{x, z_{1}, z_{2}, z_{3}\right\} \in \mathcal{H}_{3}$, so in $\mathcal{H}^{\prime}$ as well. Notice that $y, x^{\prime} \notin\left\{x, z_{1}, z_{2}, z_{3}\right\}$, $\mathcal{H}^{\prime}$ can not be contained in $\mathcal{G}_{3}$ with $B \supset\left\{x, y, x^{\prime}\right\}$ and center $y$. A contradiction. Now consider Case (c). In this case, $\left\{z_{1}, z_{2}, z_{3}, z_{4}\right\} \in \mathcal{H}_{3}$. Because it is stable at $\left\{z_{1}, z_{2}, z_{3}\right\}$ under any shift in $[n] \backslash X_{3}$, the resulting set $\left\{z_{1}, z_{2}, z_{3}, z_{4}^{\prime}\right\}$ does not contain $x$ or $y$. This contradicts that $\mathcal{H}^{\prime} \subseteq \mathcal{G}_{3}$ with $B \supset\left\{x, y, x^{\prime}\right\}$ and center $x$ or $y$.

If Case (a) or (b) happens, then any 4 -set $G \in \mathcal{G}$ satisfies $\left|G \cap X_{3}\right| \geq 2$. If Case (c) happens, since $\left\{z_{1}, z_{2}, z_{3}, z_{4}\right\}$ and $\left\{z_{1}, z_{2}, z_{3}, z_{5}\right\}$ are the only two sets disjoint from $\{x, y\}$ in $S_{x y}\left(\mathcal{H}_{3}\right)$, then every set in $\mathcal{H}_{3}$ (so in $\mathcal{G}$ ) missing $x$ and $y$ must contain $\left\{z_{1}, z_{2}, z_{3}\right\}$. If $x \in G, y \in G^{\prime}$ and $G \cap\left\{z_{1}, z_{2}, z_{3}, y\right\}=G^{\prime} \cap\left\{z_{1}, z_{2}, z_{3}, x\right\}=\emptyset$, let $F, F^{\prime} \in \mathcal{H}_{3}$ such that $G$ and $G^{\prime}$ become their resulting sets in $\mathcal{G}$ after a series of shifts in $[n] \backslash X_{3}$. By the reason that $S_{x y}\left(\mathcal{H}_{3}\right) \subseteq \mathcal{J}_{2}$ with center $x$, kernel $\left\{z_{1}, z_{2}, z_{3}\right\}$ and the set of pages $\left\{z_{4}, z_{5}\right\}$, for any set $H \in \mathcal{H}_{3}$ satisfying that $|H \cap\{x, y\}|=1$
and $H \cap\left\{z_{1}, z_{2}, z_{3}\right\}=\emptyset$, we have $\left\{z_{4}, z_{5}\right\} \subseteq H$. So $\left\{z_{4}, z_{5}\right\} \subseteq F \cap F^{\prime}$, consequently, $\left|G \cap G^{\prime}\right| \geq 2$.

Lemma 3.12 (Case 4). If we reach a 4-uniform $\mathcal{H}_{4}$ such that $S_{x y}\left(\mathcal{H}_{4}\right) \subseteq \mathcal{G}_{2}$ with core $\left\{x, x_{1}, x_{2}\right\}$, then there is a set $X_{4}=\left\{x, y, x_{1}, x_{2}, x_{3}\right\}$ such that after a series of shifts $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}\right.$ and $\left.x^{\prime}, y^{\prime} \in[n] \backslash X_{4}\right)$ to $\mathcal{H}_{4}$, we will reach a stable family $\mathcal{G}$ satisfying the conditions of Theorem 2.2. Moreover, $\left\{x, y, x_{1}, x_{3}\right\} \in \mathcal{G}$ and $G \cap X_{4} \neq \emptyset$ for any $G \in \mathcal{G}$.
Proof. Since $S_{x y}\left(\mathcal{H}_{4}\right) \subseteq \mathcal{G}_{2}$ with core $A$, by Fact 2.4 (ii), we have that $x \in A$ and $y \notin A$. Let $A=\left\{x, x_{1}, x_{2}\right\}$. By the maximality of $\left|\mathcal{H}_{4}\right|$, we may assume

$$
\begin{aligned}
& \left\{G \in\binom{X}{4}:\left\{x_{1}, x_{2}\right\} \subseteq G\right\} \subseteq \mathcal{H}_{4}, \\
& \left\{G \in\binom{X}{4}:\{x, y\} \subseteq G, G \cap\left\{x_{1}, x_{2}\right\} \neq \emptyset\right\} \subseteq \mathcal{H}_{4} .
\end{aligned}
$$

So

$$
\begin{align*}
& \left|\left\{G \in \mathcal{H}_{4}:\left\{x_{1}, x_{2}\right\} \subseteq G\right\}\right|=\binom{n-2}{2}  \tag{16}\\
& \left|\left\{G \in \mathcal{H}_{4}:\{x, y\} \subseteq G, G \cap\left\{x_{1}, x_{2}\right\} \neq \emptyset\right\}\right|=2 n-7 . \tag{17}
\end{align*}
$$

Choose a set $G=\left\{x, y, x_{1}, x_{3}\right\} \in \mathcal{H}_{4}$ and let $X_{4}:=\left\{x, y, x_{1}, x_{2}, x_{3}\right\}$. Since $S_{x y}\left(\mathcal{H}_{4}\right) \subseteq \mathcal{G}_{2}$ with core $\left\{x, x_{1}, x_{2}\right\}$, every member in $\mathcal{H}_{4}$ intersects $X_{4}$. Every member in $\mathcal{H}_{4}$ is stable at every element in $X_{4}$ under shifts $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}\right.$ and $\left.x^{\prime}, y^{\prime} \in[n] \backslash X_{4}\right)$. So $\left\{x, y, x_{1}, x_{3}\right\}$ is in the final stable family $\mathcal{G}$ and $G \cap X_{4} \neq \emptyset$ for any $G \in \mathcal{G}$. What remains is to show that performing shifts $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}\right.$ and $\left.x^{\prime}, y^{\prime} \in[n] \backslash X_{4}\right)$ to $\mathcal{H}_{4}$ will not reach Cases 1-5 in Remark 2.5.

By (16), for any $x^{\prime} \in[n] \backslash X_{4}$, there are at least $\binom{n-3}{2}$ members in $\mathcal{H}_{4}$ missing $x^{\prime}$, so we can not reach Cases 1-3.

Assume $\mathcal{H}^{\prime}:=S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime \prime}\right) \subseteq \mathcal{G}_{2}$ with core $A^{\prime}$. By (16), Fact 2.4 (ii) and Claim 3.1, $A^{\prime}=\left\{x^{\prime}, x_{1}, x_{2}\right\}$. Since $G \in \mathcal{H}^{\prime}$, and $\left|H \cap A^{\prime}\right|=1$, we get a contradiction, hence we will not reach Case 4. At last, assume $\mathcal{H}^{\prime}:=S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime \prime}\right) \subseteq \mathcal{G}_{3}$ with center and core forming a 4 -set $B$. By Fact 2.4 (iv), $x^{\prime} \in B$. By Claim 3.2 (ii) and (16), $\left\{x_{1}, x_{2}\right\} \subseteq B$ and the center must be $x_{1}$ or $x_{2}$. That is $\left\{x_{1}, x_{2}, x^{\prime}\right\} \subset B$. Since $|B|=4,|\{x, y\} \cap B|=0$ or 1. If $|\{x, y\} \cap B|=0$, then the sets containing $\{x, y\}$ in $\mathcal{H}^{\prime}$ must contain center and one point of core $A^{\prime}$, so $d_{\{x, y\}} \leq 3$. If $|\{x, y\} \cap B|=1$, then the sets containing $\{x, y\}$ in $\mathcal{H}^{\prime}$ either contain center or contain core $A^{\prime}$, so $d_{\{x, y\}} \leq n-3+1=n-2$. These members containing $\{x, y\}$ in $\mathcal{H}_{4}$ are also in $\mathcal{H}^{\prime}$, by (17), there are at least $2 n-7$, a contradiction. Hence we can not reach Case 5.
Lemma 3.13 (Case 5). If we reach a 4-uniform $\mathcal{H}_{5}$ such that $S_{x y}\left(\mathcal{H}_{5}\right) \subseteq \mathcal{G}_{3}$ with center and core $E$ forming a 4 -set $B$, then there is a set $X_{5}=\left\{x, y, x_{1}, x_{2}, x_{3}\right\}$ such that after a series of shifts $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}\right.$ and $\left.x^{\prime}, y^{\prime} \in[n] \backslash X_{5}\right)$ to $\mathcal{H}_{5}$, we will reach a stable family $\mathcal{G}$ satisfying the conditions of Theorem 2.2. Furthermore, $\left|G \cap X_{5}\right| \geq 2$ for each $G \in \mathcal{G}$.

Proof. For $S_{x y}\left(\mathcal{H}_{5}\right)$, we will meet one of the following three cases. Case (a): $x$ is the center, $y \in E$, and $E=\left\{y, x_{1}, x_{2}\right\}$. In this case, we may assume that $\left\{y, x_{1}, x_{2}, x_{3}\right\} \in$ $S_{x y}\left(\mathcal{H}_{5}\right)$ for some $x_{3} \in[n] \backslash B$. Let $X_{5}:=\left\{x, y, x_{1}, x_{2}, x_{3}\right\}$. Case (b): $x$ is the center, $y \notin E$, and $E=\left\{x_{1}, x_{2}, x_{3}\right\}$. In this case, let $X_{5}:=\left\{x, y, x_{1}, x_{2}, x_{3}\right\}$. Case (c): $x_{1}$ is the center, $x \in E$, and $E=\left\{x, x_{2}, x_{3}\right\}, y \in[n] \backslash B$. In this case, let $X_{5}:=\left\{x, y, x_{1}, x_{2}, x_{3}\right\}$. We first observe that $\left|G \cap X_{5}\right| \geq 2$ for each $G \in \mathcal{H}_{5}$ in each case.

First we consider Case (a). In this case, a member in $\mathcal{H}_{5}$ must contain $x$ or $y$. By the maximality of $\left|\mathcal{H}_{5}\right|$, we may assume

$$
\left\{G \in\binom{X}{4}:\{x, y\} \subseteq G\right\} \subseteq \mathcal{H}_{5}
$$

So

$$
\begin{equation*}
\left|\left\{G \in \mathcal{H}_{5}:\{x, y\} \subseteq G\right\}\right|=\binom{n-2}{2} \tag{18}
\end{equation*}
$$

Performing $S_{x^{\prime} y^{\prime}}$ in $[n] \backslash X_{5}$ to $\mathcal{H}_{5}$ will not reach Cases 1-3 since there are at least $\binom{n-3}{2}$ members that containing $\{x, y\}$ and missing $x^{\prime}$ in $\mathcal{H}_{5}$ and these sets are stable after $S_{x^{\prime} y^{\prime}}$ in $[n] \backslash X_{5}$ (by (18p).

Assume that $\mathcal{H}^{\prime}:=S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime \prime}\right) \subseteq \mathcal{G}_{2}$ with core $A$. By (18), Fact 2.4 (ii) and Claim 3.1, $A=\left\{x^{\prime}, x, y\right\}$. Since $S_{x y}\left(\mathcal{H}_{5}\right)$ is not EKR, $\left\{y, x_{1}, x_{2}, x_{3}\right\} \in S_{x y}\left(\mathcal{H}_{5}\right)$, $\left\{x, x_{1}, x_{2}, x_{3}\right\} \in \mathcal{H}_{5}$, so in $\mathcal{H}^{\prime}$. However, $\left|\left\{x, x_{1}, x_{2}, x_{3}\right\} \cap A\right|=1$, this is a contradiction, hence we will not reach Case 4. Assume that $\mathcal{H}^{\prime}:=S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime \prime}\right) \subseteq \mathcal{G}_{3}$ with center and core forming a 4 -set $B^{\prime}$. By (18), Fact 2.4 (iv) and Claim 3.2 (ii), $\left\{x, y, x^{\prime}\right\} \subseteq B^{\prime}$, and the center is either $x$ or $y$. In either case, the existence of $\left\{x, x_{1}, x_{2}, x_{3}\right\}$ and $\left\{y, x_{1}, x_{2}, x_{3}\right\}$ will lead to a contradiction. Hence we will not reach Case 5.

Next we consider Case (b). By the maximality of $\left|\mathcal{H}_{5}\right|$, we may assume that

$$
\left\{G \in\binom{X}{4}:\{x, y\} \subseteq G \text { and } G \cap E \neq \emptyset\right\} \subseteq \mathcal{H}_{5}
$$

and

$$
\left\{G \in\binom{X}{4}:\left\{x_{1}, x_{2}, x_{3}\right\} \subseteq G\right\} \subseteq \mathcal{H}_{5}
$$

In particular, $\left\{x, x_{1}, x_{2}, x_{3}\right\} \in \mathcal{H}_{5}$ and $\left\{y, x_{1}, x_{2}, x_{3}\right\} \in \mathcal{H}_{5}$. Computing directly, we have

$$
\begin{equation*}
\left|\left\{G \in \mathcal{H}_{5}:\{x, y\} \subseteq G, G \cap E \neq \emptyset\right\}\right|=3 n-12 \tag{19}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\left\{G \in \mathcal{H}_{5}:\left\{x_{1}, x_{2}, x_{3}\right\} \subseteq G\right\}\right|=n-3 \tag{20}
\end{equation*}
$$

Since $\left\{x, y, x_{1}, x_{2}\right\},\left\{x, y, x_{1}, x_{3}\right\},\left\{x, y, x_{2}, x_{3}\right\} \in \mathcal{H}_{5}$ and these sets miss $x^{\prime}$ and are stable after shifts $S_{x^{\prime} y^{\prime}}\left(x^{\prime}<y^{\prime}\right.$ and $\left.x^{\prime}, y^{\prime} \in[n] \backslash X_{5}\right)$, we will not reach Cases 1-3.

Assume $\mathcal{H}^{\prime}:=S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime \prime}\right) \subseteq \mathcal{G}_{2}$ with core $A$, where $x^{\prime}<y^{\prime}$ and $x^{\prime}, y^{\prime} \in[n] \backslash X_{5}$. By (19), Fact 2.4 (ii) and Claim 3.1, we have $A=\left\{x^{\prime}, x, y\right\}$. However $\left\{x, x_{1}, x_{2}, x_{3}\right\} \in \mathcal{H}^{\prime}$ and $\left|\left\{x, x_{1}, x_{2}, x_{3}\right\} \cap A\right|=1$, a contradiction, so we will not reach Case 4.

Assume that $\mathcal{H}^{\prime}:=S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime \prime}\right) \subseteq \mathcal{G}_{3}$ with center and core forming a 4 -set $B^{\prime}$. By Fact 2.4 (iv), $x^{\prime} \in B^{\prime}$. Equation (19) and Claim 3.2 (i) imply that the center must be $x$ or $y$.

By Claim 3.3 and (20), either $\left\{x_{1}, x_{2}, x_{3}\right\} \subset B^{\prime}$ or $\left|\left\{x_{1}, x_{2}, x_{3}\right\} \cap B^{\prime}\right|=2$ and one of $\left\{x_{1}, x_{2}, x_{3}\right\}$ is the center. But it's impossible to satisfy both conditions in the previous paragraph and this paragraph, hence we will not reach Case 5.

At last we consider Case (c). By the maximality of $\left|\mathcal{H}_{5}\right|$, we may assume

$$
\left\{G \in\binom{X}{4}:\left\{x_{1}, x_{2}\right\} \subseteq G\right\} \subseteq \mathcal{H}_{5} \text { and }\left\{G \in\binom{X}{4}:\left\{x_{1}, x_{3}\right\} \subseteq G\right\} \subseteq \mathcal{H}_{5}
$$

By direct computation,

$$
\begin{align*}
& \left|\left\{G \in \mathcal{H}_{5}:\left\{x_{1}, x_{2}\right\} \subseteq G\right\}\right|=\binom{n-2}{2}  \tag{21}\\
& \left|\left\{G \in \mathcal{H}_{5}:\left\{x_{1}, x_{3}\right\} \subseteq G\right\}\right|=\binom{n-2}{2} \tag{22}
\end{align*}
$$

Since there are $\binom{n-3}{2}$ sets containing $\left\{x_{1}, x_{2}\right\}$ but missing $x^{\prime}$, after performing $S_{x^{\prime} y^{\prime}}$ in $[n] \backslash X_{5}$ to $\mathcal{H}_{5}$, we will not reach Case 1-3.

If we reach Cases 4 , that is, after performing shifts in $[n] \backslash X_{5}$ to $\mathcal{H}_{5}$ repeatedly, $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right) \subseteq \mathcal{G}_{2}$ with core $A$. By (21), 22), Fact 2.4 (ii) and Claim 3.1, $x^{\prime}, x_{1}, x_{2}, x_{3} \in$ $A$, but $|A|=3$, a contradiction. If we reach Case 5, that is $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right) \subseteq \mathcal{G}_{3}$ with the center and the core forming a 4 -set $B^{\prime}$. By (21), (22), Fact 2.4 (iv) and Claim 3.1, $B^{\prime}=\left\{x_{1}, x_{2}, x_{3}, x^{\prime}\right\}$, and $x_{1}$ is the center. Recall that $\left\{x, y, x_{2}, x_{3}\right\} \in \mathcal{H}_{5}$, also in $S_{x^{\prime} y^{\prime}}\left(\mathcal{H}^{\prime}\right)$, but $\left\{x, y, x_{2}, x_{3}\right\} \cap\left\{x_{1}, x^{\prime}, y^{\prime}\right\}=\emptyset$, a contradiction, hence we cannot reach Case 5.

As remarked earlier, $\left|G \cap X_{5}\right| \geq 2$ for each $G \in \mathcal{H}_{5}$. Note that performing shifts in $[n] \backslash X_{5}$ to $\mathcal{H}_{5}$ keeps this property, so $\left|G \cap X_{5}\right| \geq 2$ for each $G \in \mathcal{G}$.

By Lemmas 3.4 to 3.13, we have shown that if one of Case 1-5 happens, then there exists a set $X_{i}$ with $\left|X_{i}\right| \leq 5$ and $\{x, y\} \subseteq X_{i}$ such that performing shifts in $[n] \backslash X_{i}$ to $\mathcal{H}_{i}$ will not result in any case of Case 1-5, so the final family is a stable family satisfying the conditions in Theorem 2.2. Furthermore, $G \cap X_{i} \neq \emptyset$ for any set $G$ in the final family. So we complete the proof of Lemma 2.6 .

### 3.2 Proof of Lemma 2.7

Proof. We first consider $k \geq 5$. In this case, we have $\left|X_{i}\right| \leq 3$ and $\left|Y_{i}\right| \geq 2 k-3$.
We first prove (ii). Assume on the contrary that there are $G$ and $G^{\prime} \in \mathcal{G}$ such that $G \cap G^{\prime} \cap Y=\emptyset$ and let $\left|G \cap G^{\prime}\right|$ be the minimum among all pairs of sets in $\mathcal{G}$
not intersecting in $Y$. Clearly $\left|G \cap G^{\prime} \cap([n] \backslash Y)\right| \geq 1$. Note that $\left|\left(G \cup G^{\prime}\right) \cap Y_{i}\right| \leq$ $\left|G \cap Y_{i}\right|+\left|G^{\prime} \cap Y_{i}\right| \leq 2 k-4$ (since $|G \cap([n] \backslash Y)| \geq 1$ and $\left|G \cap X_{i}\right| \geq 1$, so $\left|G \cap Y_{i}\right| \leq k-2$, same for $\left.G^{\prime}\right)$. But $\left|Y_{i}\right| \geq 2 k-3$, so there exists a point $a \in Y_{i} \backslash\left(G \cup G^{\prime}\right)$. Pick any point $b \in G \cap G^{\prime} \cap([n] \backslash Y)$, we have $a<b$. Notice that $\mathcal{G}$ is stable on $[n] \backslash X_{i}$, so $G^{\prime \prime}:=\left(G^{\prime} \backslash\{b\}\right) \cup\{a\} \in \mathcal{G}$. Then $G \cap G^{\prime \prime} \cap Y=\emptyset$ and $\left|G \cap G^{\prime \prime}\right|<\left|G \cap G^{\prime}\right|$, contradicting the minimality of $\left|G \cap G^{\prime}\right|$.

For (i), assume on the contrary, that $\mathcal{A}_{1} \neq \emptyset$. Let $\{x\} \in \mathcal{A}_{1}$, then there is a set $G \in \mathcal{G}$ such that $G \cap Y=\{x\}$. By (ii), for any another set $G^{\prime} \in \mathcal{G}$ we have $G \cap G^{\prime} \cap Y \neq \emptyset$, so $x \in G^{\prime}$. This implies that $\mathcal{G}$ is EKR, a contradiction, so $\mathcal{A}_{1}=\emptyset$.

Next consider for $k=4$. In this case, for $1 \leq i \leq 5,\left|X_{i}\right|=5$ and $\left|Y_{i}\right|=9-5=4$, and for $i=6,\left|X_{i}\right|=0$ and $\left|Y_{i}\right|=9$.

Claim 3.14. If $G$ and $G^{\prime}$ in $\mathcal{G}$ satisfies that $\left|Y_{i} \backslash\left(G \cup G^{\prime}\right)\right| \geq\left|G \cap G^{\prime} \cap([n] \backslash Y)\right|$, then $G \cap G^{\prime} \cap Y \neq \emptyset$.

Proof. If $G \cap G^{\prime} \cap Y=\emptyset$, then $D:=G \cap G^{\prime} \cap([n] \backslash Y) \neq \emptyset$. Since $\left|Y_{i} \backslash\left(G \cup G^{\prime}\right)\right| \geq$ $\left|G \cap G^{\prime} \cap([n] \backslash Y)\right|$, there is a subset $D^{\prime} \subseteq Y_{i} \backslash\left(G \cup G^{\prime}\right)$ with size $\left|D^{\prime}\right|=|D|$. By the definition of $Y_{i}$, all numbers in $D^{\prime}$ are smaller than $D$. Since $\mathcal{G}$ is stable on $[n] \backslash X_{i}$, $F:=\left(G^{\prime} \backslash D\right) \cup D^{\prime} \in \mathcal{G}$. However $G \cap F=\emptyset$, a contradiction to the intersecting property of $\mathcal{G}$. So $G \cap G^{\prime} \cap Y \neq \emptyset$.

Claim 3.15. $\left|\mathcal{A}_{1}\right| \leq 1 ; \mathcal{A}_{2}$ and $\mathcal{A}_{4}$ are intersecting.
Proof. Obviously, $\mathcal{A}_{4}$ is intersecting. Assume that $\left|\mathcal{A}_{1}\right| \geq 2$ and $\left\{x_{1}\right\},\left\{x_{2}\right\} \in \mathcal{A}_{1}$. Then there are $G$ and $G^{\prime}$ in $\widetilde{\mathcal{A}_{1}}$ such that $G \cap Y=\left\{x_{1}\right\}$ and $G^{\prime} \cap Y=\left\{x_{2}\right\}$. Since any set in $\mathcal{G}$ intersects with $X_{i}$ (for $\left.i \in[5]\right), x_{1}, x_{2} \in X_{i}$. So $1 \leq\left|G \cap G^{\prime} \cap([n] \backslash Y)\right| \leq$ $3<4=\left|Y_{i} \backslash\left(G \cap G^{\prime}\right)\right|$. By Claim 3.14, $G \cap G^{\prime} \cap Y \neq \emptyset$, a contradiction. Hence, $\left|\mathcal{A}_{1}\right| \leq 1$. Let $G$ and $G^{\prime}$ be in $\widetilde{\mathcal{A}_{2}}$. Then $\left|G \cap G^{\prime} \cap([n] \backslash Y)\right| \leq 2$. Since $\left|G \cap X_{i}\right| \geq 1$ and $\left|G^{\prime} \cap X_{i}\right| \geq 1$ (for $i \in[5]$ ), then $\left|Y_{i} \backslash\left(G \cup G^{\prime}\right)\right| \geq 2$. By Claim 3.14, $G \cap G^{\prime} \cap Y \neq \emptyset$, that is $\mathcal{A}_{2}$ is intersecting, as desired.

Claim 3.16. $\mathcal{A}_{1}=\emptyset$.
Proof. By Claim 3.15, $\left|\mathcal{A}_{1}\right| \leq 1$. We may assume on the contrary that $\mathcal{A}_{1}=\{\{x\}\}$ for some $x \in X_{i}$. For any $G \in \widetilde{\mathcal{A}_{1}}$ and $G^{\prime} \in \widetilde{\mathcal{A}_{j}}$ (for $j=2,3,4$ ), $G$ and $G^{\prime}$ satisfy the condiction of Claim 3.14, so $G \cap G^{\prime} \cap Y \neq \emptyset$, this implies that $x \in G^{\prime}$ and hence $\mathcal{G}$ is EKR, a contradiction.

Claim 3.17. $\mathcal{A}_{2}$ and $\mathcal{A}_{3}$ are cross-intersecting.
Proof. Let $G \in \widetilde{\mathcal{A}_{2}}$ and $G^{\prime} \in \widetilde{\mathcal{A}_{3}}$. Then $\left|G \cap G^{\prime} \cap([n] \backslash Y)\right| \leq 1$. Since any set in $\mathcal{G}$ intersects with $X_{i}$ (for $\left.i \in[5]\right),\left|Y_{i} \backslash\left(G \cup G^{\prime}\right)\right| \geq 1$. By Claim 3.14, $G \cap G^{\prime} \cap Y \neq \emptyset$, that is $\mathcal{A}_{2}$ and $\mathcal{A}_{3}$ are cross-intersecting, as desired.

Claim 3.18. $\mathcal{A}_{3}$ is intersecting.

Proof. Assume on the contrary, that there exist $A, A^{\prime} \in \mathcal{A}_{3}$ and $G, G^{\prime} \in \widetilde{\mathcal{A}_{3}}$ such that $G \cap Y=A, G^{\prime} \cap Y=A^{\prime}$ and $A \cap A^{\prime}=\emptyset$, in other words, $G \cap G^{\prime} \cap Y=\emptyset$ and $\left|G \cap G^{\prime} \cap([n] \backslash Y)\right|=1$. If $\left|\left(G \cup G^{\prime}\right) \cap Y_{i}\right| \leq 3$, by Claim 3.14, $G \cap G^{\prime} \cap Y \neq \emptyset$, a contradiction. Hence we only need to consider the following case : $\left|A \cap X_{i}\right|=$ $1,\left|A \cap Y_{i}\right|=2,\left|A^{\prime} \cap X_{i}\right|=1$ and $\left|A^{\prime} \cap Y_{i}\right|=2$. Now we show the conclusion for each case of Lemma 2.6. All sets below are inherited from the proof of Lemma 2.6 for each cooresponding case.

If we meet Cases 1 in Lemma 2.6, then by Lemma 3.4, we have that $X_{1}=$ $\left\{x, y, y^{\prime}, z_{1}, w_{1}\right\}$ or $X_{1}=\left\{x, y, y^{\prime}, z_{2}, w_{2}\right\}$, and we may assume that $G \cap X_{i}=\{x\}$ and $G^{\prime} \cap X_{i}=\{y\}$. Respectively, $\left\{y, y^{\prime}, z_{1}, w_{1}\right\}$ or $\left\{x, y^{\prime}, z_{2}, w_{2}\right\} \in \mathcal{G}$, which is disjoint from $G$ or $G^{\prime}$. A contradiction to the intersecting property of $\mathcal{G}$.

If we meet Cases 2 in Lemma 2.6, then by Lemma 3.9, we have that $X_{2}=$ $\left\{x, y, z_{1}, z_{2}, z_{3}\right\}$, and either $\left\{z_{1}, z_{2}, z_{3}, y\right\} \in \mathcal{G}$ or $\left\{z_{1}, z_{2}, z_{3}, z_{4}^{\prime}\right\} \in \mathcal{G}$ for some $y \neq z_{4}^{\prime}$. Furthermore, if $\left\{z_{1}, z_{2}, z_{3}, y\right\} \in \mathcal{G}$, then every member in $\mathcal{G}$ contains $x$ or $y$. So we may assume that $G \cap X_{i}=\{x\}$ and $G^{\prime} \cap X_{i}=\{y\}$. Then $\left\{z_{1}, z_{2}, z_{3}, y\right\} \cap G=\emptyset$, a contradiction. If $\left\{z_{1}, z_{2}, z_{3}, z_{4}^{\prime}\right\} \in \mathcal{G}$, then every other member in $\mathcal{G}$ contains $x$ or $y$, we may assume that $G \cap X_{i}=\{x\}$ and $G^{\prime} \cap X_{i}=\{y\}$. Since $\mathcal{G}$ is stable, we may assume that $z_{4}^{\prime} \in Y_{i}$. Recall that $\left|G \cap Y_{i}\right|=\left|G^{\prime} \cap Y_{i}\right|=2$, hence $\left\{z_{1}, z_{2}, z_{3}, z_{4}^{\prime}\right\}$ must be disjoint from $G$ or $G^{\prime}$, a contradiction.

If we meet Cases 3 in Lemma 2.6, then by Lemma 3.10, $\left|G \cap G^{\prime}\right| \geq 2$, a contradiction.

If we meet Cases 4 in Lemma 2.6, then by Lemma 3.12, we have that $X_{4}=$ $\left\{x, y, x_{1}, x_{2}, x_{3}\right\},\left\{x, y, x_{1}, x_{3}\right\} \in \mathcal{G}$ and $S_{x y}\left(\mathcal{H}_{4}\right) \subseteq \mathcal{G}_{2}$ with core $\left\{x, x_{1}, x_{2}\right\}$. So for every set $F$ in $\mathcal{H}_{4}$, either $\left|F \cap\left\{x, x_{1}, x_{2}\right\}\right| \geq 2$, or $F \cap\left\{x, x_{1}, x_{2}\right\}=\left\{x_{1}\right\}$ and $y \in F$, or $F \cap\left\{x, x_{1}, x_{2}\right\}=\left\{x_{2}\right\}$ and $y \in F$. In all cases, $\left|F \cap X_{4}\right| \geq 2$. Performing shifts in $[n] \backslash X_{4}$ will not change these properties, hence every set in $\mathcal{G}$ also has the same properties, in particular, $G$ and $G^{\prime}$ do. This makes a contradiction to $\left|G \cap X_{4}\right|=\left|G^{\prime} \cap X_{4}\right|=1$.

Assume that we meet Case 5 in Lemma 2.6, then by Lemma 3.13, we have that $\left|G \cap X_{5}\right| \geq 2$ for each $G \in \mathcal{G}$. This makes a contradiction to $\left|G \cap X_{5}\right|=\left|G^{\prime} \cap X_{5}\right|=1$.

At last, assume that we will not meet any of Cases 1-5 in Lemma 2.6 if we perform shifts repeatedly to $\mathcal{G}$. In this case, $Y=[2 k]$. Assume on the contrary, and let $G$ and $G^{\prime} \in \mathcal{G}$ such that $G \cap G^{\prime} \cap Y=\emptyset$ and $\left|G \cap G^{\prime}\right|$ is the minimum among all pairs of sets in $\mathcal{G}$ not intersecting in $Y$. Then $\left|G \cap G^{\prime} \cap(X \backslash Y)\right| \geq 1$. Consequently, $\left|\left(G \cup G^{\prime}\right) \cap Y\right| \leq|G \cap Y|+\left|G^{\prime} \cap Y\right| \leq 2 k-2$ since $|G \cap Y| \leq k-1$ and $\left|G^{\prime} \cap Y\right| \leq k-1$. So there exists a point $a \in Y \backslash\left(G \cup G^{\prime}\right)$. Pick any point $b \in G \cap G^{\prime} \cap(X \backslash Y)$. Note that $a<b$, then $G^{\prime \prime}:=\left(G^{\prime} \backslash\{b\}\right) \cup\{a\} \in \mathcal{G}$ since $\mathcal{G}$ is stable. It is easy to see that $G \cap G^{\prime \prime} \cap Y=\emptyset$ and $\left|G \cap G^{\prime \prime}\right|<\left|G \cap G^{\prime}\right|$, contradicting the minimality of $\left|G \cap G^{\prime}\right|$.

Since $\mathcal{G}$ is intersecting, $\mathcal{A}_{2}$ and $\mathcal{A}_{4}$ are cross-intersecting, and $\mathcal{A}_{3}$ and $\mathcal{A}_{4}$ are crossintersecting. Combining with Claims 3.15, 3.17 and 3.18, we have completed the proof of (ii).

## 4 Concluding remarks

It is natural to ask what is the maximum size of a $k$-uniform intersecting family $\mathcal{F}$ with $\tau(\mathcal{F}) \geq 3$. About this problem, Frankl [3] gave an upper bound for sufficient large $n$. To introduce the result, we need the following construction.

Construction 4.1. Let $x \in[n], Y \subseteq[n]$ with $|Y|=k$, and $Z \subseteq[n]$ with $|Z|=k-1$, $x \notin Y \cup Z, Z \cap Y=\emptyset$ and $Y_{0}=\left\{y_{1}, y_{2}\right\} \subseteq Y$. Define
$\mathcal{G}=\{A \subseteq[n]: x \in A, A \cap Y \neq \emptyset$ and $A \cap Z \neq \emptyset\} \cup\left\{Y, Z \cup\left\{y_{1}\right\}, Z \cup\left\{y_{2}\right\},\left\{x, y_{1}, y_{2}\right\}\right\}$, $F P(n, k)=\{F \subseteq[n]:|F|=k, \exists G \in \mathcal{G}$ s.t., $G \subseteq F\}$.

It is easy to see that $F P(n, k)$ is intersecting and $\tau(F P(n, k))=3$.
Theorem 4.2 (Frankl [3]). Let $k \geq 3$ and $n$ be sufficiently large integers. Let $\mathcal{H}$ be an $n$-vertex $k$-uniform family with $\tau(\mathcal{H}) \geq 3$. Then $|\mathcal{H}| \leq|F P(n, k)|$. Moreover, for $k \geq 4$, the equality holds only for $\mathcal{H}=F P(n, k)$.

It is interesting to consider what is the maximum $k$-uniform intersecting families with covering number $s \geq 4$.

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