Plug-and-play state estimation and application to distributed output-feedback model predictive control ☆

Stefano Riverso [a,c], Marcello Farina [b], Giancarlo Ferrari-Trecate [a,*]

[a] Dipartimento di Ingegneria Industriale e dell'Informazione, Università degli Studi di Pavia, via Ferrata 1, 27100 Pavia, Italy
[b] Dipartimento di Elettronica, Informazione e Bioingegneria, Politecnico di Milano, via Ponzio 34/5, 20133 Milan, Italy
[c] United Technologies Research Center Ireland, 4th Floor, Penrose Business Center, Penrose Wharf, Cork, Ireland

Received 30 June 2014
Received in revised form
5 February 2015
Accepted 8 April 2015
Available online 30 April 2015

1. Introduction

In the last years, we have witnesses a renewed interest in the development of distributed and decentralized control and state estimation methods [25,3]. This has been motivated by the ever increasing need of coordination and integration of a possibly large number of different devices. When a plant is composed of several interconnected subsystems, centralized control and state estimation architectures are inadequate. In fact, online operations, such as the transmission of measurements to a central processing unit or the simultaneous estimation of all states, can be prohibitive.

Focusing on state estimation, a large body of research has been recently devoted to the development Distributed State Estimators (DSEs) where subsystems are equipped with Local State Estimators (LSEs) connected through a communication network. While in some approaches each LSE has to reconstruct the state of the overall system (see, e.g., [2,26] and references therein for theoretical contributions and applications to power networks), in other methods LSEs are dedicated to the reconstruction of local states only [24,38,12,34,35,4,7,17,21]. In this paper we consider the latter case. In terms of communication requirements, some methods are more parsimonious than others, as they do not require *all-to-all information exchange* among LSEs. For instance, some DSEs require only a network matching the *parent–child topology* due to coupling among subsystems [12,34,35,4,7,17,21]. Furthermore, there are methods that also guarantee the fulfillment of constraints on local states [4] or estimation errors [7,17,21].

To the best of our knowledge, the DSE proposed in [21] is the first one relying on a fully distributed design procedure. In [21], boundedness and convergence of the global estimation error can be guaranteed through numerical tests that are associated with individual LSEs and that can be performed in parallel using hardware collocated with subsystems. Moreover, each test involves information about a subsystem and its parents only, hence requiring a communication net-work with the same parent–child topology used for real-time operations. Reconfiguration of the estimation scheme can be per-formed through Plug-and-Play (PnP) design, meaning that (i) when a subsystem is added to a plant, the corresponding LSE can be designed using pieces of information from parent subsystems only; (ii) in order to preserve the key properties of the whole DSE, the plugging in and out of a subsystem triggers at most the update of LSEs associated to child subsystems and (iii) the design/update of an LSE is automatized, e.g. as in our approach it is recast into an optimization problem that can be solved using local hardware. We argue that PnP design (firstly introduced in [37] for control) can be useful in the context of systems of systems [32] and cyber-physical systems [1] where, typically, the number of subsystems changes over time.

☆The research leading to these results has received funding from the European Union Seventh Framework Programme [FP7/2007-2013] under Grant agreement no. 257462 HYCON2 Network of Excellence.

* Corresponding author.
E-mail addresses: riverss@utrc.utc.com (S. Riverso), marcello.farina@polimi.it (M. Farina), giancarlo.ferrari@unipv.it (G. Ferrari-Trecate).

In this paper we extend the results presented in [21] to account for bounded measurement noise. Furthermore, we use the DSE in combination with the PnP distributed Model Predictive Control (MPC) schemes described in [30,22] to provide a novel output-feedback PnP distributed control algorithm. More specifically, we show how the tube MPC scheme in [14] can be used in PnP design to guarantee closed-loop nominal convergence and constraint satisfaction at all times.

Recently, output feedback Distributed MPC (DMPC) schemes have been proposed based on a cooperative approach (see, e.g., [36,10]) or in a non-cooperative setting (see, e.g., [7,31]). However, in contrast with PnP design, all these methods involve a centralized offline design phase.

This paper is structured as follows. In Section 2 we introduce the state estimation problem and the main assumptions on the system, while in Section 3 we describe the PnP DSE. In Section 4 we describe the PnP output-feedback DMPC strategy embedding the proposed estimation scheme. In Section 5 two case studies are discussed: the application of DSE to an array of 16 masses connected by springs and dampers, and output-feedback frequency control in a power network. In Section 6 some conclusions are drawn.

Notation: We use $a:b$ for the set of integers $\{a,a+1,\dots,b\}$. The column vector with s components $v_1,\dots,v_s$ is $\mathbf{v}=(v_1,\dots,v_s)$. The symbol $\oplus$ denotes the Minkowski sum, i.e. $A=B\oplus C$ if and only if $A=\{a: a=b+c, b\in B, c\in C\}$. Moreover, $\oplus_{i=1}^{s} G_i = G_1\oplus\cdots\oplus G_s$. The Pontryagin difference is denoted by $\ominus$, i.e. $A=B\ominus C$ iff $A=\{a: a+c\in B, \forall c\in C\}$. The symbol $\mathbf{1}_\alpha$ (resp. $\mathbf{0}_\alpha$) denotes a column vector with $\alpha\in\mathbb{N}$ elements all equal to 1 (resp. 0). The identity matrix of size n is $\mathbb{I}_n$. Given a matrix $A\in\mathbb{R}^{n\times n}$, with entries a_{ij} its entry-wise 1-norm is $||A||_1=\sum_{i=1}^{n}\sum_{j=1}^{n}|a_{ij}|$ and its Frobenius norm is $||A||_F=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^2}$. The standard Euclidean norm is denoted with $\|\cdot\|$. The pseudo-inverse of a matrix $A\in\mathbb{R}^{m\times n}$ is denoted with $A^\flat$. A matrix $A\in\mathbb{R}^{n\times n}$ is Schur stable if all its eigenvalues λ verify $|\lambda|<1$. Moreover, for $\delta>0$, $\mathcal{B}_\delta(v)=\{x\in\mathbb{R}^n: \|x-v\|<\delta\}$.

The set $\mathbb{X}\subseteq\mathbb{R}^n$ is Robust Positively Invariant (RPI) for $x(t+1)=f(x(t),w(t))$, $w(t)\in\mathbb{W}\subseteq\mathbb{R}^m$ if $x(t)\in\mathbb{X}\Rightarrow f(x(t),w(t))\in\mathbb{X}$, $\forall w(t)\in\mathbb{W}$.

2. Model of interconnected systems

We consider a discrete-time linear time-invariant system described by

$$\mathbf{x}^+=\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u}+\mathbf{D}\mathbf{d} \tag{1a}$$

$$\mathbf{y}=\mathbf{C}\mathbf{x}+\boldsymbol{\varrho} \tag{1b}$$

where $\mathbf{x}\in\mathbb{R}^n$, $\mathbf{u}\in\mathbb{R}^m$, $\mathbf{y}\in\mathbb{R}^p$, $\mathbf{d}\in\mathbb{R}^r$ and $\boldsymbol{\varrho}\in\mathbb{R}^p$ are the state, the input, the output, the model disturbance and the output disturbance, respectively, at time t and $\mathbf{x}^+$ stands for $\mathbf{x}$ at time $t+1$. Let $\mathcal{M}=1:M$ be the set of subsystem indexes. We assume that the state is composed of M state vectors $x_{[i]}\in\mathbb{R}^{n_i}$, $i\in\mathcal{M}$ such that $\mathbf{x}=(x_{[1]},\dots,x_{[M]})$, and $n=\sum_{i\in\mathcal{M}}n_i$. Similarly, the input, the output, the model disturbance and the output disturbance are composed of vectors $u_{[i]}\in\mathbb{R}^{m_i}$, $y_{[i]}\in\mathbb{R}^{p_i}$, $d_{[i]}\in\mathbb{R}^{r_i}$, $\varrho_{[i]}\in\mathbb{R}^{p_i}$, $i\in\mathcal{M}$ such that $\mathbf{u}=(u_{[1]},\dots,u_{[M]})$, $m=\sum_{i\in\mathcal{M}}m_i$, $\mathbf{y}=(y_{[1]},\dots,y_{[M]})$, $p=\sum_{i\in\mathcal{M}}p_i$, $\mathbf{d}=(d_{[1]},\dots,d_{[M]})$, $r=\sum_{i\in\mathcal{M}}r_i$ and $\boldsymbol{\varrho}=(\varrho_{[1]},\dots,\varrho_{[M]})$. We assume that system (1) is partitioned into M interconnected subsystems $\Sigma_{[i]}$, $i\in\mathcal{M}$, each described by the following dynamical equations:

$$\Sigma_{[i]}:\quad x_{[i]}^+=A_{ii}x_{[i]}+B_iu_{[i]}+\sum_{j\in\mathcal{N}_i}A_{ij}x_{[j]}+D_id_{[i]} \tag{2a}$$

$$y_{[i]}=C_ix_{[i]}+\varrho_{[i]} \tag{2b}$$

where $A_{ij}\in\mathbb{R}^{n_i\times n_j}$, $i,j\in\mathcal{M}$, $B_i\in\mathbb{R}^{n_i\times m_i}$, $D_i\in\mathbb{R}^{n_i\times r_i}$, $C_i\in\mathbb{R}^{p_i\times n_i}$. Since, in general, $A_{ij}\neq\mathbf{0}_{n_i\times n_j}$ for $j\neq i$, the subsystems are coupled through state variables (i.e., they are dynamically coupled). We say that $\Sigma_{[j]}$ is a *parent* of $\Sigma_{[i]}$ if $A_{ij}\neq\mathbf{0}_{n_i\times n_j}$, and we define with $\mathcal{N}_i=\{j\in\mathcal{M}: A_{ij}\neq\mathbf{0}_{n_i\times n_j}, i\neq j\}$ the set of parents of $\Sigma_{[i]}$. From (2b), it is assumed that $y_{[i]}$ depends on the local state $x_{[i]}$ and the local measurement noise $\varrho_{[i]}$ only, while, from (2a), the time evolution of $x_{[i]}$ directly depends only upon the local input $u_{[i]}$ and the local disturbance $d_{[i]}$. Generalizations of model (2) are discussed in **Remark 3**. We also assume that subsystems $\Sigma_{[i]}$, $i\in\mathcal{M}$ are subject to the constraints

$$d_{[i]}\in\mathbb{D}_i,\quad \varrho_{[i]}\in\mathbb{O}_i. \tag{3}$$

For the overall system (1), this is equivalent to

$$\mathbf{d}\in\mathbb{D},\quad \boldsymbol{\varrho}\in\mathbb{O}. \tag{4}$$

where $\mathbb{D}=\prod_{i\in\mathcal{M}}\mathbb{D}_i$ and $\mathbb{O}=\prod_{i\in\mathcal{M}}\mathbb{O}_i$. We introduce the following assumptions on subsystem dynamics and disturbances.

Assumption 1. (I) The pair (A_{ii},C_i) is detectable, $\forall i\in\mathcal{M}$.

(II) Constraints $\mathbb{D}_i\subset\mathbb{R}^{r_i}$ and $\mathbb{O}_i\subset\mathbb{R}^{p_i}$ are zonotopes centered at the origin. Without loss of generality [33], $\mathbb{D}_i$ can be written as

$$\begin{aligned}\mathbb{D}_i&=\{d_{[i]}\in\mathbb{R}^{r_i}:\mathcal{F}_iw_{[i]}\leq\mathbf{1}_{\overline{\upsilon}_i}\}\\&=\{d_{[i]}\in\mathbb{R}^{r_i}: d_{[i]}=\Delta_iF_i^{d},\ ||F_i^{d}||_\infty\leq1\}\end{aligned} \tag{5}$$

where $\mathcal{F}_i=(f_{i,1}^T,\dots,f_{i,\overline{\upsilon}_i}^T)\in\mathbb{R}^{\overline{\upsilon}_i\times r_i}$, $\operatorname{rank}(\mathcal{F}_i)=r_i$, $\Delta_i\in\mathbb{R}^{r_i\times\overline{r}_i}$, $F_i^d\in\mathbb{R}^{\overline{r}_i}$. Furthermore, $\mathbb{O}_i$ can be written as

$$\begin{aligned}\mathbb{O}_i&=\{\varrho_{[i]}\in\mathbb{R}^{p_i}:\mathcal{G}_i\varrho_{[i]}\leq\mathbf{1}_{\overline{\nu}_i}\}\\&=\{\varrho_{[i]}\in\mathbb{R}^{p_i}:\varrho_{[i]}=\Upsilon_iF_i^{\varrho},\ ||F_i^{\varrho}||_\infty\leq1\}\end{aligned} \tag{6}$$

where $\mathcal{G}_i=(g_{i,1}^T,\dots,g_{i,\overline{\nu}_i}^T)\in\mathbb{R}^{\overline{\nu}_i\times p_i}$, $\operatorname{rank}(\mathcal{G}_i)=p_i$, $\Upsilon_i\in\mathbb{R}^{p_i\times\overline{p}_i}$ and $F_i^{\varrho}\in\mathbb{R}^{\overline{p}_i}$.

Next we introduce assumptions that will be instrumental for developing DMPC schemes.

Assumption 2. (I) The pair (A_{ii},B_i) is stabilizable, $\forall i\in\mathcal{M}$.

(II) Subsystems $\Sigma_{[i]}$, $i\in\mathcal{M}$ are subject to the constraints

$$x_{[i]}\in\mathbb{X}_i,\quad u_{[i]}\in\mathbb{U}_i \tag{7}$$

where $\mathbb{X}_i$ and $\mathbb{U}_i$ are polytopes given by

$$\mathbb{X}_i=\{x_{[i]}\in\mathbb{R}^{n_i}: c_{x_{i,\tau}}^Tx_{[i]}\leq1,\ \forall\tau\in1:\tau_i^x\} \tag{8a}$$

$$\mathbb{U}_i=\{u_{[i]}\in\mathbb{R}^{m_i}: c_{u_{i,\tau}}^Tu_{[i]}\leq1,\ \forall\tau\in1:\tau_i^u\}, \tag{8b}$$

with $c_{x_{i,\tau}}\in\mathbb{R}^{n_i}$ and $c_{u_{i,\tau}}\in\mathbb{R}^{m_i}$.

3. DSE with PnP features

3.1. State estimation scheme

In this section we propose a DSE for system (1) similar to the one presented in [21], where measurement noise was not accounted for.

For each subsystem $\Sigma_{[i]}$, $i\in\mathcal{M}$, the corresponding LSE $\tilde{\Sigma}_{[i]}$ is defined as follows

$$\begin{aligned}\tilde{\Sigma}_{[i]}:\quad \tilde{x}_{[i]}^+&=A_{ii}\tilde{x}_{[i]}+B_iu_{[i]}-L_{ii}(y_{[i]}-C_i\tilde{x}_{[i]})\\&\quad+\sum_{j\in\mathcal{N}_i}A_{ij}\tilde{x}_{[j]}-\sum_{j\in\mathcal{N}_i}\delta_{ij}^{L}L_{ij}(y_{[j]}-C_j\tilde{x}_{[j]})\end{aligned} \tag{9}$$

where $\tilde{x}_{[i]}\in\mathbb{R}^{n_i}$ is the state estimate, $L_{ij}\in\mathbb{R}^{n_i\times p_j}$ are gain matrices and $\delta_{ij}^{L}\in\{0,1\}$. In view of (9), $\tilde{\Sigma}_{[i]}$ depends on local variables (i.e., $\tilde{x}_{[i]}$, $u_{[i]}$ and $y_{[i]}$), and on outputs and state estimates of the parent subsystems (i.e., $\tilde{x}_{[j]}$ and $y_{[j]}$, $j\in\mathcal{N}_i$). Variables δ_{ij}^{L}, $j\in\mathcal{N}_i$ are set to one if parents' outputs are used for local estimation purposes, at the price of slightly increasing the amount of transmitted information. Defining

$$e_{[i]}=x_{[i]}-\tilde{x}_{[i]}, \tag{10}$$

from (2), (9), and (10), the local error dynamics is

$$e_{[i]}^{+}=\bar{A}_{ii}e_{[i]}+\tilde{v}_{[i]} \tag{11}$$

where

$$\tilde{v}_{[i]}=\sum_{j\in\mathcal{N}_i}\bar{A}_{ij}e_{[j]}+D_i d_{[i]}+L_{ii}\varrho_{[i]}+\sum_{j\in\mathcal{N}_i}\delta_{ij}^{L}L_{ij}\varrho_{[j]} \tag{12}$$

and $\bar{A}_{ii}=A_{ii}+L_{ii}C_i$, $\bar{A}_{ij}=A_{ij}+\delta_{ij}^{L}L_{ij}C_j$, $i\neq j$. Our first goal is to solve the following problem.

Problem 1. Design in a distributed fashion LSEs $\tilde{\Sigma}_{[i]}$, for all $i\in\mathcal{M}$, that

(a) are nominally convergent, i.e. when $\mathbb{D}_i=\{\mathbf{0}_{r_i}\}$ and $\mathbb{O}_i=\{\mathbf{0}_{p_i}\}$ it holds

$$e_{[i]}(t)\to\mathbf{0}_{n_i} \text{ as } t\to\infty \tag{13}$$

(b) guarantee, for suitable initial conditions, that

$$e_{[i]}(t)\in\mathbb{E}_i,\ \forall t\geq 0 \tag{14}$$

where $\mathbb{E}_i\subseteq\mathbb{R}^{n_i}$ are zonotopes centered at the origin defined by

$$\begin{aligned}\mathbb{E}_i&=\{e_{[i]}\in\mathbb{R}^{n_i}:\mathcal{H}_i e_{[i]}\leq\mathbf{1}_{\bar{\tau}_i}\}\\&=\{e_{[i]}\in\mathbb{R}^{n_i}:e_{[i]}=\Xi_i F_i^{e},||F_i^{e}||_\infty\leq 1\}.\end{aligned} \tag{15}$$

In (15), $\mathcal{H}_i=(h_{i,1}^T,\dots,h_{i,\bar{\tau}_i}^T)\in\mathbb{R}^{\bar{\tau}_i\times n_i}$, $\operatorname{rank}(\mathcal{H}_i)=n_i$, $\Xi_i\in\mathbb{R}^{n_i\times\bar{n}_i}$ and $F_i^{e}\in\mathbb{R}^{\bar{n}_i}$.

Sets $\mathbb{E}_i$ in (15) are design parameters fixed by the user. In many applications, sets $\mathbb{E}_i$ are given a priori, on the basis of constraints, nonlinearities and requirements on the prescribed accuracy of the estimation. In alternative, if subsystem states $x_{[i]}$ are confined in a known polyhedron $\mathbb{X}_i$ including the origin, then, in the absence of any prior information concerning the initial states, one can set $\mathbb{E}_i$ as a zonotope including $\mathbb{X}_i$ and initialize the LSE state as $\tilde{x}_{[i]}=\mathbf{0}_{n_i}$.

Defining the variable $\mathbf{e}=(e_{[1]},\dots,e_{[M]})\in\mathbb{R}^{n}$, from (11) one obtains the collective dynamics of the estimation error

$$\mathbf{e}^{+}=\bar{\mathbf{A}}\mathbf{e}+\mathbf{D}\mathbf{d}+\mathbf{L}\boldsymbol{\varrho} \tag{16}$$

The matrix $\bar{\mathbf{A}}$ is composed by blocks $\bar{A}_{ij}$ and matrix $\bar{\mathbf{L}}$ is composed by blocks L_{ii}, and by blocks $\delta_{ij}^{L}L_{ij}$ for $j\neq i$. We equip system (16) with constraints $\mathbf{e}\in\mathbb{E}=\prod_{i\in\mathcal{M}}\mathbb{E}_i$ and (4).

From (16), if $\mathbf{L}$ is such that $\bar{\mathbf{A}}$ is Schur stable, then (13) holds. Moreover, if there exists an RPI set $\mathbb{S}\subseteq\mathbb{E}$ for (16) with respect to the bounded disturbances $\mathbf{Dd}$ and $\mathbf{L}\boldsymbol{\varrho}$, then $\mathbf{e}(0)\in\mathbb{S}$ guarantees property (14). In this paper we require a stronger condition: there exists a “rectangular” invariant set $\mathbb{S}$, i.e., $\mathbb{S}=\prod_{i\in\mathcal{M}}\mathbb{S}_i$, such that

(a) $\mathbb{S}_i\subseteq\mathbb{E}_i$ for all $i\in\mathcal{M}$
(b) if $e_{[i]}(0)\in\mathbb{S}_i$ for all $i\in\mathcal{M}$, then

$$e_{[i]}(t)\in\mathbb{S}_i,\quad t\geq 0. \tag{17}$$

3.2. Distributed design of LSEs

In the following we solve Problem 1 by designing matrices L_{ij} $i,j\in\mathcal{M}$ such that system (16) is nominally asymptotically stable and by guaranteeing the existence of an RPI set $\mathbb{S}_i\subseteq\mathbb{E}_i$ for each local error dynamics. Recalling (12) and assuming that $e_{[j]}^{+}\in\mathbb{E}_j$ for all $j\in\mathcal{N}_i$, we obtain that

$$\tilde{v}_{[i]}\in\tilde{\mathbb{V}}_i=\left(\bigoplus_{j\in\mathcal{N}_i}\bar{A}_{ij}\mathbb{E}_j\right)\oplus D_i\mathbb{D}_i\oplus L_{ii}\mathbb{O}_i\oplus\left(\bigoplus_{j\in\mathcal{N}_i}\delta_{ij}^{L}L_{ij}\mathbb{O}_j\right). \tag{18}$$

From Assumption 1-(II) and (15), one has that $\tilde{\mathbb{V}}_i$ is a zonotope and therefore it can be written as $\tilde{\mathbb{V}}_i=\{\tilde{v}_{[i]}\in\mathbb{R}^{n_i}:\tilde{v}_{[i]}=\Psi_i\tilde{F}_i,||\tilde{F}_i||_\infty\leq 1\}$ where $\Psi_i\in\mathbb{R}^{n_i\times\hat{n}_i}$, $\operatorname{rank}(\Psi_i)=n_i$ and $\tilde{F}_i\in\mathbb{R}^{\hat{n}_i}$. The following proposition provides the key conditions enabling distributed design of the DSE.

Proposition 1. *Let Assumption 1 holds. If, for given matrices L_{ij} and parameters δ_{ij}^{L}, $i\in\mathcal{M}$, $j\in\mathcal{N}_i$ the following conditions are fulfilled simultaneously*:

$$\bar{A}_{ii} \text{ is Schur stable},\ \forall i\in\mathcal{M} \tag{19a}$$

$$\beta_i=\sum_{j\in\mathcal{N}_i}\sum_{k=0}^{\infty}||\mathcal{H}_i\bar{A}_{ii}^{k}\bar{A}_{ij}\mathcal{H}_j^{\flat}||_\infty<1,\ \forall i\in\mathcal{M} \tag{19b}$$

$$\gamma_i=\sum_{k=0}^{\infty}||\mathcal{H}_i\bar{A}_{ii}^{k}\Psi_i||_\infty<1,\forall i\in\mathcal{M} \tag{19c}$$

then

(I) *$\bar{\mathbf{A}}$ is Schur stable;*
(II) *for all $i\in\mathcal{M}$ there exists an RPI set $\mathbb{S}_i\subseteq\mathbb{E}_i$ for (11), such that $\mathbb{S}=\prod_{i\in\mathcal{M}}\mathbb{S}_i$ is an RPI set for system (16).*

Proof. The proof is given in Appendix 8.7 of [23].□

We highlight that, for a given $i\in\mathcal{M}$, the quantity β_i in (19b) depends only upon local fixed parameters $\{A_{ii},C_i,\mathcal{H}_i,\Psi_i\}$, parents' fixed parameters $\{A_{ij},C_j,\mathcal{H}_j,\Psi_i\}_{j\in\mathcal{N}_i}$ (Ψ_i depends also on fixed parameters of parent subsystems) and local tunable parameters $\{L_{ii},\{L_{ij},\delta_{ij}^{L}\}_{j\in\mathcal{N}_i}\}$ but not on parents' tunable parameters. This implies that the choice of $\{L_{ii},\{L_{ij},\delta_{ij}^{L}\}_{j\in\mathcal{N}_i}\}$ does not influence the choice of $\{L_{jj},\{L_{jk},\delta_{jk}^{L}\}_{k\in\mathcal{N}_j}\}$, for $i\neq j$. Then, the design of LSEs can be distributed, because Problem 1 can be decomposed into the following independent design problems $\mathcal{P}_i$, $i\in\mathcal{M}$.

Problem 2 (*Problem $\mathcal{P}_i$*). Check if there exist L_{ii} and $\{L_{ij}\}_{j\in\mathcal{N}_i}$ such that $\bar{A}_{ii}$ is Schur stable, $\beta_i<1$ and $\gamma_i<1$.

Remark 1. As shown in [13], a necessary condition for the existence of RPI sets $\mathbb{S}_i$ for (11) is that

$$\tilde{\mathbb{V}}_i\subseteq\mathbb{E}_i,\forall i\in\mathcal{M} \tag{20}$$

where $\tilde{\mathbb{V}}_i$ depend upon sets $\mathbb{E}_j$, $j\in\mathcal{N}_i$, see (18). In our approach, sets $\mathbb{E}_i$ are assigned *a priori* and we implicitly assume conditions (20) are verified. However, if subsystems are added sequentially to an existing plant and LSEs are designed with the PnP procedure described in Section 3.3, conditions (20) are automatically checked and, if violated, they prevent from plugging-in subsystem $\Sigma_{[i]}$. When sets $\mathbb{E}_i$ can be arbitrarily chosen, the problem of assigning them so as to fulfill (20) arises. In the presence of loop interconnections among subsystems, this issue is not trivial and requires centralized algorithms as those proposed in [7].

The procedure for solving problems $\mathcal{P}_i$, $i\in\mathcal{M}$ is summarized in Algorithm 1, which can be executed in parallel by each subsystem using local hardware. Its main steps are described next.

Algorithm 1. Design of the LSE $\tilde{\Sigma}_{[i]}$ for subsystem $\Sigma_{[i]}$.

Input: A_{ii}, $\mathbb{D}_i$, $\mathbb{O}_i$, $\mathbb{E}_i$, $\mathcal{N}_i$, $\{A_{ij}\}_{j\in\mathcal{N}_i}$, $\{\delta_{ij}^{L}\}_{j\in\mathcal{N}_i}$.

Output: set $\mathbb{S}_i$ and state estimator $\tilde{\Sigma}_{[i]}$.

(I) Receive from parent subsystems $j\in\mathcal{N}_i$ sets $\mathbb{E}_j$ and sets $\mathbb{O}_j$ if $\delta_{ij}^{L}=1$.

(II) if $\delta_{ij}^{L}=1$, compute the matrix L_{ij}, $\forall j\in\mathcal{N}_i$ solving

$$\min_{L_{ij}}||\mathcal{H}_i\bar{A}_{ij}\mathcal{H}_j^{\flat}||_p \tag{21}$$

where p is a generic norm.

(III) compute a matrix L_{ii} such that $\beta_i < 1$ and $\gamma_i < 1$. If it does not exist **stop** (the state estimator $\tilde{\Sigma}_{[i]}$ cannot be designed);

(IV) compute an RPI set $\mathbb{S}_i \subseteq \mathbb{E}_i$ for system (16).

In Step (II), if $\delta^L_{ij} = 1$, the computation of matrices L_{ij}, $j \in \mathcal{N}_i$ is required. Since the choice of L_{ij} affects the coupling term $\overline{A}_{ij} = A_{ij} + \delta^L_{ij} L_{ij} C_j$, and hence the possibility of verifying inequalities (19), we propose to reduce the magnitude of coupling by minimizing the magnitude of $\overline{A}_{ij}$ in (21), where $\mathcal{H}_i$ and $\mathcal{H}^\flat_j$ allow us to take into account the sizes of sets $\mathbb{E}_i$ and $\mathbb{E}_j$, respectively. More precisely, it can be shown that the greater $||\mathcal{H}_i \overline{A}_{ij} \mathcal{H}^\flat_j||_p$, the more difficult is to fulfill the constraint $e_{[i]} \in \mathbb{E}_i$ (see Appendix 8.7.3 in [23]). We highlight that the minimization of $\|\mathcal{H}_i \overline{A}_{ij} \mathcal{H}^\flat_j\|_1$ in (21) amounts to a Linear Programming (LP) problem and the minimization of $\|\mathcal{H}_i \overline{A}_{ij} \mathcal{H}^\flat_j\|_F$ can be recast into a QP problem. In Step (III), for the computation of matrix L_{ii} we solve the following nonlinear optimization problem:

$$\min_{L_{ii}} \mu_i \tag{22a}$$

$$\overline{\rho}(A_{ii} + L_{ii} C_i) < 1 \tag{22b}$$

$$\beta_i < 1 \tag{22c}$$

$$\gamma_i < 1 \tag{22d}$$

where $\mu_i = \max(\beta_i, \gamma_i, \overline{\rho}(A_{ii} + L_{ii} C_i))$, $\overline{\rho}(.)$ denotes the spectral radius and constraint (22d) is needed only if $\mathbb{D}_i \neq \{\mathbf{0}_{r_i}\}$ or $\mathbb{O}_i \neq \{\mathbf{0}_{p_i}\}$. Since (22) is a nonlinear optimization problem, a suitable initialization of L_{ii} is needed, e.g. start with L_{ii} satisfying at least (22b). Note that the series in (22c) and (22d) involve only positive terms and can be easily truncated if either (22c) or (22d) is violated or summands fall below the machine precision (see also [23,29] for similar optimization problem in the case of PnP control design). The feasibility of problem (22) guarantees that the estimator $\tilde{\Sigma}_{[i]}$ can be successfully designed. In Step (IV) of Algorithm 1 we need to compute a nonempty RPI set $\mathbb{S}_i \subseteq \mathbb{E}_i$ that, in view of Proposition 1, exists if the optimization problem (22) is feasible. To this purpose, several algorithms can be used. For instance, [27] discusses the computation of ϵ-outer approximation of the minimal RPI set. The maximal RPI set can be obtained using methods in [9]. More recently, efficient procedures have been also proposed for computing polytopic [28] or zonotopic [15] RPI sets.

3.3. PnP operations

Distributed and parallel design of LSEs described in the previous section imply the DSE can be automatically updated in case subsystems are added or removed, while preserving the fundamental properties (13) and (14). This can be done by updating a limited number of existing LSEs.

3.3.1. Plugging-in operation

We start considering the addition of subsystem $\Sigma_{[M+1]}$, characterized by parameters $A_{M+1,M+1}$, C_{M+1}, $\mathbb{E}_{M+1}$, $\mathbb{D}_{M+1}$, $\mathbb{O}_{M+1}$, $\mathcal{N}_{M+1}$ and coupling terms $\{A_{M+1,j}\}_{j \in \mathcal{N}_{M+1}}$, where $\mathcal{N}_{M+1}$ identifies parents of $\Sigma_{[M+1]}$. Subsystems that will be influenced by $\Sigma_{[M+1]}$ are given by $\mathcal{S}_{M+1}$ where

$$\mathcal{S}_i = \{j : i \in \mathcal{N}_j\}$$

is the set of children of subsystem $\Sigma_{[i]}$. For designing the LSE $\tilde{\Sigma}_{[M+1]}$ we execute Algorithm 1 that needs information only from subsystems $\Sigma_{[j]}$, $j \in \mathcal{N}_{M+1}$. If Algorithm 1 stops before the last step, we declare that $\Sigma_{[M+1]}$ cannot be plugged in. Since sets $\mathcal{N}_j$, $j \in \mathcal{S}_{M+1}$ have now one more element, previously obtained matrices L_{jj}, $j \in \mathcal{S}_{M+1}$ might give $\beta_j \geq 1$ or $\gamma_j \geq 1$. Indeed, from (19b) and (19c), scalars β_j and γ_j can only increase. Furthermore, the size of the set $\tilde{\mathbb{V}}_j$ increases and therefore an RPI set $\mathbb{S}_j \subseteq \mathbb{E}_j$ must be recomputed. This means that for each $j \in \mathcal{S}_{M+1}$ the LSE $\tilde{\Sigma}_{[j]}$ must be redesigned by running Algorithm 1. Again, if Algorithm 1 stops before completion for some $j \in \mathcal{S}_{M+1}$, we declare that $\Sigma_{[M+1]}$ cannot be plugged in.

Note that LSE redesign does not propagate further in the network. Indeed, as highlighted by the input arguments of Algorithm 1, the LSE design for a subsystem depends on parent subsystems only and not on their LSEs. The addition of $\Sigma_{[M+1]}$ does not change the parents of subsystems $\Sigma_{[i]}$, $i \notin \{M+1\} \bigcup \mathcal{S}_{M+1}$. Therefore, conditions (19) are still verified for $i \notin \{M+1\} \bigcup \mathcal{S}_{M+1}$ and, even without changing the corresponding LSEs, properties (13) and (14) are guaranteed for the new estimation scheme.

3.3.2. Unplugging operation

We consider removal of subsystem $\Sigma_{[k]}$, $k \in \mathcal{M}$. Since for each $i \in \mathcal{S}_k$ the set $\mathcal{N}_i$ contains one element less, one has that β_i in (19) and γ_i in (19c) cannot increase. Furthermore, the set $\mathbb{S}_i$, chosen before the removal of system $\Sigma_{[k]}$, still verifies $\mathbb{S}_i \supseteq \tilde{\mathbb{V}}_i$ and therefore previously obtained optimizers for (21) can still be used. This means that for each $i \in \mathcal{S}_k$ the LSE $\tilde{\Sigma}_{[i]}$ does not have to be redesigned. Moreover, since for each system $\Sigma_{[j]}$, $j \notin \{k\} \bigcup \mathcal{S}_k$, the

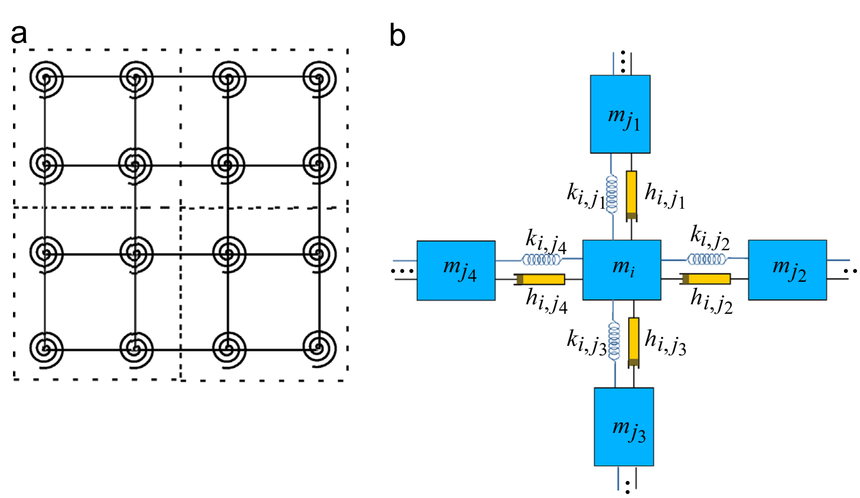

Fig. 1. Array of masses. (a) Position of the 16 masses on the plane. Dashed lines define subsystems $\Sigma_{[i]}$, $i \in \mathcal{M} = 1:4$. (b) Details of the interconnections.

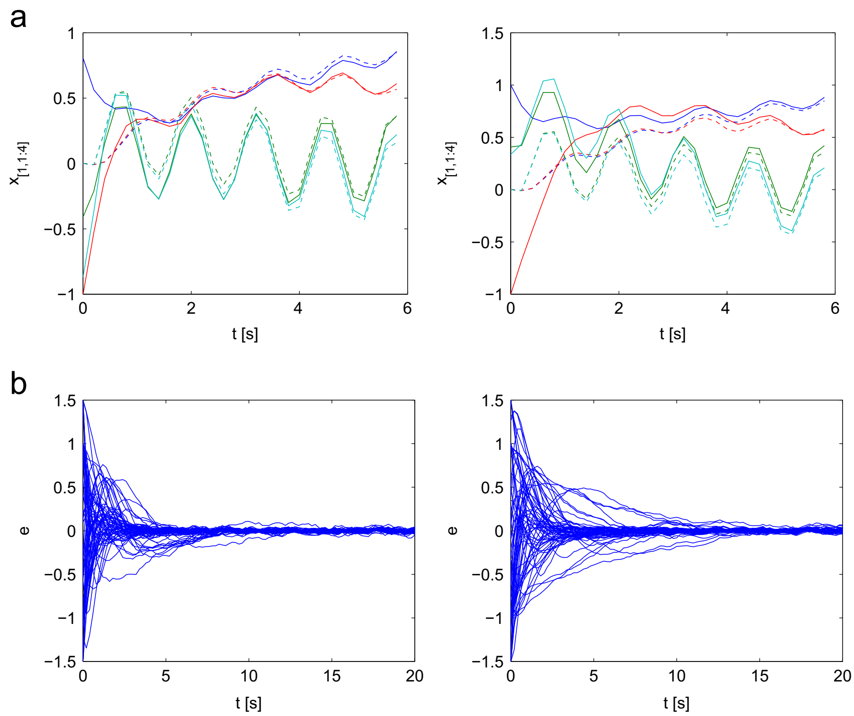

Fig. 2. State estimation results for LSE designed setting $\delta^L_{ij}=0, j\in\mathcal{N}_i$ (left panels) and $\delta^L_{ij}=1, j\in\mathcal{N}_i$ (right panels). (a) State (dashed lines) and state estimation (continuous line) of the upper left mass in Fig. 1a at time instants $t=0:6$ s. The same color has been used for a state and its estimate: cyan and green lines denote velocities while blue and red lines denote positions. (b) Estimation errors for all states at times $t=0:20$ s. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

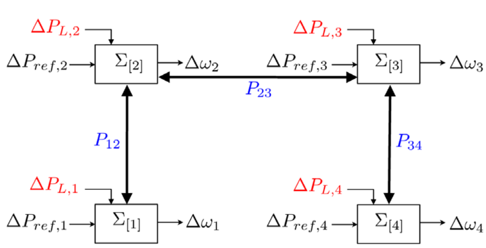

Fig. 3. PNS of Scenario 1 in [20]. Arrows represent tie-lines connecting generation areas.

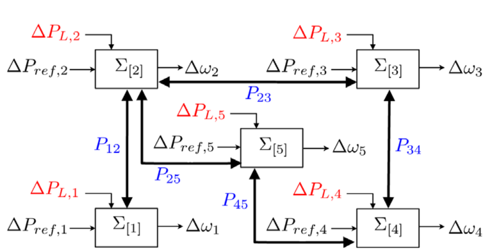

Fig. 4. PNS of Scenario 2 in [20]. The generation area $\Sigma_{[5]}$ has been added to the PNS in Fig. 3.

set $\mathcal{N}_j$ does not change, the redesign of the LSE $\tilde{\Sigma}_{[j]}$ is not required. In conclusion, the unplugging of a subsystem does not trigger the redesign of any LSE.

The previous discussion assumes that plugging-in/out affects only the coupling terms in (2). If however entries of matrices A_{ii}, B_i, C_i and D_i change when a child subsystem $\Sigma_{[k]}$ is removed, then the LSE $\tilde{\Sigma}_{[i]}$ must be obviously redesigned [22].

Remark 2. The plugging-in and unplugging operations described above guarantee the observer dynamics that is still asymptotically stable after the addition or removal of subsystems. This property is guaranteed even if the operation is performed online. However, if plugging-in/out of subsystems never stops, stability problems might arise. Indeed, as it is well known in the hybrid system literature [11], frequent and persistent switching between different modes of operation could compromise asymptotic stability of the whole plant. A remedy could be assuming a minimal dwell-time between consecutive plugging-in/out [11], which is reasonable if physical components are added or removed from the plant.

4. Output-feedback PnP DMPC

In this section we propose an output-feedback distributed control scheme by jointly using the DSE in Section 3 and the control scheme discussed in [30,22]. The main idea is to use the robust local regulators in [30,22] to control the state estimates $\tilde{x}_{[i]}$ rather than the states $x_{[i]}$. This method for generalizing state-feedback MPC to the output-feedback case has been firstly proposed in [14] for centralized control.

Each subsystem $\Sigma_{[i]}$ is equipped with the controller

$$\tilde{\mathcal{C}}_{[i]}:\quad u_{[i]}=v_{[i]}+\overline{\kappa}_i(\tilde{x}_{[i]}-\overline{x}_{[i]})+\sum_{j\in\mathcal{N}_i}\delta_{ij}K_{ij}\tilde{x}_{[j]} \tag{23}$$

where $\overline{\kappa}_i(\cdot):\mathbb{R}^{n_i}\to\mathbb{R}^{m_i}$ is a state-feedback control law, $K_{ij}\in\mathbb{R}^{m_i\times n_j}$ and $\delta_{ij}\in\{0,1\}$, $i,j\in\mathcal{M}$. Note that, if $\delta_{ij}=0$, $\forall i\in\mathcal{M}$, $\forall j\in\mathcal{N}_i$, the control scheme is completely decentralized, since each input $u_{[i]}$ depends upon state estimates of subsystem $\Sigma_{[i]}$ only. Furthermore, following [16], in (23) we set

$$v_{[i]}(t)=v_{[i]}(0|t),\qquad \overline{x}_{[i]}(t)=\hat{x}_{[i]}(0|t) \tag{24}$$

where $v_{[i]}(0|t)$ and $\hat{x}_{[i]}(0|t)$ are optimal values of the variables $v_{[i]}(0)$ and $\hat{x}_{[i]}(0)$, respectively, appearing in the MPC-i problem

$$\mathbb{P}^N_i(\tilde{x}_{[i]}(t)):\ \min_{\substack{\hat{x}_{[i]}(0)\\ v_{[i]}(0:N_i-1)}}\sum_{k=0}^{N_i-1}\ell_i(\hat{x}_{[i]}(k),v_{[i]}(k))+V_{f_i}(\hat{x}_{[i]}(N_i)) \tag{25a}$$

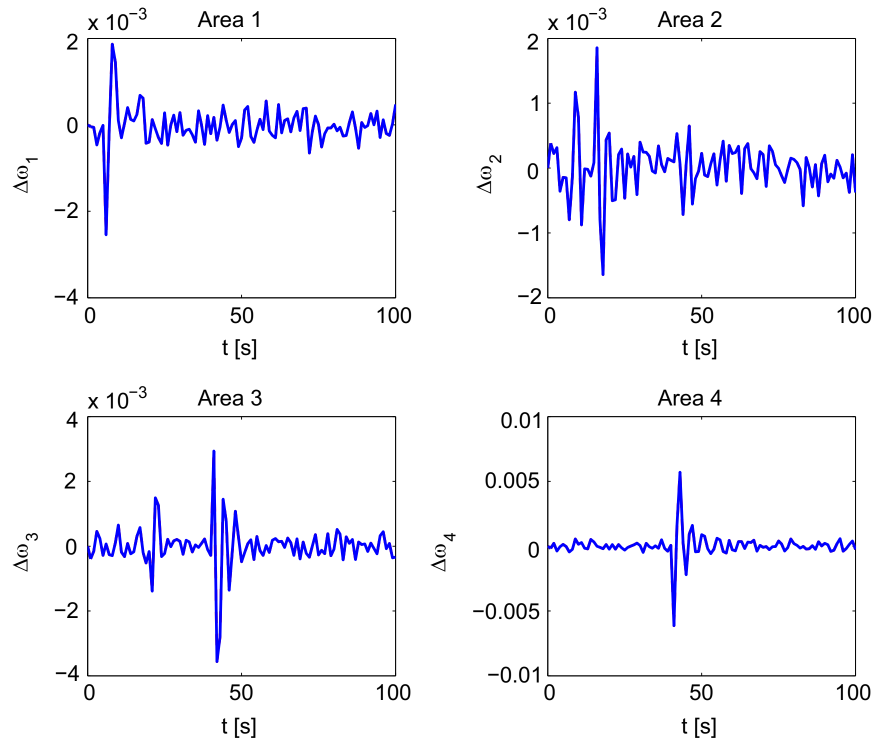

Fig. 5. Simulation Scenario 1: frequency deviation in each area controlled by the proposed output-feedback PnP DMPC.

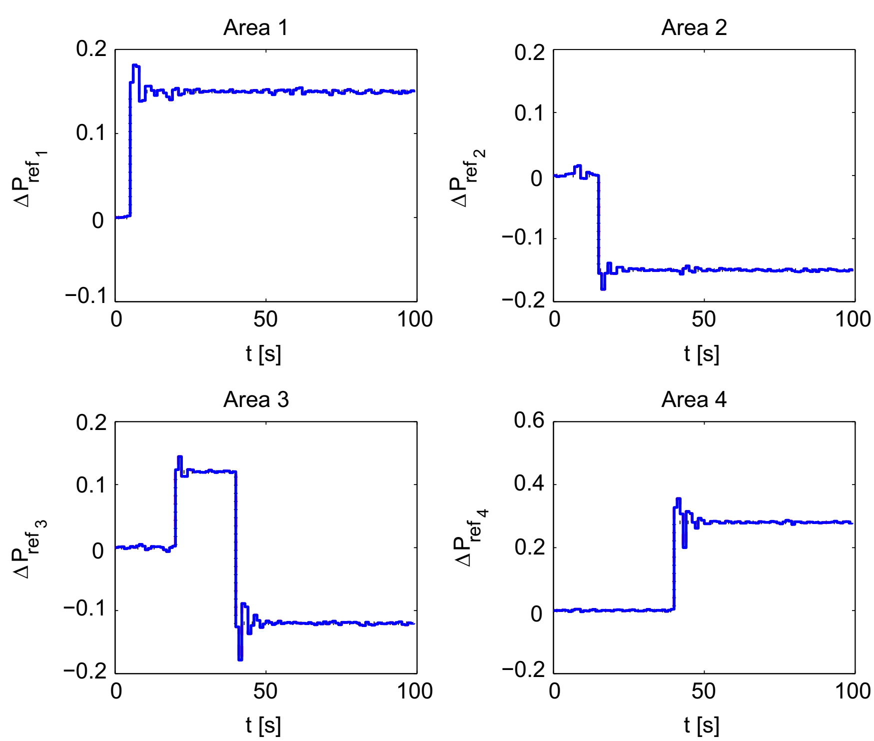

Fig. 6. Simulation Scenario 1: load reference set-point in each area controlled by the proposed output-feedback PnP DMPC.

$$\tilde{x}_{[i]}(t)-\hat{x}_{[i]}(0)\in\mathbb{Z}_i \tag{25b}$$

$$\hat{x}_{[i]}(k+1)=A_{ii}\hat{x}_{[i]}(k)+B_i v_{[i]}(k),\;\; k\in 0:N_i-1 \tag{25c}$$

$$\hat{x}_{[i]}(k)\in\hat{\mathbb{X}}_i,\quad v_{[i]}(k)\in\mathbb{V}_i,\;\; k\in 0:N_i-1 \tag{25d}$$

$$\hat{x}_{[i]}(N_i)\in\hat{\mathbb{X}}_{f_i}. \tag{25e}$$

In (25), $N_i>0$ is the control horizon and $\ell_i:\mathbb{R}^{n_i\times m_i}\to\mathbb{R}_{0+}$ is the stage cost, $V_{f_i}:\mathbb{R}^{n_i}\to\mathbb{R}_{0+}$ is the final cost and $\hat{\mathbb{X}}_{f_i}$ is the terminal set. Matrices K_{ij}, function $\overline{\kappa}_i(\cdot)$ in (23), sets $\mathbb{Z}_i$, $\hat{\mathbb{X}}_i$, $\mathbb{V}_i$, $\hat{\mathbb{X}}_{f_i}$ and functions $\ell_i(\cdot,\cdot)$ and $V_{f_i}(\cdot)$ are obtained through Algorithm 2 which is discussed next.

As for the communication requirements, from Steps (I) and (IV) of Algorithm 2 pieces of information must be transmitted to

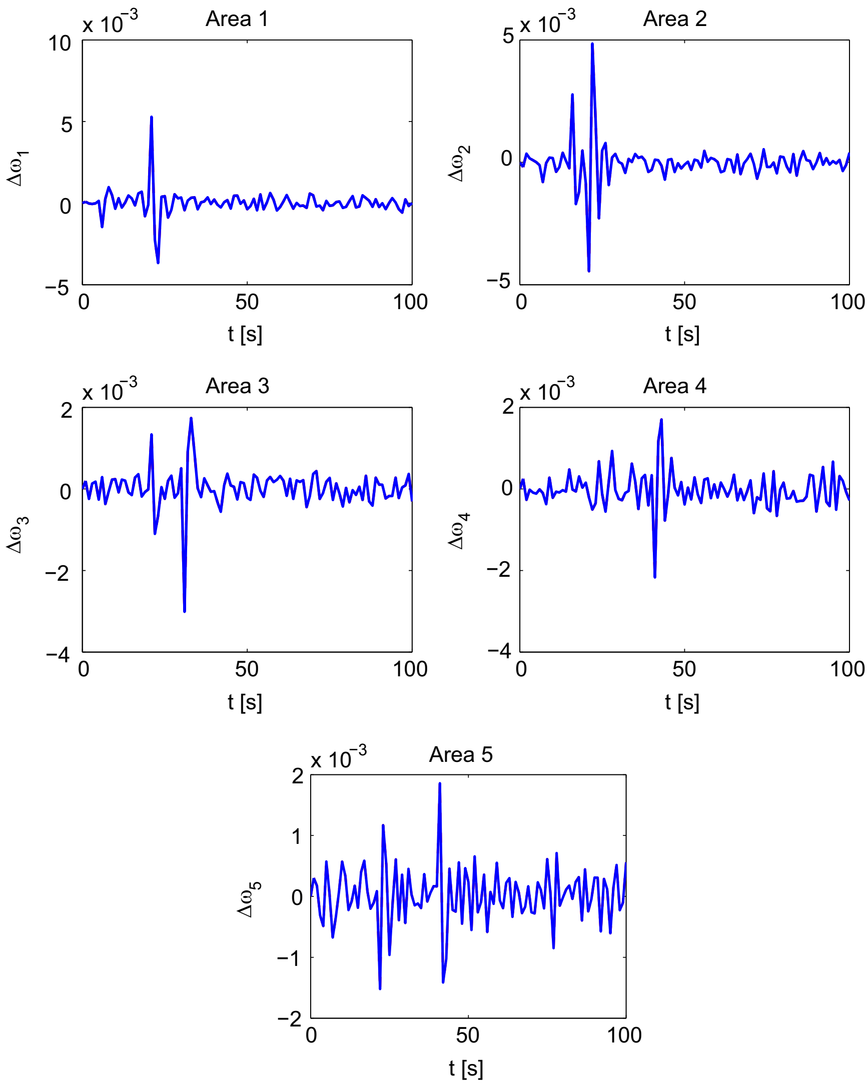

Fig. 7. Simulation Scenario 2: frequency deviation in each area controlled by the proposed output-feedback PnP DMPC.

subsystem $\Sigma_{[i]}$ only from its parents. Hence the design procedure is distributed and, similarly to the operations described in Section 3.3, plugging-in and unplugging of subsystems involve only the update of a limited number of controllers (details are given in Section IV of [29] and in Section 6.4 of [23]).

Algorithm 2. Design of an output-feedback controller for subsystem $\Sigma_{[i]}$.

Input: A_{ii}, B_i, $\mathbb{X}_i$, $\mathbb{U}_i$, $\mathbb{D}_i$, $\mathbb{O}_i$, $\mathbb{E}_i$, $\mathcal{N}_i$, $\{A_{ij}\}_{j\in\mathcal{N}_i}$, $\{\eta_{ij}\}_{j\in\mathcal{N}_i}$, $\{\delta_{ij}\}j\in\mathcal{N}_i$, $\{\delta^L_{ij}\}_{j\in\mathcal{N}_i}$

Output: output-feedback controller composed by state estimator $\tilde{\Sigma}_{[i]}$ and state-feedback controller $\tilde{\mathcal{C}}_{[i]}$

(I) Receive from parent subsystems $j\in\mathcal{N}_i$ sets $\mathbb{E}_j$ and sets $\mathbb{O}_j$ if $\delta^L_{ij}=1$.

(II) Execute Steps (II)–(IV) of Algorithm 1.

(III) Compute set $\tilde{\mathbb{X}}_i=\mathbb{X}_i\ominus\mathbb{S}_i=\{\tilde{x}_{[i]}\in\mathbb{R}^{n_i}:\tilde{\mathcal{L}}_{x_i}\tilde{x}_{[i]}\le\mathbf{1}_{\tilde{r}_i^x}\}$.

(IV) Receive from parent subsystems $j\in\mathcal{N}_i$ sets $\tilde{\mathbb{X}}_j$ and sets $\mathbb{S}_j$ if $\delta^L_{ij}=1$.

(V) For all $j\in\mathcal{N}_i$, if $\delta_{ij}=1$, compute the matrix K_{ij} solving

$$\min_{K_{ij}}||\tilde{\mathcal{L}}_{x_i}(A_{ij}+\delta_{ij}B_iK_{ij})\tilde{\mathcal{L}}_{x_j}^{\flat}||_p \tag{26a}$$

$$||K_{ij}\tilde{\mathcal{L}}_{x_j}^{\flat}||_p\le\eta_{ij} \tag{26b}$$

where p is a generic norm and scalars $\eta_{ij}>0$ are given.

(VI) Compute the set $\overline{\mathbb{U}}_i=\mathbb{U}_i\ominus\bigoplus_{j\in\mathcal{N}_i}\delta_{ij}K_{ij}\tilde{\mathbb{X}}_j$. If $\overline{\mathbb{U}}_i$ is empty, **stop** (the controller $\tilde{\mathcal{C}}_{[i]}$ cannot be designed).

(VII) Compute the set

$$\tilde{\mathbb{W}}_i=\left(\bigoplus_{j\in\mathcal{N}_i}(A_{ij}+\delta_{ij}B_iK_{ij})\tilde{\mathbb{X}}_j\right)\oplus(-L_{ii}C_i\mathbb{S}_i)$$
$$\oplus\left(\bigoplus_{j\in\mathcal{N}_i}\delta^L_{ij}(-L_{ij}C_j\mathbb{S}_j)\right)\oplus(-L_{ii}\mathbb{O}_i)\oplus\left(\bigoplus_{j\in\mathcal{N}_i}\delta^L_{ij}(-L_{ij}\mathbb{O}_j)\right)$$

and choose the set $\overline{\mathbb{Z}}_i^0$ such that $\tilde{\mathbb{X}}_i\supseteq\overline{\mathbb{Z}}_i^0\supseteq\tilde{\mathbb{W}}_i\oplus\mathcal{B}_{\omega_i}(0)$ for a sufficiently small $\omega_i>0$. If $\overline{\mathbb{Z}}_i^0$ does not exist, **stop** (the controller $\tilde{\mathcal{C}}_{[i]}$ cannot be designed).

(VIII) Check the LP feasibility condition in Step (II) of Algorithm 6.1 in [23]. If it is not verified, **stop** (the controller $\tilde{\mathcal{C}}_{[i]}$ cannot be designed).

(IX) Execute Steps (III) and (IV) of Algorithm 6.1 in [23]. They provide all quantities defining the MPC-i problem (25) and the function $\overline{\kappa}_i(\cdot)$ defined through the LP problem (6.15) in [23].

In Step (V) of Algorithm 2, the choice of terms K_{ij} results from a trade-off: on one hand, the magnitude of the coupling terms

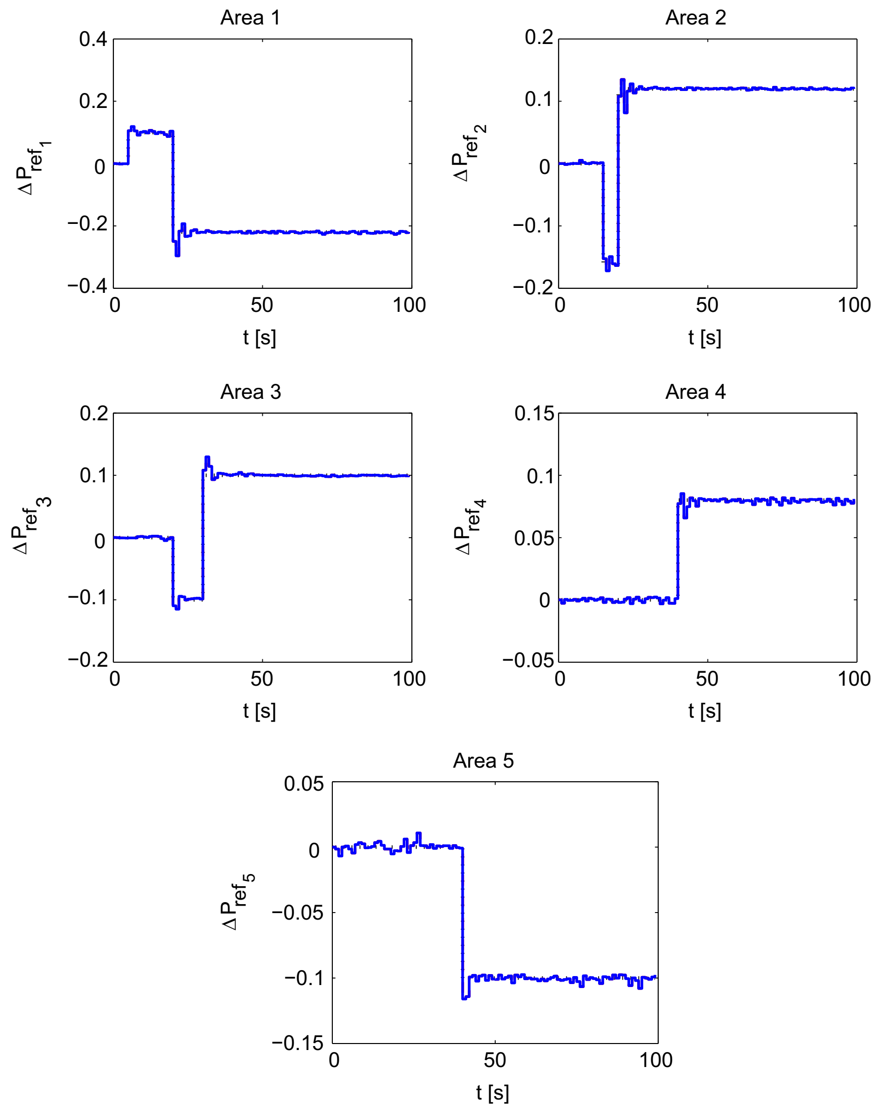

Fig. 8. Simulation Scenario 2: load reference set-point each area controlled by the proposed output-feedback PnP DMPC.

$A_{ij}+B_iK_{ij}$ must be reduced as much as possible, so as to reduce the size of set $\tilde{\mathbb{W}}_i$; on the other hand, the existence of sets $\overline{\mathbb{U}}_i$ in Step (VI) must be guaranteed. For this reason we minimize (26) but, at the same time, we bound the size of the set $K_{ij}\tilde{\mathbb{X}}_j$ with the scalar η_{ij} in (26b). We highlight that the minimization of $||\tilde{\mathcal{L}}_{x_i}(A_{ij}+B_iK_{ij})\tilde{\mathcal{L}}^{\flat}_{x_j}||_1$ in (26) is an LP problem and the minimization of $||\tilde{\mathcal{L}}_{x_i}(A_{ij}+B_iK_{ij})\tilde{\mathcal{L}}^{\flat}_{x_j}||_F$ can be recast into a QP problem.

The second task accomplished by Algorithm 2 through Steps (VII)–(IX) is the design of the MPC-i controller (25). The procedure is similar to the synthesis of local MPC controllers in [30,22] and therefore details are omitted (they can be found in [23]). Here, we just highlight that

- Steps (VIII) and (IX), which provide constraints in (25), are the most computationally expensive ones because they involve Minkowski sums and differences of polytopic sets. The interested reader is referred to Sections 6.3.1–6.3.3 in [23], where we show how to avoid burdensome computations exploiting results from [28] and how to compute a suitable function $\overline{\kappa}_i$ through LP.
- the execution of Step (IV) requires sets computed by parent subsystems during the synthesis of LSEs. Therefore, we can design local controllers $\tilde{\mathcal{C}}_{[i]}$ only after all parents of subsystem $\Sigma_{[i]}$ have terminated the execution of Algorithm 1.

The main properties of the overall closed-loop system are described in the next theorem.

Definition 1. The feasibility region for the MPC-i problem is

$$\tilde{\mathbb{X}}_i^N=\{s_{[i]}\in\tilde{\mathbb{X}}_i:(25)\text{ is feasible for }\tilde{x}_{[i]}(t)=s_{[i]}\}.$$

Theorem 1. *Let Assumptions 1 and 2 hold. Assume that the output-feedback controllers are computed using Algorithm 2 and define* $\Xi_i=(\mathbb{Z}_i\oplus\mathbb{S}_i)\times\mathbb{Z}_i$, $\Xi=\prod_{i\in\mathcal{M}}\Xi_i$. *Then, the set* Ξ *is robustly attractive for the closed-loop system with state* $\boldsymbol{\xi}=(\xi_{[1]},\ldots,\xi_{[M]})$, $\xi_{[i]}=(x_{[i]},\tilde{x}_{[i]})$, $i\in\mathcal{M}$. *Furthermore, a region of attraction for* Ξ *is* $\prod_{i\in\mathcal{M}}(\tilde{\mathbb{X}}_i^N\oplus\mathbb{S}_i)\times(\tilde{\mathbb{X}}_i^N)$. *Finally, if* $d_{[i]}=\mathbf{0}_{r_i}$ *and* $\varrho_{[i]}=\mathbf{0}_{p_i}$, *then* $\tilde{x}_{[i]}(0)\in\tilde{\mathbb{X}}_i^N$ *and* $x_{[i]}(0)-\tilde{x}_{[i]}(0)\in\mathbb{S}_i$ *imply that* $x_{[i]}(t)\to\mathbf{0}_{n_i}$ *as* $t\to\infty$.

Proof. Consider the nominal case, i.e. $d_{[i]}=\mathbf{0}_{r_i}$ and $\varrho_{[i]}=\mathbf{0}_{p_i}$. Using the results in Proposition 1, the LSEs are asymptotically stable, hence $e_{[i]}(t)\to\mathbf{0}_{n_i}$. Similarly to the proof of Theorem 6.1 in [23], one can show that $\tilde{x}_{[i]}(t)\to\mathbf{0}_{n_i}$. Since $x_{[i]}=\tilde{x}_{[i]}+e_{[i]}$, we can conclude that $x_{[i]}(t)\to\mathbf{0}_{n_i}$. If $d_{[i]}\neq\mathbf{0}_{r_i}$ and $\varrho_{[i]}\neq\mathbf{0}_{p_i}$ we use the results in Proposition 1 and in Step 1 of the proof of Theorem 6.1 in [23]. Since $e_{[i]}(t)\in\mathbb{S}_i$, $t\geq 0$ and the distance between $\tilde{x}_{[i]}(t)$ and the set $\mathbb{Z}_i$ goes to zero as $t\to+\infty$, they allow us to conclude that the set $\mathbb{Z}_i\oplus\mathbb{S}_i$ is robustly attractive, i.e. the distance between $x_{[i]}(t)$ and $\mathbb{Z}_i\oplus\mathbb{S}_i$ vanishes.□

Remark 3. For the sake of simplicity, in (2) we did not consider subsystems coupled through inputs and disturbances. Here we briefly describe a generalization to this case, where (2) is replaced by

$$\mathbf{x}_{[i]}^{+} = A_{ii}x_{[i]} + B_i u_{[i]} + \sum_{j \in \mathcal{N}_i} (A_{ij}x_{[j]} + B_{ij}u_{[j]} + D_{ij}d_{[j]}) + D_i d_{[i]}$$
$$y_{[i]} = C_i x_{[i]} + \varrho_{[i]}.$$

To account for input coupled subsystems is sufficient that each LSE $\tilde{\Sigma}_{[i]}$ receives the inputs from parent subsystems $\Sigma_{[j]}$, $j \in \mathcal{N}_i$ and therefore (9) can be rewritten as

$$\tilde{x}_{[i]}^{+} = A_{ii}\tilde{x}_{[i]} + B_i u_{[i]} - L_{ii}(y_{[i]} - C_i\tilde{x}_{[i]}) + \sum_{j \in \mathcal{N}_i} (A_{ij}\tilde{x}_{[j]} + B_{ij}u_{[j]}) - \sum_{j \in \mathcal{N}_i} \delta_{ij}^{L} L_{ij}(y_{[j]} - C_j\tilde{x}_{[j]}).$$

In order to design a DSE for an overall system where matrix $\mathbf{D}$ is not block-diagonal, we have to enlarge the set $\tilde{\mathbb{V}}_i$ in (18). In particular one has to define

$$\tilde{v}_{[i]} \in \tilde{\mathbb{V}}_i = \left(\bigoplus_{j \in \mathcal{N}_i} (\overline{A}_{ij}\mathbb{E}_j \oplus D_{ij}\mathbb{D}_j) \right) \oplus D_i\mathbb{D}_i \oplus L_{ii}\mathbb{O}_i \oplus \left(\bigoplus_{j \in \mathcal{N}_i} \delta_{ij}^{L} L_{ij}\mathbb{O}_j \right).$$

As regards the output-feedback control architecture, since the subsystems are input coupled, we can treat terms $B_{ij}\mathbb{U}_j$ as disturbances and enlarge set $\tilde{\mathbb{W}}_i$ in Step (VII) of Algorithm 2 by adding $\oplus_{j \in \mathcal{N}_i}(B_{ij}\mathbb{U}_j)$. This is idea has been proposed also in [6,18].

5. Examples

In this section we present two numerical examples. In Section 5.1 we consider an array of 16 masses and discuss the performance of the DSE only. In Section 5.2, we use a Power Network System (PNS). Modeling, design of LSEs and PnP DMPC regulators as well as simulations have been performed using the *PnPMPC-toolbox* for MatLab [19].

5.1. State estimation for an array of masses

We consider a system composed by 16 masses coupled as in Fig. 1a where the four edges connected to a point correspond to springs and dampers arranged as in Fig. 1b. Each mass $f \in 1:16$ is an LTI system with state variables $x_{[f]} = (x_{[f,1]}, x_{[f,2]}, x_{[f,3]}, x_{[f,4]})$ and input $u_{[f]} = (u_{[f,1]}, u_{[f,2]})$, where $x_{[f,1]}$ and $x_{[f,3]}$ are the displacements of mass f with respect to a given equilibrium position in the plane (equilibria lie on a regular grid), $x_{[f,2]}$ and $x_{[f,4]}$ are the horizontal and the vertical velocity of the mass f, respectively, and $100u_{[f,1]}$ (respectively $100u_{[f,2]}$) is the force applied to mass f in the horizontal (respectively, vertical) direction. The values of m_f have been extracted randomly in the interval $[5, 10]$ while spring constants and damping coefficients are identical and equal to 0.5. Each mass is equipped with local state estimation error constraints $||e_{[f,j]}||_\infty \le 1$, $j \in \{1, 3\}$ and $||e_{[f,l]}||_\infty \le 1.5$, $l \in \{2, 4\}$.

A subsystem $\Sigma_{[i]}$, $i \in \mathcal{M} = 1:4$ is a group of four masses as in Fig. 1a. Therefore each subsystem has order 16 and two parents. For each subsystem $\Sigma_{[i]}$ we have 8 outputs that are the displacements of two masses and the velocities of the other two masses.

We obtain models $\Sigma_{[i]}$ by discretizing the continuous-time models with 0.2 s sampling time, using the mE-ZOH method proposed in [5], i.e., using zero-order hold discretization for the local dynamics and treating $x_{[j]}, j \in \mathcal{N}_i$ as exogenous signals. We design an LSE $\tilde{\Sigma}_{[i]}$, $i \in \mathcal{M}$ using Algorithm 1 and initializing L_{ii} in the nonlinear optimization problem (22) as the dual LQR gain associated to matrices $\tilde{Q}_i = 0.01\mathbb{I}_{16}$ and $R_i = 100\mathbb{I}_8$. In Fig. 2 we show a simulation where each state of subsystem $\Sigma_{[i]}$, $i \in 1:4$ is affected by a disturbance $d_{[i]}$ sampled from the uniform distribution in the set $\mathbb{D}_i = \{d_{[i]} \in \mathbb{R} : |d_{[i]}| \le 0.015\}$. This has been obtained setting $D_i = \mathbf{1}_{16}$. Outputs of each subsystem are also corrupted by a disturbance $\varrho_{[i]}$ drawn uniformly in the set $\mathbb{O}_i = \{\varrho_{[i]} \in \mathbb{R}^{p_i} : |\varrho_{[i]}| \le 0.02\}$.

The left panels of Figs. 2a and b show results produced by LSE designed with $\delta_{ij}^{L} = 0, j \in \mathcal{N}_i$ while the right panels of Figs. 2a and b show the results obtained for $\delta_{ij}^{L} = 1$, $j \in \mathcal{N}_i$ and choosing norm $p = F$. Moreover, when using coupling attenuation terms L_{ij}, we want to show that some LSEs can be initialized so as to have larger initial error $e_{[i]}(0)$. This can be noticed by comparing estimation errors at time $t = 0$ in the left and right panels of Fig. 2b. In both cases, although using different initial states $\tilde{x}_{[i]}(0)$, the errors fulfill the prescribed bounds but do not converge to zero because of the persistent disturbances $d_{[i]}$ and $\varrho_{[i]}$, $i \in 1:4$.

5.2. Output-feedback control of a PNS

In this section, we apply the proposed output-feedback PnP DMPC scheme to the design of the AGC control layer for a PNS. As described in [20], the goal is to control each generation area composing the PNS so as to steer to zero the deviations from the nominal network frequency. In the following, we first design the AGC layer for the PNS of Scenario 1 in [20], composed of 4 generation areas connected as in Fig. 3. Then we show how to perform the plugging-in of an area (Scenario 2 in [20] corresponding to Fig. 4).

For a comparison with different decentralized, distributed and centralized control architectures based on state-feedback and on $\mathbf{d} = \mathbf{0}_r$, we defer the reader to [22,20].

Each generation area is a fourth-order system. We assume to measure only two states: the angular deviation $\Delta\theta_{[i]}$ and the speed deviation $\Delta\omega_{[i]}$ of each area. Moreover we add bounded disturbances on all states setting $D_i = \mathbb{I}_4$ and

$$\mathbb{D}_i = \{d_{[i]} \in \mathbb{R}^4 : ||d_{[i]}||_\infty \le 5 \cdot 10^{-4}\}$$

We require to keep the state estimation error of each area in the following set:

$$\mathbb{E}_i = \{e_{[i]} \in \mathbb{R}^4 : ||e_{[i]}||_\infty \le 0.01\}.$$

5.2.1. Scenario 1

For each system $\Sigma_{[i]}$ we synthesize the controller $\tilde{\mathcal{C}}_{[i]}$, $i \in \mathcal{M} = 1:4$ using Algorithm 2. For the design of local estimators, we set $\delta_{ij}^{L} = 1$, $i \in \mathcal{M}$, $j \in \mathcal{N}_i$. This allows us to compute matrices L_{ij} such that $\overline{A}_{ij} = \mathbf{0}_{n_i \times n_j}$. For the design of local controllers we set $\delta_{ij} = 0$, therefore we do not use the state of parent subsystems in order to reduce the coupling terms.

In Figs. 5 and 6 we show performance of output-feedback PnP DMPC. Step-like control signals in Fig. 6 are caused by load steps specified in Table 3 in [20]. In spite of these changes, deviations from the network frequency are promptly driven close to zero, as shown in Fig. 5. However, they cannot be completely rejected because, differently from the simulations conducted in [29], each area is affected by persistent disturbances.

5.2.2. Scenario 2

We consider the PNS proposed in Scenario 1 and add a fifth area connected as in Fig. 4. Therefore, the set of children of $\Sigma_{[5]}$ is $\tilde{\mathcal{S}}_5 = \{2, 4\}$. For the design of the LSE $\hat{\Sigma}_{[5]}$, and the update of $\hat{\Sigma}_{[2]}$ and $\hat{\Sigma}_{[4]}$ we set $\delta_{ij}^{L} = 1$, $i \in \{2, 4, 5\}$, $j \in \mathcal{N}_i$. This allows us to compute matrices L_{ij} such that $\overline{A}_{ij} = \mathbf{0}_{n_i \times n_j}$. For the design of local controllers we set $\delta_{ij} = 0$.

In Figs. 7 and 8 we show closed-loop simulations. As for Scenario 1, the overall performance is satisfactory as the frequency deviations are confined in a small interval around zero.

6. Conclusions

In this paper we have proposed a DSE for linear discrete-time systems for which offline design and online implementation are distributed and scalable. In particular, LSEs for subsystems that get added or removed can be designed in a PnP fashion. The DSE has been used in combination with a PnP DMPC scheme to provide a novel output-feedback PnP controller. Future work includes the extension of the proposed control scheme for tracking output reference signals and the embedding of target optimization [8] in distributed PnP controllers. Generalizations of the cooperative PnP scheme in [39] to the output-feedback case and extensions of PnP state estimation to account for noise samples with unbounded support (e.g. drawn form a Gaussian distribution) will be also considered.

References

[1] P.J. Antsaklis, B. Goodwine, V. Gupta, M.J. McCourt, Y. Wang, P. Wu, M. Xia, H. Yu, F. Zhu, Control of cyberphysical systems using passivity and dissipativity based methods, Eur. J. Control 19 (5) (2013) 379–388.
[2] R. Carli, A. Chiuso, L. Schenato, S. Zampieri, Distributed Kalman filtering based on consensus strategies, IEEE J. Select. Areas Commun. 26 (4) (2008) 622–633.
[3] P.D. Christofides, R. Scattolini, D. Muñoz de la Peña, J. Liu, Distributed model predictive control: a tutorial review and future research directions, Comput. Chem. Eng. 51 (2013) 21–41.
[4] M. Farina, G. Ferrari-Trecate, R. Scattolini, Moving-horizon partition-based state estimation of large-scale systems, Automatica 46 (5) (2010) 910–918.
[5] M. Farina, P. Colaneri, R. Scattolini, Block-wise discretization accounting for structural constraints, Automatica 49 (11) (2013) 3411–3417.
[6] M. Farina, R. Scattolini, Distributed predictive control: a non-cooperative algorithm with neighbor-to-neighbor communication for linear systems, Automatica 48 (6) (2012) 1088–1096.
[7] M. Farina, R. Scattolini, An output feedback distributed predictive control algorithm, in: Proceedings of the 50th IEEE Conference on Decision and Control, and the European Control Conference, Orlando, FL, USA, December 12–15, 2011, pp. 8139–8144.
[8] A. Ferramosca, D. Limon, A. González, Cooperative distributed MPC integrating a steady state target optimizer, in: Distributed Model Predictive Control Made Easy, Springer Netherlands, 2014, pp. 569–584.
[9] E.G. Gilbert, K.T. Tan, Linear systems with state and control constraints: the theory and application of maximal output admissible sets, IEEE Trans. Autom. Control 36 (9) (1991) 1008–1020.
[10] P. Giselsson, Output feedback distributed model predictive control with inherent robustness properties, in: Proceedings of American Control Conference 2013, Washington, DC, USA, June 17–19, 2013, pp. 1691–1696.
[11] J.P. Hespanha, A.S. Morse, Stability of Switched Systems with Average Dwell-Time, in: Proceedings of the 38th IEEE Conference on Decision and Control, Phoenix, AZ, USA, December 7–10, 1999, pp. 2655–2660.
[12] U.A. Khan, J.M.F. Moura, Distributing the Kalman filter for large-scale systems, IEEE Trans. Signal Process. 56 (10) (2008) 4919–4935.
[13] I. Kolmanovsky, E.G. Gilbert, Theory and computation of disturbance invariant sets for discrete-time linear systems, Math. Prob. Eng. 4 (4) (1998) 317–363.
[14] D.Q. Mayne, S.V. Raković, R. Findeisen, F. Allgöwer, Robust output feedback model predictive control of constrained linear systems, Automatica 42 (7) (2006) 1217–1222.
[15] S.V. Raković, Robust control of constrained discrete time systems: characterization and implementation (Ph.D. thesis), Imperial College London, University of London, 2005.
[16] S.V. Raković, D.Q. Mayne, A simple tube controller for efficient robust model predictive control of constrained linear discrete time systems subject to bounded disturbances, in: Proceedings of the 16th IFAC World Congress, Prague, Czech Republic, July 4–8, 2005, pp. 241–246.
[17] S. Riverso, D. Rubini, G. Ferrari-Trecate, Distributed bounded-error state estimation for partitioned systems based on practical robust positive invariance, in: Proceedings of the 12th European Control Conference, Zurich, Switzerland, July 17–19, 2013, pp. 2633–2638.
[18] S. Riverso, F. Sarzo, G. Ferrari-Trecate, Plug-and-play decentralized frequency regulation for power networks with FACTS devices, in: Proceedings of the IEEE SmartGridComm, Venice, Italy, November 3–6, 2014, pp. 79–84.
[19] S. Riverso, A. Battocchio, G. Ferrari-Trecate, PnPMPC: A Toolbox For MatLab, 2012, URL ⟨http://sisdin.unipv.it/pnpmpc/pnpmpc.php⟩.
[20] S. Riverso, G. Ferrari-Trecate, Hycon2 Benchmark: Power Network System, Technical Report, Dipartimento di Informatica e Sistemistica, Università degli Studi di Pavia, Pavia, Italy, 2012, arxiv:1207.2000.
[21] S. Riverso, M. Farina, R. Scattolini, G. Ferrari-Trecate, Plug-and-play distributed state estimation for linear systems, in: Proceedings of the 52nd IEEE Conference on Decision and Control, Florence, Italy, December 10–13, 2013, pp. 4889–4894.
[22] S. Riverso, G. Ferrari-Trecate, Plug-and-play distributed model predictive control with coupling attenuation, Optim. Control Appl. Meth., 36 (2015) 292–305.
[23] S. Riverso, Distributed and plug-and-play control for constrained systems (Ph. D. thesis), Università degli studi di Pavia, 2014, URL ⟨http://sisdin.unipv.it/pnpmpc/phpinclude/papers/phd_thesis_riverso.pdf⟩.
[24] A.G.O. Mutambara, Decentralized Estimation and Control for Multisensor Systems, CRC Press, Boca Raton, FL, 1998.
[25] R. Negenborn, J. Maestre (Eds.), Distributed MPC made easy, Intelligent Systems, Control and Automation: Science and Engineering, vol. 69, Springer, Dordrecht, 2014.
[26] F. Pasqualetti, R. Carli, F. Bullo, Distributed estimation via iterative projections with application to power network monitoring, Automatica 48 (5) (2012) 747–758.
[27] S.V. Raković, E.C. Kerrigan, K.I. Kouramas, D.Q. Mayne, Invariant approximations of the minimal robust positively invariant set, IEEE Trans. Autom. Control 50 (3) (2005) 406–410.
[28] S.V. Raković, M. Baric, Parameterized robust control invariant sets for linear systems: theoretical advances and computational remarks, IEEE Trans. Autom. Control 55 (7) (2010) 1599–1614.
[29] S. Riverso, M. Farina, G. Ferrari-Trecate, Plug-and-play decentralized model predictive control for linear systems, IEEE Trans. Autom. Control 58 (10) (2013) 2608–2614.
[30] S. Riverso, M. Farina, G. Ferrari-Trecate, Plug-and-Play Model Predictive Control based on robust control invariant sets, Automatica 50 (8) (2014) 2179–2186.
[31] S. Roshany-Yamchi, M. Cychowski, R.R. Negenborn, B. De Schutter, K. Delaney, J. Connell, Kalman filter-based distributed predictive control of large-scale multi-rate systems: application to power networks, IEEE Trans. Control Syst. Technol. 21 (1) (2013) 27–39.
[32] T. Samad, T. Parisini, Systems of Systems, in: T. Samad, A.M. Annaswamy (Eds.), The Impact of Control Technology, IEEE Control Systems Society, Piscataway, NJ, USA, 2011, pp. 175–183, URL⟨ieeecss.org/general/impact-control-technology⟩.
[33] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Cambridge University Press, Cambridge, UK, 1993.
[34] S.S. Stankovic, M.S. Stankovic, D.M. Stipanovic, Consensus based overlapping decentralized estimator, IEEE Trans. Autom. Control 54 (2) (2009) 410–415.
[35] S.S. Stankovic, M.S. Stankovic, D.M. Stipanovic, Consensus based overlapping decentralized estimation with missing observations and communication faults, Automatica 45 (2009) 1397–1406.
[36] B.T. Stewart, A.N. Venkat, J.B. Rawlings, S.J. Wright, G. Pannocchia, Cooperative distributed model predictive control, Syst. Control Lett. 59 (8) (2010) 460–469.
[37] J. Stoustrup, Plug & play control: control technology towards new challenges, Eur. J. Control 3–4 (15) (2009) 311–330.
[38] R. Vadigepalli, F.J. Doyle III, A distributed state estimation and control algorithm for plantwide processes, IEEE Trans. Control Syst. Technol. 11 (1) (2003) 119–127.
[39] M. Zeilinger, Y. Pu, S. Riverso, G. Ferrari-Trecate, C. Jones, Plug and play distributed model predictive control based on distributed invariance and optimization, in: Proceedings of the 52nd IEEE Conference on Decision and Control, 2013, pp. 5770–5776.