Pure pairs. X. Tournaments and the strong Erdős-Hajnal property.[1]

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Abstract

A *pure pair* in a tournament G is an ordered pair (A, B) of disjoint subsets of $V(G)$ such that every vertex in B is adjacent from every vertex in A. Which tournaments H have the property that if G is a tournament not containing H as a subtournament, and $|G| > 1$, there is a pure pair (A, B) in G with $|A|, |B| \geq c|G|$, where $c > 0$ is a constant independent of G? Let us say that such a tournament H has the *strong EH-property*

As far as we know, it might be that a tournament H has this property if and only if its vertex set has a linear ordering in which its backedges form a forest. Certainly this condition is necessary, but we are far from proving sufficiency. We make a small step in this direction, showing that if a tournament can be ordered with at most three backedges then it has the strong EH-property (except for one case, that we could not decide). In particular, every tournament with at most six vertices has the property, except for three that we could not decide. We also give a seven-vertex tournament that does not have the strong EH-property.

This is related to the Erdős-Hajnal conjecture, which in one form says that for every tournament H there exists $\tau > 0$ such that every tournament G not containing H as a subtournament has a transitive subtournament of cardinality at least $|G|^\tau$. Let us say that a tournament H satisfying this has the *EH-property.* It is known that every tournament with the strong EH-property also has the EH-property; so our result extends work by Berger, Choromanski and Chudnovsky, who proved that every tournament with at most six vertices has the EH-property, except for one that they did not decide.

1 Introduction

A *tournament* is a digraph G, with no loops, such that for every pair u, v of distinct vertices, exactly one of uv, vu is an edge. (All graphs and digraphs in this paper are finite, and have no loops or parallel edges.) Let us say a tournament G *contains* a tournament H if there is a subtournament of G isomorphic to H, and G is *H-free* otherwise. We denote the number of vertices of G by $|G|$.

The *Erdős-Hajnal conjecture* was raised as a question by Erdős and Hajnal [11, 12] and asserts that, for every graph H, there exists $\tau > 0$ such that every graph G not containing an induced subgraph isomorphic to H has a clique or stable set of cardinality at least $|G|^\tau$. Alon, Pach and Solymosi [1] showed that it is equivalent to the following assertion about tournaments:

1.1 Conjecture: *For every tournament H there exists $\tau > 0$ such that every H-free tournament G has a transitive subtournament with at least $|G|^\tau$ vertices.*

For a tournament H, if there exists $\tau > 0$ as in 1.1, we say that H has the *EH-property* or *(weak) EH-property.* Thus the conjecture says that all tournaments have the EH-property.

Let P_7 denote the Paley tournament with seven vertices; that is, its vertex set is $\{1, \ldots, 7\}$, and for all distinct $i, j \in \{1, \ldots, 7\}$, j is adjacent from i if $j - i$ is congruent to $1, 2$ or 4 modulo 7. Let P_7^- denote the tournament obtained by deleting one vertex from P_7. (It makes no difference which vertex is deleted.) Berger, Choromanski and Chudnovsky [4] showed:

1.2 *Every tournament with at most six vertices has the EH-property, except possibly for P_7^-.*

There are other classes of tournaments that have been shown to have the EH-property: see for instance [3, 6, 19].

In a graph G, a *pure pair* is a pair A, B of disjoint subsets of $V(G)$ such that either there are no edges between A, B or every vertex in A is adjacent to every vertex in B; and its *order* is $\min(|A|, |B|)$. Let us say a *pure pair* in a tournament G is an ordered pair (A, B) of disjoint subsets of $V(G)$ such that every vertex in B is adjacent from every vertex in A; and its *order* is $\min(|A|, |B|)$. And let us say a tournament H has the *strong EH-property* or *SEH property* if there exists $c > 0$ such that for every H-free tournament G with $|G| > 1$, there is a pure pair in G with order at least $c|G|$. It is easy to see that every tournament with the strong EH-property also has the EH-property, but not all tournaments have the strong EH-property; we shall see that P_7 does not. So it is natural to ask which tournaments do. (One can take the same approach for graphs – see [8].) This question seems not to have been studied to any great extent. We discussed it briefly in [8]; "heroes", defined in [2], have the strong EH-property; and Berger, Choromanski, Chudnovsky and Zerbib [5] proved that D_5 has the strong EH-property (D_5 is defined below); but we know of nothing else on the topic.

A *numbering* of a graph is an enumeration $(v_1, \ldots, v_n)$ of its vertex set; and an *ordered graph* is a graph together with some numbering. If $(v_1, \ldots, v_n)$ is a numbering of a tournament H, then the corresponding *backedge graph* of H is the ordered graph B with vertex set $V(H)$ and numbering $(v_1, \ldots, v_n)$, in which for $1 \le i < j \le n$, v_i and v_j are adjacent in B if and only if v_i is adjacent from v_j in H. Its edges are called *backedges.* A tournament can be reconstructed from a backedge graph and the corresponding numbering, and it is often convenient to work with the backedge graph rather than directly with the tournament.

Different numberings of the same tournament may result in wildly different backedge graphs, of course. For instance, D_5 is the (unique, up to isomorphism) tournament with five vertices, in which every vertex has outdegree two; and the following are two of its backedge graphs (in such figures, vertices are always numbered from left to right):

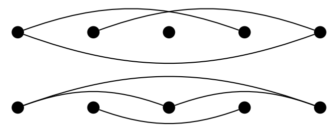

Figure 1: Two backedge graphs for D_5.

For a graph G, we denote its complement graph by $\overline{G}$; and if G is an ordered graph, $\overline{G}$ means the complement graph with the same numbering. Let us say the *reverse* $\overline{H}$ of a tournament H is obtained by reversing the direction of all edges of H. A tournament has the strong EH-property if and only if its reverse does. Note that, if under some numbering a tournament H has backedge graph B, then $\overline{B}$ is the backedge graph of $\overline{H}$ under the same numbering; and B with its numbering reversed is the backedge graph of $\overline{H}$ under the reverse numbering.

Can we hope to characterize the tournaments with the strong EH-property? A parallel question for graphs had a very satisfactory answer: we proved in [8] that:

1.3 *For a graph H, the following are equivalent:*

- *there exists $c > 0$ such that for every graph G with $|G| > 1$ not containing H or $\overline{H}$ as an induced subgraph, there is a pure pair A, B in G with order at least $c|G|$;*
- *one of $H, \overline{H}$ is a forest.*

One might hope for a parallel for this in the world of tournaments. Certainly, one half is true: we will show, in 10.2, that

1.4 *Every tournament with the strong EH-property admits a numbering for which the backedge graph is a forest.*

As far as we know, the converse to this might also be true. Initially this seemed unlikely to us, but we have tried hard to disprove it and failed, so let us pose it as a conjecture:

1.5 Conjecture: *A tournament has the strong EH-property if and only if it admits a numbering for which the backedge graph is a forest.*

This would be a beautiful analogue of 1.3, but we are far from proving it. Indeed, the tournament P_7^- has a backedge graph that is a forest with only four edges (see figure 2), and we cannot even show that it has the (weak) EH-property.

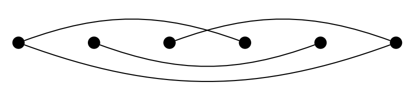

Figure 2: Backedge graph for P_7^-.

There is a recent positive result in this area: it is shown in [14] that if a tournament can be built from nothing by repeatedly adding vertices with in-degree at most one or out-degree at most one (and consequently admits an ordering in which the backedge graph is a forest), then it has the (weak) EH-property.

The first main result of this paper is:

1.6 *Let H be a tournament that admits a numbering $(v_1, \ldots, v_n)$ for which the backedge graph B has at most three edges. Suppose that H is also D_5-free, that is, there do not exist a, b, c, d, e with $1 \le a < b < c < d < e \le n$ such that $E(B) = \{v_a v_d, v_a v_e, v_b v_e\}$. Then H has the strong EH-property.*

To see the equivalence asserted in the second sentence, observe that if there exist a, b, c, d, e as stated then H contains D_5; and conversely, if H contains D_5 then there exist a, b, c, d, e as stated, since up to isomorphism there is only one backedge graph of D_5 with only three edges. (We leave the reader to check this.) Perhaps the second sentence in 1.6 (the condition about D_5) can be omitted, but that remains open.

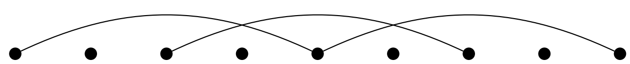

Figure 3: Backedge graph of a tournament satisfying 1.6.

For instance, 1.6 implies that the tournament with the backedge graph in figure 3 has the strong EH-property. A referee kindly told us that the methods of earlier papers would not show this. Let H_6 be the tournament with a backedge graph as in figure 4.

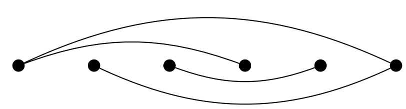

Figure 4: Backedge graph for H_6.

1.6 will be used to show our second main result, that:

1.7 *Every tournament with at most six vertices has the strong EH-property, except possibly for P_7^-, H_6 and $\overline{H_6}$.*

All except one of them (and that one is easy) either contain D_5 or admit backedge graphs with at most three edges, and so we can apply 1.6.

We remark that if we just wanted to prove that these tournaments have the (weak) EH-property, we could make use of the theorem of Alon, Pach and Solymosi [1] that the class of tournaments with the EH-property is closed under substitution, and so it would only be necessary to examine the tournaments that are not built from smaller ones by substitution. But the class with the strong EH-property is *not* closed under substitution.

The paper is organized as follows. Sections 2–6 are devoted to proving 1.6, and then we turn to 1.7. We need to prove that if $|H| \le 6$ then H has the SEH property (except for three cases).

- In sections 7 and 8, we prove that if H contains D_5 then H has the SEH property (except for two of the exceptional cases). This proof is a modification of a proof of Berger, Choromanski, Chudnovsky and Zerbib [5], who showed that D_5 itself has the strong EH-property.

- In section 9, we prove by case-by-case analysis, that every tournament with at most six vertices admits a numbering for which the backedge graph has at most three edges, except for four particular tournaments, three of which are the exceptions in 1.7 (it is easy to show that the fourth has the SEH property). So 1.7 follows from 1.6.

- In section 10 we prove 1.4, and show that P_7 does not have the SEH property. Finally, in section 11 we give two other tournaments that have the SEH property but not a "rainbow" refinement of it discussed in the proof of 1.6.

2 The strong EH-property for ordered graphs

An ordered graph G *contains* another (H say) if some induced subgraph of G, with the induced numbering, is isomorphic (as an ordered graph) to H; and if not, G is H*-free*. It is helpful to recast our problem about tournaments into the language of ordered graphs. We observe first that:

2.1 *Let G be a tournament, and let J be a backedge graph of G, with numbering $(v_1, \ldots, v_n)$.*

- *If G has a pure pair of order t, then J has a pure pair of order at least $t/2$.*

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Proof. Let (A, B) be a pure pair of order t in G, and choose $i \in \{1, \ldots, n\}$ minimum such that one of A', B' has cardinality at least $t/2$, where $A' = \{v_1, \ldots, v_i\} \cap A$ and $B' = \{v_1, \ldots, v_i\} \cap B$. Define $A'' = A \setminus A'$ and $B'' = B \setminus B'$. If $|A'| \geq t/2$, then from the minimality of i, $|B'| < t/2$, and so $|B''| \geq t/2$; and since every vertex in B is G-adjacent from every vertex in A, it follows that there are no edges of J between A' and B'', and this pair of sets is the desired pure pair of J. Similarly, if $|B'| \geq t/2$, then every vertex of B' is J-adjacent to every vertex in in A'', and so this is the desired pure pair. This proves the first assertion.

For the second, let A, B be a pure pair of order t in J, choose i as before, and define A', A'', B', B'' as before. By exchanging A, B if necessary, we may assume that $|A'| \geq t/2$; and so either (A', B'') (if there are no edges of J between A, B) or (B'', A') (if every vertex in A is J-adjacent to every vertex in B) is the desired pure pair of G. This proves the second assertion, and so proves 2.1. ∎

Alon, Pach and Solymosi [1] proved that the Erdős-Hajnal conjecture is equivalent to the same statement for ordered graphs: that is, for every ordered graph H there exists $\tau > 0$ such that every H-free ordered graph G has a clique or stable set of cardinality at least $|G|^\tau$. One can extend the "strong EH-property" to ordered graphs in the natural way, but while it makes sense to ask which tournaments have the strong EH-property, the same question for ordered graphs is unprofitable, as only very trivial ordered graphs have the property. For instance, a result of Fox [13] shows that the ordered graph with vertices v_1, v_2, v_3 numbered in this order, and edges v_1v_2, v_2v_3, does *not* have the property (see [17, 18] for related results). We can show (we omit the proof) that if an ordered graph has this property, then each of its components either has at most two vertices, or is a three-vertex path with middle vertex the first or last in the induced numbering, or is one particular four-vertex ordered path.

It is better to exclude more than one ordered graph at the same time. Let $\mathcal{A}$ be a set of ordered graphs. We say an ordered graph G is $\mathcal{A}$*-free* if G is H-free for all $H \in \mathcal{A}$; and $\mathcal{A}$ has the *strong*

EH-property if there exists $c > 0$ such that every $\mathcal{A}$-free ordered graph G with $|G| > 1$ has a pure pair of order at least $c|G|$.

To translate our question about tournaments into the language of ordered graphs, we observe that

- because of 2.1, a tournament G has a linear pure pair if and only if some (or equivalently, every) backedge graph of G has a linear pure pair (with a different constant of linearity);
- a tournament G does not contain a tournament H if and only if some (and therefore every) backedge graph of G contains none of $B_1, \ldots, B_k$, where $B_1, \ldots, B_k$ are the backedge graphs of H that arise from the different numberings of H.

Thus a tournament H has the strong EH-property if and only if the set $\mathcal{A}$ of all backedge graphs that arise from H under its different numberings has the strong EH-property.

This set $\mathcal{A}$ can be rather large. For instance, when H is D_5, it has 24 nonisomorphic backedge graphs, and we are looking at the ordered graphs that contain none of 24 specific ordered graphs. Excluding just one of them is not enough, but 24 is more than we need; the proof given in 8.3 shows that a subset of four of them already has the strong EH-property, the two shown in figure 1 and their complements. A similar thing happens for all the tournaments we can handle: we need to retain at most three (usually two) backedge graphs and their complements.

3 Blockades and rainbow subgraphs

Our goal at the moment is to show that all D_5-free tournaments that admit backedge graphs with at most three edges have the strong EH-property. We will prove that in fact they have a stronger property that we explain now.

A *blockade* in a set V is a family $\mathcal{B} = (B_i : i \in I)$ of pairwise disjoint nonempty subsets of V, where I is a finite set of integers. (We have used blockades in several papers of this series, for instance in [8].) Its *length* is $|I|$, and the minimum of $|B_i|$ $(i \in I)$ is its *width*. We write $W(\mathcal{B})$ to denote the width of $\mathcal{B}$. We call the sets B_i *blocks* of the blockade. (What matters is that the blocks are not too small. We could shrink the larger ones to make them all the same size.) We are interested in blockades of some fixed length in the vertex set of some graph, ordered graph or tournament G, in which each block contains linearly many vertices of G.

If $(v_1, \ldots, v_n)$ is a numbering of V, a blockade $(B_i : i \in I)$ *respects* the numbering if for all $i_1, i_2 \in I$ with $i_1 < i_2$, if $v_h \in B_{i_1}$ and $v_j \in B_{i_2}$ then $h < j$. In this case we say $\mathcal{B}$ is *respectful.*

Let $\mathcal{B}$ be a blockade in a set V. A graph (or ordered graph, or tournament) H with $V(H) \subseteq V$ is $\mathcal{B}$*-rainbow* if each vertex of H belongs to some block of $\mathcal{B}$, and no two vertices belong to the same block. A *copy* of a graph (or ordered graph, or tournament) H is another such object isomorphic to H.

In order to prove that a tournament has the strong EH-property, it is often easier to prove something even stronger. Let us say a tournament H has the *rainbow strong EH-property* or *RSEH-property* if there exists c with $0 < c < 1$ such that if $\mathcal{B}$ is a blockade of length at least $1/c$ in a tournament G, and there is no $\mathcal{B}$-rainbow copy of H contained in G, then there is a pure pair in G of order at least $cW(\mathcal{B})$.

3.1 *If H is a tournament with the RSEH-property then H has the strong EH-property.*

Proof. Choose $c > 0$ as in the definition of the RSEH-property; by reducing c we may assume that $k = 1/c$ is an integer. Let $c' = c^2/2$. Now let G be an H-free tournament with $|G| > 1$. We claim that G has a pure pair of order at least $c'|G|$. Since $|G| > 1$, we may assume that $|G| > 1/c'$, since otherwise a pure pair of order 1 exists and satisfies the theorem. There is a blockade $\mathcal{B}$ in G of length k, where

$$W(\mathcal{B}) \geq \lfloor |G|/k \rfloor = \lfloor c|G| \rfloor \geq c|G|/2;$$

and since G is H-free there is certainly no $\mathcal{B}$-rainbow copy of H contained in G. Thus G has a pure pair of order at least $cW(\mathcal{B}) \geq c^2|G|/2 = c'|G|$. This proves 3.1. ∎

The converse of 3.1 is not true: we will see that D_5 has the strong EH-property, but not the RSEH-property. On the other hand, we will show that:

3.2 *If H is a D_5-free tournament that admits a backedge graph with at most three edges, then H has the RSEH-property.*

Similarly, if $\mathcal{A}$ is a set of ordered graphs, we say that $\mathcal{A}$ has the *rainbow strong EH-property* or *RSEH-property* if there exists c with $0 < c < 1$ such that if $\mathcal{B}$ is a respectful blockade of length at least $1/c$ in an ordered graph G, and there is no $\mathcal{B}$-rainbow copy of any member of $\mathcal{A}$ contained in G, then there is a pure pair in G of order at least $cW(\mathcal{B})$.

Evidently we have:

3.3 *If H is a tournament and the set of all backedge graphs of H (or a subset of this set) has the RSEH-property, then H has the RSEH-property.*

An *anticomplete pair* in a graph G (possibly ordered) is a pair A, B of disjoint subsets of $V(G)$ such that there are no edges between A, B; and its *order* is $\min(|A|, |B|)$.

We need to throw a form of sparsity into this sea of definitions too: we say that a set $\mathcal{A}$ of ordered graphs has the *sparse rainbow strong EH-property* or *SRSEH-property* if there exists c with $0 < c < 1$ such that if $\mathcal{B}$ is a respectful blockade of length at least $1/c$ in an ordered graph G, and there is no $\mathcal{B}$-rainbow copy of any member of $\mathcal{A}$ contained in G, and every vertex of G has degree less than $cW(\mathcal{B})$, then there is an anticomplete pair in G of order at least $cW(\mathcal{B})$. We say c is an *SRSEH-coefficient* for $\mathcal{A}$.

Next we show that if a set of ordered graphs has the SRSEH-property then it together with its set of complement graphs has the RSEH-property. The proof will use the following theorem of [7]:

3.4 *For all $\varepsilon > 0$ and every graph P on p vertices, there exist $\gamma, \delta > 0$ such that if G is a graph containing fewer than $\gamma|G|^p$ induced labelled copies of P, then there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that one of $G[X], \overline{G}[X]$ has maximum degree at most $\varepsilon\delta|G|$.*

3.5 *Let $\mathcal{A}$ be a set of ordered graphs, and let $\mathcal{A}'$ be the set of complements of the members of $\mathcal{A}$. If $\mathcal{A}$ has the SRSEH-property then $\mathcal{A} \cup \mathcal{A}'$ has the RSEH-property.*

Proof. Choose $H \in \mathcal{A}$. By 7.2, there is a graph P such that for every numbering of P, the ordered graph that results contains H. Let $p = |P|$.

Let c' be an SRSEH-coefficient for $\mathcal{A}$. By reducing c' we may assume that $1/c'$ is an integer at least two ($1/c' = K'$ say).

Let $\varepsilon \le c'^2/2$ with $\varepsilon > 0$, and choose $\gamma, \delta > 0$ to satisfy 3.4. Choose c with $1/c$ an integer ($1/c = K$ say), such that $c \le c'\delta/2$, and $(1-cp)^p > 1-\gamma$, and $c \le \delta/c - 1/c'$. We claim that c satisfies our requirement.

(1) $\delta K/K' - 1 \ge \max(\varepsilon\delta K/c', c/c')$.

To see that $\delta K/K' - 1 \ge \varepsilon\delta K/c'$, observe that $\delta K/(2K') = \delta c'/(2c) \ge 1$, and $\delta K/(2K') \ge \varepsilon\delta K/c'$. The second part, that $\delta K/K' - 1 \ge c/c'$, is true from the choice of c. This proves (1).

Let G be an ordered graph with a blockade $\mathcal{B} = (B_1, \dots, B_K)$ that respects the numbering $(v_1, \dots, v_n)$ of G, such that there is no $\mathcal{B}$-rainbow copy of any member of $\mathcal{A} \cup \mathcal{A}'$. Let $W = W(\mathcal{B})$. We must show that there is a pure pair in G of order at least cW. We may assume that $V(G) = B_1 \cup \dots \cup B_K$, and that $|B_i| = W$ for each i, and so $|G| = KW$, and

$$B_i = \{v_j : (i-1)W < j \le iW\}$$

for $1 \le i \le K$.

The number of injections ϕ from $V(P)$ into $V(G)$ such that the vertices $\phi(v)$ $(v \in V(P))$ all belong to different blocks of the blockade, is

$$|G|(|G|-W)(|G|-2W)\cdots(|G|-(p-1)W) > (1-p/K)^p|G|^p \ge (1-\gamma)|G|^p,$$

and since none of them give an isomorphism from P to an induced subgraph of G (from the choice of P, and since there is no $\mathcal{B}$-rainbow copy of H in G), it follows that the number of induced labelled copies of P in G is less than $|G|^p - (1-\gamma)|G|^p = \gamma|G|^p$. By 3.4, there exists $X \subseteq V(G)$ with $|X| \ge \delta KW$, such that one of $G[X], \overline{G}[X]$ has maximum degree less than $\varepsilon\delta KW$; and by replacing G by $\overline{G}$ if necessary, we may assume that $G[X]$ has maximum degree less than $\varepsilon\delta KW$. By (1), there exists a real number W' such that

$$\frac{\delta K}{K'} - 1 \ge \frac{W'}{W} \ge \max\left(\varepsilon\delta K/c', \frac{c}{c'}\right).$$

The sets $B_1 \cap X, \dots, B_K \cap X$ each have cardinality at most W, but their union has cardinality at least δKW. Define $i_0 = 0$, and inductively for $s = 1, 2, \dots$ choose $i_s \in \{1, \dots, K\}$ minimum such that $|B'_s| \ge W'$, where $B'_s = \bigcup_{i_{s-1} < i \le i_s} B_i \cap X$, if such a choice is possible; and let the first value of s where the choice is impossible be $s = t+1$. Thus $B'_1, \dots, B'_t$ are defined. From the minimality of each i_s it follows that $|B'_s| \le W' + W$ for $1 \le s \le t$. From the maximality of t,

$$|X \cap (B_{i_t+1} \cup \dots \cup B_K)| < W';$$

and so $\delta KW \le |X| \le t(W'+W) + W'$. Since $\delta K/K' - 1 \ge W'/W$ it follows that $t \ge K'$.

Let $\mathcal{B}'$ be the blockade $(B'_1, \dots, B'_{K'})$; it has width at least W', and it respects the numbering $(v_1, \dots, v_n)$. Let $G' = G[B'_1 \cup \dots \cup B'_{K'}]$. Every vertex of G' has degree less than $\varepsilon\delta KW \le c'W'$ in G'. Also there is no $\mathcal{B}'$-rainbow copy of any member of $\mathcal{A}$ in G', since such a copy would also be $\mathcal{B}$-rainbow. Hence from the choice of c', there is an anticomplete pair in G' of order at least $c'W' \ge cW$. This proves 3.5. ∎

By combining 3.5, 3.3 and 3.1, we have:

3.6 *Let H be a tournament. If $\mathcal{A}$ is a set of some of the backedge graphs of H, and $\mathcal{A}$ has the SRSEH-property, then H has the RSEH-property and hence the strong EH-property.*

Proof. Let $\mathcal{A}'$ be the set of complements of the members of $\mathcal{A}$. Thus all the members of $\mathcal{A}'$ are also backedge graphs of H under appropriate numberings of H (obtained by reversing the numberings that give the members of $\mathcal{A}$). Since $\mathcal{A}$ has the SRSEH-property, 3.5 implies that $\mathcal{A} \cup \mathcal{A}'$ has the RSEH-property; and hence so does the set of all backedge graphs of H. Consequently H has the RSEH-property, by 3.3, and hence the strong EH-property, by 3.1. This proves 3.6. ∎

4 Blockades

If we start with a blockade with great length, we might hope to make a smaller, but more tightly structured, blockade by shrinking or removing some of its blocks. Here are two useful ways to make smaller blockades from larger. First, if $\mathcal{B} = (B_i : i \in I)$ is a blockade, let $I' \subseteq I$; then $(B_i : i \in I')$ is a blockade, of smaller length but of at least the same width, and we call it a *sub-blockade* of $\mathcal{B}$. Second, for each $i \in I$ let $B'_i \subseteq B_i$ be nonempty; then the sequence $(B'_i : i \in I)$ is a blockade, of the same length but possibly of smaller width, and we call it a *contraction* of $\mathcal{B}$. A contraction of a sub-blockade (or equivalently, a sub-blockade of a contraction) we call a *minor* of $\mathcal{B}$.

If H is a $\mathcal{B}$-rainbow induced subgraph, its *support* is the set of all $i \in I$ such that $V(H) \cap B_i \neq \emptyset$. If J is an ordered graph, we define the *trace* of J (relative to $\mathcal{B} = (B_i : i \in I)$) to be the set of supports of all $\mathcal{B}$-rainbow copies of J. If $\tau \geq 1$ an integer, we say $\mathcal{B}$ is *τ-support-uniform* if for every ordered graph J with $|J| \leq \tau$, either the trace of J is empty, or it consists of all subsets of I of cardinality $|J|$.

Let $0 < \kappa \leq 1$ and $\tau \geq 1$. We say $\mathcal{B}$ is *(κ, τ)-support-invariant* if for every contraction $\mathcal{B}' = (B'_i : i \in I)$ of $\mathcal{B}$ of width at least κ times the width of $\mathcal{B}$, and for every ordered graph J with $|J| \leq \tau$, the trace of J relative to $\mathcal{B}$ equals the trace of J relative to $\mathcal{B}'$.

We need a theorem of [8], the following:

4.1 *Let $k \geq 0$ and $\tau \geq 1$ be integers, and $0 < \kappa \leq 1$; then there exist an integer K with the following property. Let $\mathcal{B} = (B_1, \ldots, B_K)$ be a blockade in a graph. Then there is a minor $\mathcal{B}'$ of $\mathcal{B}$, with length k and width at least $\kappa^{2^{K+\tau^2}} W(\mathcal{B})$, such that $\mathcal{B}'$ is τ-support-uniform and (κ, τ)-support-invariant.*

In fact we have cheated a little here: the theorem of [8] defines "τ-support-uniform" and "(κ, τ)-support-invariant" using ordered trees rather than general ordered graphs. But it is not worth writing the proof out again, since exactly the same argument works for general ordered graphs, except we have to replace the bound τ^τ used in [8] for the number of ordered trees on at most τ vertices, by the bound 2^{τ^2} for the number of ordered graphs on at most τ vertices. (This only changes the multiplicative constant in 4.1.)

One important application of 4.1 is the following.

4.2 *Let $\mathcal{A}$ be a set of ordered graphs, and let $H \in \mathcal{A}$. Suppose that $V_1, V_2 \subseteq V(H)$ with $V_1 \cup V_2 = V(H)$ and $V_1 \cap V_2 = \emptyset$, such that there are no edges of H between V_1, V_2. For $i = 1, 2$, let H_i be the ordered subgraph of H induced on V_i, and let $\mathcal{A}_i = \{H_i\} \cup (\mathcal{A} \setminus \{H\})$. If both $\mathcal{A}_1, \mathcal{A}_2$ have the SRSEH-property then so does $\mathcal{A}$.*

Proof. For $i = 1, 2$, let c_i be an SRSEH-coefficient for $\mathcal{A}_i$. By reducing c_1 or c_2 we may assume that $c_1 = c_2$ ($= c'$ say), and $1/c' \geq 2|H|$, and $1/c'$ is an integer. Let $k = 1/c'$, and choose K as in 4.1, taking $\tau = |H|$ and $\kappa = 1/2$. Let $c > 0$ with $c \leq c' 2^{-2^{K+|H|^2}}$ and $c \leq 1/K$.

We claim that, if $\mathcal{B}$ is a respectful blockade of length at least $1/c$ in an ordered graph G, and there is no $\mathcal{B}$-rainbow copy in G of any member of $\mathcal{A}$, and every vertex of G has degree less than $cW(\mathcal{B})$, then there is a pure pair in G of order at least $cW(\mathcal{B})$. Let $W = W(\mathcal{B})$. By 4.1, there is a minor $\mathcal{C} = (C_1, \ldots, C_k)$ of $\mathcal{B}$, of width W' where $W' \geq 2^{-2^{K+|H|^2}} W$, such that $\mathcal{C}$ is $|H|$-support-uniform and $(1/2, |H|)$-support-invariant. If there is no $\mathcal{C}$-rainbow copy in G of any member of $\mathcal{A}_1$, then since G has maximum degree less than $cW \leq c'W'$, there is an anticomplete pair in G of order at least $c'W' \geq cW$ as required. So we may assume that there is a $\mathcal{C}$-rainbow copy in G of some member of $\mathcal{A}_1$. But there is no $\mathcal{B}$-rainbow copy in G of any member of $\mathcal{A}$, so there is a $\mathcal{C}$-rainbow copy in G of H_1, and similarly we may assume (for a contradiction) that there is a $\mathcal{C}$-rainbow copy in G of H_2.

Let $(v_1, \ldots, v_n)$ be the numbering of H, and let $I_j = \{i : v_i \in V_j\}$ for $j = 1, 2$. Since $\mathcal{C}$ is $|H|$-support-uniform, there is a $\mathcal{C}$-rainbow copy J_1 of H_1 with support I_1. For $1 \leq i \leq k$, if $i \notin I_2$ let $D_i = C_i$, and if $i \in I_2$ let D_i be the set of vertices in C_i with no neighbour in $V(J_1)$. Thus $|D_i| \geq |C_i| - cW|J_1| \geq |C_i|/2$, since

$$cW|J_1| \leq cW|H| \leq (c'|H|) \left(2^{-2^{K+|H|^2}} W\right) \leq W'/2 \leq |C_i|/2.$$

Let $\mathcal{D} = (D_1, \ldots, D_k)$. Since $\mathcal{C}$ is $|H|$-support-uniform and $(1/2, |H|)$-support-invariant, and there is a $\mathcal{C}$-rainbow copy of H_2 in G, it follows that there is a $\mathcal{D}$-rainbow copy J_2 of H_2 in G with support I_2. But there are no edges of G between $V(J_1)$ and $V(J_2)$, and so G contains a $\mathcal{C}$-rainbow copy of H, a contradiction. This proves 4.2. ∎

Let $\mathcal{A}$ be a set of ordered graphs, and for each $H \in \mathcal{A}$ let C_H be a component of H (with the induced numbering). We call the set $\{C_H : H \in \mathcal{A}\}$ a *transversal* of $\mathcal{A}$. By repeated application of 4.2, it follows that:

4.3 *Let $\mathcal{A}$ be a finite set of ordered graphs. If every transversal of $\mathcal{A}$ has the SRSEH-property then $\mathcal{A}$ has the SRSEH-property.*

For instance, let $\mathcal{A} = \{H_1, H_2\}$, where for $i = 1, 2$, H_i is an ordered graph in which two components (A_i, B_i say) have more than two vertices, and perhaps some other components have at most two vertices. In order to prove that $\mathcal{A}$ has the SRSEH-property it would suffice to show that

$$\{A_1, A_2\}, \{A_1, B_2\}, \{B_1, A_2\}, \{B_1, B_2\}$$

all have the SRSEH-property. There are other transversals, using a one- or two-vertex component of one of H_1 or H_2, but they all obviously have the SRSEH-property and we don't have to check them. But in general we do have to check all four of the transversals given. When we come to work with a tournament, the art will be to select a small number of orderings of the tournament such that *every* transversal of the corresponding set $\mathcal{A}$ of backedge graphs has the SRSEH-property, so that we can apply 4.3.

5 Some sets of ordered graphs that have the SRSEH-property

In this section we prove that certain sets of ordered graphs have the SRSEH-property, enough that for every tournament we need to handle, it has a set of backedge graphs such that all their transversals can be shown to have the SRSEH-property.

A *left-star* is an ordered graph, with numbering $(v_1, \ldots, v_n)$ say, such that $n > 0$, and v_1 is adjacent to every other vertex, and every edge is incident with v_1. If it has n vertices it is also called a *left* $(n-1)$*-star*. We start with an easy one:

5.1 *Let $\mathcal{A}$ be a set of ordered graphs, such that all components of some member of $\mathcal{A}$ are left-stars or right-stars, and all components of some member of $\mathcal{A}$ are cliques. Then $\mathcal{A}$ has the SRSEH-property.*

Proof. By 4.3 it suffices to prove the result when $\mathcal{A}$ has two members, one a left-star and one a clique. Choose t such that both these ordered graphs have at most t vertices.

Let $N \geq 0$ be an integer such that every graph on at least N vertices has either a stable set or a clique of size $t-1$. Let $c = 1/(N+1)$. Let G be an ordered graph, let $\mathcal{B} = (B_1, \ldots, B_K)$ be a respectful blockade in G of length at least $1/c$, and suppose that there is no $\mathcal{B}$-rainbow copy in G of any member of $\mathcal{A}$, and every vertex of G has degree less than $cW(\mathcal{B})$. (The last condition will not be used.) We claim that G has an anticomplete pair of order at least $cW(\mathcal{B})$. We may assume that each B_i has cardinality $W(\mathcal{B})$. For $2 \leq i \leq N+1$, we may assume that fewer than $cW(\mathcal{B})$ vertices in B_1 have no neighbour in B_i, since this set of vertices forms an anticomplete pair with B_i. Since $Nc < 1$, it follows that there is a vertex $v_1 \in B_1$ with a neighbour $v_i \in B_i$ for $2 \leq i \leq N+1$. From the choice of N, either $t-1$ of the vertices $v_2, \ldots, v_{N+1}$ form a stable set (and then G contains a $\mathcal{B}$-rainbow copy of the left-star in $\mathcal{A}$) or $t-1$ of them form a clique (and then G contains a $\mathcal{B}$-rainbow copy of the clique in $\mathcal{A}$), in either case a contradiction. This proves 5.1. ∎

We need another theorem of [8], the following:

5.2 *For every tree T, there exists $c > 0$, such that for every graph G with a blockade $\mathcal{B}$ of length at least $1/c$, if there is no $\mathcal{B}$-rainbow copy of T, and every vertex has degree less than $cW(\mathcal{B})$, then there is an anticomplete pair of order at least $cW(\mathcal{B})$.*

This is a theorem about unordered graphs, and in particular, the vertices of the $\mathcal{B}$-rainbow copy of T might be in any order. Still, we can deduce some useful results about ordered graphs from it, for instance:

5.3 *Let $\mathcal{A}$ be a set of ordered graphs that does not have the SRSEH-property. For every tree T, there is a numbering of T such that the ordered graph formed by T with this numbering contains no member of $\mathcal{A}$.*

Proof. Let T be a tree, and let $c > 0$ satisfy 5.2. Since $\mathcal{A}$ does not have the SRSEH-property, there is an ordered graph G, and a respectful blockade $\mathcal{B}$ in G, such that G has maximum degree less than $cW(\mathcal{B})$, and there is no $\mathcal{B}$-rainbow copy of any member of $\mathcal{A}$ in G, and there is no anticomplete pair in G of order at least $cW(\mathcal{B})$. By the choice of c, there is a $\mathcal{B}$-rainbow copy of the unordered graph T in G; let J be this copy, with the induced numbering. Then J contains no member of $\mathcal{A}$. This proves 5.3. ∎

A *left-spike* is an ordered graph, with numbering $(v_1, \dots, v_n)$ say, such that $n > 1$, and v_2 is adjacent to every other vertex, and every edge is incident with v_2. *Right-spikes* are defined similarly. A *monotone path* is an ordered graph, with numbering $(v_1, \dots, v_n)$ where $n > 0$, with edge set $\{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$. Left-stars, left-spikes and monotone paths are all special cases of a *left-broom*, which is an ordered tree, with numbering $(v_1, \dots, v_n)$ where $n > 0$, and with edge set

$$\{v_1v_2, v_2v_3, \dots, v_{m-1}v_m\} \cup \{v_mv_{m+1}, v_mv_{m+2}, \dots, v_mv_n\}$$

for some m with $1 \le m \le n$. A *right-broom* is defined similarly.

If $(v_1, \dots, v_n)$ is a numbering, we say v_i is *earlier* than v_j if $i < j$, and v_i is *later* than v_j if $i > j$. Sometimes we will have different graphs and digraphs with the same vertex set, and we will sometimes speak of "H-neighbour" meaning "neighbour in H", and so on. 5.2 has the following consequence:

5.4 *Let $\mathcal{A}$ be a set of ordered graphs. If either $\mathcal{A}$ contains a left-star and a right-broom, or it contains a right-star and a left-broom, then $\mathcal{A}$ has the SRSEH-property.*

Proof. Let $\mathcal{A}$ contain a left-star L and a right-broom R, both with at most $t \ge 2$ vertices. Let T be the tree in which every vertex has degree either $2t$ or one; and there is a vertex v with degree $2t$ such that every path of T with one end v, and maximal with this property, has exactly $t - 1$ edges. Take a numbering of T, making an ordered graph T'. For each vertex u of T' with degree $2t$, we may assume that at least t of its neighbours are earlier than u, since otherwise T' contains L. Let $u_0 = v$, and having chosen u_i, if $i < t$ let u_{i+1} be a T'-neighbour of u_i that is earlier than u_i. For each i with $0 \le i < t$, there are at least t T'-neighbours of u_i that are earlier than u_i, and so T' contains R (because $|R| \le t$.) From 5.3, this proves 5.4. ∎

This will suffice to handle almost all the tournaments of interest to us, but there are a few tough ones that need something extra, provided by the following three results.

Let us say a *left-bristle* is an ordered graph, with numbering $(v_1, \dots, v_n)$ where $n > 2$, where v_1 is adjacent to $v_2, \dots, v_{n-1}$, and v_n is adjacent to exactly one of $v_2, \dots, v_{n-1}$, and there are no other edges. A *right-bristle* is defined similarly.

5.5 *Let $\mathcal{A}$ be a set of ordered graphs containing a left 2-star and a right-bristle; or containing a right 2-star and a left-bristle. Then $\mathcal{A}$ has the SRSEH-property.*

Proof. Choose $t \ge 3$ such that $\mathcal{A}$ contains a left 2-star and a right-bristle R, both with at most t vertices. Let T be the tree in which every vertex has degree either $2t$ or one; and there is a vertex v such that every path of T with one end v, and maximal with this property, has exactly two edges. Thus $|T| = 4t^2 + 1$. Let c' satisfy 5.2 (with c replaced by c').

Choose $c > 0$ such that $c \le c'/(1 + c')$ and $1/c \ge 4t^2 + 2$. Let G be an ordered graph, let $\mathcal{B} = (B_1, \dots, B_K)$ be a respectful blockade in G of length at least $1/c$, and suppose that there is no $\mathcal{B}$-rainbow copy in G of any member of $\mathcal{A}$, and every vertex of G has degree less than $cW(\mathcal{B})$. Let $W = W(\mathcal{B})$. We claim that G has an anticomplete pair of order at least cW. We may assume that each B_i has cardinality W.

For $2 \le i \le 4t^2 + 2$ let C_i be the set of vertices in B_i with a neighbour in B_1. We may assume that $|B_i \setminus C_i| < cW$, since $B_i \setminus C_i$ is anticomplete to B_1, and so $|C_i| > (1 - c)W$. If there is no $(C_2,\dots,C_{4t^2+2})$-rainbow copy of T, then by 5.2, either some vertex of G has degree at least $c'(1-c)W$, or G has an anticomplete pair of order at least $c'(1-c)W$, and since $c'(1-c)\ge c$, we are done.

Thus we may assume (for a contradiction) that there is a $(C_2,\dots,C_{4t^2+2})$-rainbow copy of T, and we assume (to simplify notation) that this is T itself. Now v has $2t$ neighbours in T, and at least $2t-1$ of them are earlier than v; let $2t-1$ of them be $a_1,\dots,a_{2t-1}$, numbered in order. One neighbour of a_t is later than a_t, namely v, and so all the others are earlier than a_t, say $b_1,\dots,b_{2t-1}$, again numbered in order. Since T is $(C_2,\dots,C_{4t^2+2})$-rainbow, it follows that b_t has a neighbour $x\in B_1$. Now x is nonadjacent to $b_1,\dots,b_{t-1}$ and to $b_{t+1},\dots,b_{2t-1}$. If x is also nonadjacent to a_t, then the ordered subgraph induced on $\{x,a_t,b_1,\dots,b_{2t-1}\}$ contains R, a contradiction. Thus x is adjacent to a_t. By the same argument, x is nonadjacent to $a_1,\dots,a_{t-1}$ and to $a_{t+1},\dots,a_{2t-1}$; and it is also nonadjacent to v, since otherwise $\{x,b_t,v\}$ induces a $\mathcal{B}$-rainbow left 2-star. But then $\{x,v,a_1,\dots,a_{2t-1}\}$ contains R, a contradiction. This proves 5.5. ∎

Let us say a *crossed left-star* is an ordered graph, with a numbering $(v_1,\dots,v_n)$, such that v_1 is adjacent to $v_2,\dots,v_n$, and only one edge is not incident with v_1. (Thus it consists of a left-star and one more edge joining some pair of leaves of the left-star.) A *crossed right-star* is defined similarly.

5.6 *If $\mathcal{A}$ contains a three-vertex monotone path, and a crossed left-star, and a crossed right-star then $\mathcal{A}$ has the SRSEH-property.*

Proof. Let the crossed left-star and the crossed right-star both have at most n vertices. Let $c=1/(2n)$, let G be an ordered graph, let $\mathcal{B}=(B_1,\dots,B_K)$ be a respectful blockade in G of length $K\ge 1/c$, let W be its width, and suppose that there is no $\mathcal{B}$-rainbow copy in G of any member of $\mathcal{A}$, and every vertex of G has degree less than cW. We claim that G has an anticomplete pair of order at least cW. We may assume that each B_i has cardinality W.

For $i=2,3,\dots,2n-1$ in turn, we will inductively define $A_i\subseteq B_i$ with the following properties:

- either at least $(1-1/n)W$ vertices in B_1, or at least $(1-1/n)W$ vertices in B_{2n}, have a neighbour in A_i;
- the sets $A_2,A_3,\dots,A_i$ are pairwise anticomplete; and
- for all $j\in\{2,\dots,2n-1\}\setminus\{i\}$, fewer than cW vertices in B_j have a neighbour in A_i.

The inductive definition is as follows. Assume that $2\le i\le 2n-1$, and $A_2,\dots,A_{i-1}$ are defined. Let X be the set of vertices in B_i that have a neighbour in one of $A_2\cup\dots\cup A_{i-1}$; thus $|X|\le(i-2)cW$. Since $|B_i\setminus X|\ge cW$, we may assume that fewer than cW vertices in B_n have no neighbour in $B_i\setminus X$, since otherwise G has an anticomplete pair of order at least cW. Since $cW\le W/n$, we may choose $A_i\subseteq B_i\setminus X$ minimal such that either at least $(1-1/n)W$ vertices in B_1, or at least $(1-1/n)$ vertices in B_{2n}, have a neighbour in A_i. Since each vertex in A_i has fewer than cW neighbours in B_n, the minimality of A_i implies that at most $(1-1/n)W+cW$ vertices in B_1 have a neighbour in A_i, and hence the set Z of vertices in B_1 with no neighbour in A_i has cardinality at least $(1/n-c)W\ge cW$. Similarly the set Z' of vertices in B_{2n} with no neighbour in A_i has cardinality at least cW. Let $2\le j\le 2n-1$ with $j\ne i$. We claim that fewer than cW vertices in B_j have a neighbour in A_i. To see this, suppose that $j>i$ (the argument when $j<i$ is similar and we omit it). If $v\in B_j$ has a neighbour in A_i, then it has no neighbour in Z', since there is no $\mathcal{B}$-rainbow monotone three-vertex

path in G; and since we may assume that G has no anticomplete pair of order at least cW, and $|Z'| \geq cW$, it follows that fewer than cW vertices in B_j have a neighbour in A_i as claimed. This completes the inductive definition. In summary, we have:

- for $2 \leq i \leq 2n-1$, either at least $(1-1/n)W$ vertices in B_1, or at least $(1-1/n)$ vertices in B_{2n}, have a neighbour in A_i;
- the sets $A_2, A_3, \ldots, A_{2n-1}$ are pairwise anticomplete; and
- for all distinct $i, j \in \{2, \ldots, 2n-1\}$, fewer than cW vertices in B_j have a neighbour in A_i.

From the symmetry, we may assume that for at least $n-1$ values of $i \in \{2, \ldots, 2n-1\}$, at least $(1-1/n)W$ vertices in B_1 have a neighbour in A_i. Choose $n-1$ such values, say $i_2, \ldots, i_n$ in increasing order, and define $i_1 = 1$. Since at most W/n vertices in B_1 have no neighbour in each B_{i_s} for $2 \leq s \leq n$, there is a set $M_1 \subseteq B_1$ with $|M_1| \geq W/n$ such that every vertex in M_1 has a neighbour in each of $A_{i_2}, \ldots, A_{i_n}$.

There is a crossed left-star L in $\mathcal{A}$; let its numbering be $(v_1, \ldots, v_t)$ say where $t \leq n$, and v_a, v_b are adjacent for some a, b with $2 \leq a < b \leq t$. Let M_{i_a} be the set of all vertices in B_{i_a} with no neighbour in any of the sets A_{i_s} where $s \in \{2, \ldots, t\} \setminus \{a, b\}$, and define $M_{i_b} \subseteq B_{i_b}$ similarly. Thus $|M_{i_a}|, |M_{i_b}| \geq W - (t-1)cW \geq 2cW$. We may assume that fewer than cW vertices in M_{i_a} have no neighbour in M_1 (since $|M_1| \geq cW$) and similarly, fewer than cW vertices in M_{i_a} have no neighbour in M_{i_b}; and so some vertex $u_{i_a} \in B_{i_a}$ has a neighbour $u_1 \in M_1$ and a neighbour $u_{i_b} \in M_{i_b}$. Since there is no $\mathcal{B}$-rainbow three-vertex path in G it follows that u_1, u_{i_b} are adjacent. For each $s \in \{2, \ldots, t\} \setminus \{a, b\}$, choose $u_{i_s} \in A_{i_s}$ adjacent to u_1 (this is possible since every vertex of M_1 has a neighbour in A_{i_s}). Then the ordered subgraph induced on $\{u_{i_1}, u_{i_2}, \ldots, u_{i_t}\}$ is a $\mathcal{B}$-rainbow copy of L, a contradiction. This proves 5.6. ∎

A *left-split* is an ordered graph, with numbering $(v_1, \ldots, v_n)$, such that:

- v_1, v_2 are nonadjacent, and $\{v_3, \ldots, v_n\}$ is a clique; and
- for $3 \leq i \leq n$, v_i is adjacent to at most one of v_1, v_2.

5.7 *If $\mathcal{A}$ contains a left 2-star, and a crossed right-star, and a left-split, then $\mathcal{A}$ has the SRSEH-property.*

Proof. For $t \geq 1$, a *t-uniform crossed right-star* is a crossed right-star with numbering $(v_1, \ldots, v_n)$, where $n = 3t-1$, and v_t, v_{2t} are adjacent. Every crossed right-star is contained in a t-uniform crossed right-star for all sufficiently large t.

For $t \geq 1$, a *t-uniform left-split* is a left-split with numbering $(v_1, \ldots, v_n)$, where $n = 3t+2$, such that for $i = 3, \ldots, n$, v_1 is adjacent to v_i if i is divisible by three, and v_2 is adjacent to v_i if $i-1$ is divisible by three. Every left-split is contained in a t-uniform left-split for all sufficiently large t. Choose $t \geq 1$ such that some member of $\mathcal{A}$ is contained in a t-uniform crossed right-star, and some member of $\mathcal{A}$ is contained in a t-uniform left-split.

Choose K satisfying 4.1, taking $k = 35t+1$ and $\tau = 8t+1$ and $\kappa = 1/4$. Let $L = K + (8t+1)^2$. By 5.4, the set consisting of a left 2-star and a monotone $(8t+1)$-vertex path has the SRSEH-property. Let $c_0 > 0$ be an SRSEH-coefficient for this set.

Choose $c > 0$ with

$$c \le \min\left(c_0 4^{-2^L}, 1/K, 1/(35t+1), 4^{-2^L-1}/t\right).$$

We will show that c is an SRSEH-coefficient for $\mathcal{A}$. Let G be an ordered graph, let $\mathcal{B}'$ be a respectful blockade in G of length at least $1/c$, let W' be its width, and suppose that there is no $\mathcal{B}'$-rainbow copy in G of any member of $\mathcal{A}$. We claim that either some vertex has degree at least cW' in G, or G has an anticomplete pair of order at least cW'.

From the choice of K, since $1/c \ge K$, there is a minor $\mathcal{B} = (B_1, \ldots, B_{35t+1})$ of $\mathcal{B}'$ of width at least $4^{-2^L}W'$, such that $\mathcal{B}$ is $(8t+1)$-support-uniform and $(1/4, 8t+1)$-support-invariant. Let its width be W; we may assume that all its blocks have cardinality W. There is no $\mathcal{B}$-rainbow left 2-star, and if there is no $\mathcal{B}$-rainbow monotone $(8t+1)$-vertex path, then from the choice of c_0, there is either some vertex with degree at least $c_0 4^{-2^L}W'$ in G, or an anticomplete pair in G of order at least $c_0 2^{-2^L}W'$; and since $c_0 4^{-2^L} \ge c$, in either case this proves our claim.

So we may assume (for a contradiction) that there is a $\mathcal{B}$-rainbow monotone $(8t+1)$-vertex path. For $i = 0, 1, 2$, let C_{8ti+1} be the set of vertices in B_{8ti+1} that have at least one neighbour in each of $B_{32t+2}, \ldots, B_{35t+1}$. For each $j > 8ti+1$, at most cW' vertices in B_{8ti+1} have no neighbour in B_j; and so $|B_{8ti+1} \setminus C_{8ti+1}| \le 3tcW'$. Hence $|C_{8ti+1}| \ge W - 3tcW'$. Again, for $i = 0, 1, 2$ let D_i be the set of vertices in B_{24t+1} that belong to a $\mathcal{B}$-rainbow monotone $(8t+1)$-vertex path with support $\{8ti+1, \ldots, 8t(i+1), 24t+1\}$ and with its first vertex (the vertex in B_{8ti+1}) in C_{8ti+1}. Now $|C_{8ti+1}| \ge W - 3tcW' \ge W/4$, because $tc \le 4^{-2^L-1} \le W/(4W')$. Since $\mathcal{B}$ is $(8t+1)$-support-uniform and $(1/4, 8t+1)$-support-invariant, and there is no $\mathcal{B}$-rainbow monotone $(8t+1)$-vertex path with support $\{8ti+1, \ldots, 8t(i+1), 24t+1\}$ and with its first vertex in C_{8ti+1} and last vertex in $B_{24t+1} \setminus D_i$, it follows that $|B_{24t+1} \setminus D_i| < W/4$, and so $|D_i| > 3W/4$.

Let D_3 be the set of vertices in B_{24t+1} that belong to a $\mathcal{B}$-rainbow monotone $(8t+1)$-vertex path with support $\{24t+1, \ldots, 32t+1\}$; then by the same argument $|D_3| > 3W/4$. Since $D_0, \ldots, D_3$ all have cardinality more than $3W/4$, and they are all subsets of B_{24t+1} (which has cardinality W), there exists $w \in D_0 \cap D_1 \cap D_2 \cap D_3$. For $i = 0, 1, 2$, let P_i be a $\mathcal{B}$-rainbow monotone $(8t+1)$-vertex path with support $\{8ti+1, \ldots, 8t(i+1), 24t+1\}$, with first vertex ($u_i$ say) in C_{8ti+1} and last vertex w; and let P_3 be a $\mathcal{B}$-rainbow monotone $(8t+1)$-vertex path with support $\{24t+1, \ldots, 32t+1\}$ and first vertex w.

(1) *No vertex in any of $B_{32t+2}, \ldots, B_{35t+1}$ is adjacent to more than one of u_0, u_1, u_2.*

Suppose that $z \in B_{32t+2} \cup \cdots \cup B_{35t+1}$ is adjacent to u_a, u_b say, where $0 \le a < b \le 2$. Since there is no $\mathcal{B}$-rainbow left 2-star, it follows that z is adjacent to every vertex of $P_a \cup P_b$, and hence to every vertex of P_3. Let w' be the neighbour of w in P_a. Since P_a is induced, it has a stable set I_a of cardinality t containing w'. Each vertex in $I_a\setminus$ has at most two neighbours in $V(P_b)$ since there is no $\mathcal{B}$-rainbow left 2-star; and so there is a stable subset I_b of P_b containing w and with cardinality t, such that ww' is the only edge of $G[I_a \cup I_b]$. Each vertex in $I_a \cup I_b \setminus \{w\}$ has at most two neighbours in $V(P_3)$; so there is a stable subset I_3 of P_3 of cardinality t, containing w, and such that ww' is the only edge of $G[I_a \cup I_b \cup I_3]$. But then the ordered graph induced on $I_a \cup I_b \cup I_3 \cup \{z\}$ is a $\mathcal{B}$-rainbow copy of a t-uniform crossed right-star, a contradiction. This proves (1).

For $i = 3, \ldots, 3t+2$ choose $u_i \in B_{32t-1+i}$, adjacent to u_1 if i is divisible by three, adjacent to u_2 if $i-1$ is divisible by three, and adjacent to u_0 otherwise. (This is possible since $u_j \in C_{8tj+1}$ for $j = 0, 1, 2$.) Thus each of $u_3, \ldots, u_{3t+2}$ has exactly one neighbour in $\{u_0, u_1, u_2\}$, by (1). For

$3 \le i \le 3t+2$, u_i is adjacent to one of u_0, u_1, u_2, and hence to all of one of $V(P_0), V(P_1), V(P_2)$, since there is no $\mathcal{B}$-rainbow left 2-star; and in particular, it is adjacent to w. Thus w is adjacent to each of $u_3, \ldots, u_{3t+2}$, and consequently $\{u_3, \ldots, u_{3t+2}\}$ is a clique; and the ordered subgraph induced on $\{u_1, \ldots, u_{3t+2}\}$ is a t-uniform left-split, a contradiction. This proves 5.7. ∎

6 Tournaments that have backedge graphs with at most three edges

Let us (at last!) apply all these results to prove:

6.1 *If H is a D_5-free tournament with a backedge graph with at most three edges, then H has the RSEH-property.*

Proof. Let B be a backedge graph of H that has at most three edges, and let $(v_1, \ldots, v_n)$ be its numbering. B will have at most six vertices of positive degree, but between them there may be arbitrary sequences of vertices of degree zero, and we cannot ignore them, because we need to use B to find other backedge graphs in order to apply 4.3. We need some notation to encompass this. Let B have t vertices of positive degree, and let us number them $b_1, b_3, b_5, \ldots, b_{2t-1}$ in order; and let us label the sequences of vertices between them as $B_0, B_2, \ldots, B_{2t}$, where the sequence $(v_1, \ldots, v_n)$ is the concatenation of

$$B_0, b_1, B_2, b_3, \ldots, b_{2t-1}, B_{2t}.$$

This notation does not tell us the number of vertices in each sequence B_i, but we do not need that.

By 3.6 it suffices to show that some set of backedge graphs of H has the SRSEH-property. First, if no vertex of B has degree more than one, then every component of B is both a left-star and a right-star, and so from 5.4 and 4.3, $\{B\}$ has the SRSEH-property. Thus we may assume that some vertex has degree more than one. If some b_i is incident with every edge of B, then by moving b_i to the start of the numbering, we obtain a numbering with back-edge graph a left-star (and isolated vertices), and similarly by moving b_i to the other end of the numbering, we obtain a numbering with back-edge graph a right-star (and isolated vertices), and 5.4 and 4.3 imply that the set of these two backedge graphs has the SRSEH-property, and so H has the RSEH-property. So we may assume there is no such b_i. In particular, B has exactly three edges.

Suppose that $t = 3$, and so b_1, b_3, b_5 are pairwise adjacent. The numberings

$$B_0, b_5, b_1, B_2, b_3, B_4, B_6$$

$$B_0, B_2, b_3, B_4, b_5, b_1, B_6$$

have backedge graphs in which each component is a left-star, and each component is a right-star, respectively. Thus each transversal (of this set of two ordered graphs) consists of a left-star and a right-star, and therefore we may apply 5.4 and 4.3.

So we may assume that $t \ge 4$. Suppose that $t = 4$. Since B has three edges and no vertex is incident with all of them, the subgraph induced on $\{b_1, b_3, b_5, b_7\}$ is a four-vertex path. There are several possibilities for the order in which b_1, b_3, b_5, b_7 appear in this path, but there is some symmetry we can use to reduce the number of cases. First, there are two orders in which b_1, b_3, b_5, b_7 appear in this path, reverses of one another, and we only need list one of them. Second, we do not need to list both of two cases which can be taken one to the other by reversing the numbering of H;

since reversing the numbering of H gives a backedge graph of the reverse of H, and the result holds for H if and only if it holds for the reverse of H. Up to these two symmetries, the possibilities for the vertices of this path in order are the following:

- b_1-b_3-b_5-b_7. Apply 5.4 and 4.3 to

$$B_0, b_1, B_2, B_4, b_5, b_3, B_6, b_7, B_8$$

$$B_0, b_1, B_2, b_5, b_3, B_4, B_6, b_7, B_8.$$

- b_1-b_3-b_7-b_5. Apply 5.4 and 4.3 to

$$B_0, b_1, B_2, b_7, b_3, B_4, b_5, B_6, B_8$$

$$B_0, B_2, b_3, b_1, B_4, b_5, B_6, b_7, B_8.$$

- b_1-b_5-b_3-b_7. Apply 5.4 and 4.3 to

$$B_0, b_5, b_1, B_2, b_3, B_4, B_6, b_7, B_8$$

$$B_0, b_1, B_2, B_4, b_5, B_6, b_7, b_3, B_8.$$

- b_1-b_5-b_7-b_3. Apply 5.4 and 4.3 to

$$B_0, b_1, B_2, b_7, b_3, B_4, b_5, B_6, B_8$$

$$B_0, b_1, B_2, b_3, B_4, B_6, b_7, b_5, B_8.$$

- b_1-b_7-b_3-b_5. The numberings

$$B_0, b_1, B_2, B_4, b_5, B_6, b_7, b_3, B_8$$

$$B_0, b_1, B_2, \overline{B_4}, b_5, b_7, b_3, B_6, B_8$$

  (where $\overline{B_4}$ means B_4 with order reversed) have back-edge graphs in which every component is a right-star, and every component is either a clique or left-spike, respectively. Thus every transversal of the two consists of either a right-star and a clique (and such a transversal has the SRSEH-property by 5.1), or a right-star and a left-spike (and such a transversal has the SRSEH-property by 5.4). Consequently, the result follows from 4.3.

- b_1-b_7-b_5-b_3. Apply 5.4 and 4.3 to

$$B_0, b_1, B_2, b_3, B_4, B_6, b_7, b_5, B_8$$

$$B_0, b_1, B_2, b_3, B_4, b_7, b_5, B_6, B_8.$$

- b_3-b_1-b_7-b_5. The numberings

$$B_0, b_3, b_1, \overline{B_2}, B_4, b_5, B_6, b_7, B_8$$

$$B_0, b_1, B_2, b_3, B_4, \overline{B_6}, b_7, b_5, B_8$$

$$B_0, b_3, b_1, B_2, B_4, b_5, B_6, b_7, B_8$$

  have back-edge graphs in which each component is a right-star or clique; each component is a left-star or clique; and each component is a left-star or right-star, respectively. Hence we may apply 5.1, 5.4 and 4.3.

- b_3-b_7-b_1-b_5. Since H is D_5-free, it follows that B_4 is null. Apply 5.1 and 4.3 to

$$B_0, B_2, b_5, b_7, b_1, b_3, B_6, B_8$$

$$B_0, \overline{B_2}, b_5, b_7, b_1, b_3, \overline{B_6}, B_8$$

This completes the list of cases with $t = 4$; so $t \geq 5$, and therefore $t = 5$, since B has only three edges and some vertex has degree more than one. The subgraph induced on $\{b_1, b_3, b_5, b_7, b_9\}$ has two components, one an edge and the other a three-vertex path. The three-vertex path might be a monotone path or a left-star or a right-star. Suppose first that it is a monotone path, and so all components of B are monotone paths. Hence the claim follows if we can exhibit a numbering for which every component of the backedge graph is a left-star or left-spike, or if there is a numbering for which every component is a right-star or right-spike. If the vertices of the three-vertex path in order are v_i-v_j-v_k where $i < j < k$, and none of $v_{i+1}, \ldots, v_{j-1}$ have positive degree in B, then the numbering

$$(v_1, \ldots, v_{i-1}, v_j, v_i, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)$$

gives a backedge graph in which every component is a left-star, as required; and similarly we may assume that one of $v_{j+1}, \ldots, v_{k-1}$ has positive degree in B. Consequently the only possibility is (using the b_i, B_i notation) that the edges of B are b_1b_5, b_3b_7 and b_5b_9. The backedge graph of the numbering

$$B_0, B_2, b_3, B_4, b_5, B_6, b_7, B_8, b_1, b_9, B_{10}$$

has two components, one an edge and the other a crossed right-star; and the backedge graph of

$$B_0, b_1, b_9, B_2, b_3, B_4, b_5, B_6, b_7, B_8, B_{10}$$

again has two components, one an edge and the other a crossed left-star. Since every component of B itself is a monotone path, the claim follows from 5.6.

Thus we may assume that a component of B is either a left 2-star or a right 2-star; and from the symmetry under reversal, we may assume it is a left 2-star. Let its vertices be v_i, v_j, v_k. If none of $v_{i+1}, \ldots, v_{j-1}$ has positive degree in B, then every component of the backedge graph of the numbering

$$(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_j, v_i, v_{j+1}, \ldots, v_n)$$

is either a right-star or a right-broom, and the claim follows from 5.4 and 4.3. So we may assume that one of $v_{i+1}, \ldots, v_{j-1}$ has positive degree in B. If also some vertex earlier than v_i has positive degree, the edges of B (in the B_i, b_i notation) are b_1b_5, b_3b_7, b_3b_9; and then the non-singleton component of the backedge graph of the numbering

$$B_0, b_1, B_2, B_4, b_5, B_6, b_7, B_8, b_9, b_3, B_{10}$$

is a right-bristle, and since every component of B is a left-star with at most three vertices, the claim follows from 5.5. So we may assume that b_1 is adjacent to exactly two of b_5, b_7, b_9, and b_3 is adjacent to the third. There are three cases, but the same argument applies to each. Every component of B is a left-star with at most three vertices. The non-singleton component of the backedge graph of the numbering

$$B_0, B_2, b_3, B_4, b_5, B_6, b_7, B_8, b_9, b_1, B_{10}$$

is a crossed right-star; and the non-singleton component of the backedge graph of the numbering

$$B_0, b_1, B_2, b_3, B_4, b_9, \overline{B_8}, b_7, \overline{B_6}, b_5, B_{10}$$

is a left-split, so the claim follows from 5.7. This proves 6.1. ∎

7 Sparsity

Now we turn to the proof of 1.7. We will need the following theorem of Rödl [15]: (see for instance [9] for this version):

7.1 *For every graph H, and every $\varepsilon > 0$, there exists $\delta > 0$ such that if G is a graph with no induced subgraph isomorphic to H, there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that one of the graphs $G[X]$, $\overline{G}[X]$ has maximum degree less than $\varepsilon\delta|G|$.*

We need a version of this for ordered graphs. To obtain that, we use a theorem of Rödl and Winkler [16], that says:

7.2 *For every ordered graph H, there exists a graph P such that, for every numbering of P, the resulting ordered graph contains H.*

We deduce

7.3 *For every ordered graph H, and every $\varepsilon > 0$, there exists $\delta > 0$ such that if G is an H-free ordered graph, there exists $X \subseteq V(G)$ with $|X| \geq \delta|G|$ such that one of the graphs $G[X]$, $\overline{G}[X]$ has maximum degree less than $\varepsilon\delta|G|$.*

Proof. By 7.2, there is a graph P such that for every numbering of P, the resulting ordered graph contains H. Choose δ as in 7.1, with H replaced by P. We claim that δ satisfies the theorem. Let G be an ordered graph that does not contain H. From the choice of P, it follows that G (as an unordered graph) does not contain P as an induced subgraph; and so the result follows from the choice of δ. This proves 7.3. ∎

8 Six-vertex tournaments containing D_5

We will handle the tournaments (with at most six vertices) that contain D_5 separately from those that do not, because the arguments needed are quite different. In this section we handle those that contain D_5. Berger, Choromanski, Chudnovsky and Zerbib [5] proved that D_5 itself has the strong EH-property, and we will show that their proof method also works for what we need.

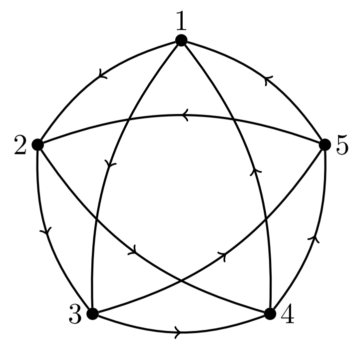

Figure 5: D_5.

Which tournaments do we need to handle? As we said, D_5 itself is handled in [5], so we are concerned with the tournaments with exactly six vertices that contain D_5. With D_5 numbered as in figure 5, if we add a new vertex, we can describe it by giving its set of out-neighbours. That might be any subset of $\{1, \dots, 5\}$; but by taking the reverse if necessary, we may assume the new vertex has at most two out-neighbours (since D_5 is isomorphic to its reverse), and from the symmetry there are only four cases that give nonisomorphic tournaments, namely

$$\emptyset, \{1\}, \{1,2\}, \{1,3\}.$$

The fourth case yields H_6, which we cannot do, so we will just show how to handle the first three. Let us give these three tournaments names: if we start with D_5 numbered as above, and add a new vertex with out-neighbour set X where $X \subseteq \{1, \dots, 5\}$ (and in-neighbour set $\{1, \dots, 5\} \setminus X$), we call the tournament we obtain D_5^X. Thus we need to handle $D_5^{\emptyset}$, $D_5^{\{1\}}$ and $D_5^{\{1,2\}}$.

We need a result proved in [5]. Let us say a digraph G is *out-simplicial* if for all distinct $v, x, y \in V(G)$ such that vx, vy are edges, at least one of xy, yx is an edge. It was proved in [5] that:

8.1 *For every out-simplicial digraph G with $|G| > 1$, either:*

- *there exist disjoint subsets $A, B \subseteq V(G)$ with $|A|, |B| \geq \lfloor |G|/6 \rfloor$, such that there are no edges of G between A, B (in either direction); or*
- *there exist disjoint subsets $A, B \subseteq V(G)$ with $|A|, |B| \geq \lfloor |G|/6 \rfloor$, such that for all $a \in A$ and $b \in B$, there is a directed path in G from a to b.*

The proof for D_5 given in [5] extends to the following:

8.2 *Let G be a tournament, and let $A_1, \dots, A_7$ be pairwise disjoint subsets of $V(G)$, each of cardinality at least W. Then either:*

- *there exist $1 \le i < j \le 7$ such that some vertex in A_i has at least $W/9$ in-neighbours in A_j, or some vertex in A_j has at least $W/9$ out-neighbours in A_i; or*
- *there is a pure pair in G with order at least $\lfloor W/6 \rfloor$; or*
- *G contains all of $D_5^{\emptyset}$, $D_5^{\{1\}}$ and $D_5^{\{1,2\}}$.*

Proof. Let B be the graph with vertex set $A_1 \cup \cdots \cup A_7$, where $uv \in E(G)$ if $u \in A_i$ and $v \in A_j$ for some $i < j$, and u is adjacent from v in G. Thus we may assume that for $1 \le i \le 7$, every vertex of B has fewer than $W/9$ B-neighbours in A_i, for otherwise the theorem holds. We may assume that B has at least one edge, since otherwise the second outcome holds; so $W \ge 10$. We begin with the following:

(1) *If there exist $v_1, v_2 \in A_1$ and $v_4, v_5 \in A_6$ such that*

$$v_1v_2, v_4v_5, v_5v_1, v_5v_2, v_4v_1, v_2v_4 \in E(G),$$

then G contains all of $D_5^{\emptyset}$, $D_5^{\{1\}}$ and $D_5^{\{1,2\}}$.

For $1 \le i \le 7$ let A'_i be the set of vertices in A_i that are not B-adjacent to any of v_1, v_2, v_4, v_5. Consequently $|A'_i| \ge 5W/9$. For each $v_3 \in A'_3$, it follows that the subtournament of G induced on $\{v_1, \dots, v_5\}$ is isomorphic to D_5. (We have chosen the numbering to match that in figure 5.) If we choose $v_3 \in A'_3$ and $v_6 \in A'_7$, not B-adjacent (this is possible since v_3 has fewer than $W/9$ B-neighbours in A'_7, and $|A'_7| \ge 5W/9 > W/9$), the set $\{v_1, \dots, v_6\}$ induces $D_5^{\emptyset}$. If we choose $v_3 \in A'_3$ and $v_6 \in A'_7$, B-adjacent (this is possible since otherwise (A'_3, A'_7) is a pure pair in G and the second outcome of the theorem holds) then $\{v_1, \dots, v_6\}$ induces $D_5^{\{1\}}$. If we choose $v_3, v_6 \in A'_3$, then $\{v_1, \dots, v_6\}$ induces $D_5^{\{1,2\}}$. This proves (1).

(2) *If there exist $v_1, v_2 \in A_1$ and $v_4 \in A_3$ and $v_5 \in A_6$ such that*

$$v_1v_2, v_4v_5, v_5v_1, v_5v_2, v_4v_1, v_2v_4 \in E(G),$$

then G contains all of $D_5^{\emptyset}$, $D_5^{\{1\}}$ and $D_5^{\{1,2\}}$.

Define $A'_1, \dots, A'_7$ as before; then again each $|A'_i| \ge 5W/9$. If we choose $v_3 \in A'_2$, then $\{v_1, \dots, v_5\}$ induces D_5. If we choose $v_3 \in A'_2$ and $v_6 \in A'_7$, not B-adjacent, then $\{v_1, \dots, v_6\}$ induces $D_5^{\emptyset}$. If we choose $v_3 \in A'_2$ and $v_6 \in A'_7$, B-adjacent, then $\{v_1, \dots, v_6\}$ induces $D_5^{\{1\}}$. If we choose $v_3, v_6 \in A'_2$, then $\{v_1, \dots, v_6\}$ induces $D_5^{\{1,2\}}$. This proves (2).

(3) *If there exist $v_1 \in A_1$, $v_5 \in A_3$ and $v_4 \in A_6$, pairwise B-adjacent, then G contains all of $D_5^{\emptyset}$, $D_5^{\{1\}}$ and $D_5^{\{1,2\}}$.*

For $1 \le i \le 7$ let A'_i be the set of vertices in A_i that are not B-adjacent to any of v_1, v_4, v_5. Thus each $|A'_i| \ge W - 3W/9 = 2W/3$. If we choose $v_3 \in A'_2$ and $v_2 \in A'_4$, B-adjacent, then $\{v_1, \dots, v_5\}$ induces D_5. If we choose $v_3 \in A'_2$ and $v_2 \in A'_4$, B-adjacent, and choose $v_6 \in A'_7$, not B-adjacent to v_2, v_3, then $\{v_1, \dots, v_6\}$ induces $D_5^{\emptyset}$. If we choose $v_3 \in A'_2$ and $v_2 \in A'_4$, B-adjacent, and $v_6 \in A'_5$, not B-adjacent to v_2, v_3, then $\{v_1, \ldots, v_6\}$ induces $D_5^{\{1\}}$. Finally, we may assume that some vertex in A'_4 has at least two B-neighbours in A'_2, because otherwise we can choose $X \subseteq A'_2$ and $Y \subseteq A'_4$ with $|X|, |Y| = \lfloor W/3 \rfloor \geq \lfloor W/6 \rfloor$ such that there are no B-edges between X, Y, and so (X, Y) is a pure pair of G and the second outcome holds. Let $v_2 \in A'_4$ have two B-neighbours $v_3, v_6 \in A'_2$; then $\{v_1, \ldots, v_6\}$ induces $D_5^{\{1,2\}}$. This proves (3).

We assume therefore that none of (1), (2), (3) apply. From now on the argument is exactly as in [5], but we give it for the reader's convenience. Let J be the digraph with vertex set A_1, in which for all distinct $u, v \in A_1$, v is J-adjacent from u if v is G-adjacent from u and there exists $w \in A_6$ B-adjacent to both u, v.

(4) *J is out-simplicial.*

Suppose that $v_1 \in A_1$ is adjacent in J to $v_2, v'_2 \in A_1$, and neither of $v_2v'_2, v'_2v_2$ is an edge of J. Choose $v_5 \in A_6$ B-adjacent to v_1, v_2, and choose $v_4 \in A_6$ B-adjacent to v_1, v'_2. Since one of v_1v_2, v_2v_1 is an edge of G, and not an edge of J, it follows that v_4 is not B-adjacent to v_2, and v_5 is not B-adjacent to v'_2. From the symmetry we may assume that v_4v_5 is an edge of G. But then v_1, v_2, v_4, v_5 satisfy the hypotheses of (1), a contradiction. This proves (4).

From 8.1, either

- there exist disjoint subsets $X, Y \subseteq A_1$ with $|X|, |Y| \geq \lfloor |A_1|/6 \rfloor$, such that there are no edges of J between X, Y (in either direction); or
- there exist disjoint subsets $X, Y \subseteq A_1$ with $|X|, |Y| \geq \lfloor |A_1|/6 \rfloor$, such that for all $x \in X$ and $y \in Y$, there is a directed path in J from x to y.

Suppose that the first holds. It follows that no vertex in A_6 has both a B-neighbour in X and a B-neighbour in Y. Thus either at least half the vertices in A_6 have no B-neighbour in X, or at least half have no B-neighbour in Y; and so G has a pure pair (P, Q) with P one of X, Y and $Q \subseteq A_6$ with $|Q| \geq |A_6|/2 \geq W/2$. Since $|P| \geq \lfloor W/6 \rfloor$, the second outcome of the theorem holds.

Thus we may assume that the second bullet holds; there exist disjoint subsets $X, Y \subseteq A_1$ with $|X|, |Y| \geq \lfloor |A_1|/6 \rfloor$, such that for all $x \in X$ and $y \in Y$, there is a directed path in J from x to y.

(5) *There do not exist $x \in X, y \in Y$ and $z \in A_3$ such that z is B-adjacent to x, and z is not B-adjacent to y.*

Suppose that such x, y, z exist. Since there is a directed path of J between x, y, there is an edge $uv \in E(J)$ such that z is B-adjacent to u and not to v. Choose $w \in A_6$ B-adjacent to both u, v (this exists from the definition of J). If z, w are not B-adjacent, the hypotheses of (2) are satisfied, and if z, w are B-adjacent then the hypotheses of (3) are satisfied, in either case a contradiction.

From (5), either half the vertices in A_3 have no B-neighbour in X, or half the vertices in A_3 are B-adjacent to all of Y; so there exists $P \subseteq A_3$ with $|P| \geq W/2$ such that one of $(X, P), (P, Y)$ is a pure pair of G, and the second outcome of the theorem holds. This proves 8.2. ∎

We deduce:

8.3 *If H is a tournament with $|H| \le 6$ that contains D_5, and H is different from $H_6, \overline{H_6}$, then H has the strong EH-property.*

Proof. As we saw, we may assume that H is one of $D_5^{\emptyset}$, $D_5^{\{1\}}$ and $D_5^{\{1,2\}}$. Let J be some backedge graph of H. By 7.3 there exists $\delta > 0$ such that if G is a J-free ordered graph, there exists $X \subseteq V(G)$ with $|X| \ge \delta|G|$ such that one of the graphs $G[X]$, $\overline{G}[X]$ has maximum degree less than $\delta|G|/126$. Let $c = \delta/84$; we will show that every H-free tournament G with $|G| > 1$ has a pure pair of order at least $c|G|$. Let G be an H-free tournament with $|G| > 1$. If $|G| \le 1/c$ then G has a pure pair of order 1 that satisfies the theorem, so we may assume that $|G| > 1/c$. Let $(u_1, \dots, u_m)$ be a numbering of G, and let B be its backedge graph. Thus B is J-free. From the choice of δ, there exists $X \subseteq V(G)$ with $|X| \ge \delta|G|$ such that one of $B[X]$, $\overline{B}[X]$ has maximum degree less than $\delta|G|/126$; and by reversing the numbering of G if necessary, we may assume that $B[X]$ has maximum degree less than $\delta|G|/126$.

Let $W = 6\lceil c|G|\rceil$. Since $c|G| \ge 1$, it follows that $\lceil c|G|\rceil \le 2c|G|$, and so

$$\delta|G|/14 = 6c|G| \le W \le 12c|G| = \delta|G|/7 \le |X|/7.$$

Choose disjoint subsets $A_1, \dots, A_7$ of X, each of cardinality W, such that for $1 \le i < j \le 7$, if $u_p \in A_i$ and $u_q \in A_j$ then $p < q$. Since G is H-free and H is one of $D_5^{\emptyset}$, $D_5^{\{1\}}$ and $D_5^{\{1,2\}}$, it follows from 8.2 (since $|W|$ is divisible by six) that either

- there exist $1 \le i < j \le 7$ such that some vertex in A_i has at least $W/9$ in-neighbours in A_j, or some vertex in A_j has at least $W/9$ out-neighbours in A_i; or
- there is a pure pair in G with order at least $W/6$.

The first is impossible since every vertex in X has degree less than $\delta|G|/126 \le W/9$ in $B[X]$. Consequently G has a pure pair of order at least $W/6 \ge c|G|$. This proves 8.3. ∎

9 Choosing a backedge graph

To complete the proof of 1.7, we need to handle the six-vertex tournaments that do not contain D_5, which is the content of this section. We will have to examine all tournaments with at most six vertices, and we will enumerate them by their backedge graphs. Each tournament may have several different backedge graphs, and we only need to examine one per tournament, so let us try to choose a good one. In this section we show that all six-vertex tournaments have backedge graphs with at most four edges; so we can handle most of them by means of 6.1, and the others are handled case by case. Let us say a numbering of a tournament H is *optimal* if it has as few backedges as possible, over all numberings of H.

9.1 *Let $(v_1, \dots, v_n)$ be an optimal numbering of a tournament H, with backedge graph B, and let $1 \le i < j \le n$. Then:*

- *v_i is B-adjacent to at most $(j-i)/2$ members of $\{v_{i+1}, \dots, v_j\}$; and v_j is B-adjacent to at most $(j-i)/2$ members of $\{v_i, \dots, v_{j-1}\}$.*
- *If $j - i \le 3$ then $B[\{v_i, \dots, v_j\}]$ has at most one edge.*

- *If $j-i=4$ then $B[\{v_i,\dots,v_j\}]$ has at most three edges. It has three only if they are v_iv_{i+4}, v_iv_{i+3} and $v_{i+1}v_{i+4}$, and then G contains D_5.*

Proof. For the first statement,

$$(v_1,\dots,v_{i-1},v_{i+1},\dots,v_j,v_i,v_{j+1},\dots,v_n)$$

is a numbering of H, and the number of its backedges is obtained from the number of backedges of $(v_1,\dots,v_n)$ by adding the number of H-out-neighbours of v_i in $\{v_{i+1},\dots,v_j\}$ and subtracting the number of H-in-neighbours of v_i in this set; and since $(v_1,\dots,v_n)$ is optimal, it follows that at least half of the vertices in $\{v_{i+1},\dots,v_j\}$ are H-out-neighbours of v_i, that is, v_i is B-adjacent to at most $(j-i)/2$ members of $\{v_{i+1},\dots,v_j\}$. This proves half of the first statement and the other half follows from symmetry.

For the second statement, let $j\le i+3$, and suppose that $B[\{v_i,\dots,v_j\}]$ has at least two edges. Hence there exist $i\le a<b\le j$ with $b-a<j-i$ such that v_a,v_b are B-adjacent; and by the first statement $b-a\ge 2$. Consequently $j-i=3$. The only pairs of vertices in $\{v_i,\dots,v_{i+3}\}$ that might be adjacent are $v_iv_{i+2}, v_{i+1}v_{i+3}$ and v_iv_{i+3}; and by the first statement, v_i has at most one B-neighbour in $\{v_{i+1},v_{i+2},v_{i+3}\}$, and v_{i+3} has at most one B-neighbour in $\{v_i,v_{i+1},v_{i+2}\}$. Thus v_i is not B-adjacent to v_{i+3}; and so v_iv_{i+2} and $v_{i+1}v_{i+3}$ are backedges. But then the numbering

$$(v_1,\dots,v_{i-1},v_{i+2},v_i,v_{i+3},v_{i+1},v_{i+1},\dots,v_n)$$

has fewer backedges, a contradiction. This proves the second statement.

For the third, let $j=i+4$, and suppose that $B[\{v_i,\dots,v_j\}]$ has at least three edges. By the second statement, $B[\{v_i,\dots,v_{j-1}\}]$ has at most one edge, and so does $B[\{v_{i+1},\dots,v_j\}]$; so $B[\{v_i,\dots,v_j\}]$ has exactly three edges, and one of them is v_iv_j, and each of the other two only appears in one of $B[\{v_i,\dots,v_{j-1}\}]$, $B[\{v_{i+1},\dots,v_j\}]$. Let the other two backedges be v_iv_b and v_av_j where $a,b\in\{i+1,\dots,j-1\}$. Now $b\in\{i+2,i+3\}$ and $a\in\{i+1,i+2\}$, from the first statement; so there are four cases.

- If $a=b=i+2$, then

$$(v_1,\dots,v_{i-1},v_{i+1},v_{i+4},v_{i+2},v_i,v_{i+3},v_{i+5},\dots,v_n)$$

has fewer backedges, a contradiction.

- If $a=i+1$ and $b=i+2$, then

$$(v_1,\dots,v_{i-1},v_{i+2},v_{i+4},v_i,v_{i+1},v_{i+3},v_{i+6},\dots,v_n)$$

has fewer backedges, a contradiction; and similarly there is a contradiction if $a=i+2$ and $b=i+3$.

- If $a=i+1$ and $b=i+3$, then the tournament induced on $\{v_i,\dots,v_{i+5}\}$ is isomorphic to D_5, and the third outcome of the theorem holds.

This proves 9.1. ∎

Figure 6 defines the tournament F_6.

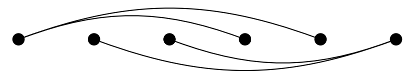

Figure 6: Backedge graph of F_6.

9.2 *Let H be a tournament with $|H| \leq 6$, and let $(v_1, \dots, v_{|H|})$ be an optimal numbering, with backedge graph B. If $|H| \leq 4$, there is at most one backedge. If $|H| = 5$, there are at most three backedges, and at most two unless $H = D_5$. If $|H| = 6$, there are at most four backedges, and at most three unless H is one of $P_7^-, H_6, \overline{H_6}$ and F_6.*

Proof. If $|H| \leq 4$ the claim is clear, and if $|H| = 5$ the claim follows from 9.1.3 (that is, from the third statement of 9.1; we will use this notation again). Thus we may assume that $|H| = 6$, and there are at least four backedges.

Suppose that $B[\{v_2, \dots, v_6\}]$ has at least three edges. By 9.1.3 it has exactly three, and they are v_2v_5, v_2v_6 and v_3v_6. All other edges of B are incident with v_1. By 9.1.1, v_1 is not B-adjacent to v_6 or to v_2, so the only possible further edges of B are v_1v_3, v_1v_4 and v_1v_5. By 9.1.1 at most two of them are present; and also by 9.1.1, not both v_1v_3, v_1v_4 are backedges. If v_1v_5 and one of v_1v_3, v_1v_4 are both present then the numbering

$$(v_3, v_4, v_5, v_1, v_6, v_2)$$

has fewer backedges, a contradiction. So exactly one of v_1v_3, v_1v_4, v_1v_5 is present. If v_1v_3 is a backedge then

$$(v_3, v_1, v_4, v_5, v_6, v_2)$$

has fewer backedges; if v_1v_4 is a backedge then

$$(v_4, v_1, v_5, v_6, v_2, v_3)$$

has fewer backedges; and if v_1v_5 is a backedge then

$$(v_5, v_1, v_6, v_2, v_3, v_4)$$

has fewer backedges, a contradiction.

Thus we may assume that $B[\{v_2, \dots, v_6\}]$ has at most two edges. Since by 9.1.1, v_1 is incident with at most two edges of B, and B has at least four edges, it follows that exactly two are incident with b_1, and B has four edges altogether. Similarly, exactly two are incident with v_6.

Suppose that v_1, v_6 are not B-adjacent. Since v_1 has two B-neighbours in $\{v_2, \dots, v_5\}$, and at most one in $\{v_2, v_3, v_4\}$ by 9.1.1, it follows that v_1v_5 is a backedge, and similarly so is v_2v_6. Each of v_1, v_6 has one further B-neighbour; let v_1v_a and v_bv_6 be backedges, where $a, b \in \{3, 4\}$. Now there are four cases, $(a, b) = (3, 4), (3, 3), (4, 4), (4, 3)$.

- If $(a, b) = (3, 4)$, then
$$(v_3, v_5, v_1, v_6, v_2, v_4)$$
has fewer backedges.

- If $(a,b) = (3,3)$ then H is isomorphic to H_6 (the numbering

$$(v_2, v_3, v_5, v_1, v_4, v_6)$$

gives the backedge graph of figure 4). Similarly if $(a,b) = (4,4)$ then H is isomorphic to $\overline{H_6}$.

- If $(a,b) = (4,3)$ then H is isomorphic to F_6 (B itself is the graph of figure 6).

Thus we may assume that v_1v_6 is a backedge. More, we may assume that for *every* optimal numbering of H, the first and last vertices are adjacent and both are incident with two backedges for that numbering. Suppose that $v_1v_3 \in E(B)$. Then

$$(v_1, v_2, v_3, v_4, v_5, v_6)$$
$$(v_2, v_3, v_1, v_4, v_5, v_6)$$
$$(v_3, v_1, v_2, v_4, v_5, v_6)$$

are all optimal numberings, and so v_6 is B-adjacent to each of v_1, v_2, v_3, contrary to 9.1.1.

Thus we may assume that v_1v_3 is not a backedge, and similarly v_4v_6 is not a backedge. Suppose that v_3v_5 is a backedge. Since v_6 is incident with two backedges, and 9.1.2 implies that v_6 has no B-neighbour in $\{v_3, v_4, v_5\}$, it follows that v_2v_6 is a backedge. Also one of v_1v_4, v_1v_5 is a backedge. If v_1v_4 is a backedge, then H is isomorphic to H_6; and if v_1v_5 is a backedge, then the numbering

$$(v_1, v_2, v_4, v_5, v_3, v_6)$$

shows that again H is isomorphic to H_6.

Thus we may assume that v_3v_5 is not a backedge, and similarly v_2v_4 is not a backedge. But there is a backedge with both ends in $\{v_2, v_3, v_4, v_5\}$, and so v_2v_5 is a backedge. Also v_1v_b is a backedge for some $b \in \{4,5\}$, and v_av_6 is a backedge for some $a \in 2,3$. There are four cases, $(a,b) = (2,5), (3,5), (2,4), (3,4)$.

- If $(a,b) = (2,5)$, then H is isomorphic to F_6, as we see from the numbering

$$(v_1, v_3, v_4, v_5, v_6, v_2).$$

- If $(a,b) = (3,5)$, the numbering

$$(v_5, v_1, v_2, v_3, v_4, v_6)$$

is optimal and yet the first and last vertices are not joined by a backedge of this numbering, a contradiction. Similarly $(a,b) \neq (2,4)$.

- If $(a,b) = (3,4)$, then H is isomorphic to P_7^-.

This proves 9.2. ∎

Let us observe also that:

9.3 *The tournament F_6 has the RSEH-property.*

Proof. Every component of the backedge graph shown in figure 6 is a left-star or right-star. Let $(v_1, \ldots, v_6)$ be the corresponding numbering: then the non-singleton component of the backedge graph of the numbering $(v_3, v_2, v_4, v_5, v_6, v_1)$ is a clique, so the result follows from 5.1. This proves 9.3. ∎

We deduce our main result 1.7, which we restate in a slightly strengthened form:

9.4 *Let H be a tournament with at most six vertices. If H is different from P_7^-, H_6 and $\overline{H_6}$ then H has the strong EH-property; and if in addition H is D_5-free then H has the rainbow strong EH-property.*

This is immediate from 9.2, 8.3, 6.1 and 9.3.

10 Forests and the Paley tournament P_7

We promised earlier to show that P_7 does not have the strong EH-property. For that, we use a variant of a theorem of Erdős [10], which we shall also need in the next section:

10.1 *Let $c > 0, g > 0$; then there exists an integer $d > 0$ such that for all sufficiently large integers n, there is a graph G with n vertices, such that:*

- *every cycle of G has length more than g;*
- *there do not exist anticomplete $A, B \subseteq V(G)$ with $|A|, |B| \geq cn$; and*
- *G has maximum degree less than d.*

Proof. Choose an integer $d > 0$ with $(dc^2/(8e))^d \geq 6$ (where e is Euler's constant). Let n be some (sufficiently) large number, and let us take a random graph G with vertex set $\{1, \ldots, 2n\}$, where i, j are adjacent independently with probability $p = 4/(c^2n)$. Let x_1 be the number of pairs (A, B) with $A, B \subseteq V(G)$, such that A, B are anticomplete, and $|A|, |B| \geq cn$. Let x_2 be the number of cycles in G of length at most g; and let x_3 be the number of vertices with degree at least d. We need to estimate the expected value $E(x_i)$ of x_i for $i = 1, 2, 3$.

First, let $A, B \subseteq V(G)$ be disjoint, with $|A|, |B| \geq cn$. The probability that there are no edges of G between A, B is at most $(1-p)^{(cn)^2} \leq e^{-pc^2n^2}$; and the number of choices of (A, B) is at most 3^{2n}. So

$$E(x_1) \leq e^{-pc^2n^2} 3^{2n} \leq n/3$$

if n is sufficiently large (since $p = 4/(c^2n)$).

The expected number of cycles of length i in G is at most $p^i(2n)^i/(2i)$, so

$$E(x_2) \leq \sum_{3 \leq i \leq g} p^i(2n)^i/(2i) \leq p^g(2n)^g/2 \leq (8/c^2)^g/2 \leq n/3.$$

For a vertex v, the probability that v has degree at least d is at most $\binom{2n}{d}p^d \leq (2pn)^d/d!$; and since $d! \geq (d/e)^d$ by Stirling's formula, it follows that the probability that v has degree at least d is at most $(8e/(c^2d))^d \leq 1/6$. So $E(x_3) \leq n/3$.

Hence the expected value of $x_1 + x_2 + x_3$ is at most n; and so there is a choice of G where $x_1 + x_2 + x_3 \leq n$. Hence by deleting n vertices appropriately we obtain a graph with n vertices as in the theorem. This proves 10.1. ∎

Now we can prove 1.4, which we restate:

10.2 *Let H be a tournament with the strong EH-property. Then there is a numbering of H such that the backedge graph is a forest. Consequently there is a partition of $V(H)$ into two subsets both inducing transitive tournaments.*

Proof. Choose $c > 0$ such that every H-free tournament G with $|G| > 1$ admits a pure pair with order at least $c|G|$. Choose d satisfying 10.1 with c replaced by $c/2$ and g replaced by $|H|$. Let $n \geq 2d/c$ be some large number, large enough that there is a graph J with n vertices, satisfying the three bullets of 10.1 with c replaced by $c/2$ and g replaced by $|H|$. Take a numbering $(v_1, \ldots, v_n)$ of J, and let G be the tournament such that J is the backedge graph of G under this numbering. Suppose that there is a pure pair (X, Y) in G with $|X|, |Y| \geq cn$. Choose i minimum such that $|\{v_1, \ldots, v_i\} \cap X| \geq cn/2$, and let $A = \{v_1, \ldots, v_i\} \cap X$ and $B = Y \cap \{v_{i+1}, \ldots, v_n\}$. Since J has maximum degree less than $d \leq cn/2$, and $v_i \in X$ is J-adjacent to every vertex in $Y \setminus B$, it follows that $|B| \geq cn/2$; and yet A, B are anticomplete, contrary to the choice of J.

Thus G has no pure pair (X, Y) in G with $|X|, |Y| \geq cn$. From the definition of c, it follows that G contains H. The backedge graph for H under the numbering induced by $(v_1, \ldots, v_n)$ has no cycles, since all cycles of J have length more than $|H|$. Hence it is a forest.

This forest is two-colourable; and each colour class induces a transitive subtournament of H, since it is a stable set of a backedge graph of H. This proves 1.4. ∎

We deduce:

10.3 *P_7 does not have the strong EH-property.*

Proof. It suffices to show that the vertex set of P_7 cannot be partitioned as in 1.4, and to show that it suffices to show that P_7 has no four-vertex transitive subtournament. But for every vertex of P_7, its three out-neighbours form a cyclic triangle. This proves 10.3. ∎

In [8] we mentioned 1.4 and several other conditions that were necessary if a tournament is to have the strong EH-property. But we subsequently observed that each of the other conditions was implied by the first; and at the moment, 1.4 is the only necessary condition we know. As we mentioned in the introduction, it might be that having a backedge graph that is a forest is necessary and sufficient for a tournament to have the strong EH-property. One piece of evidence in favour of this is the following, which follows from results of [18]:

10.4 *For a tournament H, the following are equivalent:*

- *some backedge graph of H is a forest;*
- *for every $c > 0$ there exists $\varepsilon > 0$ such that for every H-free tournament G with $|G| > 1$, there is a pure pair in G of order at least $\varepsilon|G|^{1-c}$.*

Proof. It is shown in [18] that:

(1) *If J is an ordered forest, then for all $c > 0$, there exists $\varepsilon > 0$ such that if G is an ordered graph with $|G| > 1$ that is both J-free and $\overline{J}$-free, then G contains a pure pair of order at least $\varepsilon|G|^{1-c}$.*

Now let H be a tournament. Suppose first that some backedge graph J of H is a forest. Then $\overline{J}$ is also a backedge graph of H (reversing the numbering); and if G is an H-free tournament, and B is its backedge graph, then B contains neither J nor $\overline{J}$, and so (1) implies that B has the desired pure pair, and hence, by 2.1, so does G.

For the converse, let H be a tournament for which no backedge graph is a forest. Let $c < 1/|H|$; we claim there is no ε satisfying the second bullet of the theorem. Let $\varepsilon > 0$. An argument like that of 10.1 shows that if we take a random graph J on n vertices where n is sufficiently large, in which every edge is present independently with probability $\frac{1}{2}n^{-1+1/|H|}$, then with high probability, there will be a set X of at least $n/2$ vertices in which $J[X]$ has no cycle of length at most $|H|$ and has no pure pair of order at least $\varepsilon|X|^{1-c}/2$. Number X arbitrarily, and let G be the tournament with J (and this numbering) as a backedge graph. Then G does not contain H, since if it did, the induced numbering of H would have backedge graph contained in J with a cycle of length at most $|H|$. And yet G has no pure pair of order at least $\varepsilon|G|^{1-c}$, by 2.1, and so ε does not satisfy the second bullet of the theorem. This proves 10.4. ∎

11 D_5 and P_7^- do not have the rainbow strong EH-property

We claimed earlier that D_5 does not have the RSEH-property. The same holds for P_7^-, and even excluding them both simultaneously is not enough. We will show:

11.1 *For all $c > 0$, and infinitely many integers n, there is a tournament G with n vertices, and a blockade $\mathcal{B}$ in G of length at least $1/c$, such that G has no pure pair of order at least $cW(\mathcal{B})$, and contains no $\mathcal{B}$-rainbow copy of either of D_5, P_7^-.*

To show this we need a construction as follows. Let G be an ordered graph. A *walk* in G is a sequence

$$p_0, p_1, \ldots, , p_r,$$

where $p_0, \ldots, p_r \in V(G)$ and there is an edge of G with ends p_{i-1}, p_i for $1 \le i \le r$. (We do not require $p_0, \ldots, p_r$ all to be distinct, but consecutive terms are distinct.) Its *length* is r, and its *imbalance* is $N_1 - N_2$, where N_1 is the number of $i \in \{1, \ldots, r\}$ such that p_{i-1} is before p_i in the numbering of G, and N_2 is the number of i such that p_{i-1} is after p_i. A walk is *balanced* if its imbalance is zero, and *unbalanced* otherwise; and *closed* if $p_0 = p_r$.

11.2 *Let $k \ge 1$ be an integer, and let $c > 0$. Then there is an integer D, such that for all sufficiently large integers W, there is an ordered graph J with kW vertices, and the following properties:*

- *every vertex has degree at most D;*
- *G admits a respectful blockade $\mathcal{B} = (B_1, \ldots, B_k)$ of width W;*
- *G has no pure pair of order at least cW;*
- *every closed walk in J of length at most six is balanced;*
- *there is no $\mathcal{B}$-rainbow copy in J of any of the ordered graphs shown in figure 7.*

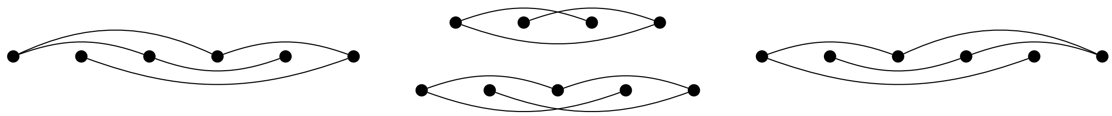

Figure 7: Ordered graphs for 11.2.

Proof. Let $c' = c/k$, and $g = 6 \cdot 3^k$. Choose d to satisfy 10.1 with c replaced by c'. Let $D = d^{3^k}$. Let W be a sufficiently large integer. Then by 10.1 there is a graph J_k with kW vertices $v_1, \dots, v_{kW}$, such that

- every cycle of J_k has length more than g;
- there is no anticomplete pair in J_k of order at least $c'kW = cW$; and
- J_k has maximum degree less than d.

For $1 \le i \le k$ let $B_i = \{v_j : (i-1)W < j \le iW\}$, and $\mathcal{B} = (B_1, \dots, B_k)$. If $u, v \in V(J_k)$, and $u \in B_i$, $v \in B_j$, we define the *$\mathcal{B}$-length* of the pair (u, v) to be $|j - i|$, and the $\mathcal{B}$-length of an edge uv is the $\mathcal{B}$-length of (u, v). We say P is a *welcoming path* if

- P is a path of length three with $V(P) \subseteq V(J_k)$, with ends s, t where s is before t in the numbering $(v_1, \dots, v_{kW})$,
- the $\mathcal{B}$-length of (s, t) is at least one;
- every edge of P has $\mathcal{B}$-length strictly greater than the $\mathcal{B}$-length of (s, t); and
- the walk of length three from s to t in P has imbalance one (note that this is different from having imbalance -1).

For $i = k-1, \dots, 1$ we define J_i as follows. We say a pair (s, t) of vertices of J_{i+1} is *i-good* if s is before t in the numbering $(v_1, \dots, v_{kW})$, and the $\mathcal{B}$-length of (s, t) is exactly i, and s, t are nonadjacent in J_{i+1}, and there is a welcoming path in J_{i+1} (not necessarily induced) with ends s, t. We construct J_i from J_{i+1} by adding an edge between s, t for every i-good pair (s, t).

Let $J = J_1$; we claim that J satisfies the theorem. First, let d_i denote the maximum degree of J_i; then since each vertex of J_i is an end vertex of at most $d_{i+1}(d_{i+1} - 1)^2$ paths of length three, it follows that $d_i \le d_{i+1}(d_{i+1} - 1)^2 + d_{i+1} \le d_{i+1}^3$. Since $d_k \le d$, it follows that J has maximum degree at most $d^{3^k} = D$.

Second, since J_k has no anticomplete pair of order at least $c'kW$, the same holds for J. Third, for every closed walk of J_i, we can replace each edge e of J_i not in J_{i+1} by a three-edge walk along the corresponding welcoming path P of J_{i+1}; and since this three-edge walk has imbalance the same as the corresponding one-edge walk along e, it follows that there is a closed walk of J_{i+1} with the same imbalance and with length at most three times as great. Since every cycle of J_k has length more than g, and so every closed walk in J_k with length at most g is balanced, it follows that every closed walk of J_i of length at most $g3^{i-k}$ is balanced, and in particular every closed walk of J with length at most $g3^{-k} = 6$ is balanced.

Fourth, we must show that J contains no $\mathcal{B}$-rainbow copy of any of the four graphs in figure 7. For this we use:

(1) *For every welcoming path of J, its ends are adjacent in J.*

Let P be a welcoming path of J, with ends s, t where s is earlier than t. Let i be the $\mathcal{B}$-length of (s, t). Since every edge of P has $\mathcal{B}$-length more than i, it follows that every such edge is an edge of J_{i+1} (because all edges added later have $\mathcal{B}$-length at most i), and so P is a welcoming path of J_{i+1}. But then s, t are adjacent in J_i and hence in J. This proves (1).

Suppose that J contains a $\mathcal{B}$-rainbow copy H of one of the four graphs in figure 7. (Thus H is induced.) Suppose first that $|H| = 4$, and let its numbering be (u_1, u_2, u_3, u_4). Thus its edges are u_1u_3, u_2u_4, u_1u_4. Let i be the $\mathcal{B}$-length of (u_2, u_3); thus $i \geq 1$ since H is $\mathcal{B}$-rainbow, and for the same reason, all three edges of H have $\mathcal{B}$-length at least $i + 1$. But then H is a welcoming path of J and its ends are nonadjacent, contrary to (1).

Now suppose that $|H| = 5$, and so its edges are u_2-u_5-u_3-u_1-u_4, where $(u_1, \dots, u_5)$ is its numbering. Let the $\mathcal{B}$-length of (u_1, u_2) be i_1, and that of (u_4, u_5) be i_2. From the symmetry we may assume that $i_1 \leq i_2$. But then all edges of the path u_1-u_3-u_5-u_2 have $\mathcal{B}$-length strictly more than i_1, since H is $\mathcal{B}$-rainbow; so it is welcoming, contrary to (1).

Finally, suppose that $|H| = 6$; and from the symmetry, we may assume that its edges are u_2-u_4-u_6-u_3-u_1-u_5, where $(u_1, \dots, u_6)$ is its numbering. Let the $\mathcal{B}$-length of (u_2, u_3) be i_1, and that of (u_5, u_6) be i_2. If $i_1 \leq i_2$, then the path u_2-u_4-u_6-u_3 is welcoming, and if $i_2 \leq i_1$ then the path u_5-u_1-u_3-u_6 is welcoming, and in either case we have a contradiction to (1). This proves 11.2. ∎

We deduce 11.3, which we restate:

11.3 *For all $c > 0$, and infinitely many integers n, there is a tournament G with n vertices, and a blockade $\mathcal{B}$ in G of length at least $1/c$, such that G has no pure pair of order at least $cW(\mathcal{B})$, and there is no $\mathcal{B}$-rainbow copy of either of D_5, P_7^-. Conequently D_5, P_7^- do not have the RSEH-property.*

Proof. Let $k = \lceil 2/c \rceil$, and choose D, and $W > 2D/c$ sufficiently large that the construction J of 11.2 exists with c replaced by $c/2$. Let G be the tournament with backedge graph J. Since every vertex of J has degree at most $D < cW/2$, it follows that J has no pure pair of order at least $cW/2$, and so G has no pure pair of order at least cW, by 2.1. By examining all the backedge graphs of D_5 (there are 24 of them) and all the backedge graphs of P_7^- (there are 240 of them) we observe that each of them contains an unbalanced cycle of length at most five, or one of the ordered graphs of figure 7. Consequently there is no $\mathcal{B}$-rainbow copy in J of any backedge graph of D_5 or of P_7^-, and so G contains no $\mathcal{B}$-rainbow copy of D_5 or of P_7^-. This proves 11.3. ∎

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