# On the Connectivity and Diameter of Geodetic Graphs 

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#### Abstract

A graph $G$ is geodetic if between any two vertices there exists a unique shortest path. In 1962 Ore raised the challenge to characterize geodetic graphs, but despite many attempts, such characterization still seems well beyond reach. We may assume, of course, that $G$ is 2 -connected, and here we consider only graphs with no vertices of degree 1 or 2 . We prove that all such graphs are, in fact 3-connected. We also construct an infinite family of such graphs of the largest known diameter, namely 5 .


## 1 Introduction

In this work we consider loopless, undirected graphs $G=(V, E)$. We think of a path in $G$ as a sequence of vertices $P=\left(v_{1}, \ldots v_{k}\right)$, and the subpath of $P$ from $v_{j}$ to $v_{l}$ is denoted $P\left(v_{j}, v_{l}\right)$. The length of a path $P$ is denoted $|P|$, and path concatenation is denoted by $*$. The distance between $u$ and $v$ is $d_{G}(v, u)=d(v, u)$, and is the length of a shortest $u, v$ path. For clarity, we occasionally add an index indicating the graph in which some parameter or quantity is calculated.

The notion of geodetic graphs was introduced by Ore [6] as a natural extension of trees: a tree is a graph in which between any two vertices there exists a unique simple path, and hence a unique shortest path (i.e - a geodesic). Ore purposed an extended definition, and asked in which simple graphs geodesics are unique. Some simple examples are trees, complete graphs, and odd-length cycles. Specifically, Ore raised the challenge to characterize geodetic graphs. Despite many attempts, a complete characterization still seems beyond reach.

There are easy necessary and sufficient properties for a graph to be geodetic, the following can be easily proved:

Claim 1.1. A graph $G$ is not geodetic if and only if it contains an even circuit $C$ with two vertices $u, v \in C$ such that $d_{C}(u, v)=d_{G}(u, v)=\frac{|C|}{2}$.

Claim 1.2. A graph $G$ is geodetic if and only if each block of $G$ is geodetic.
Here a block is a maximal 2-connected component of $G$. There is clearly no loss in generality if we restrict our attention to 2 -connected graphs. By Claim 1.2, since a vertex of degree 1 is a block, it suffices to assume all vertices have degree no less than 2. Moreover, the following claim is given in [12]:

Claim 1.3. Let $G$ be a geodetic graph, and let $P=v_{1}, \ldots, v_{k}$ be a path with $\operatorname{deg}\left(v_{1}\right), \operatorname{deg}\left(v_{k}\right) \geq 3$ and $\forall i \neq 1, k \operatorname{deg}\left(v_{i}\right)=2$. Then $P$ is the $v_{1}, v_{k}$ geodesic.

[^0]By this claim, one can replace any path that consists of degree 2 vertices by a single weighted edge. Therefore the question of geodeticity gains a more arithmetic flvaour. Since our emphasis is combinatorial and geometric we will concentrate on graphs whose smallest degree is at least 3.

There are not many families of geodetic graphs that are classified in full. The only constructive classifications known presently are planar geodetic graphs [12], which is extended to a classification of geodetic graphs homeomorphic to a complete graph [11]. There is a considerable body of work classifying geodetic graphs of diameter $2[10,8,1]$, including a classification of such graphs. While some constructions are known, we do not know that they exhaust all possible geodetic graphs of diameter 2. In this sense, the classification is lacking. Naturally the question turns to higher diameters. Some properties of geodetic graphs of diameter 3 are known [7]. However, to the best of our knowledge the following is unknown:
Problem 1. Do there exist geodetic blocks $G$ of diameter 3 with $\delta(G) \geq 3$ ?
Here $\delta(G)$ is the minimal degree of $G$. Progress on this problem has been very slow. Bridgland [2] constructed a family of geodetic blocks of diameter 4 and arbitrarily large minimal degree. This construction was later generalized in several ways using block designs [9], yielding a family of geodetic blocks of diameter 5 . This construction has the largest diameter presently known. Despite many attempts, we were unable to retrieve the latter paper. We therefore present these constructions along a different proof of their geodeticity. A main problem that we raise is:

Problem 2. What is the largest possible diameter of a geodetic block with minimal degree $\geq 3$ ? Can it be arbitrarily large?

In the journey to classification, other properties of geodetic graphs were discovered. A graph is called self centered if its diameter equals its radius. Geodetic blocks of diameter 2 are known to have this property [10]. Likewise, for blocks of diameter 3 [7]. Some connections to other graph properties were explored, namely by Zelinka [15], Gorovoy and Zamiaikou [4], and connections to other fields such as algebra and group theory [3, 5]. We continue these lines of research, resulting in our main theorem:

Theorem 1.1. Every 2 -connected geodetic graph $G$ with $\delta(G) \geq 3$ must be 3 -connected. This lower bound is tight as shown by the Petersen Graph.

## 2 Geodetic Graphs and Connectivity

In this section we prove Theorem 1.1. From here on we assume $G$ is geodetic with $\delta(G) \geq 3$. We denote the (unique) $v, u$ geodesic in $G$ by $\pi(v, u)$, and by convention we enumerate its vertices in order from $v$ to $u$. Arguing by contradiction, let $S=\{x, y\}$ be a vertex cut for which $d(x, y)$ is as small as possible, denote this distance by $\ell$. We denote by $\Pi=\pi(x, y)$ the $x, y$ geodesic, with vertices $\Pi=x, x_{1}, x_{2}, \ldots, x_{\ell-1}, y$. Let $A_{1} \ldots A_{k}$ the connected components of $G \backslash S$. Clearly, $x_{1}, x_{2}, \ldots, x_{\ell-1}$ all belong to the same connected component of $G \backslash S$, say they are in $A_{1}$.

Lemma 2.1. For every $i$, if $u, v \in A_{i}$, then $\pi_{G}(u, v)$ is contained in $A_{i} \cup \Pi$.
Proof. If $\pi_{G}(u, v)$ is not contained in $A_{i} \cup \Pi$, then it must leave $A_{i}$ and come back. But the only way to exit $A_{i}$ is via $x$ or $y$. But then $\pi_{G}(u, v)=\pi_{G}(u, x) * \Pi * \pi_{G}(y, v)$, since $\Pi$ is the $x, y$ geodesic. Thus $\pi_{G}(u, v)$ is contained in $A_{i} \cup \Pi$, as claimed.

Lemma 2.2. The graph $G \backslash S$ has exactly two connected components, i.e., $k=2$.
Proof. Suppose toward contradiction that $A_{1}, A_{2}, A_{3} \neq \emptyset$. Let $\pi_{i}$ to be an $x, y$ geodesic in the subgraph induced by $A_{i} \cup S$. (in particular $\pi_{1}=\Pi$ ). At least two of the integers $\left|\pi_{1}\right|,\left|\pi_{2}\right|,\left|\pi_{3}\right|$ have the same parity, so the corresponding paths form a cycle $C$ of even length. We consider two cases:

1. $C=\pi_{1} * \pi_{2}^{-1}$ : Since $\left|\pi_{2}\right| \geq\left|\pi_{1}\right|+2$ we can find two vertices $v_{0}, v_{1} \in A_{2}$ which are antipodal points on $C$. However, $\pi_{2}\left(v_{0}, v_{1}\right)=\pi_{G}\left(v_{0}, v_{1}\right)$ by Lemma 2.1. But $\pi_{C}\left(v_{0}, x\right) * \Pi * \pi_{C}\left(y, v_{1}\right)$ is another $v_{0}, v_{1}$ path of the same length, contradicting geodeticity.
2. $C=\pi_{2} * \pi_{3}^{-1}$ : Let $v_{0}, v_{1}$ be $C$ - antipodal points with $v_{0} \in A_{2}, v_{1} \in A_{3}$. The two arcs of $C$ that $v_{0}, v_{1}$ define are two $v_{0}, v_{1}$ paths of equal length. By assumption $G$ is geodetic so $\pi_{G}\left(v_{0}, v_{1}\right)$ differs from both these paths. But $\pi_{G}\left(v_{0}, v_{1}\right)$ must traverse either $x$ or $y$. This means e.g., that $\left|\pi_{G}\left(v_{0}, x\right)\right|<\left|\pi_{2}\left(v_{0}, x\right)\right|$ contrary to the assumption that $\pi_{2}$ is a shortest $x, y$ path in $A_{2} \cup S$.

A graph contraction between two graphs $\Gamma, \Omega$, is a function $f: V(\Gamma) \rightarrow V(\Omega)$ such that for any $v u \in E(\Gamma)$, either $f(v) f(u) \in E(\Omega)$ or $f(v)=f(u)$. Clearly, for any $u, v \in V(\Gamma)$ it holds that $d_{\Omega}(f(v), f(u)) \leq d_{\Gamma}(v, u)$. Therefore if $\Gamma$ is connected, so is $f[\Gamma]$.

Let $\mathbb{Z}_{\infty}^{2}$ be the graph with vertex set $\mathbb{Z}^{2}$ where $p, q \in \mathbb{Z}^{2}$ are neighbors whenever $\|p-q\|_{\infty}=$ 1. We denote coordinates in this plane by $(\xi, \eta)$ and employ this graph as a visualization tool for $G$. This is accomplished using the mapping $\varphi: V(G) \rightarrow \mathbb{Z}^{2}$, where

$$
\varphi(v)= \begin{cases}(d(x, v), d(y, v)) & v \in A_{1} \cup S \\ (-d(y, v)+\ell,-d(x, v)+\ell) & v \in A_{2} \cup S\end{cases}
$$

(Recall that $\ell=d(x, y)$ ). We denote the image of $G$ in $\mathbb{Z}_{\infty}^{2}$ by $\Omega=\varphi[G]$. Clearly $\varphi$ is a graph contraction, and therefore $\varphi\left[A_{i}\right]$ is a connected subgraph of $\Omega$. For any $j$ define

$$
R_{j}=\left\{v \in A_{1} \mid d(v, y)-d(v, x)=j\right\} \quad L_{j}=\left\{v \in A_{2} \mid d(v, y)-d(v, x)=j\right\}
$$

For ease of notation, we add $x$ to $R_{\ell}, L_{\ell}$ and $y$ to $R_{-\ell}, L_{-\ell}$. We denote $R=\bigcup_{j=-\ell}^{\ell} R_{j}$ and $L=\bigcup_{j=-\ell}^{\ell} L_{j}$. By Lemma $2.2 G=R \cup L$, and $\varphi\left[L_{j}\right], \varphi\left[R_{j}\right]$ are included in the straight line $\{(\xi, \xi+j) \mid \xi \in \mathbb{Z}\}$.


Figure 1: How $\varphi$ maps a 2-connected graph. $\Pi=\left(x, x_{1}, y\right)$ are the black vertices, $R_{1}$ 's vertices are the gray square vertices and $L_{-1}$ is the white square vertex.

In using $\varphi(G)$ as a visualization tool, we keep in mind that $\varphi$ is not injective. We note that $\varphi[\Pi]$ is the interval between $(0, \ell)$ and $(\ell, 0)$, and that by geodeticity $\varphi^{-1}((j, \ell-j))$ is the $j$-th vertex in $\Pi$. In drawing $\varphi[R]$ and $\varphi[L]$, we note that $\varphi[R]$ is "to the right" of $\varphi[\Pi]$, and $\varphi[L]$ is "to the left" of $\varphi[\Pi]$.

Lemma 2.3. If $u, v$ are neighbors in $G, u \in R_{j}, v \in R_{k}$, then $|k-j| \leq 2$. Moreover, if $|k-j|=2$, then the edge $\varphi(v) \varphi(u)$ is one of the two $(\xi, \xi+j) \sim(\xi \pm 1, \xi+j \mp 1)$.
Proof. A simple application of the triangle inequality.
Lemma 2.4. Suppose $R_{j}, R_{j-2} \neq \emptyset$ whereas $R_{j-1}=\emptyset$, then there exists an edge between $R_{j}$ and $R_{j-2}$.
Proof. Since $A_{1}$ is connected, there must be a path between $R_{j}$ and $R_{j-2}$. Consider a shortest such path. It must completely reside in $R_{j} \cup R_{j-1} \cup R_{j-2}$, and since $R_{j-1}=\emptyset$, its first step outside of $R_{j}$, must be to a vertex in $R_{j-2}$, as claimed.

The previous two lemmas clearly apply to $L$ as well.
Lemma 2.5. If $|j|<\ell$, then either $R_{j}$ or $L_{-j}$ must be empty.
Proof. We show that if there exist vertices $v \in R_{j}, u \in L_{-j}$, then $G$ is not geodetic, because $d(u, v)$ is realized by two distinct paths. Any $u, v$ path must clearly traverse either $x$ or $y$. But the assumption that $|j|<\ell$ implies that the shortest $u, x, v$ path cannot traverse $y$ and the shortest $u, y, v$ path cannot traverse $x$. In particular, the shortest $u, x, v$ path and $u, y, v$ path in $G$ are distinct. Moreover, they have the same length, because the shortest length of a $u, y, v$ resp. $u, x, v$ paths is $d(u, y)+d(y, v)$ resp. $d(u, x)+d(x, v)$. Since $d(u, y)-d(u, x)=j=d(v, x)-d(v, y)$, they have the same length, as claimed.

It follows from Lemma 2.5 that at least one of $R_{0}, L_{0}$ must be empty. We denote below $\Pi=x, x_{1}, x_{2}, \ldots, x_{\ell-1}, y$. In particular, if $\ell=1$, then $x_{1}=y$. We need the following lemma.
Lemma 2.6. If both $R_{\ell-1}=R_{\ell-3}=\emptyset$, then $R_{\ell}=\{x\}$. In particular, $x_{1}$ is the only neighbor of $x$ in $A_{1} \cup \Pi$.
Proof. The vertex $x_{1}$ has degree at least 3 , and must therefore have a neighbor in $R$. But by assumption $R_{\ell-1}=R_{\ell-3}=\emptyset$, so $x_{1}$ must have a neighbor in $R_{\ell-2}$. Consequently, $x_{1}$ is not the only vertex in $R_{\ell-2}$. The graph $\varphi\left[A_{1}\right]$ is connected, and since $\varphi^{-1}(0, \ell)=\{x\}$, it does not contain the vertex $(0, \ell)$, nor the edge $(0, \ell) \sim(1, \ell-1)$. Let us assume toward contradiction that $R_{\ell}$ is comprised not only of $x$. Then $\varphi\left[R_{\ell}\right] \backslash\{(0, \ell)\}$ and $R_{\ell-2}$, are nonempty whereas $R_{\ell-1}=\emptyset$. By Lemma 2.4, there exists an edge $(\xi, \xi+\ell) \sim(\xi+1, \xi+\ell-1)$. Let $v_{\xi} u_{\xi} \in E(G)$ be a pre-image of this edge. Namely, $\varphi\left(v_{\xi}\right)=(\xi, \xi+\ell), \varphi\left(u_{\xi}\right)=(\xi+1, \xi+\ell-1)$. Clearly $\pi\left(v_{\xi}, x\right)$ must be contained in $R_{\ell}$. Moreover, $\pi\left(u_{\xi}, y\right)$ does not go through $x$ because $d\left(u_{\xi}, y\right)-d\left(u_{\xi}, x\right)=\ell-2$. Therefore, $\pi\left(v_{\xi}, x\right) * \Pi$ and $v_{\xi} * \pi\left(u_{\xi}, y\right)$ are two distinct $v_{\xi}, y$ geodesics - contrary to the assumption of geodeticity.

A similar lemma can be proved for $R_{-\ell}$ : If both $R_{1-\ell}=R_{3-\ell}=\emptyset$, then $R_{-\ell}=\{y\}$. In particular, $x_{\ell-1}$ is the only neighbor of $y$ in $A_{1} \cup \Pi$.


Figure 2: Illustration of Lemma 2.6. The existence of the dotted edge follows from the assumption that $x$ is not the only vertex of $R_{\ell}$.

We can now prove Theorem 1.1.
Theorem 1.1. Every 2 -connected geodetic graph $G$ with $\delta(G) \geq 3$ must be 3 -connected. This lower bound is tight as shown by the Petersen Graph.

Proof. The argument runs as follows: The sets $R_{j}$ are nonempty for every other value of $j$. To wit, $R_{\ell-2 j}$ is nonempty for any $1 \leq j \leq \ell-1$, since $R_{\ell-2 j}$ contains the vertex $x_{j}$. But then Lemma 2.5 implies that $L_{\ell-2 j}=\emptyset$, see Figure 3b. We claim that $L_{\ell-1}$ and $L_{1-\ell}$ are nonempty, thus repeated application of Lemma 2.4 results in $L_{k} \neq \emptyset$ for any $k \not \equiv \ell \bmod 2$. Therefore, $R_{k}=\emptyset$ for such $k$. Now we satisfy the assumptions of Lemma 2.6, so $R_{\ell}=\{x\}$. But then $\left\{x_{1}, y\right\}$ is a vertex cut. If $x y \in E(G)$, then $x_{1}=y$, contrary to $G$ being 2 -connected. Otherwise, $d\left(x_{1}, y\right)<d(x, y)$ - contrary to the minimality of $\ell$.

It is left to justify that $L_{\ell-1}, L_{1-\ell}$ are nonempty. The neighbors of $x$ in $A_{2}$ reside in $L_{\ell}, L_{\ell-1}$. If $x$ does not have a neighbor in $L_{\ell}$, we are done, so suppose $x$ has a neighbor in $L_{\ell}$, and $y$ has a neighbor $v_{y} \in L_{-\ell} \cup L_{1-\ell}$. Since $A_{2}$ is connected, there must be a path connecting these vertices. Consider a shortest such path, and its first step outside of $L_{\ell}$. It cannot be a vertex in $L_{\ell-2}$, so it must be in $L_{\ell-1}$.


Figure 3: Illustration of the proof: Black lines represent diagonals which are known to be nonempty. Dotted lines stand for the presently undecided cases. A diagonal whose status is decided becomes black if proven nonempty, and deleted if empty.

## 3 Geodetic blocks of diameter 4 and 5

In this section, we construct two families of geodetic blocks, the graphs in which have diameter 4 and 5 . We need some notation first. Let $v$ be a vertex in a graph $G=(V, E)$ and let $i \geq 0$ be an integer. Clearly $|d(v, x)-d(v, y)| \leq 1$ for every edge $e=x y$ in $G$. We say that $e$ is $v$-horizontal resp. $v$-vertical if $d(v, x)=d(v, y)$ resp. $|d(v, x)-d(v, y)|=1$. Note that every edge in every shortest $v \rightarrow u$ path is $v$-vertical.

Proposition 3.1. $G$ is geodetic if and only if for every $v \in V$ the $v$-vertical edges form a spanning tree.

Proof. It is easy to see that the $v$-vertical edges form a spanning subgraph in every connected graph. Suppose that $G$ is not geodetic, and let us find some vertex $w$ such that the $w$-vertical edges in $G$ do not form a spanning tree, as there is a cycle comprised of $w$-vertical edges. Indeed, since $G$ is not geodetic, it has vertices $v, u$ with two distinct shortest paths between them $\pi \neq \pi^{\prime}$. But all the edges in $\pi, \pi^{\prime}$ are $v$-vertical, and together they contain a cycle, as claimed.

Conversely let $v$ be a vertex in a geodetic graph $G$, and let $T$ be a BFS tree rooted at $v$. Clearly all edges in $T$ are $v$-vertical, and as we show, all edges $x y \notin T$ are $v$-horizontal. Indeed, if $x y$ is $v$-vertical, with $d(v, x)=i$ and $d(v, y)=i+1$, this yields two distinct shortest paths from $v$ to $y$.

We mostly follow the notation and terminology of [13]. Let $q$ be a prime power, and let $P G_{2}(q)$ and $A G_{2}(q)$ be a projective resp. affine plane of order $q$. The point sets of these geometries is denoted by $P$, and their sets of lines (blocks) by $\mathcal{L}$. Their point-line incidence relation is denoted by $\mathcal{I} \subset P \times \mathcal{L}$. Elements of $\mathcal{I}$ are called Flags. The Levi Graph of an incidence structure $\mathbb{S}=(P, \mathcal{L}, \mathcal{I})$, denoted Levi $(\mathbb{S})$ is the bipartite graph with vertex sets $P \sqcup \mathcal{L}$, where $p$ and $L$ are neighbors iff $p$ is incident with $L$.

The Flag graph of $\mathbb{S}$ is denoted $\operatorname{Flag}(\mathbb{S})$. Its vertex set is $P \sqcup \mathcal{I}$. It has two kinds of edges: between a point $p$ and flag $(p, L)$. In addition $(p, L) \sim\left(p^{\prime}, L\right)$ for every two points $p, p^{\prime}$ of the same line $L$. As we show below $\operatorname{Flag}(\mathbb{S})$ is geodetic, and has diameter 4 when $\mathbb{S}=P G_{2}(q)$ and 5 if $\mathbb{S}=A G_{2}(q)$. To fix ideas, associated with each $L \in \mathcal{L}$ in Levi( $\left.\mathbb{S}\right)$ is a porcupine, a clique of size $|L|$, plus an edge $p \sim(p, L)$ emanating from $(p, L)$ for every $p \in L$. Therefore, to every simple path $Q$ in $\operatorname{Levi}(\mathbb{S})$ there corresponds a simple path $\hat{Q}$ in $\operatorname{Flag}(\mathbb{S})$. Namely, if $Q$ traverses through $L$, that is $\left[p, L, p^{\prime}\right]$, the corresponding steps in $\hat{Q}\left[p,(p, L),\left(p^{\prime}, L\right), p^{\prime}\right]$, which is a simple path in $\operatorname{Flag}(\mathbb{S})$. Recall that a graph is 2-connected if and only if every two of its vertices lie on a simple cycle. We conclude:

Corollary 1. If Levi( $\mathbb{S}$ ) is 2 -connected, then so is $\operatorname{Flag}(\mathbb{S})$.

### 3.1 Properties of $\operatorname{Flag}\left(A G_{2}(q)\right)$

Recall the following properties of $A G_{2}(q)$ :

1. It has $q^{2}$ points and $q^{2}+q$ lines.
2. Every point is incident with $q+1$ lines
3. Every line has $q$ points
4. If a point $p$ is not in a line $L$, then there exists a unique line $L^{\prime}$ with $p \in L^{\prime}$ such that $L$ and $L^{\prime}$ are disjoint. The common practice is to say that $L$ and $L^{\prime}$ are parallel and denote $L \| L^{\prime}$.

We denote $\mathcal{F}(q)=F l a g\left(A G_{2}(q)\right)$. Properties 1 and 3 imply that the number of vertices in $\left.\mathcal{F}(q)\right)$ is $q^{3}+2 q^{2}$. We denote by $L_{\alpha, \beta}$ the unique line that contains the two points $\alpha, \beta$.

Proposition 3.2. $\mathcal{F}(q)$ is 2-connected.

Proof. We exhibit a simple cycle through any two vertices in $A G_{2}(q)$. Consider all cycles of the form

$$
\left(L_{1}, x_{1}, L_{x_{1}, x_{2}}, x_{2}, L_{2}, x, L_{1}\right)
$$

where $L_{1}, L_{2}$ are two distinct intersecting lines with $L_{1} \cap L_{2}=x$, and $x_{1}, x_{2}$ are points in $L_{1}, L_{2}$ respectively, other than $x$. These yield cycles through any two vertices in $\mathcal{F}(q)$ other than a pair of parallel lines. Finally, here is a simple cycle through two parallel lines $L_{1}, L_{2}$

$$
\left(L_{1}, x_{1}, L_{x_{1}, x_{2}}, x_{2}, L_{2}, x_{2}^{\prime}, L_{x_{1}^{\prime}, x_{2}^{\prime}}, x_{1}^{\prime}, L_{1}\right)
$$

Here $x_{1}, x_{1}^{\prime}, x_{2}, x_{2}^{\prime}$ are distinct points on $L_{1}, L_{2}$. This completes the proof.
Theorem 3.3. $\mathcal{F}(q)$ is geodetic and of diameter 5 .
Proof. We show that the collection of $v$-vertical edges form a tree for every vertex $v$ in $\mathcal{F}(q)$. There are just two cases to consider: $v=\mathbf{p}$, a point in $P$ or a flag $v=(\mathbf{p}, \mathbf{L}) \in \mathcal{I}$. Let $\mathcal{V}(i)$ be the number of $v$-vertical edges $\alpha \beta \in E$ of height $i$, i.e., $\left\{(d(v, \alpha), d(v, \beta)\}=\{i-1, i\}\right.$. Let $N_{i}(v)$ be the $i$-sphere centered at $v$, that is $N_{i}(v)=\{u \in V \mid d(v, u)=i\}$. We analyze $N_{i}(v)$ in either case.
$\mathbf{v}=\mathbf{p}:$

1. Clearly $N_{1}(\mathbf{p})=\{(\mathbf{p}, L) \in \mathcal{I}\}$, so $\mathcal{V}(1)=\left|N_{1}(\mathbf{p})\right|=q+1$.
2. Each $(\mathbf{p}, L) \in N_{1}(\mathbf{p})$ is adjacent to all the vertices in the clique defined by $L$. This contributes $q-1$ vertical edges. Also, $N_{2}(\mathbf{p})=\{(x, L) \mid \mathbf{p} \in L, x \neq p\}$, since these cliques are disjoint. It also follows that $\mathcal{V}(2)=\left|N_{1}(p)\right| \cdot(q-1)=\left(q^{2}-1\right)$.
3. Let $(x, L) \in N_{2}(\mathbf{p})$. The neighbors of $(x, L)$ are $x$ and $(y, L)$ for $y \neq x$ in $L$. All the latter are in $N_{2}(\mathbf{p})$, so every $(x, L) \in N_{2}(\mathbf{p})$ contributes exactly one vertex to $N_{3}(\mathbf{p})$. Consequently, $N_{3}(v)=\{x \mid x \neq \mathbf{p}\}$, and $\mathcal{V}(3)=\mathcal{V}(2)=\left(q^{2}-1\right)$.
4. Each $x \neq \mathbf{p}$ is incident with $q+1$ lines, only one of which is $L_{\mathbf{p}, x}$. Therefore, $x$ is adjacent to $q$ flags $\left(x, L^{\prime}\right)$, one for each $L^{\prime} \neq L_{\mathbf{p}, x}$. These flags are clearly distinct, therefore $N_{4}(v)=\left\{\left(x, L^{\prime}\right) \mid L^{\prime} \neq L_{\mathbf{p}, x}\right\}$ and $\mathcal{V}(4)=q \cdot \mathcal{V}(3)=q\left(q^{2}-1\right)$.

The calculation checks:

$$
\overbrace{(q+1)}^{\mathcal{V}(1)}+\overbrace{\left(q^{2}-1\right)}^{\mathcal{V}(2)}+\overbrace{\left(q^{2}-1\right)}^{\mathcal{V}(3)}+\overbrace{q\left(q^{2}-1\right)}^{\mathcal{V}(4)}=q^{3}+2 q^{2}-1=|V(\mathcal{F}(q))|-1 .
$$



Figure 4: $\mathcal{F}(3)$, as seen from $v=\mathbf{p}$. The colors have no mathematical significance and are only intended for better visibility.
$\mathbf{v}=(\mathbf{p}, \mathbf{L}): \quad$ This case is a bit trickier. Let $x \neq \mathbf{p}$ be a point on $\mathbf{L}$. Let $y$ be a point not on $\mathbf{L}$, and $\ell_{y}$ the unique line through $y$ that is parallel to $\mathbf{L}$. Finally, we denote by $\left\{\mathbf{L}_{x}^{i}\right\}_{i=0}^{q}$ the lines through $x$, where $\mathbf{L}=\mathbf{L}_{x}^{0}$.

1. $N_{1}(v)=\{\mathbf{p}\} \sqcup\{(x, \mathbf{L}) \mid x \neq \mathbf{p}\}$, so $\mathcal{V}(1)=1+(q-1)=q$. We refer below to descendants of $\mathbf{p}$ as the left vertices, and to descendants of $\{(x, \mathbf{L}) \mid x \neq \mathbf{p}\}$ as right vertices (see Figure 5).
2. The neighbors of $\mathbf{p}$ other than $(\mathbf{p}, \mathbf{L})$ are $\left\{\left(\mathbf{p}, \mathbf{L}_{\mathbf{p}}^{i}\right)\right\}_{i \neq 0}$, so the left side in $N_{1}(v)$ contributes $q$ edges of height 2 . The right vertices in $N_{1}(v)$ form a clique, with a porcupine structure. Therefore, each vertex of this clique contributes a single height 2 edge, namely $(x, \mathbf{L}) \sim x$. So $N_{2}((\mathbf{p}, \mathbf{L}))=\left\{\left(\mathbf{p}, \mathbf{L}_{\mathbf{p}}^{i}\right)\right\}_{i \neq 0} \sqcup\{x \mid \mathbf{p} \neq x \in \mathbf{L}\}$ and $\mathcal{V}(2)=q+(q-1)$.
3. Each left vertex in $N_{2}(v)$ is adjacent to a clique $\left\{\left(y, \mathbf{L}_{\mathbf{p}}^{i}\right)\right\}_{y \neq \mathbf{p}}$ of $q-1$ flags, so each $\left(\mathbf{p}, \mathbf{L}_{\mathbf{p}}^{i}\right)$ contributes $(q-1)$ distinct edges of height 3 . As for the right side, the edges $x \sim\left(x, \mathbf{L}_{x}^{i}\right)$ have height 3 , and each $x \neq \mathbf{p}$ contributes $q$ of them.
So, $N_{3}(v)=\left\{\left(y, \mathbf{L}_{\mathbf{p}}^{i}\right) \mid y \neq p, i \neq 0\right\} \sqcup\left\{\left(x, \mathbf{L}_{x}^{i}\right) \mid x \neq \mathbf{p}, i \neq 0\right\}$ and $\mathcal{V}(3)=q(q-1)+q(q-1)$.
4. For every $y \notin \mathbf{L}$, there holds $L_{\mathbf{p}, y}=\mathbf{L}_{\mathbf{p}}^{i}$ for some $i \neq 0$. Therefore, the left side vertices of $N_{3}(v)$ are partitioned into cliques in $N_{3}(v)$, and cover all $y \notin \mathbf{L}$. Hence each vertex contributes a unique edge $\left(y, L_{\mathbf{p}, y}\right) \sim y$ of height 4. As for the right side: Every edge $\left(x, \mathbf{L}_{x}^{i}\right)$ of height 4 is adjacent to the clique $\left\{\left(y, L_{x}^{i}\right) \mid y \neq x\right\}$. There are exactly $q-1$ such vertices for each $\left(x, \mathbf{L}_{x}^{i}\right)$. So $N_{4}(v)=\{y \mid y \notin \mathbf{L}\} \sqcup\left\{\left(y, \mathbf{L}_{x}^{i}\right) \mid x \in \mathbf{L}, y \notin \mathbf{L}, i \in[q]\right\}$ and $\mathcal{V}(4)=q(q-1)+q(q-1)^{2}$.
5. The edges $\left(y, L_{x}^{i}\right) \sim y$ are horizontal (between the two sides of $N_{4}(v)$ ), so the only 5 vertical edges are of the form $y \sim\left(y, \ell_{y}\right)$ - and since $\ell_{y}$ is unique, $\mathcal{V}(5)=q(q-1)$ and $N_{5}(v)=\left\{\left(y, \ell_{y}\right) \mid y \notin \mathbf{L}\right\}$.

Once again, the calculation checks:

$$
\overbrace{1+(q-1)}^{\mathcal{V}(1)}+\overbrace{q+(q-1)}^{\mathcal{V}(2)}+\overbrace{q(q-1)+q(q-1)}^{\mathcal{V}(3)}+\overbrace{q(q-1)+q(q-1)^{2}}^{\mathcal{V}(4)}+\overbrace{q(q-1)}^{\mathcal{V}(5)}=q^{3}+2 q^{2}-1 .
$$

This analysis also establishes that $\mathcal{F}(q)$ has diameter 5 . such edges. Each $x \in L$ lies on $q$ lines other than $L$, so each $x$ contributes $p$ additional 3 -vertical edges of the form $x\left(x, L^{\prime}\right)$ for a total of $q(q-1)$ such edges. Each such $L^{\prime}$ contains $p-1$ points other than $x$, and said points d not lie on $L$. Thus each $\left(x, L^{\prime}\right)$ contributes $p-1$ vertical edges of the form $\left(x, L^{\prime}\right)\left(y, L^{\prime}\right)$, for a total of $q \cdot(q-1)^{2}$ edges. Neighbors of $\left(y, L^{\prime}\right)$ are the points $y$, which we are of distance 4 from $(p, L)$ - as we see in the fllowing part.


Figure 5: $\mathcal{F}(3)$, as seen from $v=(\mathbf{p}, \mathbf{L})$.

### 3.2 Properties of $\operatorname{Flag}\left(P G_{2}(q)\right)$

Recall the following properties of $P G_{2}(q)$ :

1. It has $q^{2}+q+1$ points and the same number of lines.
2. Every point is incident with $q+1$ lines, and every line has $q+1$ points.

These properties imply that $\mid V\left(F \operatorname{lag}\left(P G_{2}(q)\right) \mid=(q+1)^{3}+1\right.$.
Theorem 3.4. $\operatorname{Flag}\left(P G_{2}(q)\right)$ is 2 -connected, geodetic and has diameter 4 .
Proof. By Corollary 1, if $\operatorname{Levi}\left(P G_{2}(q)\right)$ is 2-connected, then so is $\operatorname{Flag}\left(P G_{2}(q)\right)$. To show that $\operatorname{Levi}\left(P G_{2}(q)\right)$ is 2-connected, it suffices to use the first cycle that is described in the proof of Proposition 3.2. The proof of geodeticity is a slight adaptation of the proof of Theorem 3.3. The numbers change somewhat, and what's more, since no lines are parallel, we do not reach $N_{5}(v)$ in the second case. Consequently, the diameter is 4 .

## 4 Discussion

Geodetic blocks with $\delta(G) \geq 3$ (which, by Theorem 1.1, are 3 connected) seem hard to find. Specifically, we ask:

1. We recall our question whether geodetic blocks of $\delta(G) \geq 3$ can have arbitrarily large diameter. Also, can they have arbitrarily large girth? The two questions are closely related since the diameter of a graph is at least half its girth.
2. So far, we only know of three cubic geodetic blocks - the Petersen Graph, $K_{4}$ and $F l a g\left(P G_{2}(2)\right)$. Is this list exhaustive? Is the number of such graphs finite?
3. Is the study of geodetic graphs related to structural graph theory, and more concretely to the family of even-hole-free graphs [14]? The origin of this question is this: If $u, v$ are two antipodal vertices in an induced even cycle (aka an even hole) $C$ in a geodetic graph, then the $u v$ geodetic is not included in $C$.

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