Stability of intersecting families^{*}

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Abstract

The celebrated Erdős–Ko–Rado theorem [1] states that the maximum intersecting k-uniform family on [n] is a full star if $n \ge 2k + 1$. Furthermore, Hilton-Milner [9] showed that if an intersecting k-uniform family on [n] is not a subfamily of a full star, then its maximum size achieves only on a family isomorphic to $HM(n,k) := \left\{ G \in {[n] \choose k} : 1 \in G, G \cap [2, k+1] \neq \emptyset \right\} \cup \left\{ [2, k+1] \right\}$ if n > 2k and $k \ge 4$, and there is one more possibility in the case of k = 3. Han and Kohayakawa [8] determined the maximum intersecting k-uniform family on [n] which is neither a subfamily of a full star nor a subfamily of the extremal family in Hilton-Milner theorm, and they asked what is the next maximum intersecting k-uniform family on [n]. Kostochka and Mubayi [11] gave the answer for large enough n. In this paper, we are going to get rid of the requirement that n is large enough in the result by Kostochka and Mubayi [11] and answer the question of Han and Kohayakawa [8].

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1 Introduction

For a positive interge n, let $[n] = \{1, 2, ..., n\}$ and $2^{[n]}$ be the family of all subsets of [n]. An *i*-element subset $A \subseteq [n]$ is called an *i*-set. For $0 \leq k \leq n$, let $\binom{[n]}{k}$ denote the collection of all *k*-sets of [n]. A family $\mathcal{F} \subseteq \binom{[n]}{k}$ is called *k*-uniform. For a family $\mathcal{F} \subseteq 2^{[n]}$, we say \mathcal{F} is *intersecting* if for any two distinct sets F and F' in \mathcal{F} we have $|F \cap F'| \geq 1$. In this paper, we always consider a *k*-uniform intersecting family on

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[n]. The following celebrated theorem of Erdős–Ko–Rado determines the maximum intersecting family.

For $x \in [n]$ denote $\mathcal{F}_x := \{F \in {[n] \choose k} : x \in F\}$ by the *full star* centered at x. We say \mathcal{F} is EKR if \mathcal{F} is contained in a full star.

Theorem 1.1 (Erdős–Ko–Rado [1]). Let $n \ge 2k$ be integer and \mathcal{F} be a k-uniform intersecting family of subsets of [n]. Then

$$|\mathcal{F}| \le \binom{n-1}{k-1}.$$

Moreover, when n > 2k, equality holds if and only if \mathcal{F} is a full star.

The theorem of Hilton-Milner determines the maximum size of non-EKR families.

Theorem 1.2 (Hilton–Milner [9]). Let $k \ge 2$ and $n \ge 2k$ be integers and $\mathcal{F} \subseteq {\binom{[n]}{k}}$ be an intersecting family. If \mathcal{F} is not EKR, then

$$|\mathcal{F}| \le \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Moreover, for n > 2k and $k \ge 4$, equality holds if and only if \mathcal{F} is isomorphic to

$$HM(n,k) := \left\{ G \in \binom{[n]}{k} : 1 \in G, G \cap [2,k+1] \neq \emptyset \right\} \cup \left\{ [2,k+1] \right\}.$$

For the case k = 3, there is one more possibility, namely

$$\mathcal{T}(n,3) := \left\{ F \in \binom{[n]}{3} : |F \cap [3]| \ge 2 \right\}.$$

We say a family \mathcal{F} is HM if it is isomorphic to a subfamily of HM(n, k). We say that 1 is the *center* of HM(n, k).

Let $E \subseteq [n]$ be an *i*-set and $x \in [n]$. We define

$$\mathcal{G}_i := \left\{ G \in \binom{[n]}{k} : E \subseteq G \right\} \cup \left\{ G \in \binom{[n]}{k} : x \in G \text{ and } G \cap E \neq \emptyset \right\}.$$

We call x the *center*, and E the *core* of \mathcal{G}_i for $i \geq 3$. With a slight tweaking, we call $\{x\} \cup E$ the *core* of \mathcal{G}_2 . Note that $\mathcal{G}_k = HM(n, k)$.

For a (k-1)-set E, a point $x \in [n] \setminus E$, and an *i*-set $J \subset [n] \setminus (E \cup \{x\})$, we denote

$$\mathcal{J}_{i} := \left\{ G \in \binom{[n]}{k} : E \subseteq G \text{ and } G \cap J \neq \emptyset \right\} \cup \left\{ G \in \binom{[n]}{k} : J \cup \{x\} \subseteq G \right\}$$
$$\cup \left\{ G \in \binom{[n]}{k} : x \in G, G \cap E \neq \emptyset \right\}.$$

We call x the center, E the kernel, and J the set of pages.

For two k-sets E_1 and $E_2 \subseteq [n]$ with $|E_1 \cap E_2| = k - 2$, and $x \in [n] \setminus (E_1 \cup E_2)$, we define

$$\mathcal{K}_2 := \{ G \in \binom{[n]}{k} : x \in G, G \cap E_1 \neq \emptyset \text{ and } G \cap E_2 \neq \emptyset \} \cup \{ E_1, E_2 \},\$$

and call x the *center* of \mathcal{K}_2 .

In [8], Han and Kohayakawa obtained the size of a maximum non-EKR, non-HM intersecting family.

Theorem 1.3 (Han–Kohayakawa [8]). Suppose $k \geq 3$ and $n \geq 2k + 1$ and let \mathcal{H} be an intersecting k-uniform family on [n]. Furthermore, assume that \mathcal{H} is neither EKR nor HM, if k = 3, $\mathcal{H} \not\subseteq \mathcal{G}_2$. Then

$$|\mathcal{H}| \le \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} + 2.$$

For k = 4, equality holds if and only if $\mathcal{H} = \mathcal{J}_2$, \mathcal{G}_2 or \mathcal{G}_3 . For every other k, equality holds if and only if $\mathcal{H} = \mathcal{J}_2$.

Han and Kohayakawa [8] proposed the following question.

Question 1.4. Let $n \ge 2k + 1$. What is the maximum size of an intersecting family \mathcal{H} that is neither EKR nor HM, and $\mathcal{H} \not\subseteq \mathcal{J}_2$ (in addition $\mathcal{H} \not\subseteq \mathcal{G}_2$ and $\mathcal{H} \not\subseteq \mathcal{G}_3$ if k = 4)?

Regarding this question, Kostochka and Mubayi [11] showed that the answer is $|\mathcal{J}_3|$ for sufficiently large n. In fact they proved that the maximum size of an intersecting family that is neither EKR, nor HM, nor contained in \mathcal{J}_i for each $i, 2 \leq i \leq k-1$ (nor in $\mathcal{G}_2, \mathcal{G}_3$ for k = 4) is $|\mathcal{K}_2|$ for all large enough n. In paper [11], they also established the structure of almost all intersecting 3-uniform families. Sometimes, it is relatively easier to get extremal families under the assumption that n is large enough. For example, Erdős matching conjecture [2] states that for a k-uniform family \mathcal{F} on finite set $[n], |\mathcal{F}| \leq \max\{\binom{k(s+1)-1}{k}, \binom{n}{k} - \binom{n-s}{k}\}$ if there is no s+1 pairwise disjoint members of \mathcal{F} and $n \geq (s+1)k$, and it was proved to be true for large enough n in [2]. There has been a lot of recent studies for small n. Up to now, the best condition on n was given by Frankl in [5, 6] that $n \geq k(2s+1) - s$, for $(s+1)k \leq n \leq k(2s+1) - s - 1$.

As mentioned by Han and Kohayakawa in [8], for $k \ge 4$, the bound in Theorem 1.3 can be deduced from Theorem 3 in [9] which was established by Hilton and Milner in 1967. However, family \mathcal{H} in Question 1.4 does not satisfy the hypothesis of Theorem 3 in [9] for $k \ge 4$. This makes Question 1.4 more interesting. In this paper, we answer Question 1.4. We are going to get rid of the requirement that n is large enough in the result by Kostochka and Mubayi [11]. As in the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3, we will apply the shifting method. The main difficulty in our proof is to guarantee that we can get a *stable* family which is not EKR, not HM, $\not\subseteq \mathcal{J}_2$ (in addition $\not\subseteq \mathcal{G}_2, \not\subseteq \mathcal{G}_3$ if k = 4) after performing a series of shifts to a family which is not EKR, not HM, $\not\subseteq \mathcal{J}_2$ (in addition $\not\subseteq \mathcal{G}_2, \not\subseteq \mathcal{G}_3$ if k = 4). Our main result is as follows.

Theorem 1.5. Let $k \ge 4$ and $\mathcal{H} \subseteq {\binom{[n]}{k}}$ be an intersecting family which is neither *EKR* nor *HM*. Furthermore, $\mathcal{H} \not\subseteq \mathcal{J}_2$ (in addition $\mathcal{H} \not\subseteq \mathcal{G}_2$ and $\mathcal{H} \not\subseteq \mathcal{G}_3$ if k = 4). (*i*) If $2k + 1 \le n \le 3k - 3$, then

$$|\mathcal{H}| \le \binom{n-1}{k-1} - 2\binom{n-k-1}{k-1} + \binom{n-k-3}{k-1} + 2$$

Moreover, the equality holds only for $\mathcal{H} = \mathcal{K}_2$ if $k \geq 5$, and $\mathcal{H} = \mathcal{K}_2$ or \mathcal{J}_3 if k = 4. (ii) If $n \geq 3k - 2$, then

$$|\mathcal{H}| \le \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} - \binom{n-k-3}{k-3} + 3.$$

Moreover, for k = 5, the equality holds only for $\mathcal{H} = \mathcal{J}_3$ or \mathcal{G}_4 . For every other k, equality holds only for $\mathcal{H} = \mathcal{J}_3$.

In Section 2, we will give the proof of Theorem 1.5. The proofs of some crucial lemmas for the proof of Theorem 1.5 are given in Section 3.

2 Proof of Theorem 1.5

In this section, we always assume that \mathcal{H} is a maximum intersecting family which satisfies the conditions of Theorem 1.5, that is, \mathcal{H} is not EKR, not HM, $\mathcal{H} \not\subseteq \mathcal{J}_2$ (in addition $\mathcal{H} \not\subseteq \mathcal{G}_2, \mathcal{H} \not\subseteq \mathcal{G}_3$ if k = 4). By direct calculation, we have the following fact.

Fact 2.1. (i) Suppose that there is $x \in [n]$ such that there are only 2 sets, say, E_1 and $E_2 \in \mathcal{H}$ missing x. If $|E_1 \cap E_2| = k - i$ and $i \geq 2$, then

$$|\mathcal{H}| \le \binom{n-1}{k-1} - 2\binom{n-k-1}{k-1} + \binom{n-k-i-1}{k-1} + 2$$
$$\le \binom{n-1}{k-1} - 2\binom{n-k-1}{k-1} + \binom{n-k-3}{k-1} + 2.$$
(1)

The equality in (1) holds if and only if $|E_1 \cap E_2| = k - 2$, that is $\mathcal{H} = \mathcal{K}_2$. (ii) By the definiton of \mathcal{J}_i , we have

$$|\mathcal{J}_3| = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} - \binom{n-k-3}{k-3} + 3.$$
(2)

(iii) Comparing the right hand sides of (1) and (2), we can see that if $2k + 1 \le n \le 3k - 3$, then $|\mathcal{K}_2| \ge |\mathcal{J}_3|$, the equality holds only for k = 4; and if $n \ge 3k - 2$, then $|\mathcal{K}_2| < |\mathcal{J}_3|$.

By Fact 2.1, we may assume that for any x, at least 3 sets in \mathcal{H} do not contain x. To show Theorem 1.5, it is sufficient to show the following result.

Theorem 2.2. Let $k \ge 4, n \ge 2k + 1$ and $\mathcal{H} \subseteq {\binom{[n]}{k}}$ be an intersecting family which is not EKR, not HM and $\mathcal{H} \not\subseteq \mathcal{J}_2$ (in addition $\mathcal{H} \not\subseteq \mathcal{G}_2, \mathcal{H} \not\subseteq \mathcal{G}_3$ if k = 4). Moreover, for any $x \in [n]$, there are at least 3 sets in \mathcal{H} not containing x. Then

$$|\mathcal{H}| \le \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} - \binom{n-k-3}{k-3} + 3.$$

Moreover if $k \neq 5$, the equality holds only for $\mathcal{H} = \mathcal{J}_3$; if k = 5, the equality holds for $\mathcal{H} = \mathcal{J}_3$ or \mathcal{G}_4 .

From now on, we always assume that \mathcal{H} is a maximum intersecting family which satisfies the conditions of Theorem 2.2, that is \mathcal{H} is not EKR, not HM, $\mathcal{H} \not\subseteq \mathcal{J}_2$ (in addition $\mathcal{H} \not\subseteq \mathcal{G}_2, \mathcal{H} \not\subseteq \mathcal{G}_3$ if k = 4) and for any $x \in [n]$, there are at least 3 sets in \mathcal{H} not containing x.

We first give some definition related to the shifting method. For x and $y \in [n], x < y$, and $F \in \mathcal{F}$, we call the following operation a *shift*:

$$S_{xy}(F) = \begin{cases} (F \setminus \{y\}) \cup \{x\}, & \text{if } x \notin F, y \in F \text{ and } (F \setminus \{y\}) \cup \{x\} \notin \mathcal{F}; \\ F, & \text{otherwise.} \end{cases}$$

We say that F is stable under the shift S_{xy} if $S_{xy}(F) = F$. If $z \in F$ and $z \in S_{xy}(F)$ still, we say that F is stable at z after the shift S_{xy} . For a family \mathcal{F} , we define

$$S_{xy}(\mathcal{F}) = \{ S_{xy}(F) : F \in \mathcal{F} \}.$$

Clearly, $|S_{xy}(\mathcal{F})| = |\mathcal{F}|$. We say that \mathcal{F} is *stable* if $S_{xy}(\mathcal{F}) = \mathcal{F}$ for all $x, y \in [n]$ with x < y.

An important property shown in [4] is that if \mathcal{F} is intersecting, then $S_{xy}(\mathcal{F})$ is still intersecting. Let us rewrite is as a remark.

Remark 2.3. [4] If \mathcal{F} is a maximum intersecting family, then $S_{xy}(\mathcal{F})$ is still a maximum intersecting family.

This property guarantees that performing shifts repeatedly to a maximum intersecting family will yield a stable maximum intersecting family. The main difficulty we need to overcome is to guarantee that we can get a stable maximum intersecting family with further properties: not EKR, not HM, $\not\subseteq \mathcal{J}_2$ (in addition $\not\subseteq \mathcal{G}_2, \not\subseteq \mathcal{G}_3$ if k = 4). The following facts and lemmas are for this purpose.

Fact 2.4. The following properties hold. (i) If $S_{xy}(\mathcal{H})$ is EKR (or HM), then x must be the center. (ii) If $S_{xy}(\mathcal{H}) \subseteq \mathcal{G}_2$, then the core is $\{x, x_1, x_2\}$ for some $x_1, x_2 \in [n] \setminus \{x, y\}$. (iii) If $S_{xy}(\mathcal{H}) \subseteq \mathcal{J}_2$, then x is the center. (iv) If $S_{xy}(\mathcal{H}) \subseteq \mathcal{G}_3$, then x is the center or x is in the core. *Proof.* For (i) and (ii), Han and Kohaykawa proved them in [8]. We prove (iii) and (iv) only.

For (iii), suppose that $S_{xy}(\mathcal{H}) \subseteq \mathcal{J}_2$ at center $z \in [n] \setminus \{x\}$. Since $\mathcal{H} \not\subseteq \mathcal{J}_2$ at z, there are at least three sets E_1, E_2 and E_3 in \mathcal{H} missing z, after doing the shift S_{xy} , these 3 sets still miss z, so $S_{xy}(\mathcal{H})$ is not contained in \mathcal{J}_2 center at z.

For (iv), let $S_{xy}(\mathcal{H}) \subseteq \mathcal{G}_3$ at center x_0 and core $E = \{x_1, x_2, x_3\}$, and let $B = \{x_0, x_1, x_2, x_3\}$. Since $\mathcal{H} \not\subseteq \mathcal{G}_3$, there is a set $G \in \mathcal{H}$ that satisfies one of the following two cases: (a) $\{y, x_0\} \subseteq G, G \cap E = \emptyset$; (b) $y \in G, x_0 \notin G, |G \cap E| \in \{1, 2\}$. If (a) holds, then $x \neq x_0$ and x must be in the core, $y \notin B$. If (b) holds, then either $x = x_0$ is the center or x is in the core and $y \notin B$.

Remark 2.5. By Fact 2.4, if applying $S_{x'y'}(x' < y')$ repeatedly to \mathcal{H} , we may reach a family which belong to one of the following cases.

Case 1: a family \mathcal{H}_1 such that $S_{xy}(\mathcal{H}_1)$ is EKR with center x;

Case 2: a family \mathcal{H}_2 such that $S_{xy}(\mathcal{H}_2)$ is HM with center x;

Case 3: a family \mathcal{H}_3 such that $S_{xy}(\mathcal{H}_3) \subseteq \mathcal{J}_2$ with center x;

Case 4: a family \mathcal{H}_4 such that $S_{xy}(\mathcal{H}_4) \subseteq \mathcal{G}_2$ with core $\{x, x_1, x_2\}$ for some $\{x_1, x_2\} \in X \setminus \{x, y\}$ (k = 4 only);

Case 5: a family \mathcal{H}_5 such that $S_{xy}(\mathcal{H}_5) \subseteq \mathcal{G}_3$ with center x or x being in the core (k = 4 only);

Case 6: a stable family \mathcal{H}_6 satisfies the conditions of Theorem 2.2, that is we will not meet Cases 1-5 after doing all shifts.

By Remark 2.3, we know that for any shift S_{xy} on [n] we have $|S_{xy}(\mathcal{H})| = |\mathcal{H}|$ and $S_{xy}(\mathcal{H})$ is also intersecting. We hope to get a stable family satisfying the conditions of Theorem 2.2 after some shifts, that is neither EKR, nor HM, nor contained in \mathcal{J}_2 (nor in \mathcal{G}_2 , \mathcal{G}_3 if k = 4). By Fact 2.1, we can assume that a family \mathcal{G} obtained by performing shifts to \mathcal{H} has the property that for any x, at least 3 sets in \mathcal{G} do not contain x. What we are going to do is: If any case of Cases 1-5 happens, we will not perform S_{xy} . Instead we will adjust the shifts as shown in Lemma 2.6 to guarantee that the terminating family is a stable family satisfying the conditions of Theorem 2.2. We will prove the following two crucial lemmas in Section 3.

Lemma 2.6. Let $i \in [5]$. If we reach \mathcal{H}_i in Case i in Remark 2.5, then there is a set $X_i \subseteq [n]$ with $|X_i| \leq 5$ (when $k \geq 5$, $|X_i| \leq 3$ for $i \in [3]$), such that after a series of shifts $S_{x'y'}(x' < y' \text{ and } x', y' \in [n] \setminus X_i)$ to \mathcal{H}_i , we can reach a stable family satisfying the conditions of Theorem 2.2. Moreover, for any set G in the final family \mathcal{G} , we have $G \cap X_i \neq \emptyset$.

From now on, let X_i be the corresponding sets in Lemma 2.6 for $1 \le i \le 5$ and $X_6 = \emptyset$. For $k \ge 5$ and $i \in \{1, 2, 3, 6\}$, let Y_i be the set of the first $2k - |X_i|$ elements of $[n] \setminus X_i$, and for k = 4 and $i \in \{1, 2, 3, 4, 5, 6\}$, let Y_i be the first $9 - |X_i|$ elements of $[n] \setminus X_i$. Let $Y = Y_i \cup X_i$, then $|Y_i| \ge 2k - 4$ and |Y| = 2k if $k \ge 5$. If k = 4 then |Y| = 9. Let

$$\mathcal{A}_i := \{ G \cap Y : G \in \mathcal{G}, |G \cap Y| = i \},\$$

$$\widetilde{\mathcal{A}}_i := \{G : G \in \mathcal{G}, |G \cap Y| = i\}.$$

Lemma 2.7. Let \mathcal{G} be the final stable family guaranteed by Lemma 2.6 satisfying the conditions of Theorem 2.2, and let X_i be inherit from Lemma 2.6. In other words, \mathcal{G} is stable; \mathcal{G} is neither EKR, nor HM, nor contained in \mathcal{J}_2 (nor in \mathcal{G}_2 , \mathcal{G}_3 if k = 4); for any $x \in [n]$, there are at least 3 sets in \mathcal{G} not containing x; and $G \cap X_i \neq \emptyset$ for any $G \in \mathcal{G}$. Then

(i) $\mathcal{A}_1 = \emptyset$.

(ii) For all G and $G' \in \mathcal{G}$, we have $G \cap G' \cap Y \neq \emptyset$, or equivalently, $\bigcup_{i=2}^{k} \mathcal{A}_i \cup \mathcal{G}$ is intersecting.

2.1 Quantitative Part of Theorem 2.2

Lemma 2.8. For k = 4, we have $|\mathcal{A}_1| = 0$, $|\mathcal{A}_2| \le 3$, $|\mathcal{A}_3| \le 18$ and $|\mathcal{A}_4| \le 50$. For $k \ge 5$, we have

$$|\mathcal{A}_{i}| \leq \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} - \binom{k-3}{i-3}, \ 1 \leq i \leq k-1,$$
$$|\mathcal{A}_{k}| \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1} - \binom{k-1}{k-1} - \binom{k-2}{k-2} - \binom{k-3}{k-3} + 3.$$

Proof. By Lemma 2.7 (i), we have $|\mathcal{A}_1| = 0$.

First consider k = 4. If $|\mathcal{A}_2| \ge 4$, since \mathcal{A}_2 is intersecting, it must be a star. Let its center be x. Since $\mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$ is intersecting, \mathcal{A}_3 must be a star with center x and there is at most one set in \mathcal{A}_4 missing x, this implies that \mathcal{G} is EKR or HM, which contradicts the fact that \mathcal{G} is neither EKR nor HM.

Suppose that $|\mathcal{A}_3| \geq 19$. By Theorem 1.3, \mathcal{A}_3 must be EKR, HM or \mathcal{G}_2 .

If \mathcal{A}_3 is EKR with center x, then since \mathcal{G} is not EKR and $\mathcal{A}_1 = \emptyset$, there must exist $G \in \mathcal{G}$, such that either $x \notin G$ and $G \cap Y \in \mathcal{A}_2$, or $x \notin G$ and $G \cap Y \in \mathcal{A}_4$. If the former holds, by the intersecting property of $\mathcal{A}_2 \cup \mathcal{A}_3$, every set in \mathcal{A}_3 must contain at least one of the elements in $G \cap Y$, so $|\mathcal{A}_3| \leq 13$, a contradiction. Otherwise, the latter holds and \mathcal{A}_2 is a star with center x, and all sets of \mathcal{G} missing x lie in Y completely. Recall that the number of these sets is at leat 3, say $x \notin G_1, G_2, G_3 \in \mathcal{G}$. Since \mathcal{G} is not \mathcal{G}_3 , it's impossible that G_1, G_2, G_3 form a 3-star (each member contains a fixed 3-set). If any two sets in G_1, G_2, G_3 intersect at 3 vertices, then G_1, G_2, G_3 must be a 2-star. Since $\mathcal{A}_3 \cup \mathcal{A}_4$ is intersecting, calculating directly the number of triples of Y containing x and intersecting with G_1, G_2 and G_3 , we have $|\mathcal{A}_3| \leq 16$, a contradiction. Otherwise, there are two members, w.l.o.g., say, G_1, G_2 , such that $|G_1 \cap G_2| = 2$. Since $\mathcal{A}_3 \cup \mathcal{A}_4$ is intersecting, calculating directly the number of triples of Y containing x and intersecting with G_1 and G_2 , we have $|\mathcal{A}_3| \leq 17$, also a contradiction.

If \mathcal{A}_3 is HM with center x, let $\{z_1, z_2, z_3\} \in \mathcal{A}_3$. By Theorem 1.2, we have $|\mathcal{A}_3| \leq 19$, so we may assume $|\mathcal{A}_3| = 19$ and \mathcal{A}_3 is isomorphic to HM(9,3). Suppose

that there is a set G such that $x \notin G, G \cap Y \in \mathcal{A}_2$, w.l.o.g., assume $z_1 \notin G$. Since $|Y \setminus (\{x, z_1, z_2, z_3\} \cup G)| \geq 3$, there is $a \in Y \setminus (\{x, z_1, z_2, z_3\} \cup G)$ such that $\{x, z_1, a\} \cap G = \emptyset$. By the intersecting property of $\mathcal{A}_3 \cup \mathcal{A}_4$, we have $\{x, z_1, a\} \notin \mathcal{A}_3$, so $|\mathcal{A}_3| < 19$, a contradiction. Now we may assume that \mathcal{A}_2 is a star with center x. Since \mathcal{G} is neither HM nor contained in \mathcal{G}_3 , there must be a 4-set G in \mathcal{A}_4 such that either $x \notin G$ and $1 \leq |G \cap \{z_1, z_2, z_3\}| \leq 2$, w.l.o.g., assume $z_1 \notin G$ or $x \in G$ and $|G \cap \{z_1, z_2, z_3\}| = 0$. But since $\mathcal{A}_3 \cup \mathcal{A}_4$ is intersecting, the latter case will not happen. Assume the former holds. Since $|Y \setminus (\{x, z_1, z_2, z_3\} \cup G)| \geq 2$, there is $a \in Y \setminus (\{x, z_1, z_2, z_3\} \cup G)$ such that $\{x, z_1, a\} \cap G = \emptyset$. By the intersecting property of $\mathcal{A}_3 \cup \mathcal{A}_4$, we have $\{x, z_1, a\} \notin \mathcal{A}_3$, so $|\mathcal{A}_3| < 19$.

At last, assume that $\mathcal{A}_3 \subseteq \mathcal{G}_2$ with core $\{x_1, x_2, x_3\}$. Since \mathcal{A}_3 is intersecting, by calculating the number of triples in Y containing at least 2 vertices in core $\{x_1, x_2, x_3\}$, we have $|\mathcal{A}_3| \leq 19$, so we may assume that $|\mathcal{A}_3| = 19$. Since $\mathcal{G} \not\subseteq \mathcal{G}_2$, there exists a set $G \in \mathcal{G}$ such that $|G \cap \{x_1, x_2, x_3\}| \leq 1$. w.l.o.g., let $G \cap \{x_1, x_2\} = \emptyset$. Since $|Y \setminus (\{x_1, x_2, x_3\} \cup G)| \geq 2$, we can pick $a \in Y \setminus (\{x_1, x_2, x_3\} \cup G)$ such that $G \cap$ $\{x_1, x_2, a\} = \emptyset$. By the intersecting property of $\mathcal{A}_3 \cup \mathcal{G}$, we have $\{x_1, x_2, a\} \not\in \mathcal{A}_3$, hence $|\mathcal{A}_3| \leq 18$, as desired.

So we have proved that $|\mathcal{A}_3| \leq 18$ for k = 4.

Next, we prove $|\mathcal{A}_4| \leq 50$. On the contrary, suppose that $|\mathcal{A}_4| \geq 51$. By Theorem 1.3, \mathcal{A}_4 must be EKR, HM, or contained in \mathcal{J}_2 , \mathcal{G}_2 or \mathcal{G}_3 .

Suppose that \mathcal{A}_4 is EKR at x. Since \mathcal{G} is not EKR and $\mathcal{A}_1 = \emptyset$, there must exist $G \in \mathcal{G}$ such that either $x \notin G$ and $G \cap Y \in \mathcal{A}_2$ or $x \notin G$ and $G \cap Y \in \mathcal{A}_3$. If the former holds, since $\mathcal{A}_2 \cup \mathcal{A}_4$ is intersecting, by calculating the number of 4-sets in Y containing x and intersecting with $G \cap Y$ directly, we have $|\mathcal{A}_4| \leq 36$. If the latter holds, since $\mathcal{A}_3 \cup \mathcal{A}_4$ is intersecting, by calculating the number of 4-sets in Y containing x and intersecting with $G \cap Y$ directly, we have $|\mathcal{A}_4| \leq 46$.

Suppose that \mathcal{A}_4 is HM at x. Since \mathcal{G} is not HM at x, there exists $G \in \mathcal{G}$ such that either $x \notin G$ and $G \cap Y \in \mathcal{A}_2$ or $x \notin G$ and $G \cap Y \in \mathcal{A}_3$, since \mathcal{A}_4 is HM at x and $\mathcal{A}_2 \cup \mathcal{A}_4$ (or $\mathcal{A}_3 \cup \mathcal{A}_4$) is intersecting, by calculating the number of 4-subsets containing x and intersecting with $G \cap Y$, and adding 1 set not containing x, we have $|\mathcal{A}_4| \leq 37$ (or $|\mathcal{A}_4| \leq 47$).

Suppose that $\mathcal{A}_4 \subseteq \mathcal{G}_2$ with core $\{x_1, x_2, x_3\} = A$. By calculating the number of 4subsets in Y containing at least 2 of $\{x_1, x_2, x_3\}$, we have $|\mathcal{A}_4| \leq 51$, so we may assume $|\mathcal{A}_4| = 51$. Since $\mathcal{G} \not\subseteq \mathcal{G}_2$, there exists a set G in \mathcal{G} such that $|G \cap A| \leq 1, G \cap Y \in \mathcal{A}_2$ or \mathcal{A}_3 . w.l.o.g., let $G \cap \{x_1, x_2\} = \emptyset$. Since $|Y \setminus (A \cup G)| \geq 2$, we can pick $a, b \in Y \setminus (A \cup G)$ such that $(G \cap Y) \cap \{x_1, x_2, a, b\} = \emptyset$. By the intersecting property of $\mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$, we have $\{x_1, x_2, a, b\} \notin \mathcal{A}_4$. Hence $|\mathcal{A}_4| \leq 50$, as desired.

Suppose that $\mathcal{A}_4 \subseteq \mathcal{G}_3$ with core $\{x_1, x_2, x_3\}$ and center x. By direct calculation, $|\mathcal{A}_4| \leq 51$, so we may assume $|\mathcal{A}_4| = 51$ and $\mathcal{A}_4 = \mathcal{G}_3$. Since $\mathcal{G} \not\subseteq \mathcal{G}_3$, there must be $G \in \mathcal{G}$ and $G \cap Y \in \mathcal{A}_2$ or \mathcal{A}_3 , such that either $x \notin G$ and $\{x_1, x_2, x_3\} \not\subseteq G \cap Y$ or $x \in G$ and $\{x_1, x_2, x_3\} \cap (G \cap Y) = \emptyset$. By the intersecting property of $\mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$, in either case, we have $\mathcal{A}_4 \neq \mathcal{G}_3$ and $|\mathcal{A}_4| < 51$.

At last, suppose that $\mathcal{A}_4 \subseteq \mathcal{J}_2$ with center x, kernel $\{x_1, x_2, x_3\}$ and the set of

pages $\{x_4, x_5\}$. By Theorem 1.4, we may assume $|\mathcal{A}_4| = 51$ and $\mathcal{A}_4 = \mathcal{J}_2$. Since $\mathcal{A}_2 \cup \mathcal{A}_3 \cup \mathcal{A}_4$ is intersecting, there is no member in \mathcal{A}_2 or \mathcal{A}_3 avoiding x. And each member in \mathcal{A}_2 must interset with $\{x_1, x_2, x_3\}$, each member in \mathcal{A}_3 must interset with $\{x_1, x_2, x_3\}$ or contain $\{x_4, x_5\}$, to satisfy these conditions, G must be contained in \mathcal{J}_2 , a contradiction.

So we have proved that $\mathcal{A}_4 \leq 50$ for k = 4.

Next consider $k \ge 5$. Suppose on the contrary that there exists $i \in \{2, \ldots, k-1\}$ such that

$$|\mathcal{A}_{i}| > \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} - \binom{k-3}{i-3}.$$
(3)

Note that for i = 2,

$$\binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} - \binom{k-3}{i-3} = k-1.$$

If $|\mathcal{A}_2| \geq k \ (k \geq 5)$, then \mathcal{A}_2 is EKR, moreover, since $\mathcal{A}_2 \cup \mathcal{G}$ is intersecting, \mathcal{G} must be EKR or HM, a contradiction. Hence $|\mathcal{A}_2| \leq k - 1$, as desired.

Now consider $i \geq 3$. Under the assumption (3), we claim that

$$|\mathcal{A}_i| > \binom{2k-1}{i-1} - \binom{2k-i-1}{i-1} - \binom{2k-i-2}{i-2} + 2.$$
(4)

Let us explain inequality (4). We write

$$\binom{2k-i-2}{i-2} = \binom{2k-i-3}{i-2} + \binom{2k-i-3}{i-3}.$$
(5)

For $k \ge 5$ and $3 \le i \le k - 1$, we have

$$\binom{2k-1-i}{i-1} - \binom{k-1}{i-1} = \binom{k-1}{i-2} + \binom{k}{i-2} + \dots + \binom{2k-2-i}{i-2} \ge 4, \quad (6)$$

$$\binom{2k-i-3}{i-2} - \binom{k-2}{i-2} \ge 0, \quad \binom{2k-i-3}{i-3} - \binom{k-3}{i-3} \ge 0, \tag{7}$$

Combining (3), (5), (6) and (7), we obtain (4). Since \mathcal{A}_i is intersecting, we may assume, by Theorem 1.3 that \mathcal{A}_i is EKR or HM or for i = 3, $\mathcal{A}_i \subseteq \mathcal{G}_2$.

Case (i): \mathcal{A}_i is EKR or HM at center x.

In this case \mathcal{A}_i contains at most 1 *i*-set missing *x*. Recall that there are at least three sets missing *x* in \mathcal{G} . Pick three sets $G_1, G_2, G_3 \in \mathcal{G}$ missing *x*. Denote





Clearly, $t + t_1 + t_4 + t_5 \leq k$, $t + t_2 + t_4 + t_6 \leq k$, $t + t_3 + t_5 + t_6 \leq k$. By Lemma 2.7 $\mathcal{A}_i \cup \{G_1 \cap Y, G_2 \cap Y, G_3 \cap Y\}$ is intersecting. Applying Inclusion-Exclusion principle, we have

$$\mathcal{A}_{i} \leq \binom{2k-1}{i-1} - \binom{2k-1-t-t_{1}-t_{4}-t_{5}}{i-1} - \binom{2k-1-t-t_{2}-t_{4}-t_{6}}{i-1} \\ -\binom{2k-1-t-t_{3}-t_{5}-t_{6}}{i-1} + \binom{2k-1-t-t_{1}-t_{2}-t_{4}-t_{5}-t_{6}}{i-1} \\ + \binom{2k-1-t-t_{1}-t_{3}-t_{4}-t_{5}-t_{6}}{i-1} + \binom{2k-1-t-t_{2}-t_{3}-t_{4}-t_{5}-t_{6}}{i-1} \\ -\binom{2k-1-t-t_{1}-t_{2}-t_{3}-t_{4}-t_{5}-t_{6}}{i-1} + c, \end{cases}$$
(8)

where c = 0 (if \mathcal{A}_i is EKR) or 1 (if \mathcal{A}_i is HM). Denote the right side of equality (8) by f. We rewrite it as

$$f = \binom{2k-1}{i-1} - \binom{2k-2-t-t_1-t_4-t_5}{i-2} - \dots - \binom{2k-1-t-t_1-t_3-t_4-t_5-t_6}{i-2} - \binom{2k-2-t-t_2-t_4-t_6}{i-2} - \dots - \binom{2k-1-t-t_1-t_2-t_4-t_5-t_6}{i-2} - \binom{2k-2-t-t_3-t_5-t_6}{i-2} - \dots - \binom{2k-1-t-t_2-t_3-t_4-t_5-t_6}{i-2} - \binom{2k-1-t-t_2-t_3-t_4-t_5-t_6}{i-2} + c.$$

$$(9)$$

We can see that the right side of (9), consequently (8) does not decrease as $t+t_1+t_4+t_5, t+t_2+t_4+t_6, t+t_3+t_5+t_6$ increase. Since $t+t_1+t_4+t_5, t+t_2+t_4+t_6, t+t_3+t_5+t_6 \le k$, we can substitute $t+t_1+t_4+t_5 = k, t_2+t_4+t_6 = k-t, t_3+t_5+t_6 = k-t$ into

inequality (8), and this will not decrease f. So we have

$$\begin{aligned} |\mathcal{A}_{i}| &\leq \binom{2k-1}{i-1} - 3\binom{k-1}{i-1} + \binom{t+t_{4}-1}{i-1} + \binom{t+t_{5}-1}{i-1} + \binom{t+t_{6}-1}{i-1} \\ &- \binom{t+t_{5}-t_{2}-1}{i-1} + c \\ &= \binom{2k-1}{i-1} - 3\binom{k-1}{i-1} + \binom{t+t_{4}-1}{i-1} + \binom{t+t_{6}-1}{i-1} + \binom{t+t_{5}-2}{i-2} \quad (10) \\ &+ \dots + \binom{t+t_{5}-t_{2}-1}{i-2} + c \\ &\triangleq g. \end{aligned}$$

Clearly, g does not decrease as $t+t_4, t+t_5, t+t_6$ increase and $t+t_4 \le k-1, t+t_5 \le k-1$ $t+t_6 \le k-1$. If $t+t_5 - t_2 - 1 \ge k-3$, then

$$\begin{aligned} |\mathcal{A}_{i}| &\leq \binom{2k-1}{i-1} - 3\binom{k-1}{i-1} + 3\binom{k-2}{i-1} - \binom{k-3}{i-1} + c \\ &= \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} - \binom{k-3}{i-3} + c. \end{aligned}$$

The equality holds only if t = k - 1, $t_1 = t_2 = t_3 = 1$, $t_4 = t_5 = t_6 = 0$. If $t + t_5 - t_2 - 1 \le k - 4$ (*), then $t \le k - 2$ since t = k - 1 implies $t_5 = 0$ and combining with (*), we have $t_2 \ge 2$, so $t + t_2 \ge k + 1$, a contradiction. Since $t + t_4 \le k - 1$, $t + t_5 \le k - 1$ and $t + t_6 \le k - 1$, by (9) and (10), taking $t + t_1 + t_4 + t_5 = k$, $t + t_2 + t_4 + t_6 = k$, $t + t_3 + t_5 + t_6 = k$ and $t + t_4 = k - 1$, $t + t_5 = k - 1$, $t + t_6 = k - 1$ (this implies that t = k - 2, $t_4 = t_5 = t_6 = 1$ and $t_1 = t_2 = t_3 = 0$) does not decrease f. So

$$g \leq \binom{2k-1}{i-1} - 3\binom{k-1}{i-1} + 3\binom{k-2}{i-1} - \binom{k-2}{i-1} + c$$

= $\binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} - \binom{k-3}{i-3} - \binom{k-3}{i-2} + c$
 $\leq \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} - \binom{k-3}{i-3} - 2 + c.$

 So

$$|\mathcal{A}_i| \le \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} - \binom{k-3}{i-3} + c.$$

To reach c = 1, there is a set A in \mathcal{A}_i not containing x. Let G_1 be such that $G_1 \cap Y = A$. So $|G_1 \cap Y| = i \leq k - 1$. This implies that $t + t_1 + t_4 + t_5 \leq k - 1$. In view of (8) and (9), $|\mathcal{A}_i|$ strictly decreases as $t + t_1 + t_4 + t_5$ strictly decreases. So we have

$$|\mathcal{A}_i| \le \binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} - \binom{k-3}{i-3},$$

as desired.

Case (ii): For i = 3, $\mathcal{A}_i \subseteq \mathcal{G}_2$ with core, say $\{x_1, x_2, x_3\}$.

By direct calculation, we have $|\mathcal{A}_3| \leq 3(2k-3) + 1 = 6k - 8$. When $k \geq 5$, we have

$$6k - 8 < \binom{2k-1}{2} - \binom{k-1}{2} - \binom{k-2}{1} - \binom{k-3}{0},$$

as desired.

Lemma 2.9. Let \mathcal{G} be the final stable family as in Lemma 2.7. Then

$$|\mathcal{G}| \le \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} - \binom{n-k-3}{k-3} + 3.$$

Proof. Note that for any $A \in \mathcal{A}_i$, there are at most $\binom{n-|Y|}{k-i}$ k-sets in \mathcal{G} containing A. For k = 4, we have

$$|\mathcal{G}| \leq \sum_{i=1}^{4} |\mathcal{A}_i| {n-9 \choose 4-i}.$$

By Lemma 2.8,

$$\begin{aligned} |\mathcal{G}| &\leq 3 \binom{n-9}{2} + 18 \binom{n-9}{1} + 50 \\ &= \frac{3}{2}n^2 - \frac{21}{2}n + 23 \\ &= \binom{n-1}{3} - \binom{n-5}{3} - \binom{n-6}{2} - \binom{n-7}{1} + 3. \end{aligned}$$
(11)

For $k \geq 5$, we have

$$\begin{aligned} |\mathcal{G}| &\leq \sum_{i=1}^{k} |\mathcal{A}_{i}| \binom{n-2k}{k-i} \\ &\leq 3 + \sum_{i=1}^{k} \left(\binom{2k-1}{i-1} - \binom{k-1}{i-1} - \binom{k-2}{i-2} - \binom{k-3}{i-3} \right) \binom{n-2k}{k-i} \\ &= \binom{n-1}{k-1} - \binom{n-k-1}{k-1} - \binom{n-k-2}{k-2} - \binom{n-k-3}{k-3} + 3. \end{aligned}$$
(12)

By Lemma 2.9, we have obtained the quantitative part of Theorem 2.2.

2.2 Uniqueness Part of Theorem 2.2

Let \mathcal{G} be a k-uniform family such that the equality holds in Lemma 2.9 .We first show the structure of \mathcal{G} .

Theorem 2.10. Let \mathcal{G} be a family as in Lemma 2.9 such that the equality holds. If k = 5, then $\mathcal{G} = \mathcal{J}_3$ or \mathcal{G}_4 ; if $k \neq 5$, then $\mathcal{G} = \mathcal{J}_3$.

Proof. To make the equalities (11) and (12) hold, we must get all the equalities in Lemma 2.8. So $|\mathcal{A}_2| = k - 1$. By Lemma 2.7, \mathcal{A}_2 is intersecting, so \mathcal{A}_2 is a star, say with center x and leaves $\{x_1, x_2, \ldots, x_{k-1}\}$, or a triangle on $\{x, y, z\}$ (only for k = 4). First consider k = 4. If \mathcal{A}_2 is a triangle, then $\mathcal{G} = \mathcal{G}_2$, a contradiction. Otherwise, \mathcal{A}_2 is a star, this implies that all sets in \mathcal{G} missing x must contain $\{x_1, x_2, x_3\}$, and the number of such sets is at least 3. Then either $\mathcal{G} = \mathcal{G}_3$ or $\mathcal{G} = \mathcal{J}_i, 3 \leq i \leq k - 1$. By the assumption that $\mathcal{G} \not\subseteq \mathcal{G}_3$, the former is impossible, and the latter implies $\mathcal{G} = \mathcal{J}_3$. Hence, the equality in (21) holds only if $\mathcal{G} = \mathcal{J}_3$. For $k \geq 5$, \mathcal{A}_2 must be a star. Similarly, in this condition, we have either $\mathcal{G} = \mathcal{G}_{k-1}$ or $\mathcal{G} = \mathcal{J}_i, 3 \leq i \leq k - 1$. In particular, for k = 5, we can see that the extremal value of $|\mathcal{G}|$ can be achieved by $|\mathcal{G}_4|$ and $|\mathcal{J}_3|$, and for k > 5, by $|\mathcal{J}_3|$ only.

We will use some results in [8]. We say two families \mathcal{G} and \mathcal{F} are *cross-intersecting* if for any $G \in \mathcal{G}$ and $F \in \mathcal{F}$, $G \cap F \neq \emptyset$. We say that a family \mathcal{F} is *non-separable* if \mathcal{F} cannot be partitioned into the union of two cross-intersecting non-empty subfamilies.

Proposition 2.11. ([8]) Let $r \ge 2$. Let Z be a set of size $m \ge 2r + 1$ and let $A \subseteq Z$ such that $|A| \in \{r - 1, r\}$. Let \mathcal{B} be an r-uniform family on Z such that $\mathcal{B} = \{B \subseteq Z : 0 < |B \cap A| < |A|\}$. Then \mathcal{B} is non-separable.

Lemma 2.12. ([8]) Let \mathcal{F} be a k-uniform intersecting family. If $k \geq 3$ and $S_{xy}(\mathcal{F}) \in \{\mathcal{J}_2, \mathcal{G}_{k-1}, \mathcal{G}_2\}$, then \mathcal{F} is isomorphic to $S_{xy}(\mathcal{F})$.

Combining with Theorem 2.10 and Lemma 2.12, the uniqueness part of Theorem 2.2 will be completed by showing the following lemma.

Lemma 2.13. Let \mathcal{F} be a k-uniform intersecting family. If $k \geq 4$ and $S_{xy}(\mathcal{F}) = \mathcal{J}_3$, then \mathcal{F} is isomorphic to \mathcal{J}_3 .

Proof. Assume that $S_{xy}(\mathcal{F}) = \mathcal{J}_3$ with center x_0 , kernel E and the set of pages $\{x_1, x_2, x_3\}$. That is

$$\mathcal{J}_3 = \{G : \{x_0, x_1, x_2, x_3\} \subseteq G\} \cup \{G : x_0 \in G, G \cap E \neq \emptyset\} \cup \{E \cup \{x_1\}, E \cup \{x_2\}, E \cup \{x_3\}\}$$

Define

$$\mathcal{B}_x := \{ G \in \mathcal{J}_3 : x \in G, y \notin G, (G \setminus x) \cup y \notin \mathcal{J}_3 \}, \\ \mathcal{C}_x := \{ G \in \mathcal{B}_x : G \in \mathcal{F} \},$$

$$\mathcal{D}_x := \{ G \in \mathcal{B}_x : G \notin \mathcal{F} \}, \\ \mathcal{B}' := \{ G \setminus \{ x \} : G \in \mathcal{B}_x \}, \\ \mathcal{C}' := \{ G \setminus \{ x \} : G \in \mathcal{C}_x \}, \\ \mathcal{D}' := \{ G \setminus \{ x \} : G \in \mathcal{D}_x \}.$$

Then $\mathcal{B}_x = \mathcal{C}_x \sqcup \mathcal{D}_x$ and $\mathcal{B}' = \mathcal{C}' \sqcup \mathcal{D}'$. The definition of \mathcal{D}_x implies that for any $G \in \mathcal{D}_x$, $G \setminus \{x\} \cup \{y\} \in \mathcal{F}$, and the definition of \mathcal{C}_x implies that for any $G \in \mathcal{C}_x$, $G \setminus \{x\} \cup \{y\} \notin \mathcal{F}$. Clearly, only the sets in \mathcal{D}_x are in $S_{xy}(\mathcal{F}) \setminus \mathcal{F}$. If $\mathcal{D}_x = \emptyset$, then $S_{xy}(\mathcal{F}) = \mathcal{F} = \mathcal{J}_3$, and if $\mathcal{C}_x = \emptyset$, then \mathcal{F} is still \mathcal{J}_3 with center y. On the other hand, notice that \mathcal{C}_x and $\{G \setminus \{x\} \cup \{y\} : G \in \mathcal{D}_x\}$ are cross intersecting, so \mathcal{C}' and \mathcal{D}' are cross intersecting. We are going to prove that \mathcal{B}' is non-separable, this means that $\mathcal{C}' = \emptyset$ or $\mathcal{D}' = \emptyset$, and hence $\mathcal{C}_x = \emptyset$ or $\mathcal{D}_x = \emptyset$, we can conclude the proof. So what remains is to show the following claim.

Claim 2.14. \mathcal{B}' is non-separable.

Proof. We say the shift $S_{xy} : \mathcal{F} \to \mathcal{J}_3$ is trivial if $\mathcal{B}_x = \emptyset$. Let $Z := [n] \setminus \{x, y\}$. If r = k - 1, then $|Z| \ge 2k + 1 - 2 = 2r + 1$.

Let $T_1 := \{x_0\}, T_2 := E, T_3 := \{x_1, x_2, x_3\}, T_4 := [n] \setminus (T_1 \cup T_2 \cup T_3).$

Since for $x, y \in T_i$ or for $x \in T_i, y \in T_j, i > j$, the shift is trivial, we only need to consider the following three cases.

Case (i): $x = x_0$ and $y \in T_2 \cup T_3 \cup T_4$.

If $y \in T_3$, let A = E, then $\mathcal{B}' = \{B \subseteq Z : 0 < |B \cap A| < |A|\}$. By Proposition 2.11, \mathcal{B}' is non-separable. If $y \in T_2 \cup T_4$, let $A := E \setminus \{y\}$, then $|A| \in \{r - 1, r\}$. Assume that \mathcal{B}' has a partition $\mathcal{B}'_1 \cup \mathcal{B}'_2$ such that \mathcal{B}'_1 and \mathcal{B}'_2 are cross-intersecting. We now partition \mathcal{B}' into three parts $\mathcal{P}_1 \sqcup \mathcal{P}_2 \sqcup \mathcal{P}_3$, where

$$\mathcal{P}_1 := \{ B \subseteq Z : 0 < |B \cap A| < |A| \},\$$
$$\mathcal{P}_2 := \{ B \in \mathcal{B}' : B \cap A = \emptyset \} = \{ T_3 \cup F : F \subseteq T_4 \setminus \{y\}, |F| = k - 4 \},\$$

and

$$\mathcal{P}_3 := \{ B \in \mathcal{B}' : A \subseteq B \} = \begin{cases} \{ A \cup \{ z \} : z \in T_4 \}, & y \in T_2; \\ \{ A \}, & y \in T_4. \end{cases}$$

Obviously, $\mathcal{P}_1 \neq \emptyset$. By Proposition 2.11, \mathcal{P}_1 is non-separable. For any $P \in \mathcal{P}_2$, and any $a \in A$, we have $|Z \setminus \{a\}| \geq 2r$, then in \mathcal{P}_1 we can always find $P' \subseteq Z \setminus (\{a\} \cup P)$ such that $0 < |P' \cap A| < |A|$ and $P \cap P' = \emptyset$. This implies that P and P' must be in the same \mathcal{B}'_i (i = 1 or 2)(recall that we assumed that \mathcal{B}' has a partition $\mathcal{B}'_1 \cup \mathcal{B}'_2$ such that \mathcal{B}'_1 and \mathcal{B}'_2 are cross-intersecting), hence \mathcal{P}_1 and \mathcal{P}_2 are in the same \mathcal{B}'_i . For any $P \in \mathcal{P}_3$, we have $|P \cap T_4| \leq 1$. Since $|T_4| \geq k - 2$, there is a (k - 4)-set $F \subseteq T_4 \setminus \{y\}$, such that $P \cap F = \emptyset$. Note that $P' := F \cup T_3 \in \mathcal{P}_2$ and $P' \cap P = \emptyset$, so \mathcal{P}_2 and \mathcal{P}_3 are in the same \mathcal{B}'_i . Hence $\mathcal{B}' = \mathcal{B}'_1$ or \mathcal{B}'_2 , as desired.

Case (ii): $x \in T_2$ and $y \in T_3 \cup T_4$.

Let $E_i := (E \cup \{x_i\}) \setminus \{x\}, i = 1, 2, 3.$ If $y \in T_4$, then

$$\mathcal{B}' = \{E_1, E_2, E_3\} \cup \left\{ G \in \binom{[n] \setminus \{x\}}{k-1} : x_0 \in G, G \cap E = \emptyset, |G \cap T_3| \le 2, y \notin G \right\}.$$

Since $|T_4 \setminus \{y\}| \ge k - 3$, there is $P \in \mathcal{B}' \setminus \{E_1, E_2\}$, such that $P \cap E_1 = P \cap E_2 = \emptyset$. Hence, E_1 and E_2 belong to the same part \mathcal{B}'_i . Similarly, E_1 and E_3 belong to the same part. Thus E_1, E_2 and E_3 are in the same \mathcal{B}'_i . Moreover, for any $P' \in \mathcal{B}' \setminus \{E_1, E_2, E_3\}$, because $|P' \cap \{x_1, x_2, x_3\}| \le 2$, we have $P' \cap E_1 = \emptyset$, or $P' \cap E_2 = \emptyset$ or $P' \cap E_3 = \emptyset$. Hence, \mathcal{B}' is non-separable, as desired.

If $y \in T_3$, w.l.o.g., let $y = x_1$. Then

$$\mathcal{B}' = \{E_2, E_3\} \cup \left\{ G \in \binom{[n] \setminus \{x\}}{k-1} : x_0 \in G, G \cap E = \emptyset, |G \cap T_3| \le 1, y \notin G \right\}.$$

Since $|T_4| \ge k-2$, there exists $P \in \mathcal{B}' \setminus \{E_2, E_3\}$ such that $P \cap T_3 = \emptyset$, then $P \cap E_2 = \emptyset$, and $P \cap E_3 = \emptyset$, this implies that E_2 and E_3 are in the same \mathcal{B}'_i . Because $|G \cap T_3| \le 1$ and $G \cap E = \emptyset$, it's not hard to see that each $P \in \mathcal{B}' \setminus \{E_2, E_3\}$ is disjoint from one of E_2 and E_3 . Hence \mathcal{B}' is non-separable.

Case (iii): $x \in T_3$ and $y \in T_4$. w.l.o.g., let $x = x_1$. Under this condition,

$$\mathcal{B}' = \{E\} \cup \left\{ G \in \binom{[n] \setminus \{x\}}{k-1} : \{x_0, x_2, x_3\} \subseteq G, G \cap E = \emptyset, y \notin G \right\}.$$

Since E is disjoint from every other set in $\mathcal{B}' \setminus \{E\}, \mathcal{B}'$ is non-separable.

The proof of Lemma 2.13 is complete.

3 Proofs of Lemma 2.6 and Lemma 2.7

3.1 Proof of Lemma 2.6

We first show the following preliminary results. For a family $\mathcal{F} \subseteq 2^{[n]}$ and $x_1, x_2, x_3 \in [n]$, let $d_{\{x_1,x_2\}}$ be the number of sets containing $\{x_1, x_2\}$ in \mathcal{F} , and $d_{\{x_1,x_2,x_3\}}$ be the number of sets containing $\{x_1, x_2, x_3\}$ in \mathcal{F} .

Claim 3.1. Let $\mathcal{F} \subseteq \mathcal{G}_2$ be a 4-uniform family with core A satisfying $d_{\{x_1,x_2\}} > 2n-7$. Then $\{x_1, x_2\} \subseteq A$.

Proof. If $\{x_1, x_2\} \subseteq [n] \setminus A$, then a set in \mathcal{F} containing $\{x_1, x_2\}$ must have two elements from A, so $d_{(x_1, x_2)} \leq 3$, a contraction. If $|\{x_1, x_2\} \cap A| = 1$, then a set in \mathcal{F} containing $\{x_1, x_2\}$ must have at least one element from A, so $d_{(x_1, x_2)} \leq 2n - 7$, a contraction again. So $\{x_1, x_2\} \subseteq A$, as desired. \Box Claim 3.2. Let $\mathcal{F} \subseteq \mathcal{G}_3$ be a 4-uniform family with center x and core E and let $B = \{x\} \cup E$. (i) If $d_{\{x_1,x_2\}} \geq 3n - 12$, then $x \in \{x_1, x_2\}$.

(ii) If $d_{\{x_1,x_2\}} > 3n - 12$, then $\{x_1,x_2\} \subseteq B$ and $x \in \{x_1,x_2\}$.

Proof. For (i), assume that $x \notin \{x_1, x_2\}$. If $\{x_1, x_2\} \cap B = \emptyset$, then the sets containing $\{x_1, x_2\}$ must contain the center x and another vertex from core E, so $d_{(x_1, x_2)} \leq 3 < 3n - 12$, a contradiction. So $\{x_1, x_2\} \subseteq E$ or $|\{x_1, x_2\} \cap E| = 1$. If the former holds, then the sets containing $\{x_1, x_2\}$ must contain the center x or contain the core E, so $d_{(x_1, x_2)} \leq (n - 3) + (n - 4) = 2n - 7 < 3n - 12$, a contradiction. If the latter holds, w.l.o.g., let $\{x_1, x_2\} \cap E = \{x_1\}$, then the sets containing $\{x_1, x_2\}$ must contain the center x or just the set $E \cup \{x_2\}$, so $d_{(x_1, x_2)} \leq (n - 3) + 1 < 3n - 12$, also a contradiction. Hence, $x \in \{x_1, x_2\}$, as desired.

For (ii), we have shown that $x \in \{x_1, x_2\}$ by (i), w.l.o.g, let $x_1 = x$ be the center. If $x_2 \notin E$, then the sets containing $\{x_1, x_2\}$ must intersect with E, so $d_{(x_1, x_2)} \leq {\binom{n-2}{2} - \binom{n-5}{2}} = 3n - 12$, a contradiction to that $d_{\{x_1, x_2\}} > 3n - 12$, so $x_2 \in E$, that is $\{x_1, x_2\} \subseteq B$, as desired.

Claim 3.3. Fix n > 6. Let $\mathcal{F} \subseteq \mathcal{G}_3$ be a 4-uniform family with center x and core E and let $B = \{x\} \cup E$. If $d_{\{x_1, x_2, x_3\}} \ge n - 3$, then either $\{x_1, x_2, x_3\} \subset B$ or $|\{x_1, x_2, x_3\} \cap B| = 2$ with $x \in \{x_1, x_2, x_3\}$.

Proof. Suppose on the contrary that neither $\{x_1, x_2, x_3\} \subset B$ nor $|\{x_1, x_2, x_3\} \cap B| = 2$ with $x \in \{x_1, x_2, x_3\}$. Since $\mathcal{F} \subseteq \mathcal{G}_3$, it's easy to see that if $\{x_1, x_2, x_3\} \subseteq [n] \setminus B$, then $d_{\{x_1, x_2, x_3\}} = 0$, so $1 \leq |\{x_1, x_2, x_3\} \cap B| \leq 2$. First consider that $|\{x_1, x_2, x_3\} \cap B| = 1$. If $\{x_1, x_2, x_3\} \cap B = \{x\}$, then the sets containing $\{x_1, x_2, x_3\}$ in \mathcal{F} must intersect with E, so $d_{\{x_1, x_2, x_3\}} \leq 3 < n - 3$, a contradiction. If $|\{x_1, x_2, x_3\} \cap E| = 1$, then the set containing $\{x_1, x_2, x_3\}$ in \mathcal{F} must contain x, so $d_{\{x_1, x_2, x_3\}} \leq 1 < n - 3$, also a contradiction. Hence $|\{x_1, x_2, x_3\} \cap B| = 2$. By hypothesis, $|\{x_1, x_2, x_3\} \cap E| = 2$, w.l.o.g., let $\{x_1, x_2, x_3\} \cap E = \{x_1, x_2\}$, then $d_{\{x_1, x_2, x_3\}} \leq 2$ since the possible sets in \mathcal{F} containing $\{x_1, x_2, x_3\}$ are $\{x_1, x_2, x_3\} \cup \{x\}$ and $E \cup \{x_3\}$, a contradiction. \Box

Proof of Lemma 2.6. We first consider that $k \geq 5$.

In Case 1, i.e., $S_{xy}(\mathcal{H}_1)$ is EKR with center x, we take $X_1 = \{x, y\}$. In Case 2, since $S_{xy}(\mathcal{H}_2)$ is HM at center x, let $E = \{z_1, z_2, \ldots, z_k\}$ be the only member missing x, and without loss of generality, we assume $z_1 \neq y$, and take $X_2 = \{x, y, z_1\}$. In Case 3, $S_{xy}(\mathcal{H}_3) \subseteq \mathcal{J}_2$ with center x, kernal $\{z_1, z_2, \ldots, z_{k-1}\}$. Without loss of generality, we assume $z_1 \neq y$, and take $X_3 = \{x, y, z_1\}$. We can see that for any set $G \in \mathcal{H}_i$, $G \cap X_i \neq \emptyset$, for i = 1, 2, 3. After the shifts $S_{x'y'}$ for all $x' < y', x', y' \in [n] \setminus X_i$ to \mathcal{H}_i , the resulting family \mathcal{H}'_i satisfies that for every set $G' \in \mathcal{H}'_i$, $G' \cap X_i \neq \emptyset$. By the maximality of $|\mathcal{H}|$, we may assume that all k-sets containing X_i (i = 1, 2, 3) are in \mathcal{H} , so is in \mathcal{H}_i . These sets will keep stable after any shift $S_{x'y'}$, so there are at least $\binom{n-3}{k-2}$ (or $\binom{n-4}{k-3}$) > 2 sets missing x' in \mathcal{H}'_i . Fact 2.4 (i), (ii) and (iii) implies that \mathcal{H}'_i is neither EKR nor HM nor contained in \mathcal{J}_2 . We are done for $k \geq 5$. We now assume that k = 4. We will complete the proof by showing the following Lemmas corresponding to Cases 1-5 in Remark 2.5

Lemma 3.4 (*Case 1*). If we each a 4-uniform family \mathcal{H}_1 such that $S_{xy}(\mathcal{H}_1)$ is EKR at x, then there is a set $X_1 = \{x, y, y', z, w\}$ such that after a series of shifts $S_{x'y'}$ (x' < y' and $x', y' \in [n] \setminus X_1$) to \mathcal{H}_1 , we will reach a stable family \mathcal{G} satisfying the conditions of Theorem 2.2. Moreover, $\{y, y', z, w\}$ or $\{x, y', z, w\}$ is in \mathcal{G} . Furthermore, $G \cap \{x, y\} \neq \emptyset$ for any $G \in \mathcal{G}$.

Proof. Since $S_{xy}(\mathcal{H}_1)$ is EKR, for any $F \in \mathcal{H}_1$, we have $F \cap \{x, y\} \neq \emptyset$. Any set obtained by performing shifts $[n] \setminus \{x, y\}$ to a set in \mathcal{H}_1 still contains x or y. We will show Claims 3.5, 3.6 and 3.8 implying Lemma 3.4.

Claim 3.5. Performing shifts in $[n] \setminus \{x, y\}$ to \mathcal{H}_1 repeatedly will not reach Cases 1-3 in Remark 2.5.

Proof. Since $S_{xy}(\mathcal{H}_1)$ is EKR, for any $G \in \mathcal{H}_1$, we have $G \cap \{x, y\} \neq \emptyset$. By the maximality of $|\mathcal{H}|$ ($|\mathcal{H}_1|$ as well), we have

$$\left\{ G \in \binom{[n]}{k} : \{x, y\} \subseteq G \right\} \subseteq \mathcal{H}_1,$$
$$|\{G \in \mathcal{H}_1 : \{x, y\} \subseteq G\}| = \binom{n-2}{2}.$$
(13)

All these sets containing $\{x, y\}$ are stable after performing $S_{x'y'}$ $(x' < y', x', y' \notin \{x, y\})$. So there are still at least $\binom{n-3}{2} > 2$ sets missing x' after $S_{x'y'}$, so we will not reach *Case 1-3*.

Claim 3.6. If performing some shifts in $[n] \setminus \{x, y\}$ repeatedly to \mathcal{H}_1 reaches \mathcal{H}_4 in Case 4($S_{x'y'}(\mathcal{H}_4) \subseteq \mathcal{G}_2$), then there exists $X_1 = \{x, y, y', z, w\}$ such that performing shifts in $[n] \setminus X_1$ repeatedly to \mathcal{H}_4 will not reach Cases 1-5 as in Remark 2.5, and $\{y, y', z, w\}$ or $\{x, y', z, w\}$ is in the final stable family \mathcal{G} .

Proof. Assume that after some shifts in $[n] \setminus \{x, y\}$ to \mathcal{H}_1 , we get \mathcal{H}_4 such that $S_{x'y'}(\mathcal{H}_4) \subseteq \mathcal{G}_2$ with core A. Since there are $\binom{n-2}{2}$ sets containing $\{x, y\}$ in \mathcal{H}_1 and they are stable (so in \mathcal{H}_4), and $\binom{n-2}{2} > 2n-7$ $(n \geq 6)$, by Fact 2.4 (ii) and Claim 3.1, $A = \{x', x, y\}$. Since $S_{x'y'}(\mathcal{H}_4) \subseteq \mathcal{G}_2$ with core $\{x', x, y\}$, there exists $\{y, y', z_1, w_1\}$ (or $\{x, y', z_2, w_2\}$) in \mathcal{H}_4 . Let $X_1 := \{x, y, y', z_1, w_1\}$ (or $X_1 := \{x, y, y', z_2, w_2\}$). Clearly, any set containing $\{x, y\}$ and missing $x'' \in [n] \setminus X_1$ are stable after performing shifts in $[n] \setminus X_1$ repeatedly to \mathcal{H}_4 , so performing shifts $S_{x''y''}, x'', y'' \in [n] \setminus X_1$ to \mathcal{H}_4 will not reach Cases 1-3. If we reach Case 4, that is we get a family \mathcal{H}'_4 , such that $S_{x''y''}(\mathcal{H}'_4) \subseteq \mathcal{G}_2$ with core A'. By Fact 2.4 and Claim 3.1, we have $A' = \{x'', x, y\}$. However, $\{y, y', z_1, w_1\}$ (or $\{x, y', z_2, w_2\}$) is stable under all the shifts in $[n] \setminus X_1$, so it is still in $S_{x''y''}(\mathcal{H}'_4)$, contradicting that $S_{x''y''}(\mathcal{H}'_4) \subseteq \mathcal{G}_2$ with core $\{x'', x, y\}$. Thus we can not reach Case 4.

Now assume that after some shifts in $[n] \setminus X_1$ to \mathcal{H}_4 , we get \mathcal{H}_5 such that $S_{x''y''}(\mathcal{H}_5) \subseteq \mathcal{G}_3$ with center and core forming a 4-set *B* for some x'' and $y'' \in [n] \setminus X_1$. By Fact 2.4 (iv), (13) and Claim 3.2 (ii), we have $\{x, y, x''\} \subseteq B$. Since there are $\binom{n-2}{2}$ sets which contain $\{x, y\}$ in \mathcal{H}_1 (so in $S_{x''y''}(\mathcal{H}_5)$), we have one of the following cases:

(*) x is the center, and y is in the core;

(**) y is the center, and x is in the core.

Recall that there exists $\{y, y', z_1, w_1\}$ or $\{x, y', z_2, w_2\}$ in \mathcal{H}_4 . We will meet one of the following three cases:

(a) There is no set $G \in \mathcal{H}_4$ such that $G \cap \{x, y, x'\} = \{x\}$. So there exists $\{y, y', z_1, w_1\} \in \mathcal{H}_4$, and all sets containing $\{x', x\}$ in $S_{x'y'}(\mathcal{H}_4)$ are originally in \mathcal{H}_4 . Take $X_1 := \{x, y, y', z_1, w_1\}$. By the maximality of $|\mathcal{H}|$ (so is $|\mathcal{H}_4|$), there are $\binom{n-2}{2}$ sets containing $\{x', x\}$ in \mathcal{H}_4 (so in $S_{x''y''}(\mathcal{H}_5)$ as well). This implies that $x' \in E$, and x is the center. However, $\{y, y', z_1, w_1\}$ is contained in $S_{x''y''}(\mathcal{H}_5)$, a contraction to that $S_{x''y''}(\mathcal{H}_5) \subseteq \mathcal{G}_3$ with center x and core $\{y, x', x''\}$.

(b) There is no set $G \in \mathcal{H}_4$ such that $G \cap \{x, y, x'\} = \{y\}$. So there exists $\{x, y', z_2, w_2\} \in \mathcal{H}_4$, and all sets containing $\{x', y\}$ in $S_{x'y'}(\mathcal{H}_4)$ are originally in \mathcal{H}_4 . Take $X_1 := \{x, y, y', z_2, w_2\}$. By the maximality of $|\mathcal{H}|$ (so is $|\mathcal{H}_4|$), there are $\binom{n-2}{2}$ sets containing $\{x', y\}$ in \mathcal{H}_4 , so in $S_{x''y''}(\mathcal{H}_5)$. This implies that $x' \in E$ and y is the center for $S_{x''y''}(\mathcal{H}_5)$. However, $\{x, y', z_2, w_2\}$ is in $S_{x''y''}(\mathcal{H}_5)$, contradicting to that $S_{x''y''}(\mathcal{H}_5) \subseteq \mathcal{G}_3$ at center y and core $\{x, x', x''\}$.

(c) There are both $\{y, y', z_1, w_1\}$ and $\{x, y', z_2, w_2\}$ in \mathcal{H}_4 . We choose $X_1 := \{x, y, y', z_1, w_1\}$ first. Assume that (*) happens. Since $\{y, y', z_1, w_1\}$ is still in $S_{x''y''}(\mathcal{H}_5)$, this contradicts that $S_{x''y''}(\mathcal{H}_5) \subseteq \mathcal{G}_3$ with center x and $\{y, x''\}$ contained in the core. So we assume that (**) happens. Let $B = \{x, y, x'', u\}$ for some u. If u = x', then the existence of $\{y, y', z_1, w_1\}$ makes a contradiction again. Now consider $u \neq x'$.

Claim 3.7. If $u \neq x'$, then u = y'.

Proof. Assume on the contrary that $u \neq y'$. We have shown that $S_{x''y''}(\mathcal{H}_5)$ can not be contained in \mathcal{J}_2 at center y, then there are at least 3 sets containing $\{x, u, x''\}$. Although $\{x, x', x'', u\}$ and $\{x, y', x'', u\}$ may be two such sets, there must be $\{x, u, x'', v\} \in$ $S_{x''y''}(\mathcal{H}_5)$ for some $v \in [n] \setminus \{x, y, u, x', y', x''\}$. However, every set in \mathcal{H}_4 contains $\{x, y\}$, or $\{x', x\}$, or $\{x', y\}$, or $\{x, y'\}$, or $\{y, y'\}$ by recalling that $S_{x'y'}(\mathcal{H}_4) \subseteq \mathcal{G}_2$ with core $\{x, y, x'\}$, so is every set in $S_{x''y''}(\mathcal{H}_5)$ since $x'', y'' \in [n] \setminus \{x, y, y', z_1, w_1\}$, a contradiction.

By Claim 3.7, we have that $S_{x''y''}(\mathcal{H}_5) \subseteq \mathcal{G}_3$ at center y and core $\{x, x'', y'\}$. This time, we change X_1 to $X'_1 := \{x, y, y', z_2, w_2\}$. Similar to the lines in the first paragraph of the proof of Claim 3.6, we will not reach *Cases 1-4* after performing shifts $S_{x'y'}$ in $[n] \setminus X'_1$. If we reach *Case 5*, that is, after some shifts in $[n] \setminus X'_1$ to \mathcal{H}_4 , we get \mathcal{H}'_5 such that $S_{x''y''}(\mathcal{H}'_5) \subseteq \mathcal{G}_3$ with center and core forming a 4-set B'for some $x''', y''' \in [n] \setminus X'_1$. By the previous analysis, $B' = \{x, y, x''', y'\}$, and we only need to consider the case that x is the center (If y is the center, since $\{y, y', z_2, w_2\}$ is still in $S_{x'''y''}(\mathcal{H}'_5)$, this contradicts that $S_{x''y''}(\mathcal{H}'_5) \subseteq \mathcal{G}_3$ with center y and core $\{x, y', x'''\}$. We have shown that $S_{x'y'}(\mathcal{H}_5)$ can not be contained in \mathcal{G}_2 with core $\{x, y, y'\}$, so there is $G \in S_{x''y''}(\mathcal{H}_5)$ such that $G \cap \{x, y\} = \emptyset$ or $G \cap \{x, y'\} = \emptyset$ or $G \cap \{y, y'\} = \emptyset$. Since $S_{x''y''}(\mathcal{H}_5) \subseteq \mathcal{G}_3$ with core $\{x, x'', y'\}$ and center y, Gmust contain x or y. If $G \cap \{y, y'\} = \emptyset$, it contradicts that $S_{x''y''}(\mathcal{H}_5) \subseteq \mathcal{G}_3$ with core $\{x, x'', y'\}$ and center y. So there is $G \in S_{x''y''}(\mathcal{H}_5)$ such that $G \cap \{x, y'\} = \emptyset$. After shifts in $[n] \setminus X'_1$ to G, we get G' missing x and y' still. This contradicts that $S_{x'''y'''}(\mathcal{H}'_5) \subseteq \mathcal{G}_3$ with core $\{y, x''', y'\}$ and center x. Hence, we will not reach Case 5.

In summary, we have shown that there exists X_1 in the form of $\{x, y, y', z, w\}$ such that performing shifts in $[n] \setminus X_1$ repeatedly to \mathcal{H}_4 will not reach *Cases 1-5* as in Remark 2.5. Moreover, $\{y, y', z, w\}$ or $\{x, y', z, w\}$ is in the final stable family \mathcal{G} . This completes the proof of Claim 3.6.

Claim 3.8. If performing some shifts in $[n] \setminus \{x, y\}$ repeatedly to \mathcal{H}_1 does not reach Cases1-4, but reaches \mathcal{H}_5 in Case 5 ($S_{x'y'}(\mathcal{H}_5) \subseteq \mathcal{G}_3$), then there exists X_1 in the form of $\{x, y, y', z, w\}$ such that performing shifts in $[n] \setminus X_1$ repeatedly to \mathcal{H}_4 will not reach Cases 1-5 as in Remark 2.5. Moreover, $\{y, y', z, w\}$ or $\{x, y', z, w\}$ is in the final stable family \mathcal{G} .

Proof. Suppose that we get some \mathcal{H}_5 such that $S_{x'y'}(\mathcal{H}_5) \subseteq \mathcal{G}_3$ with center and core forming a 4-set B. By (13) and Claim 3.2, the center must be x or y, and $\{x, y\} \subset B$. By Fact 2.4 (iv), $X' \in B$ and $y' \notin B$. Let $B = \{x, y, x', z\}$. We consider the case that x is the center, the proof for y being the center is similar.

Since $S_{x'y'}(\mathcal{H}_5) \subseteq \mathcal{G}_3$, and recall that we are under Case 1, every set in \mathcal{H}_5 intersects $\{x, y\}$, there exists $\{y, y', z, w\}$ (or $\{x, y', z_1, z_2\}$) $\in \mathcal{H}_5$. And by the maximality of $|\mathcal{H}|$ (so is $|\mathcal{H}_5|$), we may assume that all the sets containing $\{x, z\}$ in $S_{x'y'}(\mathcal{H}_5)$ are originally in \mathcal{H}_5 . Let $X_1 := \{x, y, y', z, w\}$ (or $\{x, y, y', z_1, z_2\}$). Similar to the analysis in the first paragraph of the proof of Claim 3.6, for any shifts $S_{x''y''}$ to \mathcal{H}_5 in $[n] \setminus X_1$, we won't reach *Cases 1-4*. If we reach *Case 5* again, then the resulting family $S_{x''y''}(\mathcal{H}'_5)$ (x'' and $y'' \in [n] \setminus X_1$) must be contained in \mathcal{G}_3 with core $\{y, x'', z\}$ and center x. However $\{y, y', z, w\}$ (or $\{x, y', z_1, z_2\}$) is still in $S_{x''y''}(\mathcal{H}'_5)$, and misses x'' and x (or $\{x'', z, y\} \cap \{x, y', z_1, z_2\} = \emptyset$), contradicting that the family $S_{x''y''}(\mathcal{H}'_5) \subseteq \mathcal{G}_3$ with core $\{y, x'', z\}$ and center x. So we will not achieve *Case 5*, as desired.

By Claims 3.5, 3.6 and 3.8, we have shown that if we reach a 4-uniform family \mathcal{H}_1 such that \mathcal{H}_1 is EKR, then there exists a set X_1 with $|X_1| \leq 5$ and $\{x, y\} \subseteq X_1$ such that performing shifts $S_{x'y'}$ in $[n] \setminus X_1$ repeatedly to \mathcal{H}_1 will result in a stable family satisfying the conditions of Lemma 3.4. This completes the proof of Lemma 3.4. \Box

Lemma 3.9 (Case 2). If we each a 4-uniform family \mathcal{H}_2 such that $S_{xy}(\mathcal{H}_2)$ is HM at x, then there is a set $X_2 = \{x, y, z_1, z_2, z_3\}$ such that after a series of shifts $S_{x'y'}$ (x' < y' and $x', y' \in [n] \setminus X_2$) to \mathcal{H}_2 , we will reach a stable family \mathcal{G} satisfying the conditions of Theorem 2.2. Moreover, $\{z_1, z_2, z_3, y\}$ or $\{z_1, z_2, z_3, z_4'\} \in \mathcal{G}$. Furthermore, if $\{z_1, z_2, z_3, y\} \in \mathcal{G}$, then every member in \mathcal{G} contains x or y. If $\{z_1, z_2, z_3, z_4'\} \in \mathcal{G}$, then every other member in \mathcal{G} contains x or y.

Proof. Note that $S_{xy}(\mathcal{H}_2)$ contains exactly one set, say, $G_0 = \{z_1, z_2, z_3, z_4\}$, that misses x. W.l.o.g., let $z_1, z_2, z_3 \neq y$. Let $X_2 := \{x, y, z_1, z_2, z_3\}$. By the maximality of $|\mathcal{H}_2|$, we may assume

$$\left\{G \in \binom{X}{4} : \{x, y\} \subseteq G, G \cap G_0 \neq \emptyset\right\} \subseteq \mathcal{H}_2.$$

If $y \in G_0$, that is, $y = z_4$, then

$$|\{G \in \mathcal{H}_2 : \{x, y\}\}| = \binom{n-2}{2},\tag{14}$$

Otherwise, $y \notin G_0$. We have

$$|\{G \in \mathcal{H}_2 : \{x, y\}\}| = 4n - 18.$$
(15)

In particular, $\{x, y, z_1, z_2\}$, $\{x, y, z_1, z_3\}$ and $\{x, y, z_2, z_3\}$ are in \mathcal{H}_2 . Assume that applying shifts in $[n] \setminus X_2$ to \mathcal{H}_2 , we get \mathcal{H}' , such that $S_{x'y'}(\mathcal{H}')$ is EKR or HM or contained in \mathcal{J}_2 at center x'. However, the three sets $\{x, y, z_1, z_2\}$, $\{x, y, z_1, z_3\}$ and $\{x, y, z_2, z_3\}$ are still in $S_{x'y'}(\mathcal{H}')$ and they miss x', a contradiction. Thus we will not reach *Cases 1-3*.

Assume we reach *Case* 4 as in Remark 2.5, i.e., $S_{x'y'}(\mathcal{H}') \subseteq \mathcal{G}_2$ with core A. By (14), (15), Claim 3.1 and Fact 2.4 (ii), we have $A = \{x, y, x'\}$. However $\{z_1, z_2, z_3\} \cap \{x, y, x', y'\} = \emptyset$, after a series of shifts of $[n] \setminus X_2$ to $G_0 = \{z_1, z_2, z_3, z_4\}$, we get the resulting set $G'_0 \in \mathcal{H}'$ satisfying that $|G'_0 \cap| \leq \{x, y, x', y'\}$ 1, a contradiction to that $S_{x'y'}(\mathcal{H}') \subseteq \mathcal{G}_2$ with core $\{x, y, x'\}$. Thus we will not reach *Case* 4.

At last, assume $S_{x'y'}(\mathcal{H}') \subseteq \mathcal{G}_3$ as in Remark 2.5 (*Case 5*) with center and core forming a 4-set *B*. By Fact 2.4 (iv), $x' \in B$. By Claim 3.2 (ii) and (14), (15), there are at least 4n - 18 > 3n - 12 (n > 6) sets containing $\{x, y\}$, so $\{x, y, x'\} \subset B$. And if $\{x, y\} \subset E$, then the number of sets containing $\{x, y\}$ in \mathcal{H}' is at most 2n - 7, which is smaller than 4n - 18, this contradicts to (15). Thus the resulting family can only have center x or center y. First assume $y \in G_0$, that is $y = z_4$ and $G_0 = \{y, z_1, z_2, z_3\}$. This implies that $\{x, z_1, z_2, z_3\} \in \mathcal{H}_2$. Both $\{x, z_1, z_2, z_3\}$ and $G_0 = \{z_1, z_2, z_3, z_4\}$ are stable under shifts $S_{x'y'}(x' < y' \text{ and } x', y' \in [n] \setminus X_2$), so both of them are in $S_{x'y'}(\mathcal{H}')$. Since $x, x' \notin G_0$ and $S_{x'y'}(\mathcal{H}') \subseteq \mathcal{G}_3$ with $B \supset \{x, y, x'\}$, x can not be the center. But if y is the center, since $x', y \notin \{x, z_1, z_2, z_3\}$, also a contradiction. Next assume $y \notin G_0$. Notice that $\{z_1, z_2, z_3\} \cap \{x, y, x', y'\} = \emptyset$, after a series of shifts of $[n] \setminus X_2$ to G_0 , the resulting set $G'_0 \in S_{x'y'}(\mathcal{H}')$ satisfies that $G'_0 \cap \{x, y\} = \emptyset$, also contradicts that $S_{x'y'}(\mathcal{H}') \subseteq \mathcal{G}_3$ with $B \supset \{x, y, x'\}$, hence we will not reach *Case 5*.

Notice that if $y \in G_0$, we have $\{x, z_1, z_2, z_3\} \in \mathcal{H}_2$ and $G_0 = \{y, z_1, z_2, z_3\} \in \mathcal{H}_2$. Note that $\{z_1, z_2, z_3, y\}$ is stable under shifts $S_{x'y'}$ $(x' < y' \text{ and } x', y' \in [n] \setminus X_2)$, so $G_0 = \{z_1, z_2, z_3, y\} \in \mathcal{G}$. In this case, every member in \mathcal{H}_2 contains x or y, Since every member in \mathcal{H}_2 is stable at x and y, every member in \mathcal{G} contains x or y. If $y \notin G_0$, then $G'_0 = \{z_1, z_2, z_3, z'_4\} \in \mathcal{G}$ for some $z'_4 \neq y$, and this is the only set in \mathcal{G} that disjoint from set $\{x, y\}$. **Lemma 3.10** (*Case 3*). If we each a 4-uniform \mathcal{H}_3 such that $S_{xy}(\mathcal{H}_3) \subseteq \mathcal{J}_2$ at center x, kernel E and the set of pages J, then there is a set $X_3 = \{x, y, z_1, z_2, z_3\}$ such that after a series of shifts $S_{x'y'}$ (x' < y' and $x', y' \in [n] \setminus X_3$) to \mathcal{H}_3 , we will reach a stable family \mathcal{G} satisfying the conditions of Theorem 2.2 and $G \cap X_3 \neq \emptyset$ for any $G \in \mathcal{G}$. Moreover, either $|G \cap X_3| \geq 2$ for any $G \in \mathcal{G}$, or $|G \cap G'| \geq 2$ if $G \cap X_3 = \{x\}$ and $G' \cap X_3 = \{y\}$.

Proof. We will meet one of the following three cases. Case (a): $y \in E$. In this case, let $E = \{y, z_1, z_2\}, J = \{z_3, z_4\}$ and $X_3 := \{x, y, z_1, z_2, z_3\}$. Case (b): $y \in J$. In this case, let $E = \{z_1, z_2, z_3\}, J = \{y, z_4\}$ and $X_3 := \{x, y, z_1, z_2, z_3\}$. Case (c): $y \in [n] \setminus (E \cup J \cup \{x\})$. In this case, let $E = \{z_1, z_2, z_3\}, J = \{z_4, z_5\}$ and $X_3 := \{x, y, z_1, z_2, z_3\}$.

In each of the above three cases, by the maximality of $|\mathcal{H}|$ ($|\mathcal{H}_3|$ as well), $\{x, y, z_1, z_2\}$, $\{x, y, z_1, z_3\}$, $\{x, y, z_2, z_3\}$ are in \mathcal{H}_3 , and they are stable after a series of shifts in $[n] \setminus X_3$, so we will not reach *Cases 1-3* after performing shifts in $[n] \setminus X_3$. Assume that applying shifts in $[n] \setminus X_3$ to \mathcal{H}_3 , we get \mathcal{H}'' , such that $\mathcal{H}' := S_{x'y'}(\mathcal{H}'') \subseteq \mathcal{G}_2$ with core A. Similarly, by the maximality of $|\mathcal{H}_3|$ and direct computation, we have the following claim:

Claim 3.11. There are at least $\binom{n-2}{2}$, 4n - 18, 3n - 11 members that contain $\{x, y\}$ in Cases (a), (b), (c) respectively.

Notice that $\binom{n-2}{2}$, 4n - 18, 3n - 11 > 2n - 7. By Claim 3.1, Claim 3.11 and Fact 2.4 (ii), $A = \{x, y, x'\}$. In Case (a) or (b), we can see that $\{y, z_1, z_2, z_3\} \in \mathcal{H}'$, $|\{y, z_1, z_2, z_3\} \cap A| = 1$, a contradiction. In Case (c), we have $\{z_1, z_2, z_3, z_4\} \in \mathcal{H}_3$, after some shifts in $[n] \setminus X_3$, it becomes F in \mathcal{H}' , and $|F \cap A| \leq 1$, a contradiction to that $\mathcal{H}' \subseteq \mathcal{G}_2$ with core $\{x, y, x'\}$. Thus we will not reach *Case 4* after performing shifts in $[n] \setminus X_3$ repeatedly.

At last, we assume that $\mathcal{H}' := S_{x'y'}(\mathcal{H}'') \subseteq \mathcal{G}_3$ with center and core forming a 4-set *B*. By Claim 3.2, Claim 3.11 and Fact 2.4 (iv), we have $\{x, y, x'\} \subseteq B$, and the center of \mathcal{H}' must be *x* or *y*. In Cases (a) and (b), we have $\{y, z_1, z_2, z_3\} \in \mathcal{H}_3$, so in \mathcal{H}' . Since $x, x' \notin \{y, z_1, z_2, z_3\}$, \mathcal{H}' can not be contained in \mathcal{G}_3 with $B \supset \{x, y, x'\}$ and center *x*. Since $\{x, z_1, z_2, z_3\} \in \mathcal{H}_3$, so in \mathcal{H}' as well. Notice that $y, x' \notin \{x, z_1, z_2, z_3\}$, \mathcal{H}' can not be contained in \mathcal{G}_3 with $B \supset \{x, y, x'\}$ and center *y*. A contradiction. Now consider Case (c). In this case, $\{z_1, z_2, z_3, z_4\} \in \mathcal{H}_3$. Because it is stable at $\{z_1, z_2, z_3\}$ under any shift in $[n] \setminus X_3$, the resulting set $\{z_1, z_2, z_3, z'_4\}$ does not contain *x* or *y*. This contradicts that $\mathcal{H}' \subseteq \mathcal{G}_3$ with $B \supset \{x, y, x'\}$ and center *x* or *y*.

If Case (a) or (b) happens, then any 4-set $G \in \mathcal{G}$ satisfies $|G \cap X_3| \geq 2$. If Case (c) happens, since $\{z_1, z_2, z_3, z_4\}$ and $\{z_1, z_2, z_3, z_5\}$ are the only two sets disjoint from $\{x, y\}$ in $S_{xy}(\mathcal{H}_3)$, then every set in \mathcal{H}_3 (so in \mathcal{G}) missing x and y must contain $\{z_1, z_2, z_3\}$. If $x \in G$, $y \in G'$ and $G \cap \{z_1, z_2, z_3, y\} = G' \cap \{z_1, z_2, z_3, x\} = \emptyset$, let $F, F' \in \mathcal{H}_3$ such that G and G' become their resulting sets in \mathcal{G} after a series of shifts in $[n] \setminus X_3$. By the reason that $S_{xy}(\mathcal{H}_3) \subseteq \mathcal{J}_2$ with center x, kernel $\{z_1, z_2, z_3\}$ and the set of pages $\{z_4, z_5\}$, for any set $H \in \mathcal{H}_3$ satisfying that $|H \cap \{x, y\}| = 1$ and $H \cap \{z_1, z_2, z_3\} = \emptyset$, we have $\{z_4, z_5\} \subseteq H$. So $\{z_4, z_5\} \subseteq F \cap F'$, consequently, $|G \cap G'| \ge 2$.

Lemma 3.12 (*Case 4*). If we reach a 4-uniform \mathcal{H}_4 such that $S_{xy}(\mathcal{H}_4) \subseteq \mathcal{G}_2$ with core $\{x, x_1, x_2\}$, then there is a set $X_4 = \{x, y, x_1, x_2, x_3\}$ such that after a series of shifts $S_{x'y'}$ (x' < y' and $x', y' \in [n] \setminus X_4$) to \mathcal{H}_4 , we will reach a stable family \mathcal{G} satisfying the conditions of Theorem 2.2. Moreover, $\{x, y, x_1, x_3\} \in \mathcal{G}$ and $G \cap X_4 \neq \emptyset$ for any $G \in \mathcal{G}$.

Proof. Since $S_{xy}(\mathcal{H}_4) \subseteq \mathcal{G}_2$ with core A, by Fact 2.4 (ii), we have that $x \in A$ and $y \notin A$. Let $A = \{x, x_1, x_2\}$. By the maximality of $|\mathcal{H}_4|$, we may assume

$$\left\{ G \in \begin{pmatrix} X \\ 4 \end{pmatrix} : \{x_1, x_2\} \subseteq G \right\} \subseteq \mathcal{H}_4,$$
$$\left\{ G \in \begin{pmatrix} X \\ 4 \end{pmatrix} : \{x, y\} \subseteq G, G \cap \{x_1, x_2\} \neq \emptyset \right\} \subseteq \mathcal{H}_4$$

So

$$|\{G \in \mathcal{H}_4 : \{x_1, x_2\} \subseteq G\}| = \binom{n-2}{2}, \tag{16}$$

$$|\{G \in \mathcal{H}_4 : \{x, y\} \subseteq G, G \cap \{x_1, x_2\} \neq \emptyset\}| = 2n - 7.$$
(17)

Choose a set $G = \{x, y, x_1, x_3\} \in \mathcal{H}_4$ and let $X_4 := \{x, y, x_1, x_2, x_3\}$. Since $S_{xy}(\mathcal{H}_4) \subseteq \mathcal{G}_2$ with core $\{x, x_1, x_2\}$, every member in \mathcal{H}_4 intersects X_4 . Every member in \mathcal{H}_4 is stable at every element in X_4 under shifts $S_{x'y'}$ (x' < y' and $x', y' \in [n] \setminus X_4$). So $\{x, y, x_1, x_3\}$ is in the final stable family \mathcal{G} and $G \cap X_4 \neq \emptyset$ for any $G \in \mathcal{G}$. What remains is to show that performing shifts $S_{x'y'}$ (x' < y' and $x', y' \in [n] \setminus X_4$) to \mathcal{H}_4 will not reach *Cases 1-5* in Remark 2.5.

By (16), for any $x' \in [n] \setminus X_4$, there are at least $\binom{n-3}{2}$ members in \mathcal{H}_4 missing x', so we can not reach *Cases 1-3*.

Assume $\mathcal{H}' := S_{x'y'}(\mathcal{H}'') \subseteq \mathcal{G}_2$ with core A'. By (16), Fact 2.4 (ii) and Claim 3.1, $A' = \{x', x_1, x_2\}$. Since $G \in \mathcal{H}'$, and $|H \cap A'| = 1$, we get a contradiction, hence we will not reach *Case 4*. At last, assume $\mathcal{H}' := S_{x'y'}(\mathcal{H}'') \subseteq \mathcal{G}_3$ with center and core forming a 4-set B. By Fact 2.4 (iv), $x' \in B$. By Claim 3.2 (ii) and (16), $\{x_1, x_2\} \subseteq B$ and the center must be x_1 or x_2 . That is $\{x_1, x_2, x'\} \subset B$. Since |B| = 4, $|\{x, y\} \cap B| = 0$ or 1. If $|\{x, y\} \cap B| = 0$, then the sets containing $\{x, y\}$ in \mathcal{H}' must contain center and one point of core A', so $d_{\{x,y\}} \leq 3$. If $|\{x, y\} \cap B| = 1$, then the sets containing $\{x, y\}$ in \mathcal{H}' either contain center or contain core A', so $d_{\{x,y\}} \leq n - 3 + 1 = n - 2$. These members containing $\{x, y\}$ in \mathcal{H}_4 are also in \mathcal{H}' , by (17), there are at least 2n - 7, a contradiction. Hence we can not reach *Case 5*.

Lemma 3.13 (Case 5). If we reach a 4-uniform \mathcal{H}_5 such that $S_{xy}(\mathcal{H}_5) \subseteq \mathcal{G}_3$ with center and core E forming a 4-set B, then there is a set $X_5 = \{x, y, x_1, x_2, x_3\}$ such that after a series of shifts $S_{x'y'}$ (x' < y' and $x', y' \in [n] \setminus X_5$) to \mathcal{H}_5 , we will reach a stable family \mathcal{G} satisfying the conditions of Theorem 2.2. Furthermore, $|G \cap X_5| \ge 2$ for each $G \in \mathcal{G}$. Proof. For $S_{xy}(\mathcal{H}_5)$, we will meet one of the following three cases. Case (a): x is the center, $y \in E$, and $E = \{y, x_1, x_2\}$. In this case, we may assume that $\{y, x_1, x_2, x_3\} \in S_{xy}(\mathcal{H}_5)$ for some $x_3 \in [n] \setminus B$. Let $X_5 := \{x, y, x_1, x_2, x_3\}$. Case (b): x is the center, $y \notin E$, and $E = \{x_1, x_2, x_3\}$. In this case, let $X_5 := \{x, y, x_1, x_2, x_3\}$. Case (c): x_1 is the center, $x \in E$, and $E = \{x, x_2, x_3\}, y \in [n] \setminus B$. In this case, let $X_5 := \{x, y, x_1, x_2, x_3\}$. We first observe that $|G \cap X_5| \ge 2$ for each $G \in \mathcal{H}_5$ in each case.

First we consider Case (a). In this case, a member in \mathcal{H}_5 must contain x or y. By the maximality of $|\mathcal{H}_5|$, we may assume

$$\left\{G \in \binom{X}{4} : \{x, y\} \subseteq G\right\} \subseteq \mathcal{H}_5$$

So

$$|\{G \in \mathcal{H}_5 : \{x, y\} \subseteq G\}| = \binom{n-2}{2}.$$
(18)

Performing $S_{x'y'}$ in $[n] \setminus X_5$ to \mathcal{H}_5 will not reach *Cases 1-3* since there are at least $\binom{n-3}{2}$ members that containing $\{x, y\}$ and missing x' in \mathcal{H}_5 and these sets are stable after $S_{x'y'}$ in $[n] \setminus X_5$ (by (18)).

Assume that $\mathcal{H}' := S_{x'y'}(\mathcal{H}'') \subseteq \mathcal{G}_2$ with core A. By (18), Fact 2.4 (ii) and Claim 3.1, $A = \{x', x, y\}$. Since $S_{xy}(\mathcal{H}_5)$ is not EKR, $\{y, x_1, x_2, x_3\} \in S_{xy}(\mathcal{H}_5)$, $\{x, x_1, x_2, x_3\} \in \mathcal{H}_5$, so in \mathcal{H}' . However, $|\{x, x_1, x_2, x_3\} \cap A| = 1$, this is a contradiction, hence we will not reach *Case 4*. Assume that $\mathcal{H}' := S_{x'y'}(\mathcal{H}'') \subseteq \mathcal{G}_3$ with center and core forming a 4-set B'. By (18), Fact 2.4 (iv) and Claim 3.2 (ii), $\{x, y, x'\} \subseteq B'$, and the center is either x or y. In either case, the existence of $\{x, x_1, x_2, x_3\}$ and $\{y, x_1, x_2, x_3\}$ will lead to a contradiction. Hence we will not reach *Case 5*.

Next we consider Case (b). By the maximality of $|\mathcal{H}_5|$, we may assume that

$$\left\{G \in \binom{X}{4} : \{x, y\} \subseteq G \text{ and } G \cap E \neq \emptyset\right\} \subseteq \mathcal{H}_5$$

and

$$\left\{G \in \binom{X}{4} : \{x_1, x_2, x_3\} \subseteq G\right\} \subseteq \mathcal{H}_5.$$

In particular, $\{x, x_1, x_2, x_3\} \in \mathcal{H}_5$ and $\{y, x_1, x_2, x_3\} \in \mathcal{H}_5$. Computing directly, we have

$$|\{G \in \mathcal{H}_5 : \{x, y\} \subseteq G, G \cap E \neq \emptyset\}| = 3n - 12$$
(19)

and

$$|\{G \in \mathcal{H}_5 : \{x_1, x_2, x_3\} \subseteq G\}| = n - 3.$$
(20)

Since $\{x, y, x_1, x_2\}, \{x, y, x_1, x_3\}, \{x, y, x_2, x_3\} \in \mathcal{H}_5$ and these sets miss x' and are stable after shifts $S_{x'y'}$ $(x' < y' \text{ and } x', y' \in [n] \setminus X_5)$, we will not reach *Cases 1-3*.

Assume $\mathcal{H}' := S_{x'y'}(\mathcal{H}'') \subseteq \mathcal{G}_2$ with core A, where x' < y' and $x', y' \in [n] \setminus X_5$. By (19), Fact 2.4 (ii) and Claim 3.1, we have $A = \{x', x, y\}$. However $\{x, x_1, x_2, x_3\} \in \mathcal{H}'$ and $|\{x, x_1, x_2, x_3\} \cap A| = 1$, a contradiction, so we will not reach *Case 4*.

Assume that $\mathcal{H}' := S_{x'y'}(\mathcal{H}'') \subseteq \mathcal{G}_3$ with center and core forming a 4-set B'. By Fact 2.4 (iv), $x' \in B'$. Equation (19) and Claim 3.2 (i) imply that the center must be x or y.

By Claim 3.3 and (20), either $\{x_1, x_2, x_3\} \subset B'$ or $|\{x_1, x_2, x_3\} \cap B'| = 2$ and one of $\{x_1, x_2, x_3\}$ is the center. But it's impossible to satisfy both conditions in the previous paragraph and this paragraph, hence we will not reach *Case 5*.

At last we consider Case (c). By the maximality of $|\mathcal{H}_5|$, we may assume

$$\left\{G \in \binom{X}{4} : \{x_1, x_2\} \subseteq G\right\} \subseteq \mathcal{H}_5 \text{ and } \left\{G \in \binom{X}{4} : \{x_1, x_3\} \subseteq G\right\} \subseteq \mathcal{H}_5.$$

By direct computation,

$$|\{G \in \mathcal{H}_5 : \{x_1, x_2\} \subseteq G\}| = \binom{n-2}{2},$$
 (21)

$$|\{G \in \mathcal{H}_5 : \{x_1, x_3\} \subseteq G\}| = \binom{n-2}{2}.$$
 (22)

Since there are $\binom{n-3}{2}$ sets containing $\{x_1, x_2\}$ but missing x', after performing $S_{x'y'}$ in $[n] \setminus X_5$ to \mathcal{H}_5 , we will not reach *Case 1-3*.

If we reach *Cases 4*, that is, after performing shifts in $[n] \setminus X_5$ to \mathcal{H}_5 repeatedly, $S_{x'y'}(\mathcal{H}') \subseteq \mathcal{G}_2$ with core *A*. By (21), (22), Fact 2.4 (ii) and Claim 3.1, $x', x_1, x_2, x_3 \in A$, but |A| = 3, a contradiction. If we reach *Case 5*, that is $S_{x'y'}(\mathcal{H}') \subseteq \mathcal{G}_3$ with the center and the core forming a 4-set *B'*. By (21), (22), Fact 2.4 (iv) and Claim 3.1, $B' = \{x_1, x_2, x_3, x'\}$, and x_1 is the center. Recall that $\{x, y, x_2, x_3\} \in \mathcal{H}_5$, also in $S_{x'y'}(\mathcal{H}')$, but $\{x, y, x_2, x_3\} \cap \{x_1, x', y'\} = \emptyset$, a contradiction, hence we cannot reach *Case 5*.

As remarked earlier, $|G \cap X_5| \ge 2$ for each $G \in \mathcal{H}_5$. Note that performing shifts in $[n] \setminus X_5$ to \mathcal{H}_5 keeps this property, so $|G \cap X_5| \ge 2$ for each $G \in \mathcal{G}$. \Box

By Lemmas 3.4 to 3.13, we have shown that if one of *Case 1-5* happens, then there exists a set X_i with $|X_i| \leq 5$ and $\{x, y\} \subseteq X_i$ such that performing shifts in $[n] \setminus X_i$ to \mathcal{H}_i will not result in any case of *Case 1-5*, so the final family is a stable family satisfying the conditions in Theorem 2.2. Furthermore, $G \cap X_i \neq \emptyset$ for any set G in the final family. So we complete the proof of Lemma 2.6.

3.2 Proof of Lemma 2.7

Proof. We first consider $k \ge 5$. In this case, we have $|X_i| \le 3$ and $|Y_i| \ge 2k - 3$.

We first prove (ii). Assume on the contrary that there are G and $G' \in \mathcal{G}$ such that $G \cap G' \cap Y = \emptyset$ and let $|G \cap G'|$ be the minimum among all pairs of sets in \mathcal{G}

not intersecting in Y. Clearly $|G \cap G' \cap ([n] \setminus Y)| \ge 1$. Note that $|(G \cup G') \cap Y_i| \le |G \cap Y_i| + |G' \cap Y_i| \le 2k - 4$ (since $|G \cap ([n] \setminus Y)| \ge 1$ and $|G \cap X_i| \ge 1$, so $|G \cap Y_i| \le k - 2$, same for G'). But $|Y_i| \ge 2k - 3$, so there exists a point $a \in Y_i \setminus (G \cup G')$. Pick any point $b \in G \cap G' \cap ([n] \setminus Y)$, we have a < b. Notice that \mathcal{G} is stable on $[n] \setminus X_i$, so $G'' := (G' \setminus \{b\}) \cup \{a\} \in \mathcal{G}$. Then $G \cap G'' \cap Y = \emptyset$ and $|G \cap G''| < |G \cap G'|$, contradicting the minimality of $|G \cap G'|$.

For (i), assume on the contrary, that $\mathcal{A}_1 \neq \emptyset$. Let $\{x\} \in \mathcal{A}_1$, then there is a set $G \in \mathcal{G}$ such that $G \cap Y = \{x\}$. By (ii), for any another set $G' \in \mathcal{G}$ we have $G \cap G' \cap Y \neq \emptyset$, so $x \in G'$. This implies that \mathcal{G} is EKR, a contradiction, so $\mathcal{A}_1 = \emptyset$.

Next consider for k = 4. In this case, for $1 \le i \le 5$, $|X_i| = 5$ and $|Y_i| = 9 - 5 = 4$, and for i = 6, $|X_i| = 0$ and $|Y_i| = 9$.

Claim 3.14. If G and G' in \mathcal{G} satisfies that $|Y_i \setminus (G \cup G')| \ge |G \cap G' \cap ([n] \setminus Y)|$, then $G \cap G' \cap Y \neq \emptyset$.

Proof. If $G \cap G' \cap Y = \emptyset$, then $D := G \cap G' \cap ([n] \setminus Y) \neq \emptyset$. Since $|Y_i \setminus (G \cup G')| \ge |G \cap G' \cap ([n] \setminus Y)|$, there is a subset $D' \subseteq Y_i \setminus (G \cup G')$ with size |D'| = |D|. By the definition of Y_i , all numbers in D' are smaller than D. Since \mathcal{G} is stable on $[n] \setminus X_i$, $F := (G' \setminus D) \cup D' \in \mathcal{G}$. However $G \cap F = \emptyset$, a contradiction to the intersecting property of \mathcal{G} . So $G \cap G' \cap Y \neq \emptyset$.

Claim 3.15. $|\mathcal{A}_1| \leq 1$; \mathcal{A}_2 and \mathcal{A}_4 are intersecting.

Proof. Obviously, \mathcal{A}_4 is intersecting. Assume that $|\mathcal{A}_1| \geq 2$ and $\{x_1\}, \{x_2\} \in \mathcal{A}_1$. Then there are G and G' in $\widetilde{\mathcal{A}}_1$ such that $G \cap Y = \{x_1\}$ and $G' \cap Y = \{x_2\}$. Since any set in \mathcal{G} intersects with X_i (for $i \in [5]$), $x_1, x_2 \in X_i$. So $1 \leq |G \cap G' \cap ([n] \setminus Y)| \leq$ $3 < 4 = |Y_i \setminus (G \cap G')|$. By Claim 3.14, $G \cap G' \cap Y \neq \emptyset$, a contradiction. Hence, $|\mathcal{A}_1| \leq 1$. Let G and G' be in $\widetilde{\mathcal{A}}_2$. Then $|G \cap G' \cap ([n] \setminus Y)| \leq 2$. Since $|G \cap X_i| \geq 1$ and $|G' \cap X_i| \geq 1$ (for $i \in [5]$), then $|Y_i \setminus (G \cup G')| \geq 2$. By Claim 3.14, $G \cap G' \cap Y \neq \emptyset$, that is \mathcal{A}_2 is intersecting, as desired.

Claim 3.16. $A_1 = \emptyset$.

Proof. By Claim 3.15, $|\mathcal{A}_1| \leq 1$. We may assume on the contrary that $\mathcal{A}_1 = \{\{x\}\}$ for some $x \in X_i$. For any $G \in \widetilde{\mathcal{A}}_1$ and $G' \in \widetilde{\mathcal{A}}_j$ (for j = 2, 3, 4), G and G' satisfy the condiction of Claim 3.14, so $G \cap G' \cap Y \neq \emptyset$, this implies that $x \in G'$ and hence \mathcal{G} is EKR, a contradiction.

Claim 3.17. A_2 and A_3 are cross-intersecting.

Proof. Let $G \in \widetilde{\mathcal{A}}_2$ and $G' \in \widetilde{\mathcal{A}}_3$. Then $|G \cap G' \cap ([n] \setminus Y)| \leq 1$. Since any set in \mathcal{G} intersects with X_i (for $i \in [5]$), $|Y_i \setminus (G \cup G')| \geq 1$. By Claim 3.14, $G \cap G' \cap Y \neq \emptyset$, that is \mathcal{A}_2 and \mathcal{A}_3 are cross-intersecting, as desired.

Claim 3.18. A_3 is intersecting.

Proof. Assume on the contrary, that there exist $A, A' \in \mathcal{A}_3$ and $G, G' \in \mathcal{A}_3$ such that $G \cap Y = A, G' \cap Y = A'$ and $A \cap A' = \emptyset$, in other words, $G \cap G' \cap Y = \emptyset$ and $|G \cap G' \cap ([n] \setminus Y)| = 1$. If $|(G \cup G') \cap Y_i| \leq 3$, by Claim 3.14, $G \cap G' \cap Y \neq \emptyset$, a contradiction. Hence we only need to consider the following case : $|A \cap X_i| = 1, |A \cap Y_i| = 2, |A' \cap X_i| = 1$ and $|A' \cap Y_i| = 2$. Now we show the conclusion for each case of Lemma 2.6. All sets below are inherited from the proof of Lemma 2.6 for each cooresponding case.

If we meet *Cases 1* in Lemma 2.6, then by Lemma 3.4, we have that $X_1 = \{x, y, y', z_1, w_1\}$ or $X_1 = \{x, y, y', z_2, w_2\}$, and we may assume that $G \cap X_i = \{x\}$ and $G' \cap X_i = \{y\}$. Respectively, $\{y, y', z_1, w_1\}$ or $\{x, y', z_2, w_2\} \in \mathcal{G}$, which is disjoint from G or G'. A contradiction to the intersecting property of \mathcal{G} .

If we meet *Cases* 2 in Lemma 2.6, then by Lemma 3.9, we have that $X_2 = \{x, y, z_1, z_2, z_3\}$, and either $\{z_1, z_2, z_3, y\} \in \mathcal{G}$ or $\{z_1, z_2, z_3, z_4'\} \in \mathcal{G}$ for some $y \neq z_4'$. Furthermore, if $\{z_1, z_2, z_3, y\} \in \mathcal{G}$, then every member in \mathcal{G} contains x or y. So we may assume that $G \cap X_i = \{x\}$ and $G' \cap X_i = \{y\}$. Then $\{z_1, z_2, z_3, y\} \cap G = \emptyset$, a contradiction. If $\{z_1, z_2, z_3, z_4'\} \in \mathcal{G}$, then every other member in \mathcal{G} contains x or y, we may assume that $G \cap X_i = \{x\}$ and $G' \cap X_i = \{y\}$. Since \mathcal{G} is stable, we may assume that $z_4' \in Y_i$. Recall that $|G \cap Y_i| = |G' \cap Y_i| = 2$, hence $\{z_1, z_2, z_3, z_4'\}$ must be disjoint from G or G', a contradiction.

If we meet Cases 3 in Lemma 2.6, then by Lemma 3.10, $|G \cap G'| \ge 2$, a contradiction.

If we meet *Cases 4* in Lemma 2.6, then by Lemma 3.12, we have that $X_4 = \{x, y, x_1, x_2, x_3\}$, $\{x, y, x_1, x_3\} \in \mathcal{G}$ and $S_{xy}(\mathcal{H}_4) \subseteq \mathcal{G}_2$ with core $\{x, x_1, x_2\}$. So for every set F in \mathcal{H}_4 , either $|F \cap \{x, x_1, x_2\}| \ge 2$, or $F \cap \{x, x_1, x_2\} = \{x_1\}$ and $y \in F$, or $F \cap \{x, x_1, x_2\} = \{x_2\}$ and $y \in F$. In all cases, $|F \cap X_4| \ge 2$. Performing shifts in $[n] \setminus X_4$ will not change these properties, hence every set in \mathcal{G} also has the same properties, in particular, G and G' do. This makes a contradiction to $|G \cap X_4| = |G' \cap X_4| = 1$.

Assume that we meet *Case 5* in Lemma 2.6, then by Lemma 3.13, we have that $|G \cap X_5| \ge 2$ for each $G \in \mathcal{G}$. This makes a contradiction to $|G \cap X_5| = |G' \cap X_5| = 1$.

At last, assume that we will not meet any of Cases 1-5 in Lemma 2.6 if we perform shifts repeatedly to \mathcal{G} . In this case, Y = [2k]. Assume on the contrary, and let Gand $G' \in \mathcal{G}$ such that $G \cap G' \cap Y = \emptyset$ and $|G \cap G'|$ is the minimum among all pairs of sets in \mathcal{G} not intersecting in Y. Then $|G \cap G' \cap (X \setminus Y)| \ge 1$. Consequently, $|(G \cup G') \cap Y| \le |G \cap Y| + |G' \cap Y| \le 2k - 2$ since $|G \cap Y| \le k - 1$ and $|G' \cap Y| \le k - 1$. So there exists a point $a \in Y \setminus (G \cup G')$. Pick any point $b \in G \cap G' \cap (X \setminus Y)$. Note that a < b, then $G'' := (G' \setminus \{b\}) \cup \{a\} \in \mathcal{G}$ since \mathcal{G} is stable. It is easy to see that $G \cap G'' \cap Y = \emptyset$ and $|G \cap G''| < |G \cap G'|$, contradicting the minimality of $|G \cap G'|$. \Box

Since \mathcal{G} is intersecting, \mathcal{A}_2 and \mathcal{A}_4 are cross-intersecting, and \mathcal{A}_3 and \mathcal{A}_4 are cross-intersecting. Combining with Claims 3.15, 3.17 and 3.18, we have completed the proof of (ii).

4 Concluding remarks

It is natural to ask what is the maximum size of a k-uniform intersecting family \mathcal{F} with $\tau(\mathcal{F}) \geq 3$. About this problem, Frankl [3] gave an upper bound for sufficient large n. To introduce the result, we need the following construction.

Construction 4.1. Let $x \in [n]$, $Y \subseteq [n]$ with |Y| = k, and $Z \subseteq [n]$ with |Z| = k - 1, $x \notin Y \cup Z$, $Z \cap Y = \emptyset$ and $Y_0 = \{y_1, y_2\} \subseteq Y$. Define

$$\mathcal{G} = \{A \subseteq [n] : x \in A, A \cap Y \neq \emptyset \text{ and } A \cap Z \neq \emptyset\} \cup \{Y, Z \cup \{y_1\}, Z \cup \{y_2\}, \{x, y_1, y_2\}\} \in FP(n, k) = \{F \subseteq [n] : |F| = k, \exists G \in \mathcal{G} \text{ s.t.}, G \subseteq F\}.$$

It is easy to see that FP(n,k) is intersecting and $\tau(FP(n,k)) = 3$.

Theorem 4.2 (Frankl [3]). Let $k \geq 3$ and n be sufficiently large integers. Let \mathcal{H} be an n-vertex k-uniform family with $\tau(\mathcal{H}) \geq 3$. Then $|\mathcal{H}| \leq |FP(n,k)|$. Moreover, for $k \geq 4$, the equality holds only for $\mathcal{H} = FP(n,k)$.

It is interesting to consider what is the maximum k-uniform intersecting families with covering number $s \ge 4$.

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