



Distributed randomized algorithms for opinion formation, centrality computation and power systems estimation: A tutorial overview[☆]



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ABSTRACT

In this tutorial paper, we study three specific applications: opinion formation in social networks, centrality measures in complex networks and estimation problems in large-scale power systems. These applications fall under a general framework which aims at the construction of algorithms for distributed computation over a network. The two key ingredients of *randomization* and *time-averaging* are used, together with a local gossip communication protocol, to obtain convergence of these distributed algorithms to the global synchronous dynamics.

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1. Introduction

Over the past decade, the *networks paradigm* has emerged as a central theme in the systems and control community. Networks are now viewed as a research area which includes various applications of significant interest, such as the rapid spread of financial crises and epidemics, aggregation of human behavior and the growth of the Internet, just to name a few topics which are influencing our daily lives. Other scientific communities such as computer science, physics, applied mathematics, biology and social sciences have been involved in the research on “complex networks” since a long time, see e.g. [29]. The tools that are available within systems and control have a distinguishing feature, which is the key synergy of *uncertainty, dynamics and feedback*, which is not available elsewhere. This synergy provides the ultimate objective of this research: to develop feedback mechanisms which deal with dynamic models describing uncertain systems connected through a network with limited communication capacity. Moving on from the control of a given dynamic system, possibly nonlinear and uncertain, to the control of a large number of

systems connected through a network is certainly a major step forward and leads to *networked control systems* [5].

More specifically, one of the focal points of the research in recent years has been the study of consensus and coordination of multi-agent systems by means of a graph-theoretic setting which represents a network [16,53]. Significant research has been also performed over the years to develop tools and algorithms for distributed computation and optimization [7,58], distributed estimation based on relative measurements [6,23], clock synchronization of wireless sensor networks [35] and optimal deployment of robotic networks [15].

In this tutorial paper, we study three problems coming from different areas of control engineering: opinion formation, centrality computation and estimation problems in power systems, which are described in Sections 2, 3 and 4, respectively. Our contribution is to recast these applications under a general framework and show some strong similarities in the qualitative picture using tools previously devised in [64]. In particular, in [64] we provide technical results on ergodic properties of distributed dynamics over networks based on a combination of randomization and time-averaging techniques (see also the conference papers [31,65,66]). Moreover, compared to the former paper, the distributed algorithms developed for power systems estimation problems are new.

Randomization has proved to be a useful ingredient, complementary to classical robustness techniques [62], when dealing with control of uncertain systems [69]. In network dynamics, randomization is also quite natural and has the objective to improve the

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overall performance of the system, for example when distributed and asynchronous algorithms should be developed. On the other hand, time-averaging has been widely employed in optimization problems, for example to improve convergence speed of stochastic approximation algorithms [63].

The general framework previously mentioned concentrates on a class of randomized affine dynamics that do not enjoy an equilibrium point, but are stable on average. This stability property guarantees that the dynamics, although affected by persistent random oscillations, possesses an ergodic behavior that can be readily exploited in many network-based dynamics where randomization is (apparently) an obstacle to obtain convergence. In particular, we consider dynamics where nodes interact in randomly chosen pairs, following a so-called gossip protocol [12]. This protocol is particularly appealing, for example, when dealing with sensor networks where battery consumption is a significant concern or with social networks where individuals may discuss various topics in pairs or in small groups.

As a consequence of these observations, the desired convergence property, which holds in expectation, can be recovered by each node of the network through a process of time-averaging. Remarkably, time-averages can be computed *locally* by each node and, in some cases, even without access to a common clock to obtain the *global* synchronous dynamics.

Many network algorithms can be randomized in such a way that the dynamics converges (almost surely) to the same limit of the synchronous dynamics. Nevertheless, examples of randomized algorithms that do not converge have also recently appeared in the literature. Such algorithms require an additional “smoothing” operation in order to converge, and in our framework, this goal is simply obtained by means of time-averaging.

We now briefly describe the three applications studied in this paper; additional details are given subsequently in the general description of these applications.

The first application, see Section 2, arises in social sciences and it is focused on the mechanisms of opinion formation, which plays a significant role in many other areas such as economy, finance, biology and epidemiology. The model we consider is based on the concept of stubborn agents [38], which leads to a disagreement of opinions, and it is an alternative to classical models where the objective is to reach a consensus of opinions between several individuals. Compared to [64], the contribution is a new model of communication between agents and, thus, of opinion evolution. More precisely, we consider symmetric pairwise randomized dynamics. At each time step, a randomly chosen pair of agents update its opinion as a convex combination of its own opinion, the opinion of one of its neighbors, and the so-called prejudice. We show that, even though the resulting dynamics persistently oscillates, its average is a stable opinion profile, which is not a consensus of opinions.

The second application, see Section 3, deals with the computation of centrality measures. In particular, we discuss various measures often used in complex networks [29]: degree centrality, closeness, betweenness and eigenvector centralities, and we compare them by means of a simple illustrative example. We also remark that the eigenvector centrality is closely related to the PageRank algorithm for ranking websites in order of importance [13]. In particular, for PageRank, we study a distributed randomized algorithm and show its convergence properties. This algorithm is based on link randomization, and it is an alternative to other algorithms previously proposed in [42], which are based on node randomization.

The third application, see Section 4, deals with estimation problems in power systems where the large-scale power grids are geographically distributed. The topology of the grid is represented as a network, where each node corresponds to a bus in the

grid and the edges connecting the nodes represent the transmission lines. Using a linearized model, a (weighted) least squares approach is generally used to determine the states of the grid including the voltage magnitudes and phase angles at the buses [2,57]. However, this computation may not be practical if it is done centrally for some real-time control and monitoring applications especially when the grid size becomes larger. Hence, distributed computation has become an active area of research [40]. Here, we design new distributed randomized algorithm for estimation based on partitioning of the grid and establish its convergence properties. Simulation results showing the performance of the proposed algorithm are given for the classical IEEE 14-bus test system [24].

Finally, in Section 5, we provide brief conclusions.

1.1. General notation

We begin our work by fixing some notation and by reviewing some definitions of graph theory. We denote the sets of real and nonnegative integer with the symbols $\mathbb{R}$ and $\mathbb{Z}_{\geq 0}$, respectively. The notation $|\cdot|$ is used to indicate either the cardinality of a set or the absolute value of a real number. We denote column vectors with small letters, and matrices with capital letters. The symbol e_i is the vector with the i th entry equal to 1 and all the other elements equal to 0, and we write $\mathbf{1}$ for the vector with all entries equal to 1. A matrix A is row-stochastic (column-stochastic) when its entries are nonnegative and $A\mathbf{1} = \mathbf{1}$ ($A^T\mathbf{1} = \mathbf{1}$). A matrix A is said to be Schur stable if the absolute value of all its eigenvalues is smaller than 1. Given the sequence of real vectors $\{x(k)\}_{k \in \mathbb{Z}_{\geq 0}}$, we denote its time-average, also known as Cesàro average or Polyak average in some contexts, with

$$\bar{x}(k) := \frac{1}{k} \sum_{\ell=0}^{k-1} x(\ell). \quad (1)$$

A directed graph is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. We say that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an undirected graph if $(i, j) \in \mathcal{E}$ implies that (j, i) is also an edge in $\mathcal{E}$. The set of neighbors of $i \in \mathcal{V}$ is denoted as $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The degree of a node $i \in \mathcal{V}$ is $|\mathcal{N}_i|$. A path in a graph is a sequence of edges which connect a sequence of vertices. In an undirected graph $\mathcal{G}$ the nodes i and j are said connected if there exists a path from i to j . A graph is said to be connected if every pair of vertices in the graph is connected. To any matrix $P \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ with non-negative entries, we associate a directed graph $\mathcal{G}_P = (\mathcal{V}, \mathcal{E}_P)$ by putting $(i, j) \in \mathcal{E}_P$ if and only if $P_{ij} > 0$. The matrix P is said to be adapted to graph $\mathcal{G}$ if $\mathcal{G}_P \subseteq \mathcal{G}$.

2. Opinion formation

The study of opinion formation in social networks has increasingly attracted the attention of the control community in the past decades [8]. The definition of social network is often associated to describe the complex networked systems that play a fundamental role in a number of different fields such as economy, finance, biology, epidemiology, and sociology. In general terms, a social system consists of a large number of agents/individuals whose interactions may induce the emergence of collective behaviors. The global behavior of the network shows often complexity features that can be seen as the result of the addition of the many individual behaviors and the rapid diffusion of information facilitated by the topology of the interconnections.

Researchers devoted their attention on various models for the dynamics of continuous opinions by analytical analysis as well as via numerical simulation. Pioneering works appeared in the early 1950s [34] in order to give theoretical explanations of complex

phenomena observed empirically in populations. Most of the literature focused on models (see [27] and subsequent papers [22,73]) describing how the group can reach agreement on a common opinion by pooling their individual beliefs.

The need of modeling the disagreement among individuals in a society became a central question in [1,38]. We can distinguish two main lines of research: opinion-dependent limitations in the network connectivity and linear opinion dynamics with stubborn agents or antagonistic agents [4,71,44]. The first line of research has seen a growth of models involving “bounded confidence” between the agents: if the opinions of two agents are too far apart, they do not influence each other. These models typically are nonlinear models and lead to a fragmentation of opinions: the agents split into clusters, and each group reaches an internal consensus. Influential models have been defined in [28,41], and their understanding has been recently deepened in the control community, which has studied evolutions both in discrete time [8,17] and in continuous time [9,19], possibly including heterogeneous agents [54] and randomized updates [78,77].

Although interesting and motivated, these “bounded confidence” models do not seem to be sufficient to explain the persistence of disagreement in real societies. In fact empirical evidence suggests that the disagreement is not the consequence of lack of communication but persists over the time in spite of persistent contacts and interactions between agents. Instead, a persistent disagreement is more likely a consequence of the agents being unable, or unwilling, to change their opinions, no matter what the other agents' opinions are. This observation has been made by social scientists, as in the models introduced in [38,37]. These models are based on the assumption of synchronous rounds of interaction: agents can share and update their opinions in a synchronous fashion. This is not a realistic assumption in many contexts. It is more plausible to imagine that individuals can meet at different times and that some random process determines the possible interactions among the agents (see [56,25,3,76]). These dynamics can be expressed as linear models and their theoretical analysis can be carried out by using techniques from stochastic processes [25,3] and from game theory [21,39].

In this section we consider a *pairwise randomized dynamics* such that at each time step a randomly chosen pair of agents update its opinion to a convex combination of its own opinion, the opinion of one of its neighbors, and its own initial opinion or “prejudice”. We show that under suitable conditions the resulting dynamics persistently oscillates, its average is a stable opinion profile, which is not a consensus. This means that the expected beliefs of an agent will not in general achieve, even asymptotically, an agreement with the other agents in the society. Furthermore, we show that the oscillations of opinions are ergodic in nature, so that the averages along sample paths are equivalent to the ensemble averages. The dynamics considered in this section allows the agents to have a continuum of degrees of obstinacy, rather than a dichotomy stubborn/non-stubborn as treated in [3].

2.1. Friedkin and Johnsen's model

Here we recall Friedkin and Johnsen's model [38], which is a unified framework that includes various models for the dynamics of continuous opinions. More precisely, it investigates the mechanisms in an interacting group that can lead to a consensus or to a fragmentation of opinions. We consider a set of agents $\mathcal{V}$ among whom some process of opinion formation takes places. In this process, the agents take typically into account the opinions of the others in forming its own opinion. The process of forming the actual opinion can be repeated over the time and can be described by a discrete time dynamical system.

Mathematically, the potential interactions are encoded by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, which we refer to as the *social network*. To avoid trivialities, we assume that $|\mathcal{V}| > 1$. We assume that each agent $i \in \mathcal{V}$ is endowed with a state $x_i(k)$ with $k \in \mathbb{Z}_{\geq 0}$. For instance if $x_i(k) \in [0, 1]$, 1 may represent extreme positive and 0 negative *belief* or *opinion*, respectively. An edge $(i, j) \in \mathcal{E}$ means that agent j may directly influence the belief of agent i . For each agent i , we denote its neighborhood with $\mathcal{N}_i := \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. Let $W \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ be a nonnegative matrix which defines the strength of the interactions ($W_{ij} = 0$ if $(i, j) \notin \mathcal{E}$) and Λ be a diagonal matrix describing how sensitive each agent is to the opinions of the others, based on interpersonal influences. We assume that W is row-stochastic, i.e., $W\mathbf{1} = \mathbf{1}$, and we set $\Lambda = I - \text{diag}(W)$, where $\text{diag}(W)$ collects the self-weights given by the agents. The dynamics of opinions $x(k)$ proposed in [38] is

$$x(k+1) = \Lambda W x(k) + (I - \Lambda)u, \quad (2)$$

with $x(0) = u$ and $u \in \mathbb{R}^{\mathcal{V}}$. The input vector u is the main feature of this model, and marks its difference with, for instance, the mentioned models which are based on bounded confidence. In other terms, this model assumes that each agent in the network is socially influenced by other agents but adheres to other opinions with a certain susceptibility and to its own preconceived opinion with a certain level of obstinacy. The nodes' actions in (2) can be also interpreted at each time step as myopically optimizing a quadratic cost function of the form

$$F(x_i; x_{\mathcal{N}_i}) = \frac{1}{2}(1 - w_{ii}) \sum_{j \in \mathcal{N}_i} w_{ij}(x_i - x_j)^2 + \frac{1}{2}w_{ii}(x_i - u_i)^2$$

and the dynamics in (2) as the best-response strategy in an underlying game with natural payoffs [39].

2.1.1. Consensus versus disagreement

We are interested in characterizing the asymptotic behavior of (2) and investigating whether the dynamics will converge to a consensus or to a disagreement. Remarkably, the properties of the asymptotic behavior are determined by the graph structure and by the stubbornness levels of the agents. The limit behavior of the opinions is described in the following result.

Proposition 1 (*Opinion convergence*). *The following facts hold true:*

1. If $\text{diag}(W) = 0$ (i.e., $\Lambda = I$) and the graph is aperiodic and has a globally reachable node, then the dynamics in (2) leads asymptotically to a consensus: for every $\epsilon > 0$ there exists a constant C_ϵ such that

$$\|x(k+1) - \mathbf{1}\pi^\top u\| \leq C_\epsilon(\mu + \epsilon)^k,$$

where π is the invariant probability of matrix W (i.e. $\pi^\top W = \pi^\top$), and μ is the second largest eigenvalue of the matrix W .

2. If $\text{diag}(W) \neq 0$ and for any node $\ell \in \mathcal{V}$ there exists a path from ℓ to a node m such that $W_{mm} > 0$, then the opinions converge to

$$x^* := \lim_{k \rightarrow +\infty} x(k) = (I - \Lambda W)^{-1}(I - \Lambda)u.$$

Moreover, the convergence is geometric with a rate equal to the largest eigenvalue of ΛW .

It should be noticed that if $\mathcal{G}$ is strongly connected, then the assumptions in Proposition 1 are satisfied. The case with no stubborn agents ($\text{diag}(W) = 0$) is well studied in the literature [22] and under suitable requirements on the graph associated to the matrix W , all agents' opinion converge to a common value, which is a convex combination of the initial opinions with weights given by the invariant probability of W . On the other hand, if each agent is influenced by at least one stubborn agent (that has

positive $W_{\ell\ell}$), then for any initial profile the opinion dynamics leads asymptotically to an equilibrium point that can be computed from the weights, the obstinacy levels and the initial opinions. Let $V := (I - \Lambda W)^{-1}(I - \Lambda)$, which is referred to as the total effects matrix in [38]. Since W is stochastic, we observe that under the assumption of Proposition 1 also V is stochastic: this means that the limit opinion of each agent is a convex combination of the preconceived opinions of the group $x'_i = \sum_j V_{ij} u_j$. As a special case, the asymptotic opinion profile is a consensus if the initial opinions are at consensus. Although the above result says that the convergence to the equilibrium happens exponentially fast, it does not give any insight on how the limit point depends on the graph properties and on the stubbornness levels of the agents. It is worth mentioning that for specific cases the equilibrium can be characterized in terms of voltage in electrical networks [39].

2.1.2. Example

Here, we briefly describe the experiment conducted in [38] to illustrate the above model. We consider a group of four agents and study how opinions are formed through interactions:

1. The agents are presented with an issue on which opinions can range, say, from 1 to 100.
2. Each agent has a private initial opinion on the issue $x(0) = [25 \ 25 \ 75 \ 85]^\top$.
3. The agents can communicate with other agents individually to discuss the issue and update their opinions based on their own initial opinions and observing the opinions of their neighbors. The matrix W which determines the influence network for this group is given by

$$W = \begin{bmatrix} .220 & .120 & .360 & .300 \\ .147 & .215 & .344 & .294 \\ 0 & 0 & 1 & 0 \\ .090 & .178 & .446 & .286 \end{bmatrix},$$

where the entries represent the distribution of relative interpersonal influences on the issue. Note that agent 3 in this example is “totally stubborn”, meaning that it does not change its opinion at all during the evolution. This matrix is obtained from the experiment data $x(0)$, x' , and the estimate of the relative interpersonal influences. We take $\Lambda = I - \text{diag}(W) = \text{diag}(.780, .785, 0, .714)$; the entries represent the agents' susceptibilities to interpersonal influence. The off-diagonal entries of W are the weights of the influence by the others.

4. After a pre-specified period of discussion (e.g., 20 min), they report on their final opinions that may or may not be in agreement.

It is easy to verify that the assumptions in Point 2. of Proposition 1 are satisfied. The matrix V is

$$V = \begin{bmatrix} .280 & .045 & .551 & .124 \\ .047 & .278 & .549 & .126 \\ 0 & 0 & 1 & 0 \\ .030 & .048 & .532 & .390 \end{bmatrix}.$$

This total effects matrix indicates the influence of each agent on every other agent in the final opinions through the flow of direct and indirect interactions. For example, $V_{23} = .549$ shows that almost 55% of the final opinion of agent 2 is determined by agent 3.

The evolution of the opinions is illustrated by the simulations in Fig. 1. The final opinion profile is $x' = [60 \ 60 \ 75 \ 75]^\top$.

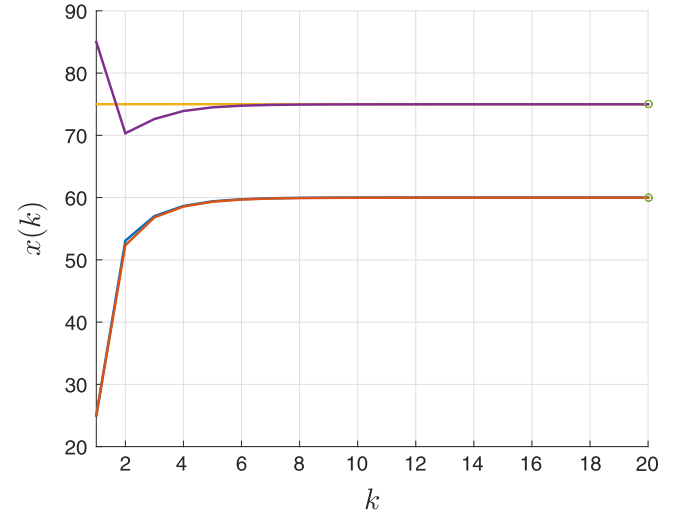


Fig. 1. Evolution of the opinion dynamics (2) in the example of Section (2.1.2). The opinions x converge to the limit x' (marked by green circles). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

2.2. Randomized dynamics for opinion formation

The Friedkin and Johnsen's model is based on the assumption of synchronous rounds of interaction: agents can share and update their opinions at the same time. This assumption is clearly not realistic in many situations. The lack of a more precise model for the inter-agent interactions is acknowledged in [38] by saying that “it is obvious that interpersonal influences do not occur in the simultaneous way that is assumed”. It is more plausible to imagine that interactions occur at different times for instance in a pairwise fashion. This motivates us to analyze a different opinion dynamics where typically some random process determines the possible interactions among the agents. We now describe a random dynamics that considers the main feature of Friedkin and Johnsen's model and introduces a more realistic model of the communication process among the agents. In our model agents interact in pairs in such a way that their opinions become a convex combination as in the original dynamics (2).

Each agent $i \in \mathcal{V}$ starts with an initial belief $x_i(0) = u_i \in \mathbb{R}$. At every time instant $k \in \mathbb{Z}_{\geq 0}$ an edge is randomly activated with uniform probability over $\mathcal{E}$. The opinions of the other agents remain unchanged, while nodes i and j exchange their opinions and produce a new belief according to the following equations:

$$\begin{aligned} x_i(k+1) &= h_i \left((1 - \gamma_{ij}) x_i(k) + \gamma_{ij} x_j(k) \right) + (1 - h_i) u_i \\ x_j(k+1) &= h_j \left((1 - \gamma_{ji}) x_j(k) + \gamma_{ji} x_i(k) \right) + (1 - h_j) u_j \\ x_{\ell}(k+1) &= x_{\ell}(k) \quad \forall \ell \in \mathcal{V} \setminus \{i\}. \end{aligned} \quad (3)$$

It worth remarking that we assume the edges to be chosen for the update according to a uniform distribution. This choice is made for simplicity, but the analysis of the model can easily be extended to different distributions. The weighting coefficients h_i and γ_{ij} satisfy the following assumption.

Assumption 1. Let the diagonal matrix H be defined by $H_{ii} = h_i$ and the matrix Γ defined by $\Gamma_{ij} = \gamma_{ij}$. We assume that (i) $h_i \in [0, 1]$ for all $i \in \mathcal{V}$; (ii) Γ is row-stochastic, i.e., for all i and j in $\mathcal{V}$ it holds $\gamma_{ij} \geq 0$, $\sum_{\ell} \gamma_{i\ell} = 1$; and (iii) $\gamma_{ij} = 0$ if j is not a neighbor of i .

As a consequence of this assumption, we observe that at all times the opinions of the agents are convex combinations of their initial prejudices. Note that if an edge of the form (i, i) is sampled at time

k , then

$$x_i(k+1) = h_i x_i(k) + (1-h_i)u_i,$$

that is, the opinion of agent i moves back closer to its preconceived opinion. Also note that if $h_i=0$, then agent i is totally stubborn, whereas if $h_i=1$, then agent i is totally open-minded: we may say that $1-h_i$ is proportional to the obstinacy of the agent.

2.2.1. Consensus versus opinion fluctuations

We ask whether a stable opinion profile arises as a result of such local interactions and if the agents reach an agreement or if a disagreement persists over the time. We start by looking at the expected evolution of the random vector by taking expectations on both sides of (3)

$$\mathbb{E}[x(k+1)] = \left(I - \frac{2}{|\mathcal{E}|} (D(I-H) + H(I-\Gamma)) \right) \mathbb{E}[x(k)] + \frac{2}{|\mathcal{E}|} D(I-H)u, \quad (4)$$

where the right-hand side holds because the matrices are selected independently and D is the degree matrix of $\mathcal{G}$, a diagonal matrix whose diagonal entry is equal to the degree $d_i = |\mathcal{N}_i|$.

It is easy to see that if $H=I$ and $\mathcal{G}$ has a globally reachable node then the expected evolution of the opinions vector will follow a Markov chain and will converge in expectation to $\pi^\top u$ where π is the stationary distribution of Γ . On the other hand, if each agent is influenced by at least one stubborn, then the opinions will converge to the value

$$x^* := \lim_{k \rightarrow \infty} \mathbb{E}[x(k)] = (D(I-H) + H(I-\Gamma))^{-1} D(I-H)u. \quad (5)$$

Further precise convergence properties can be found in the following theorem [64].

Theorem 1 (Opinion convergence versus ergodicity). *The following statements are true.*

1. If $H=I$ and $\mathcal{G}$ has a globally reachable node, then the dynamics (3) converges to a consensus almost surely and in the mean-square sense.
2. If for each node ℓ in $\mathcal{V}$, there exists a path in the graph associated to Γ from ℓ to a node m such that $h_m \neq 1$, the dynamics (3) is almost surely and mean-square ergodic, i.e. the sequence of time-averaged opinions $\bar{x}(k)_{k \geq 0}$ is such that $\lim_{k \rightarrow \infty} \bar{x}(k) = x^*$ almost surely and in mean square sense. More precisely,

$$\mathbb{E}[\|\bar{x}(k) - x^*\|_2^2] \leq \frac{\chi}{k}, \quad (6)$$

where the constant $\chi > 0$ depends on the graph.

2.3. Relating gossip and synchronous dynamics

We analyze now the relationship between the gossip dynamics and the original dynamics in (2) more carefully. From a purely mathematical point of view, we observe the following fact.

Proposition 2. *If H and Γ are chosen as*

$$h_i = \begin{cases} 1 - (1 - \lambda_{ii})/d_i & \text{if } d_i \neq 1 \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

$$\gamma_{ij} = \begin{cases} \frac{d_i(1-h_i) + h_i - (1 - \lambda_{ii}w_{ij})}{h_i} & \text{if } i=j, d_i \neq 1 \\ \frac{\lambda_{ii}w_{ij}}{h_i} & \text{if } i \neq j, d_i \neq 1 \\ 1 & \text{if } i=j, d_i = 1 \\ 0 & \text{if } i \neq j, d_i = 1 \end{cases} \quad (8)$$

then Γ and H satisfy Assumption (1) and the expected dynamics (4)

can be written as

$$\mathbb{E}[x(k+1)] = \left(I - \frac{2}{|\mathcal{E}|} \right) \mathbb{E}[x(k)] + \frac{2}{|\mathcal{E}|} (\Lambda W \mathbb{E}[x(k)] + (I - \Lambda)u). \quad (9)$$

Furthermore, $x^* = x'$.

It should be noticed that under the assumption that Γ and H are chosen as in Proposition 2, then the expected dynamics (9) is a “lazy” (slowed down) version of the Friedkin and Johnsen’s dynamics associated to the matrix W . Hence, Theorem 1 shows that the average dynamics $\mathbb{E}[x(k)]$ converges to the limit opinions of the original model (2). This relationship between the two dynamics provides an additional justification and a new perspective on the model originally proposed by Friedkin and Johnsen. Furthermore, we observe that Proposition 1 can be immediately deduced as a corollary of Theorem 1.

From a sociological perspective (7) and (8) postulate a specific form of interaction for individuals in pairwise meetings, which is reflected on average by Friedkin and Johnsen’s dynamics. Since by (7) $h_i > \lambda_{ii}$, we observe that individuals display a lower obstinacy during pairwise interaction.

2.3.1. Example 1 (continued)

In this subsection, we continue with the example of four agents. Let the weight matrices Γ and H in the update (2) be chosen according to (7) and (8). Then we have

$$H = \text{diag}(.945, .946, .000, .928),$$

$$\Gamma = \begin{bmatrix} .356 & .099 & .297 & .248 \\ .122 & .349 & .285 & .244 \\ 0 & 0 & 1.0000 & 0 \\ .069 & .137 & .343 & .451 \end{bmatrix},$$

$$D = \text{diag}(4, 4, 1, 4).$$

Suppose that at time k , as a result of gossiping, the edge between agents 1 and 2 is chosen. In this case, the dynamics (2) can be written in the matrix form as follows:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} h_1(1-\gamma_{12}) & h_1\gamma_{12} & 0 & 0 \\ h_2(1-\gamma_{21}) & h_2\gamma_{21} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(k) \\ &+ \begin{bmatrix} 1-h_1 & 0 & 0 & 0 \\ 0 & 1-h_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} u \\ &= \begin{bmatrix} .851 & .094 & 0 & 0 \\ .83 & .115 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(k) \\ &+ \begin{bmatrix} .055 & 0 & 0 & 0 \\ 0 & .054 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} u. \end{aligned}$$

For other edge choices, similar expressions can be obtained. We see in simulations (Fig. 2) that the states oscillate, but the time averages converge, as predicted by our results (Fig. 3).

3. Centrality computation

Discerning the importance of nodes in a network is a crucial question in network science and has thus attracted a lot of attention. The question can be formalized by defining a *centrality measure*, which is a function of the nodes in the directed graph

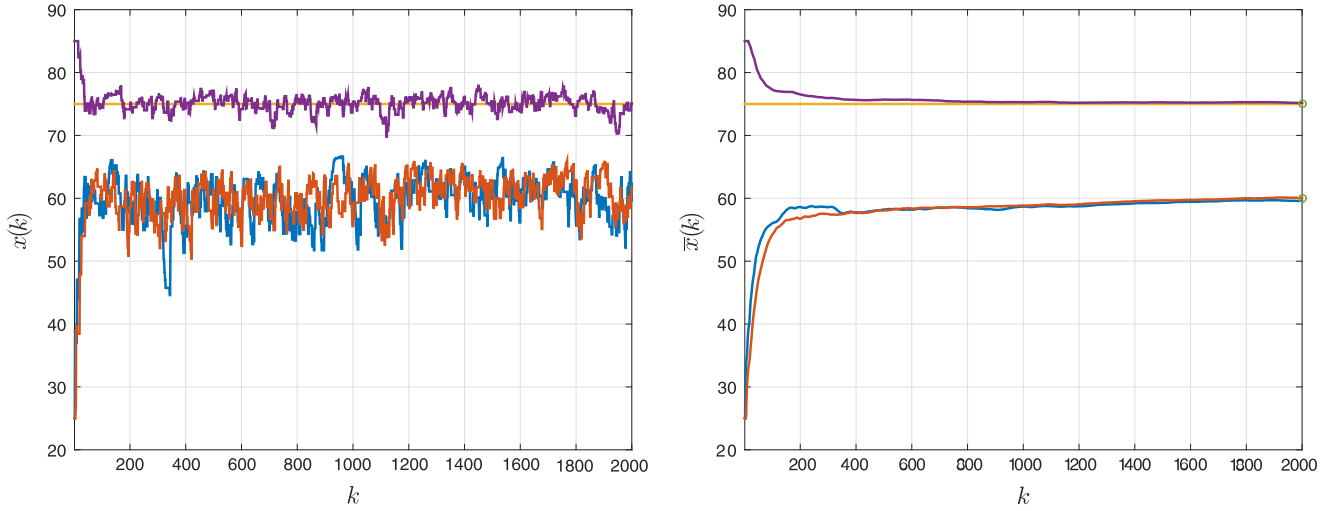


Fig. 2. Evolution of the opinions in the example of Section 2.3.1. The belief process x (left plot) oscillates persistently in a bounded interval. As the belief process is ergodic, the time averages (right plot) converge, when time goes to infinity, to the limit of Friedkin and Johnsen's dynamics (marked by blue circles). (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

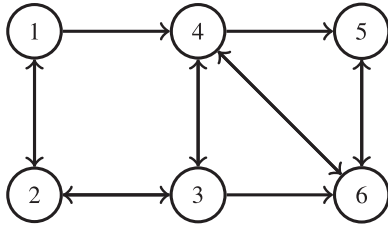


Fig. 3. Simple example of directed graph.

$\mathcal{G}=(\mathcal{V},\mathcal{E})$ representing the network. We begin this section by quickly reviewing some of such measures, before concentrating on the specific case of PageRank centrality. Afterwards, we will consider algorithms for their computation.

3.1. Centrality notions and PageRank

Depending on the applications, different notions of importance may be of interest. The simplest centrality measure is arguably the degree of the nodes, that is the number of its neighbors. This degree centrality is a purely local notion, meaning that it only depends on the neighborhood of the node. Likely, such a limited perspective would fail in many cases to capture the actual role of the node in the network. Consequently, more refined definitions have been proposed: classical examples include closeness, betweenness, and eigenvector centralities.

According to a closeness centrality, a node is more central if it is closer to most of the other nodes [33]: closeness is thus defined as the average distance from all other nodes

$$C_i = \left(\sum_{j \in \mathcal{V}} d(i,j) \right)^{-1}.$$

This definition depends on the chosen distance $d(\cdot, \cdot)$ between nodes: it can be, for instance, the length of the shortest path or the resistance distance, as measured on a corresponding electrical network where all edges are seen as resistors. Sometimes a normalization constant, depending on the size of the graph $n=|\mathcal{V}|$, can be used to facilitate comparisons across different networks: as this is not our goal here, we shall not introduce any normalization.

Betweenness, instead, evaluates a node higher if it belongs to many paths between other nodes. More precisely, one can

consider [32,75]

$$B_i = \sum_{j,k \in \mathcal{V}, j \neq k \neq i} \frac{\text{\#shortest paths from } j \text{ to } k \text{ that contain } i}{\text{\#shortest paths from } j \text{ to } k}.$$

This definition can be extended to include all possible paths between nodes, instead of just the shortest ones [68,59]. Furthermore, even if we do not describe edge centralities here, we note that an analogous definition of edge betweenness can be given [32].

Eigenvector centrality assigns centrality according to the entries of the leading eigenvector of a suitable weighted adjacency matrix associated to the graph. Examples in this class have been proposed by Bonacich [10] and by Friedkin [36], among others. Also the PageRank algorithm at the core of Google's search engine is based on this approach. In the specific case of PageRank, we study a network consisting of web pages [13]. This network is represented by a graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where the set of vertices corresponds to the web pages and edges represent the links between the pages, i.e., the edge $(i,j) \in \mathcal{E}$, if page i has an outgoing link to page j , or in other words, page j has an incoming link from page i . The PageRank value of a web page is a real number in $[0, 1]$, which is defined next. Let us denote $\mathcal{N}_i = \{h \in \mathcal{V} : (i,h) \in \mathcal{E}\}$ and $n_i = |\mathcal{N}_i|$ for each node $i \in \mathcal{V}$, and A the matrix such that

$$A_{ij} = \begin{cases} 1/n_j & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise.} \end{cases}$$

This matrix describes the behavior of a user that is randomly surfing the web from one page to another, following the links between them. When the surfer is at page j , it moves to any of the n_j pages that link to it. Note that a hyper-textual link in page i pointing to page j is actually travelled across from j to i by the user. Under the assumption that all pages have outgoing links, we have the property $\sum_i A_{ij} = 1$, that is, A is a column-stochastic matrix. If we now let $m \in (0,1)$ and denote $n=|\mathcal{V}|$, we define another column-stochastic matrix

$$M = (1-m)A + \frac{m}{n} \mathbf{1}\mathbf{1}^\top. \quad (10)$$

As the matrix A represents a random walk across the web, the choice of positive m corresponds to introducing a chance of “teleportation” of the web surfer in the matrix M : in practice, the value $m=.15$ is used. The PageRank of the graph $\mathcal{G}$ is the vector x_{pgr}^* such that $Mx_{\text{pgr}}^* = x_{\text{pgr}}^*$ and $\mathbf{1}^\top x_{\text{pgr}}^* = 1$. This vector has the interpretation of the fraction of time spent on each page by the

random surfer, on average and asymptotically. We refer to [43] for more discussion and to [14,48] for introductions to the mathematical details behind this claim.

3.1.1. Example

Different centrality measures may give very different answers regarding the importance of the nodes. This obvious fact is a direct consequence of their definitions, which are meant to describe different phenomena: e.g., being quickly reachable or being instrumental to the communication between the other nodes. The literature has indeed extensively discussed criteria and methods to select the relevant centrality measures [32,11,59]. Differences are apparent even in simple networks. To this goal we choose a small example of graph that has been used to study connections between webpages in [43]. On this graph, we compute all the centrality measures defined above. If we compare the obtained numerical values, reported in Table 1, we see that the most relevant nodes can be node 6 (according to PageRank), node 4 (according to betweenness), nodes 4 and 6 (according to closeness), or node 4 (according to degree).

3.2. Algorithms for centrality computation

In contemporary engineering research, we witness a growing tendency to phrase questions about network systems as definitions of centrality measures. For instance, [49,30] have shown that closeness measures can describe the ability for a node to optimally spread information across a network, when information propagates via noisy communications. Hence, effective algorithms to compute centrality measures are becoming more and more important.

As we mentioned, the most significant definitions of centrality are functions of the full network. Consequently, in order to compute the centrality of a given node, information about other parts of the network is necessary. How much information must be gathered? Can this information be elaborated locally by a node or by an observer with partial information, or instead computations need to be run by an external agent with full information? The latter perspective was the standard in the classical works, but researchers are currently looking for distributed algorithms, which compute centrality measures when the available information is partial and limited by privacy concerns or by other restrictions.

Within the systems and control community, interesting results in this direction have been obtained under the assumption that the graph is a tree (i.e., has no path from a node to itself). Under this assumption, both betweenness [74] and closeness [61] can be computed distributedly. The same fact is true for more complex measures of influence defined in opinion dynamics [72]. On the contrary, the approach to PageRank initiated in [42] and illustrated below does not make any restrictive assumption on the graph.

Table 1

Values of different centrality measures in the graph of Fig. 3. Note that in the definition of closeness we choose the distance $d(i,j)$ to be the length of the shortest path from j to i . Degree values are not reported in this table.

Node	Closeness	Betweenness	PageRank
1	.066	.833	.061
2	.09	.5	.085
3	.111	2.166	.122
4	.142	3.666	.214
5	.111	0	.214
6	.142	.5	.302

Let us first recall that the PageRank vector can be computed through the recursion

$$x(k+1) = Mx(k) = (1-m)Ax(k) + \frac{m}{n}\mathbf{1}, \quad (11)$$

provided the initial condition satisfies $\mathbf{1}^\top x(0) = 1$. Indeed, the reader may easily check that teleportation guarantees the Schur stability of $(1-m)A$ and thus implies that the sequence in (11) converges to

$$x_{\text{pr}}^* = (I - (1-m)A)^{-1} \frac{m}{n} \mathbf{1}.$$

However, this method requires a large number of synchronous operations at each time step. With the goal of reducing these requirements, randomized methods for PageRank computation have been studied in several recent papers and a survey on this topic has recently appeared [43]. In the remainder of this section, we describe an example of an “edge-based” randomized algorithm, originally presented in [64]. As only one edge is activated at each time, the algorithm is asynchronous and its communication requirements are minimal.

With this discussion in mind, we are ready to describe the algorithm from [64]. Each node $i \in \mathcal{V}$ holds a pair of states $(x_i, \bar{x}_i)$. For every time step k an edge $\theta(k)$ is sampled from a uniform distribution over $\mathcal{E}$ (note that sampling is independent at each time k). Then, the states are updated as follows:

$$x_i(k+1) = (1-r) \left(1 - \frac{1}{n_i}\right) x_i(k) + \frac{r}{n} \quad (12a)$$

$$x_j(k+1) = (1-r) \left(x_j(k) + \frac{1}{n_i} x_i(k)\right) + \frac{r}{n} \quad (12b)$$

$$x_h(k+1) = (1-r)x_h(k) + \frac{r}{n} \quad \text{if } h \neq i, j \quad (12c)$$

and

$$\bar{x}_\ell(k+1) = \frac{k\bar{x}_\ell(k) + x_\ell(k+1)}{k+1} \quad \forall \ell \in \mathcal{V} \quad (13)$$

where $r \in (0, 1)$ is a design parameter that will be precised later. In fact, $\bar{x}$ is just the time-averaged state as defined in (1).

The update in (12) can also be formally rewritten in a vector form as

$$x(k+1) = P(k)x(k) + u(k),$$

where

$$P(k) = (1-r)A(k), \quad u(k) = \frac{r}{n}\mathbf{1}.$$

Here $A(k)$ and $P(k)$ are random matrices which are determined by the choice of $\theta(k) = (i, j)$

$$A(k) = I + \frac{1}{n_i}(e_j e_i^\top - e_i e_i^\top).$$

More precisely, $A(k)$ is uniformly distributed over the set of matrices $\{I + \frac{1}{n_i}(e_j e_i^\top - e_i e_i^\top) : (i, j) \in \mathcal{E}\}$. This compact rewriting is convenient in proving the convergence of the algorithm. Indeed,

$$\begin{aligned} \mathbb{E}[A(k)] &= \frac{1}{|\mathcal{E}|} \sum_{(i,j) \in \mathcal{E}} \left(I + \frac{1}{n_i}(e_j e_i^\top - e_i e_i^\top) \right) \\ &= \left(1 - \frac{1}{|\mathcal{E}|} \right) I + \frac{1}{|\mathcal{E}|} A. \end{aligned}$$

The expectation of the states evolves through

$$\begin{aligned} \mathbb{E}[x(k+1)] &= (1-r)\mathbb{E}[A(k)]\mathbb{E}[x(k)] + \frac{r}{n}\mathbf{1} \\ &= (1-r) \left(\left(1 - \frac{1}{|\mathcal{E}|} \right) I + \frac{1}{|\mathcal{E}|} A \right) \mathbb{E}[x(k)] + \frac{r}{n}\mathbf{1}. \end{aligned}$$

Similar to what we argued before, the update matrix is stable and thus the dynamics converges

$$\lim_{k \rightarrow \infty} \mathbb{E}[x(k)] = \left(I - \left(1 - r - \frac{1}{|\mathcal{E}|} + \frac{r}{|\mathcal{E}|} \right) I - \frac{1-r}{|\mathcal{E}|} A \right)^{-1} \frac{r}{n} \mathbf{1}.$$

This limit vector is equal to x_{pgr}^* if r satisfies $\frac{m}{1-m} = \frac{r}{1-r} |\mathcal{E}|$. The stability and the consistency of the expected dynamics are essential to complete the proof of the following result, which establishes the convergence of $\bar{x}(k)$ to $\mathbb{E}[x(k)]$: we refer the reader to [64] for the remaining details about this ergodicity property.

Theorem 2 (Ergodic PageRank convergence). *Let us consider the dynamics (12) and (13) with*

$$r = \frac{m}{m - |\mathcal{E}|m + |\mathcal{E}|},$$

where $x(0)$ is a stochastic vector. Then, the time-averaged sequence $\{\bar{x}(k)\}_{k \geq 0}$ is such that $\lim_{k \rightarrow \infty} \bar{x}(k) = x_{\text{pgr}}^*$ almost surely and in mean square sense. More precisely,

$$\mathbb{E}[\|\bar{x}(k) - x_{\text{pgr}}^*\|_2^2] \leq \frac{C}{k}, \quad (14)$$

where the constant $C > 0$ depends on the graph and on m .

The estimate (14) highlights a drawback of using time-averaging in connection with randomization. Indeed, convergence of time-averages is not exponential, as for the synchronous dynamics (11), but polynomial: the distance from the limit value is proportional to $1/k$. This drawback stimulates research towards exponentially fast algorithms. Likely, effective algorithms can be constructed by endowing the nodes with some memory capabilities, taking advantage of the so-called asynchronous iteration method from numerical analysis [7, Section 6.2]; see also [42, Section VII] for a more extended discussion and [18] for recent results.

4. Power systems state estimation

In this section, we turn our attention to applying the distributed randomized algorithms approach to a more engineering setting of power systems. Such systems have gained significant attention in the recent years from various systems areas including power, control, communications, and computer sciences. This rise has been strongly motivated by the fast growing number of renewable generation sources and the introduction of competitive electricity markets. These factors will continue to make power systems more complex in the future grids, which is often referred to as the smart grid. For stable operation and lowering the cost, real-time diagnostic and monitoring are highly desirable.

The large-scale power grids physically have network structures due to the transmission lines and are distributed geographically. These aspects provide an interesting platform for applying networked control techniques. In this section, we focus on the so-called power systems state estimation problem, which is a static estimation problem at the transmission systems level where the power generated at power plants is sent to distant substations at high voltage [2,57].

The problem can be expressed as a least squares estimation one and is classic in the power systems literature, studied in the last few decades. The current changes in the grid operation however provide new incentives to carry out its computation in a more distributed manner. The grids are becoming larger in their size, as companies who in the past operated their grids independently started to connect them, so that they can cooperate in the presence of fluctuations in the power generated by renewables. Therefore, the size of the states to be estimated has grown and it is not feasible and unnecessary to communicate all measurements among the different companies as not all state information is

needed by individual operators. That is, it is realistic in a very large-scale system to compute only part of the states at each estimator. It is also added that power systems estimation has been considered from the viewpoint of cyber security in, e.g., [50,67,55,20] where false data injections to sensor measurements are found to affect the estimations based on least squares methods in a stealthy manner.

Over the years, there have been many efforts in developing distributed algorithms for state estimation; for a review on this subject, see [40] and the references therein. It is common that the overall estimator requires a global coordinator which collects the estimation data from others to obtain the final result (see, e.g., [45,47]). Algorithms that do not employ such a coordinator have been proposed in, for example, [26,46] in the power systems literature, but other recent distributed estimation techniques are applicable as well [51]. In the control literature, the work of [60] presents a projection-based algorithm motivated by the multi-agent consensus literature, which is guaranteed to converge in finite time, but is used for computation of the entire state at every node. The work [70] takes account of errors in clock synchronization among the sensors and extends the distributed optimization techniques of [46]. More recently, in [52], the iterative computation of the least squares type was reconsidered from the distributed computation viewpoint. In particular, the convergence performance is shown to be improved by finding optimal parameters in the algorithm which can also be computed in a distributed fashion.

We extend our framework of distributed randomized approach to the least squares estimation for power systems. In particular, our algorithm can be viewed as a randomized version of [52] which is based on deterministic and synchronous communication among the nodes. An important advantage of the setting in this subsection is that the nodes can communicate in a fully asynchronous fashion. More concretely, in computing the time averages of the states, each node may do so by using its own counting of the number of updates made so far (and not global clock). It turns out that such asynchrony cannot be exploited in some other problems. We would also like to highlight that due to the generality of the least squares estimation problem, the results of this section can be extended to similar problems that appear in other networked control applications including sensor localization [6] and clock synchronization [35] for sensor networks.

4.1. State estimation problem

In this section, we give a brief introduction on the modeling of power systems and discuss the state estimation problem.

In Fig. 4, the simple transmission grid model known as the IEEE 14-bus test system is shown [24]. Each bus denoted by a horizontal bar represents a power plant with generators, a substation, or a load. The buses are connected by multiple transmission lines, forming a power network. The objective of the power systems state estimation is to compute the estimates of the unmeasured states of the buses based on the relations among the power injections, voltage magnitudes, and phase angles. The estimation results provide important information to be used for, e.g., control and fault detection of the power system, and prediction of power consumption. In state estimation, measurements of the power flow among the buses as well as power generated or consumed comprise the main inputs. For the buses employing phasor measurement units (PMUs), the voltage magnitude and the phase information can be directly obtained. PMUs share synchronized clocks through the use of GPSs, and moreover measurements can be made at fast sampling rates with high precision.

Consider a power grid model consisting of n buses [2,57,55]. We express the topology of the grid as the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Each node in the graph corresponds to a bus in the grid,

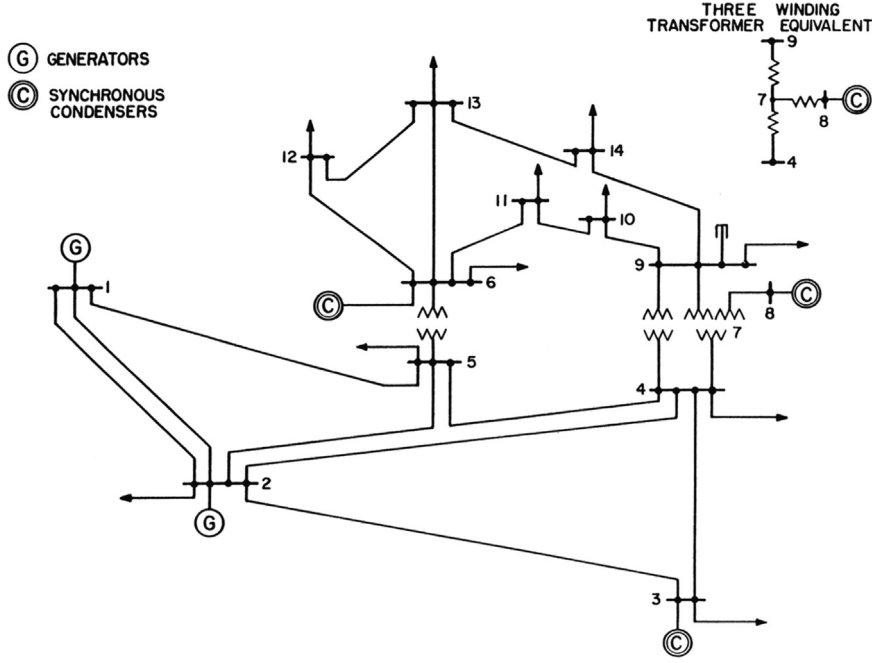


Fig. 4. IEEE 14-bus test system.

and the edges connecting the nodes represent the transmission lines.

For each bus i , denote the total active power injection by P_i and the total reactive power injection by Q_i . Let V_i be its voltage magnitude and let θ_i be its phase angle. For the phase, bus n will be considered the reference bus, and hence we assume $\theta_n = 0$. Then, at bus k , the total complex currents injected is written as

$$I_i = \sum_{k \in \mathcal{N}_i} Y_{ik} V_k^{j\theta_k}, \quad (15)$$

where $\mathcal{N}^i$ denotes the set of bus indices neighboring bus i , Y_{ik} is the admittance of the transmission line connecting bus i and k . Based on this expression, the complex power injected at bus i , that is, the difference between the power generated and consumed, becomes

$$P_i + jQ_i = V_i^{j\theta_i} I_i^*, \quad (16)$$

where $*$ indicates the complex conjugate. Now, observe that by substituting (15) into (16), this grid model is fully described by the voltage magnitude and the phase angles. Hence, as the state vector, we introduce $\tilde{x} = [V_1 \dots V_n \theta_1 \dots \theta_{n-1}]^T \in \mathbb{R}^{2n-1}$.

Let $\tilde{z}$ be the $\tilde{m}$ -dimensional measurement vector with $\tilde{m} > 2n-1$. The output equation can be written as

$$\tilde{z} = h(\tilde{x}) + \tilde{v}, \quad (17)$$

where $h(\cdot)$ is a nonlinear vectorial function and $\tilde{v} \in \mathbb{R}^{\tilde{m}}$ is a Gaussian noise whose entries are independent with mean 0 and covariance matrix $\tilde{R} > 0$.

In the state estimation problem, we find the state x from the relation in (17). This problem can in general be solved via the weighted least squares method. Estimation based on least squares is commonly accepted since it can be efficiently computed even for large-scale systems by, for example, the Newton–Gauss method. In practice, however, this can still take time and may not be as fast as required for some real-time control and monitoring purposes.

So, while the full state estimation above is carried out, an approximated version of the state is computed through linearization. In particular, the so-called DC model is employed under the assumption that the voltage magnitudes are at the reference (unit)

value, $V_i \equiv 1$ for all i . This linearization is based on the Jacobian matrix $H \in \mathbb{R}^{m \times (n-1)}$ of $h(\cdot)$ (consisting of only the relevant entries). Then the phase angle vector $x = [\theta_1 \dots \theta_{n-1}]^T \in \mathbb{R}^{n-1}$ is modeled by

$$z = Hx + v,$$

where $z \in \mathbb{R}^m$ is the measurement with $m > n$ and $v \in \mathbb{R}^m$ is the Gaussian noise with 0 mean and covariance matrix R . Clearly, we must assume observability in the form that the Jacobian matrix H has column full rank. It should be noted that due to the power network structure, this matrix typically is sparse with a limited number of nonzero entries.

Under the DC model, the state estimation problem in the least squares setting can be formulated by

$$x^* = \arg \min_x (z - Hx)^T R^{-1} (z - Hx). \quad (18)$$

This problem has a closed-form solution given by

$$x^* = L^{-1}u, \quad (19)$$

where $L = H^T R^{-1} H$ and $u = H^T R^{-1} z$. Due to the assumption on H , the matrix L is nonsingular.

One way to compute this solution x^* is through the gradient-based iterative algorithm (e.g., [7]) given by

$$x(k+1) = (I - \tau L)x(k) + \tau u, \quad (20)$$

where $x(k) \in \mathbb{R}^{n-1}$ is the estimated state at time $k \in \mathbb{Z}_{\geq 0}$ and the parameter τ is selected from the interval $(0, 2\|L\|^{-1})$; such a τ guarantees the matrix $I - \tau L$ to be Schur stable.

4.2. Distributed algorithm based on grouping

We now consider the distributed computation of the state estimates. We start with grouping of the nodes by partitioning $\mathcal{V}$ into N groups $\mathcal{V}_i \subset \mathcal{V}$ for $i \in \tilde{\mathcal{V}} = \{1, 2, \dots, N\}$, that is, $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$ if $i \neq j$ and $\mathcal{V}_i \neq \emptyset$ for all i . With a slight abuse of notation, we reorder the entries in the vectors x , z , and u and then partition them

accordingly as

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_N \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix}, \quad (21)$$

where $x_i \in \mathbb{R}^{n_i}$ with $n_i = |\mathcal{V}_i|$, $z_i \in \mathbb{R}^{m_i}$, and $u_i \in \mathbb{R}^{n_i}$ consist of the states, the measurements, and the input vector (from (20)) of nodes $j \in \mathcal{V}_i$ for each group i . Similarly, we partition the Jacobian matrix H appearing in (18) into submatrices H_{ij} whose sizes are $m_i \times n_j$.

The notion of neighbors must be introduced based on the structure of the matrix H in a slightly more general fashion than that for the original undirected graph $\mathcal{G}$. Denote by $\tilde{\mathcal{G}}$ the graph induced by the partitioned matrix H whose node set is $\tilde{\mathcal{V}}$; its edge set $\tilde{\mathcal{E}}$ is determined via $(i, j) \in \tilde{\mathcal{E}}$ if and only if $H_{ij} \neq 0$. Since the partitioned matrix H does not in general reflect the original graph $\mathcal{G}$, the induced graph $\tilde{\mathcal{G}}$ is directed. For each group $i \in \tilde{\mathcal{V}}$, the set of neighboring groups with outgoing edges is given by $\tilde{\mathcal{N}}_i = \{j \in \tilde{\mathcal{V}} : H_{ji} \neq 0\}$; such groups are called the out-neighbors. On the other hand, the set of its neighbors with incoming edges, i.e., its in-neighbors, is denoted by $\tilde{\mathcal{M}}_i = \{j \in \tilde{\mathcal{V}} : H_{ij} \neq 0\}$. We assume that group i has the knowledge of both $\tilde{\mathcal{N}}_i$ and $\tilde{\mathcal{M}}_i$.

In what follows, we outline the distributed randomized algorithm for the computation of the least square estimate in (19). Each group $i \in \tilde{\mathcal{V}}$ keeps a triple of states $(x_i, \kappa_i, \bar{x}_i)$. Their roles can be explained as follows: $x_i(k)$ is the basic estimate of the partial state x_i^* for the nodes in group i at time k ; $\kappa_i(k)$ keeps the number of updates made by the group so far by time k ; and $\bar{x}_i(k)$ is the smoothed version of the estimate obtained by time averaging. For the randomization, let $\eta(k) \in \tilde{\mathcal{V}}$ be the random variable specifying the group initiating the update by sending its own state to its neighbors at time k .

As the initialization stage, the following steps are performed at each group i : the initial states are set as $(x_i, \kappa_i, \bar{x}_i) = (0, 0, 0)$. The input vector u_i is computed by collecting the measurements z_ℓ from its out-neighbors $\ell \in \tilde{\mathcal{N}}_i$ as

$$u_i = (H^\top R^{-1} z)_i = \sum_{\ell \in \tilde{\mathcal{N}}_i} H_{\ell i}^\top R_\ell^{-1} z_\ell.$$

Then, at time $k \in \mathbb{Z}_{\geq 0}$, one group $j \in \tilde{\mathcal{V}}$ is chosen to initiate updates; this is specified by $\eta(k) = j$, where j is randomly chosen with a uniform probability distribution.

This group j sends its current estimate $x_j(k)$ to its out-neighbors $\ell \in \tilde{\mathcal{N}}_j$. Then, group ℓ computes the vector $y_\ell^{(ij)} := H_{\ell i}^\top R_\ell^{-1} H_{\ell j} x_j(k)$. This is further sent to its in-neighbors $i \in \tilde{\mathcal{M}}_\ell$ whose states are updated by

$$x_i(k+1) = x_i(k) - \tau \sum_{\ell \in \tilde{\mathcal{N}}_i \cap \tilde{\mathcal{N}}_j} y_\ell^{(ij)} + \tau u_i, \quad i \in \tilde{\mathcal{M}}_\ell, \quad \ell \in \tilde{\mathcal{N}}_j. \quad (22)$$

For other nodes, their states remain unchanged as

$$x_i(k+1) = x_i(k), \quad i \notin \tilde{\mathcal{M}}_\ell, \quad \ell \in \tilde{\mathcal{N}}_j. \quad (23)$$

The local time is updated via

$$\kappa_i(k+1) = \begin{cases} \kappa_i(k) + 1 & \text{if } i \in \tilde{\mathcal{M}}_\ell, \quad \ell \in \tilde{\mathcal{N}}_j, \\ \kappa_i(k) & \text{otherwise.} \end{cases} \quad (24)$$

The time averages of the states are computed as

$$\bar{x}_i(k+1) = \begin{cases} \frac{1}{\kappa_i(k+1)} (\kappa_i(k) \bar{x}_i(k) + x_i(k+1)) & \text{if } i \in \tilde{\mathcal{M}}_\ell, \quad \ell \in \tilde{\mathcal{N}}_j, \\ \bar{x}_i(k) & \text{otherwise.} \end{cases} \quad (25)$$

As we see above, this algorithm is fully asynchronous in the sense that the groups need not maintain their clocks to be synchronized due to the local counter $\kappa_i(k)$ employed here for

computing the state time average $\bar{x}(k)$. This aspect is in contrast with those introduced in the previous two sections, where the time average requires the global time k .

For this asynchronous distributed algorithm, we can establish an ergodicity property based on [64] as stated in the following theorem. Analysis of the average dynamics of this algorithm can be performed similarly to that carried out in the previous section.

Theorem 3 (Least squares convergence). *Let us consider the dynamics Eqs. (22)–(25). Then, the time-average sequence $\{\bar{x}_i(k)\}_{k \in \mathbb{Z}_{\geq 0}}$ of each group $i \in \tilde{\mathcal{V}}$ is such that $\lim_{k \rightarrow \infty} \bar{x}_i(k) = x_i^*$ almost surely and in mean square sense.*

As we described in the introduction of this section, in the context of the power systems state estimation problem, it is reasonable and common to partition the original grid graph $\mathcal{G}$ into groups and then place local estimators at the group level. This setting brings three differences in the algorithm compared to those in the previous sections. First, though the original graph $\mathcal{G}$ is undirected, the underlying graph $\tilde{\mathcal{G}}$ among the group nodes is directed. This is due to the structure in the matrix H representing the relation from the states to the measurements. Second, the matrix H has the consequence that even though only one group initiates an update, multiple nodes are affected via the two-hop communication involving intermediate nodes as can be seen in (22). This aspect is somewhat hidden if we only look at the update scheme in (20) based on the matrix L . To see this point more clearly, from the centralized scheme we can easily write out the iteration for the state corresponding to group i . Let $L_{ij} \in \mathbb{R}^{n_i \times n_j}$ be the (ij) block of the matrix L when partitioned in accordance with the state vector x_i s in (21). Then, the centralized update scheme (20) is expressed as

$$\begin{aligned} x_i(k+1) &= x_i(k) - \tau L_{ij} x_j(k) + \tau u_i \\ &= x_i(k) - \tau \sum_{\ell=1}^N H_{\ell i}^\top R_\ell^{-1} H_{\ell j} x_j(k) + \tau u_i \\ &= x_i(k) - \tau \sum_{\ell \in \tilde{\mathcal{N}}_i \cap \tilde{\mathcal{N}}_j} H_{\ell i}^\top R_\ell^{-1} H_{\ell j} x_j(k) + \tau u_i. \end{aligned}$$

In comparison, observe that in the distributed scheme (22), randomization is made to select only one group j , which will then trigger the updates. Finally, as already mentioned, the local clock $\kappa_i(k)$ may be used in this problem for the time averaging. This scheme has the advantage that not only the clocks need not be synchronized, but also the updates at each group can take place at random times without any fixed sampling period.

4.3. Simulation

We illustrate the proposed distributed randomized algorithm through a numerical example using the IEEE 14-bus system in Fig. 4.

First, the grid is partitioned into four groups as shown in Fig. 5. The buses are grouped as $\mathcal{V}_1 = \{1, 2, 5\}$, $\mathcal{V}_2 = \{3, 4, 7, 8\}$, $\mathcal{V}_3 = \{6, 11, 12, 13\}$, and $\mathcal{V}_4 = \{9, 10, 14\}$. There are in total 27 measurements. Among the sensors, there are three PMUs located at buses 1, 4, and 12, indicated by the gray nodes in Fig. 5. Each group has between 5 and 8 measurements, and these include the power flows over internal lines within the group, the power injections at the buses (except for bus 5 in group 1), and the PMUs. The graph induced by the partitioned matrix H is shown in Fig. 6, which is a directed graph for this case. For measurements obtained from PMUs, Gaussian noises are added with zero means and standard deviations of .02. For other measurements, the standard deviations of the Gaussian noises are set as .66 of the absolute value of the measured value plus a fixed bias of 1.6×10^{-3} as considered in the commonly used toolbox Matpower [79] for Matlab.

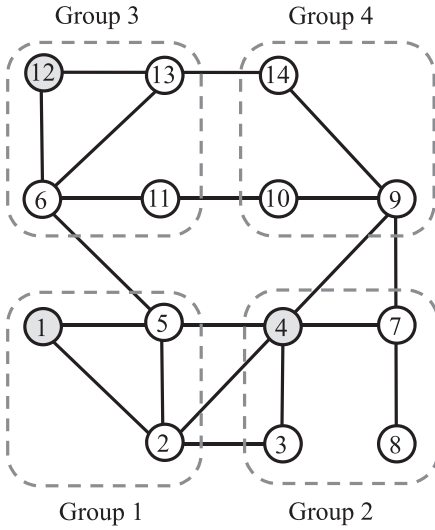


Fig. 5. IEEE 14-bus test system with grouping.

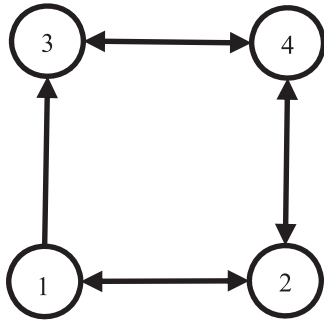


Fig. 6. The induced graph $\tilde{G}$ representing the relation among groups through the partitioned matrix H .

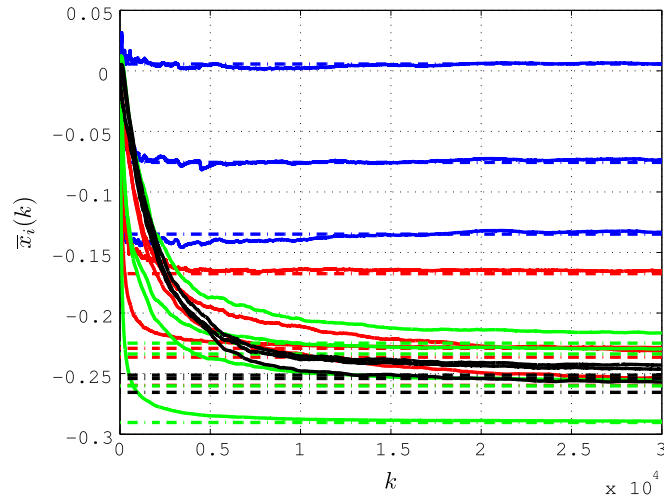


Fig. 7. Time responses of the time-average states (in solid lines) and the least-squares values (in dash-dot horizontal lines) for 14 buses. The colored lines indicate the groups to which the buses belong. (For interpretation of the references to color in this figure caption, the reader is referred to the web version of this paper.)

The time responses of the average states $\bar{x}_i$ are shown in Fig. 7 as the solid lines. There, four colors, red, blue, green, and black are used to indicate the phase angles of buses in the four groups. The dash-dot horizontal lines indicate the least-squares solution x_i^* . It is clear that the time average asymptotically reaches the desired values.

5. Conclusion

In this paper, we have studied a general framework which has the objective to construct algorithms, based on the interplay of randomization and time-averaging, for distributed computation over a network. Several open problems will be the objective of future research, including in particular the development of similar algorithms for other systems and control applications, such as sensor localization and clock-synchronization, and the derivation of fully-distributed algorithms for the computation of centrality measures in complex networks.

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