

# STABILIZATION OF THE WAVE EQUATION WITH MOVING BOUNDARY

KAÏS AMMARI, AHMED BCHATNIA, AND KARIM EL MUFTI

**ABSTRACT.** We deal with the wave equation with assigned moving boundary ( $0 < x < a(t)$ ) upon which Dirichlet-Neuman boundary conditions are satisfied, here  $a(t)$  is assumed to move slower than the light and periodically. We give a feedback which guarantees the exponential decay of the energy. The proof relies on a reduction theorem [1, 14]. At the end we give a remark on the moving-pointwise stabilization problem.

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## 1. INTRODUCTION AND MAIN RESULT

In this note, we analyze the stabilization property of solutions for the wave equation with a moving boundary. More precisely, we consider the following system:

$$(1.1) \quad \begin{cases} u_{tt} - u_{xx} = 0 & \text{for } 0 < x < a(t), t > 0, \\ u(0, t) = 0 & \text{and } u_t(a(t), t) + f(t)u_x(a(t), t) = 0, t > 0, \\ u(x, 0) = \phi(x), u_t(x, 0) = \psi(x), & 0 < x < a(0), \end{cases}$$

$(\phi, \psi) \in H_l^1((0, a(0))) \times L^2((0, a(0)))$ , where

$$H_l^1((0, a(0))) = \{v \in H^1((0, a(0))), v(0) = 0\}.$$

Here  $a$  is a strictly positive real function which is continuous, 1-periodic and  $f \in L^\infty(\mathbb{R}_+^*)$ .

Denote by

$$E_u(t) = \frac{1}{2} \int_0^{a(t)} \left[ |u_t(x, t)|^2 + |u_x(x, t)|^2 \right] dx$$

the energy of the field  $u$ . Our major concern will be to detect the feedback  $f(t)$  necessary to obtain the exponential decay of  $E_u(t)$ .

We start with some notations and known results. Let  $\text{Lip}(\mathbb{R})$  be the space of Lipschitz continuous functions on  $\mathbb{R}$ . We shall denote the Lipschitz constant of a function  $F$  by

$$L(F) := \sup_{x, y \in \mathbb{R}, x \neq y} \left| \frac{F(x) - F(y)}{x - y} \right|.$$

On the existence of solutions to the system (1.1), we refer the reader to [9]. We have the following proposition:

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**Proposition 1.1.** *If  $a \in \text{Lip}(\mathbb{R})$ ,  $L(a) \in [0, 1)$ ,  $a > 0$ ,  $f \in L^\infty(\mathbb{R}_+^*)$  and  $(\phi, \psi) \in H_l^1((0, a(0))) \times L^2((0, a(0)))$ , denote by  $Q := (0, a(t)) \times \mathbb{R}_+$  and  $Q_\tau := (0, a(t)) \times (0, \tau)$ ,  $\tau \in \mathbb{R}_+$ . There exists a unique weak solution<sup>1</sup>  $u$  of the system (1.1). Moreover there exists  $h \in H_{loc}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$  such that*

$$(1.2) \quad u(x, t) = h(t + x) - h(t - x) \quad \text{a.e. in } Q,$$

and  $u \in L^\infty(Q) \cap H^1(Q_\tau)$ .

We denote by  $D_p$  the set of continuous functions and strictly increasing of the form  $x + g(x)$ , where  $g(x)$  is a 1-periodic continuous function.

We recall the following results.

**Proposition 1.2.** ([10, Herman] and [12, Yamaguchi]) *Let  $a$  be a 1-periodic function. Then*

$$(1.3) \quad F := (I + a) \circ (I - a)^{-1}$$

belongs to  $D_p$ . Moreover, the rotation number  $\rho(F)$  defined by

$$\rho(F) = \lim_{n \rightarrow \infty} \frac{F^n(x) - x}{n}$$

exists, and the limit is equal for all  $x \in \mathbb{R}$ .

After, we construct a transformation of the time-dependent domain  $[0, a(t)] \times \mathbb{R}$  onto  $[0, \rho(F)/2] \times \mathbb{R}$  that preserves the D'Alembertian form of the wave equation. For this purpose, we use the following proposition:

**Proposition 1.3.** ([10, Herman, section II]) *Assume that  $a(t)$  is a 1-periodic function,  $a(t) > 0$ ,  $a \in \text{Lip}(\mathbb{R})$  such that  $L(a) \in [0, 1)$ . Assume also that  $|a'(t)| < 1$  for all  $t \in \mathbb{R}$  and  $\rho(F) \in \mathbb{R} \setminus \mathbb{Q}$  such that there exists a function  $H \in D_p$  and*

$$(1.4) \quad H^{-1} \circ F \circ H(\xi) = \xi + \rho(F).$$

Our main result is stated now as follows:

**Theorem 1.4** (Exponential stability). *Let*

$$(1.5) \quad f(t) = \frac{(\mu - 1)H'(a(t) + t) + (\mu + 1)H'(-a(t) + t)}{(1 - \mu)H'(a(t) + t) + (\mu + 1)H'(-a(t) + t)}$$

where  $\mu$  is a nonnegative constant and assume that there exist  $\lambda_1 > 0$  and  $\lambda_2 > 0$  such that

$$(1.6) \quad \lambda_1 \leq H'(t) \leq \lambda_2, \quad t \in \mathbb{R}.$$

Then, in the case where  $\mu \neq 1$ , there exists a positive constant  $C$  such that

$$(1.7) \quad E_u(t) \leq Ce^{-\omega t} E_u(0),$$

for every solution  $u$  of (1.1) with initial data  $(\phi, \psi) \in H_l^1((0, a(0))) \times L^2((0, a(0)))$

and where  $\omega = \ln \left( \frac{1 + \mu}{1 - \mu} \right)$ .

In the case  $\mu = 1$  which corresponds to  $f(t) = 1$ , we obtain

$$(1.8) \quad E_u(t) = 0, \quad \text{for all } t \geq T_0 =: (I + a)^{-1} \circ H^{-1} \left( \frac{3\rho(F)}{2} \right).$$

We give an example where assumption (1.6) is guaranteed (see [1, 8] for more details).

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<sup>1</sup> $u \in H^1(Q_\tau)$  is called a weak solution of (1.1) if  $u_{tt} - u_{xx} = 0$  in  $\mathcal{D}'(Q)$  and the boundary conditions are satisfied.

**Example 1.5.** Let  $a$  be continuous and 1-periodic on  $\mathbb{R}$ ,  $a > 0$ , be such that

$$(1.9) \quad a(t) := \begin{cases} \alpha t + \frac{\alpha(1-\alpha)(1+\beta)}{2(\alpha-\beta)} & \text{if } \frac{\alpha(1+\beta)}{2(\alpha-\beta)} \leq t \leq \frac{\alpha(1+\beta)-2\beta}{2(\alpha-\beta)}, \\ \beta t - \beta + \frac{\alpha(1-\beta^2)}{2(\alpha-\beta)} & \text{if } \frac{\alpha(1+\beta)-2\beta}{2(\alpha-\beta)} \leq t \leq \frac{\alpha(3+\beta)-2\beta}{2(\alpha-\beta)}, \end{cases}$$

with  $-1 < \beta < 0 < \alpha < 1$ . The function  $F$  is defined by:

$$F(x) := (I + a) \circ (I - a)^{-1}(x) = \begin{cases} l_1 x + F_0 & \text{if } 0 \leq x \leq x_0, \\ l_2 x + F_0 + 1 - l_2 & \text{if } x_0 < x < 1, \end{cases}$$

with  $l_1 := \frac{1+\alpha}{1-\alpha}$ ,  $l_2 := \frac{1+\beta}{1-\beta}$ ,  $F_0 := \frac{l_2(l_1-1)}{l_1-l_2}$  and  $x_0 := \frac{1-l_2}{l_1-l_2}$ .

We extend  $F$  through the formula:  $F(x+1) = F(x) + 1$  for any  $x \in \mathbb{R}$ . Also the rotation number is given by the expression:

$$(1.10) \quad \rho(F) = \frac{\ln l_1}{\ln\left(\frac{l_1}{l_2}\right)},$$

and the function  $H$  given by (1.4) is done by

$$H(x) = h_0 \ln(|x + h_1|) + h_2,$$

where  $h_0 = \frac{1}{\ln\left(\frac{l_1}{l_2}\right)}$ ,  $h_1 = \frac{l_2}{l_1-l_2}$  and  $h_2 = -\ln(h_1)$ .  $H$  satisfies the inequalities:

$$(1.11) \quad \frac{1}{\ln\left(\frac{l_1}{l_2}\right)} \frac{l_1 - l_2}{l_1} \leq H'(x) \leq \frac{1}{\ln\left(\frac{l_1}{l_2}\right)} \frac{l_1 - l_2}{l_2}.$$

Here  $f(t) = \frac{2a(t) + 2\mu t + 2\mu h_1}{2\mu a(t) + 2t + 2h_1}$  and  $T_0 = (I + a)^{-1} \circ H^{-1}\left(\frac{3\rho(F)}{2}\right)$ .

**Remark 1.6.** For the moving pointwise stabilization of the wave equation, see section 3.

## 2. PROOF OF THE MAIN RESULT

Before starting the proof, we begin by defining a domain transformation

$$\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2,$$

using  $H$  given by (1.4), as follows:

$$(2.12) \quad \begin{cases} \xi = (H(x+t) - H(-x+t))/2, \\ \tau = (H(x+t) + H(-x+t))/2, \end{cases}$$

for  $(x, t) \in \mathbb{R}^2$ .

**Proposition 2.1.** ([13, Yamaguchi]) The transformation  $\Phi$  is a bijection of  $[0, a(t)] \times \mathbb{R}$  to  $[0, \rho(F)/2] \times \mathbb{R}$  and  $\Phi$  maps the boundaries  $x = 0$  and  $x = a(t)$  onto the boundaries  $\xi = 0$  and  $\xi = \rho(F)/2$ .

**Proposition 2.2.** ([13, Yamaguchi]) Let  $u(x, t)$  satisfying  $(\partial_t^2 - \partial_x^2)u(x, t) = 0$  and  $V(\xi, \tau)$  defined by  $u(\Phi^{-1}(\xi, \tau))$ . Then the following identity holds

$$(\partial_t^2 - \partial_x^2)u(x, t) = K(\xi, \tau)(\partial_\tau^2 - \partial_\xi^2)V(\xi, \tau)$$

where  $K(\xi, \tau)$  is defined by

$$4H' \circ H^{-1}(\xi + \tau)H' \circ H^{-1}(-\xi + \tau) \circ H^{-1}(\xi + \tau).$$

Now we consider the system:

$$(2.13) \quad \begin{cases} u_{tt} - u_{xx} = 0 & \text{for } 0 < x < a(t), t > 0, \\ u(0, t) = 0 & \text{and } u_t(a(t), t) + f(t)u_x(a(t), t) = 0, t > 0, \\ u(x, 0) = \phi_1(x), u_t(x, 0) = \psi_1(x), & 0 < x < a(0) \end{cases}$$

where  $f(t) = \frac{(\mu-1)H'(a(t)+t) + (\mu+1)H'(-a(t)+t)}{(1-\mu)H'(a(t)+t) + (\mu+1)H'(-a(t)+t)}$  and  $\mu$  is a nonnegative constant.

**Proposition 2.3.** *The transformation of the system (2.13) is*

$$(2.14) \quad \begin{cases} v_{\tau\tau} - v_{\xi\xi} = 0 & \text{for } 0 < \xi < \rho(F)/2, \tau > 0, \\ v(0, \tau) = 0 & \text{and } v_{\tau}(\rho(F)/2, \tau) + \mu v_{\xi}(\rho(F)/2, \tau) = 0, \tau > 0, \\ v(\xi, 0) = \phi(\xi), v_{\tau}(\xi, 0) = \psi(\xi), & 0 < \xi < \rho(F)/2. \end{cases}$$

*Proof.* We have:

$$(2.15) \quad \begin{cases} u_x(a(t), t) = V_{\xi}(\rho(F)/2, \tau) \xi_x(a(t), t) + V_{\tau}(\rho(F)/2, \tau) \tau_x(a(t), t), \\ u_t(a(t), t) = V_{\xi}(\rho(F)/2, \tau) \xi_t(a(t), t) + V_{\tau}(\rho(F)/2, \tau) \tau_t(a(t), t). \end{cases}$$

Starting from (2.13) and make use of:

$$\xi_x = (\partial_x \xi) = (\partial_t \tau) = \tau_t = [H'(x+t) + H'(-x+t)]/2,$$

$$\xi_t = (\partial_t \xi) = (\partial_x \tau) = \tau_x = [H'(x+t) - H'(-x+t)]/2,$$

we conclude:

$$u_t(a(t), t) + \frac{\xi_t - \mu \tau_t}{\mu \tau_x - \xi_x} u_x(a(t), t) = 0.$$

So we have

$$\xi_t u_x(a(t), t) - \xi_x u_t(a(t), t) + \mu(\tau_x u_t(a(t), t) - \tau_t u_x(a(t), t)) = 0.$$

Finally we get:  $v_{\tau}(\rho(F)/2, \tau) + \mu v_{\xi}(\rho(F)/2, \tau) = 0$ .

Note that  $f(t) = 1$  in the special case  $\mu = 1$ . □

The next lemma shows that the energy of the solution (1.1) and the energy of the corresponding static system are equivalent.

**Lemma 2.4.** ([1, Ammari et al.]) *Under the assumption (1.6), there are two positive constants  $C_1$  and  $C_2$  such that*

$$(2.16) \quad C_1 E_V \left( H(a(t)+t) - \frac{\rho(F)}{2} \right) \leq E_u(t) \leq C_2 E_V \left( H(a(t)+t) - \frac{\rho(F)}{2} \right),$$

where  $E_V(\tau)$  is the energy of the field  $V$  defined by:

$$E_V(\tau) = \int_0^{\rho(F)/2} (|V_{\xi}(\xi, \tau)|^2 + |V_{\tau}(\xi, \tau)|^2) d\xi.$$

Note that we can write  $H(a(t)+t) = a(t)+t+g(a(t)+t)$ , where  $g(x)$  is a 1-periodic continuous function.

Finally, the stabilization of the system (1.1) is a direct combination of Proposition 2.3, Lemma 2.4 and the following Lemma 2.5.

**Lemma 2.5.** ([7, Cox and Zuazua]) *For  $\mu \neq 1$ , there exists a positive constant  $C$  such that*

$$(2.17) \quad E_V(\tau) \leq C e^{-\ln(|\frac{1+\mu}{1-\mu}|) \tau} E_V(0), \forall \tau > 0,$$

where  $V$  is the solution of the following system:

$$(2.18) \quad \begin{cases} V_{\tau\tau} - V_{\xi\xi} = 0 & \text{for } 0 < \tau < \rho(F)/2, \tau > 0, \\ V(0, \tau) = 0 & \text{and } V_{\tau}(\rho(F)/2, \tau) + \mu V_{\xi}(\rho(F)/2, \tau) = 0, \tau > 0, \\ V(\xi, 0) = \phi(\xi), V_{\tau}(\xi, 0) = \psi(\xi), & 0 < \xi < \rho(F)/2. \end{cases}$$

It is well known that for  $\mu = 1$ ,  $E_V(t) = 0$  for all  $t \geq \rho(F)$ , see [7] for more details.

## 3. MOVING POINTWISE STABILIZATION

We consider the following problem:

$$(3.19) \quad \begin{cases} u_{tt} - u_{xx} + [f_1(t)u_t + f_2(t)u_x]\delta_{a(t)} = 0 & \text{for } 0 < x < b(t), t > 0, \\ u(0, t) = 0 & \text{and } u(b(t), t) = 0, t > 0, \\ u(x, 0) = \phi_1(x), u_t(x, 0) = \psi_1(x) & 0 < x < b(0). \end{cases}$$

The aim of this section is to determine the functions  $f_1$ ,  $f_2$  and  $b$  to get after transformation the vibrations of a string with the static pointwise damping and conclude the asymptotic behavior of the energy.

**Proposition 3.1.** *The transformation of the system:*

$$(3.20) \quad \begin{cases} u_{tt} - u_{xx} + K \left( \frac{H(a(t)+t)-H(-a(t)+t)}{2}, \frac{H(a(t)+t)+H(-a(t)+t)}{2} \right) \\ \cdot \left[ \left( \frac{1}{H'(a(t)+t)} - \frac{1}{H'(-a(t)+t)} \right) u_t + \left( \frac{1}{H'(a(t)+t)} + \frac{1}{H'(-a(t)+t)} \right) u_x \right] \delta_{a(t)} = 0 \\ \text{for } 0 < x < b(t) = \Lambda_t^{-1}(1) - t, t > 0, \\ u(0, t) = 0 \quad \text{and} \quad u(b(t), t) = 0, t > 0, \\ u(x, 0) = \phi_1(x), u_t(x, 0) = \psi_1(x), \quad 0 < x < b(0) \end{cases}$$

is

$$(3.21) \quad \begin{cases} v_{\tau\tau} - v_{\xi\xi} + v_{\tau} \delta_{\frac{\rho(F)}{2}} = 0 & \text{for } 0 < \xi < 1, \tau > 0, \\ v(0, \tau) = 0 \quad \text{and} \quad v(1, \tau) = 0, \tau > 0, \\ v(\xi, 0) = \phi(\xi), v_{\tau}(\xi, 0) = \psi(\xi), \quad 0 < \xi < 1, \end{cases}$$

where  $\Lambda_t$  is defined by  $\Lambda_t(y) = \frac{H(y+t)-H(-y+t)}{2}$ .

*Proof.* We recall that if  $u(x, t)$  satisfying  $(\partial_t^2 - \partial_x^2)u(x, t) = 0$  and  $v(\xi, \tau)$  defined by  $u(\Phi^{-1}(\xi, \tau))$ . Then the following identity holds

$$(3.22) \quad (\partial_t^2 - \partial_x^2)u(x, t) = K(\xi, \tau)(\partial_{\tau}^2 - \partial_{\xi}^2)v(\xi, \tau)$$

where  $K(\xi, \tau)$  is defined by

$$4H' \circ H^{-1}(\xi + \tau)H' \circ H^{-1}(-\xi + \tau) \circ H^{-1}(\xi + \tau).$$

On the other hand, make use of (2.15) we obtain:

$$(3.23) \quad \begin{aligned} v_{\tau} &= \frac{\xi_t u_x - \xi_x u_t}{\xi_t^2 - \xi_x^2} \\ &= \left( \frac{1}{H'(a(t)+t)} - \frac{1}{H'(-a(t)+t)} \right) u_t + \left( \frac{1}{H'(a(t)+t)} + \frac{1}{H'(-a(t)+t)} \right) u_x. \end{aligned}$$

We combine (3.22) and (3.23) to obtain the result of the Proposition 3.1.  $\square$

**Remark 3.2.** *If we return to the Example 1.5 and after some computation we get that the transformation of the system:*

$$(3.24) \quad \begin{cases} u_{tt} - u_{xx} + \left( \frac{8}{a(t)+t+h_1} \cdot \frac{a(t)+t}{-a(t)+t+h_1} \right) \cdot [a(t)u_t + (t+h_1)u_x] \delta_{a(t)} = 0 \\ \text{for } 0 < x < b(t) = (t+h_1)e^{\frac{h_2}{h_0}} \tanh\left(\frac{1}{h_0}\right), t > 0, \\ u(0, t) = 0 \quad \text{and} \quad u(b(t), t) = 0, t > 0, \\ u(x, 0) = \phi_1(x), u_t(x, 0) = \psi_1(x), \quad 0 < x < b(0), \end{cases}$$

is the system (3.21).

As above and according to [4, 5] we have the following:

- $\lim_{t \rightarrow +\infty} E_u(t) = 0, \forall (\phi_1, \psi_1) \in H_0^1(0, b(0)) \times L^2(0, b(0)) \Leftrightarrow \rho(F) \notin \mathbb{Q}$ .
- For any  $\rho(F)/2 \in (0, 1)$  the system (3.20) is not exponentially stable in  $H_0^1(0, b(0)) \times L^2(0, b(0))$ .

- For all  $\rho(F)/2 \in \mathcal{S}$ <sup>2</sup> and for all  $(\phi_1, \psi_1) \in \mathcal{D}$  we have according to [2, 3] that there exists  $C > 0$  such that:

$$E_u(t) \leq \frac{C}{t}, \forall t > 0.$$

- If  $\varepsilon > 0$  then, for almost all  $\rho(F)/2 \in (0, 1)$  and for all  $(\phi_1, \psi_1) \in \mathcal{D}$  we have according to [2, 3] that there exists  $C > 0$  such that:

$$E_u(t) \leq \frac{C}{t^{1+\varepsilon}}, \forall t > 0,$$

where

$$\mathcal{D} := \{(\varphi, \psi) \in [H^2(0, b(0)) \cap H_0^1(0, b(0))] \times H_0^1(0, b(0)), \\ \left( \frac{1}{H'(a(0))} - \frac{1}{H'(-a(0))} \right) \psi(a(0)) + \left( \frac{1}{H'(a(0))} + \frac{1}{H'(-a(0))} \right) \frac{d\varphi}{dx}(a(0)) = 0 \}.$$

#### REFERENCES

- [1] K. Ammari, A. Bchatnia and K. El Mufti, A remark on observability of the wave equation with moving boundary, *Journal of Applied Analysis*, in press, **1** (2017).
- [2] K. Ammari and M. Tucsnak, Stabilization of second order evolution equations by a class of unbounded feedbacks, *ESAIM Control Optim. Calc. Var.*, **6** (2001), 361–386.
- [3] K. Ammari and S. Nicaise, *Stabilization of elastic systems by collocated feedback*, Lecture Notes in Mathematics, 2124. Springer, Cham, 2015.
- [4] K. Ammari, A. Henrot and M. Tucsnak, Asymptotic behaviour of the solutions and optimal location of the actuator for the pointwise stabilization of a string, *Asymptot. Anal.*, **28** (2001), 215–240.
- [5] ———, Optimal location of the actuator for the pointwise stabilization of a string, *C. R. Acad. Sci. Paris Sr. I Math.*, **330** (2000), 275–280.
- [6] J. W. S. Cassals, *An introduction to Diophantine Approximation*, Cambridge University Press, Cambridge (1966).
- [7] S. Cox and E. Zuazua, The Rate at which energy decays in a string damped at one end, *Comm. Partial Differential Equations.*, **19** (1994), 213–243.
- [8] N. Gonzalez, An example of pure stability for the wave equation with moving boundary, *J. Math. Anal. Appl.*, **228** (1998), 51–59.
- [9] ———, *L'équation des ondes dans un domaine dépendant du temps*, Ph.D. Thesis, University of Toulon and Czech Technical University, 1997.
- [10] M. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, *Publ. Math. I.H.E.S.*, **49** (1979), 5–234.
- [11] S. Lang, *Introduction to diophantine approximations*, Addison Wesley, New York (1966).
- [12] M. Yamaguchi, Periodic solutions of nonlinear equations of string with periodically oscillating boundaries, *Funkcialaj. Ekvacioj.*, **45** (2002), 397–416.
- [13] M. Yamaguchi and H. Yoshida, Nonhomogeneous string problem with periodically moving boundaries, *Fields Inst. Commun.*, **25** (2000), 565–574.
- [14] J. C. Yoccoz, Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne, *Ann. Sci. École Norm. Sup.*, **17** (1984), 333–359.

UR ANALYSIS AND CONTROL OF PDEs, UR13ES64, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF MONASTIR, UNIVERSITY OF MONASTIR, 5019 MONASTIR, TUNISIA  
E-mail address: `kais.ammari@fsm.rnu.tn`

UR ANALYSE NON-LINÉAIRE ET GÉOMÉTRIE, UR13ES32, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF TUNIS, UNIVERSITY OF TUNIS EL MANAR, TUNISIA  
E-mail address: `ahmed.bchatnia@fst.utm.tn`

UR ANALYSIS AND CONTROL OF PDEs, UR13ES64, ISCAE, UNIVERSITY OF MANOUBA, TUNISIA  
E-mail address: `karim.elmufti@iscae.rnu.tn`

<sup>2</sup>Denote by  $\mathcal{S}$  the set of all numbers  $\xi \in (0, 1)$  such that  $\xi \notin \mathbb{Q}$  and if  $[0, a_1, \dots, a_n, \dots]$  is the expansion of  $\xi$  as a continued fraction, then  $(a_n)$  is bounded. Let us notice that  $\mathcal{S}$  is obviously uncountable and, by classical results on diophantine approximation (cf. [6], p. 120), its Lebesgue measure is equal to zero. In particular, by EulerLagrange theorem (see Lang [11], p. 57)  $\mathcal{S}$  contains all  $\xi \in (0, 1)$  such that  $\xi$  is an irrational quadratic number (i.e. satisfying a second degree equation with rational coefficients).