

Reconfigurable formations of quadrotors on Lissajous curves for surveillance applications

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Abstract

This paper proposes trajectory planning strategies for online reconfiguration of a multi-agent formation on a Lissajous curve. In our earlier work [2], a multi-agent formation with constant parametric speed was proposed in order to address multiple objectives such as repeated collision-free surveillance and guaranteed sensor coverage of the area with ability for rogue target detection and trapping. This work addresses the issue of formation reconfiguration within this context. In particular, smooth parametric trajectories are designed for the purpose using calculus of variations. These trajectories have been employed in conjunction with a simple local cooperation scheme so as to achieve collision-free reconfiguration between different Lissajous curves. A detailed theoretical analysis of the proposed scheme is provided. These surveillance and reconfiguration strategies have also been validated through simulations in MATLAB® for agents performing parametric motion along the curves, and by Software-In-The-Loop simulation for quadrotors. In addition, they are validated experimentally with a team of quadrotors flying in a motion capture environment.

Keywords: Lissajous curves, Multi-agent systems, Calculus of variations, Formation reconfiguration, Area surveillance

1. Introduction

The autonomous area surveillance task involves planning paths for a single/multiple autonomous agents with a limited sensing range. This could be done so as to ensure that all points in an area of interest are viewed /sensed repeatedly in finite time, with guaranteed detection of a target of interest in the area being monitored. Multi-agent systems offer several advantages over single agent systems for repeated coverage and target detection tasks. The most significant the search and detection of a rogue element, and reduction of the time required for a single agent, which is alleviated by the parallelism implicit in multi-agent implementations. Another notable advantage is the increased robustness due to redundancy. Some major application domains are searching for threats [28], surveillance ([21], [24]), and so on. Some preliminary results in regard to the proposed strategy were presented in [2], where a trajectory plan was proposed for a multi-agent formation on a Lissajous curve in order to achieve the following objectives simultaneously:

- O1 Complete and periodic coverage of the rectangular region of interest.
- O2 Collision-free patrolling for agents having finite non-zero size with agent speed bounded above by V_{max} .
- O3 Finite time entrapment and detection of a rogue element held at the center of the region.

The prior work in [2] considered a non-cooperating group of agents on a Lissajous curve and exploited the geometric properties of the curve to meet the above objectives simultaneously. Also, in [2] a sufficient upper bound on the agent size was derived in order to guarantee collision-free motion of the multi-agent formation.

In this paper, we extend the work of [2] to a cooperating reconfigurable multi-agent formation on Lissajous curves. Our proposed reconfigurable multi-agent formation guarantees smooth and collision-free trajectories for formation reconfiguration, considering the following operations:

- 1) Agent addition, 2) Agent removal, 3) Agent replacement.

The key features of the proposed reconfiguration strategy are:

1. A connected communication graph between the formation agents having limited communication range for cooperation.
2. Parametric trajectories for smooth acceleration, deceleration and transitions between Lissajous curves.
3. Cooperative assignment schemes for each reconfiguration operation to ensure collision-free multi-agent transitions from one Lissajous curve to another.
4. Collision-free trajectories are designed for agents having finite non-zero size satisfying the size bound derived in [2].

The proposed strategy has potential applications to target search, repeated surveillance, monitoring and mapping of dynamic environments, area sweeping for cleaning, spraying, etc.

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The rest of this paper is organized as follows: Section 2 recalls some related works. Section 3 discusses some preliminaries about Lissajous curves and the design of smooth trajectories using the calculus of variations approach. Section 4 discusses the theoretical details of the proposed surveillance strategy and gives a systematic algorithm to implement it, recalling some earlier work from [2]. Section 5 discusses the proposed formation reconfiguration strategies for addition, removal and replacement of agents in the multi-agent formation proposed for the surveillance strategy discussed in Section 4. Section 6 validates the proposed surveillance and reconfiguration strategies through simulations and experiment. Section 7 concludes the paper.

2. Literature survey

The novelty of the work presented here is in simultaneously addressing multiple surveillance objectives, while ensuring collision-free paths for the multiple agents having a finite non-zero size. In literature, several methods have been proposed to address the aerial surveillance problem [13, 12, 3, 23, 11]. [13] takes an approach based on temporal logic. [12] proposes a strategy based on parametrized curves using splines. In contrast, [3] takes a ‘planning’ viewpoint based upon casting the problem as a partially observed Markov decision process (POMDP). [23] have addressed the surveillance problem with a particle swarm optimization based approach. The article [11] takes into account energy considerations leading to a distinctive estimation/optimization problem. Our proposed surveillance strategy is based on tracking a parametric curve, viz., the Lissajous curve, that lends itself to a clean analytic way of path planning with a priori guarantees in certain performance criteria.

While complete area coverage is subsumed in our objective, our scheme goes well beyond it towards surveillance and detection of rogue elements. Coverage by itself has been extensively researched as a stand-alone theme. We recall a few relevant works here. One of the approaches for single agent area coverage is cell decomposition (discussed in [5]) wherein an area is divided into cells and these cells are searched systematically using zig-zag scan patterns. Occupancy grid based strategies studied in [29], use a distance transform to assign a specific numeric value to each free grid element starting from a ‘goal’ point, and a pseudo-gradient descent approach to generate a coverage path from ‘start’ to ‘goal’. The grid based algorithms such as Spiral Spanning Tree Coverage Algorithm [7] and the Backtracking Spiral Algorithm [9] ensure that the robot returns to its starting grid location after completion of the coverage task. Thus these algorithms can be used for repeated coverage tasks. Some of these approaches have also been extended to multi-agent scenarios ([30], [15]). Another problem relevant to patrolling is target search and detection. Recent works in this direction include: gradient based strategies for multi-UAV search [8], Voronoi partition based strategy using an uncertainty map [10], and game theoretic search strategies [26]. In comparison to these approaches, the proposed strategy gives

deterministic guarantee on repeated complete coverage and target detection in finite time.

Closer in spirit to our approach are the works based on well defined geometric curves such as raster scanning zigzag paths used in cell based schemes discussed in [5], [15], etc., and space filling curves such as the Hilbert curve for multi-agent coverage in [25], and for non-uniform priority based coverage in [22] and [20]. These curves often involve sharp turns and require additional path planning to return to the ‘start’ position to repeat the task. However, Lissajous curves have a simple parametric form and an appropriate choice of describing variables results in a smooth periodic curve called a ‘non-degenerate Lissajous curve’ ([6]) of prescribed mesh density within a rectangle of any dimensions. The parametric form of the Lissajous curve simplifies the expression for a moving reference point that can be tracked by a robot. The problem of optimal choice of a Lissajous curve for multi-agent persistent monitoring of 2-D spaces has been addressed in [14], where each agent is assigned a separate Lissajous curve. Unlike [14], the aim of this work is to explore the advantage of multiple agents following a single curve so that each agent covers the entire area over the period of time. This feature allows robustness against failure and fast coverage due to parallelism. It also makes this approach suitable for surveillance tasks where each agent is equipped with different types of sensors.

In this work we also propose collision-free online reconfiguration trajectories for the multi-agent formation on Lissajous curves proposed in [2]. We highlight some literature in broadly related areas with similar motivations regarding multi-agent formation reconfiguration. [27] have proposed a decentralised trajectory planner which guarantees convergence of a multi-robot formation to a centralised trajectory. Additionally, they use a simple rule based assignment methodology for collision-free formation shape reconfiguration. In [1], a quadrotor team is employed for building a structure. For this task, collision-free routes to shared resources, such as battery charging stations, are computed using reserved passing lanes and reservation systems.

The problem of agent removal for recharging or refuelling in long endurance missions has been posed as a problem of scheduling and goal-reassignment task in [19] and [18]. [19] present a heuristic method to solve a Mixed Integer Linear program (MILP) formulation in order to efficiently cover a set of targets with agent removal for recharging. In [18], this approach has been extended considering an energy aware optimisation objective with an initially uncertain energy expenditure model.

In [16] and [17], a Mixed Integer Quadratic Program (MIQP) formulation has been used to plan piecewise smooth collision-free trajectories for formation reconfiguration. Of these, [16] use integer constraints to enforce collision avoidance for formation reconfiguration of a heterogeneous team of quadrotors. In [17] trajectories are computed for online substitution of quadrotors in a multi-quadrotor formation. In [19, 18, 17] and [16] the reported experiments have been conducted by implementing the proposed trajectory planners with appropriate optimisation solvers on a central computer which commands the robots in flight.

Other recent approaches of formation reconfiguration and

control include the Virtual Rigid Body abstraction in [31] and the use of path planning algorithms based on sequential convex programming (SCP) in [4]. In [31], collision-free trajectories are obtained to maintain a fixed relative quadrotor formation in a manoeuvre and also to switch between a sequence of quadrotor formations. In [4], an incremental sequential convex programming algorithm has been proposed for finding feasible collision-free trajectories for quadrotor teams. These are computed in near real time on a central computer.

Note that these works depend significantly more on centralised processing as compared to our proposed reconfiguration scheme.

3. Preliminaries

Let the dimensions of rectangular environment be $2A \times 2B$ with $A, B \in \mathbb{R}$. We consider the Lissajous curves with parametric equation

$$x(s(t)) = A \cos(as(t)), y(s(t)) = B \sin(bs(t)), \quad (1)$$

where s is the parameter, a and $b \in \mathbb{N}$ are co-prime positive integer constants (having common factors, results in the same Lissajous curve, e.g., $a : b = 1 : 2, 4 : 8$ and $12 : 24$). The coordinates X and Y are defined along the directions parallel to the sides of the rectangle and the origin is chosen to be the center of the rectangular region.

For the work presented in this paper, only non-degenerate Lissajous curves (refer [6]) are considered, with the property that its entire curve length is traversed only once along a single direction by the running parameter s in the parametric period of $[0, 2\pi)$. To ensure non-degeneracy of (1), a must be an odd integer. The properties of non-degenerate Lissajous curves used to derive some results in this paper are stated with proofs in the online supplement¹. The points on the Lissajous curve which are encountered twice within a complete traversal of the curve are called intersection points (red and green points in Fig. 1), and the points where the Lissajous curve touches the boundaries of the rectangular region of interest are called boundary points (black and magenta points in Fig. 1). Together, the intersection and boundary points are referred to as node points. To guarantee smooth collision-free motion of the agents while ensuring that the speed of the agents is bounded above by V_{max} (the maximum allowable speed of the agents), smooth transition trajectories are used. The trajectories are designed to be at least twice continuously differentiable, using calculus of variations approach.

For this purpose, we consider the solutions of the calculus of variations problem given in Lemma 1:

Lemma 1. *The optimal function minimizing the integral $\int_0^{T_f} \frac{\dot{g}^2(t)}{2} dt$ for fixed end time $T_f > 0$, subject to the following sets of boundary conditions, is as follows:*

¹<https://drive.google.com/open?id=0B41bdPZ-BnshS25qcVc3T1NnV1U>

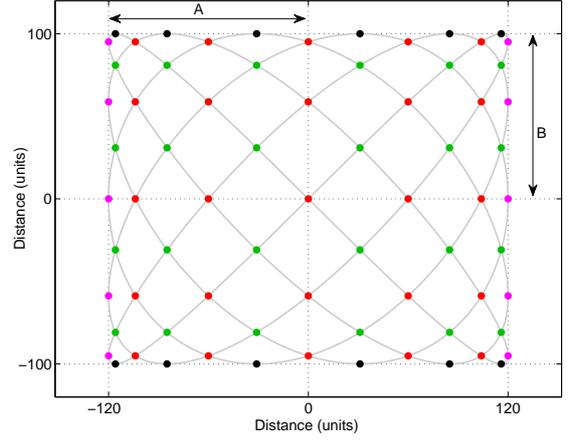


Figure 1: Non-degenerate Lissajous curve with $a = 5, b = 6, A = 120, B = 100$

C1 For $g(0) = g_0, \dot{g}(0) = \dot{g}_0, \ddot{g}(0) = 0, g(T_f) = \text{free}, \dot{g}(T_f) = \dot{g}_f, \ddot{g}(T_f) = 0$, it is:

$$g^*(t) = (\dot{g}_f - \dot{g}_0) \left(-\frac{t^4}{2T_f^3} + \frac{t^3}{T_f^2} \right) + \dot{g}_0 t + g_0.$$

C2 For $g(0) = g_0, \dot{g}(0) = 0, \ddot{g}(0) = 0, g(T_f) = g_f, \dot{g}(T_f) = 0, \ddot{g}(T_f) = 0$, it is:

$$g^*(t) = (g_f - g_0) \left(10T_f^2 - 15T_f t + 6t^2 \right) \frac{t^3}{T_f^3} + g_0.$$

The proof for Lemma 1 is given in the appendix.

Notation : We define the following notations for brevity of representation: T_0, T_f are start and end times of a parametric trajectory. $T_p = T_f - T_0$ is the time period of the transition trajectory. $\Delta t = t - T_0$ where t is the current time. $\Delta g = g_f - g_0$ where g_0, g_f are initial and final parametric positions at times T_0 and T_f respectively. Similarly $\Delta \dot{g} = \dot{g}_f - \dot{g}_0$, where \dot{g}_f, \dot{g}_0 are initial and final parametric speeds at times T_0 and T_f respectively.

Lemma 1 gives the template for designing the following trajectories for the reconfiguration strategy:

Monotone transition trajectory

To design a monotone transition trajectory for smooth acceleration or deceleration for a parameter $g_m(t)$ from an initial parameter value and speed to a final parameter value and speed, we use the solution for the free end state and fixed end time boundary conditions C1 of the calculus of variations problem discussed in Lemma 1. In this case the boundary conditions on parameter value and speed are $g_m(T_0) = g_0, \dot{g}_m(T_0) = \dot{g}_0$ and $\dot{g}_m(T_f) = \dot{g}_f$, and the terminal value of $g_m(T_f)$ is free. Thus parameter trajectory $g_m(t)$ and its derivatives are as follows:

$$g_m(t) = \Delta \dot{g} \left(-\frac{\Delta t^4}{2T_p^3} + \frac{\Delta t^3}{T_p^2} \right) + \dot{g}_0 \Delta t + g_0, \quad (2)$$

$$\dot{g}_m(t) = \Delta \dot{g} \left(-2\frac{\Delta t^3}{T_p^3} + \frac{3\Delta t^2}{T_p^2} \right) + \dot{g}_0, \quad (3)$$

$$\ddot{g}_m(t) = 6\Delta\dot{g}\left(-\frac{\Delta t^2}{T_p^3} + \frac{\Delta t}{T_p^2}\right), \quad (4)$$

$$\ddot{\ddot{g}}_m(t) = 6\Delta\dot{g}\left(-2\frac{\Delta t}{T_p^3} + \frac{1}{T_p^2}\right). \quad (5)$$

Some properties of the $g_m(t)$ and its derivatives are summarised as follows:

1. $\ddot{g}_m(T_0) = 0$, $\ddot{g}_m(T_f) = 0$ implying constant parametric speed at the beginning and the end of the transition trajectory in the time window $[T_0, T_f]$.
2. From (4), $t = T_0$ and T_f are the solutions of $\ddot{g}_m(t) = 0$, and they are the only extremizers of $\dot{g}_m(t)$. For $\dot{g}_0 < \dot{g}_f$ from (5), $\ddot{\ddot{g}}_m(T_f) < 0$ and $\ddot{\ddot{g}}_m(T_0) > 0$ which implies $\dot{g}_m(t)$ is monotonically increasing in $[T_0, T_f]$ with minimum value \dot{g}_0 at $t = T_0$ and maximum value \dot{g}_f at $t = T_f$. By similar arguments, for $\dot{g}_f < \dot{g}_0$, $\dot{g}_m(t)$ is monotonically decreasing in $[T_0, T_f]$ with minimum value \dot{g}_f at $t = T_f$ and maximum value \dot{g}_0 at $t = T_0$.

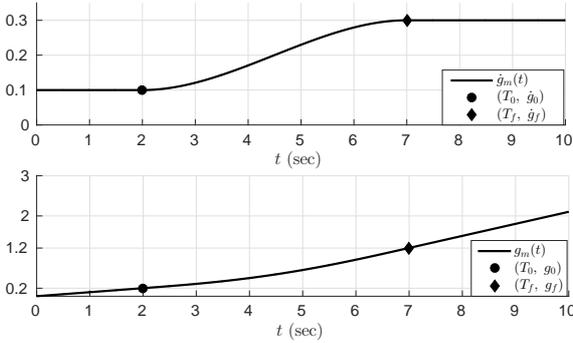


Figure 2: A monotone transition trajectory example where: $T_0 = 2$ sec, $g_0 = 0.2$, $g_f = 1.2$, $\dot{g}_0 = 0.1/\text{sec}$, $\dot{g}_f = 0.3/\text{sec}$. Thus $T_p = 5$ sec and $T_f = 7$ sec

The value of the free terminal parameter $g_m(T_f)$ depends on the fixed end time $T_f = T_0 + T_p$. Thus it depends on the width of the time interval T_p specified for the transition. As a result, to obtain a desired final parameter value $g_m(T_f) = g_f$, the appropriate transition interval size T_p , and T_f can be calculated by substituting $t = T_f$ (i.e., $\Delta t = T_f - T_0$) in (2) and we get

$$T_p = \frac{2(g_f - g_0)}{\dot{g}_f + \dot{g}_0} \Rightarrow T_f = T_0 + \frac{2(g_f - g_0)}{\dot{g}_f + \dot{g}_0}. \quad (6)$$

An example of this class of trajectories is shown in Fig. 2. The parameters that completely specify such a monotone trajectory are

$$T_0, g_0, g_f, \dot{g}_0 \text{ and } \dot{g}_f. \quad (7)$$

Symmetric transition trajectory

In order to move a parameter g_s from the initial boundary conditions $g_s(T_0) = g_0$, $\dot{g}_s(T_0) = 0$ and $\ddot{g}_s(T_0) = 0$, to the final boundary condition $g_s(T_f) = g_f$, $\dot{g}_s(T_f) = 0$, and $\ddot{g}_s(T_f) = 0$ (i.e., parameter is stationary at initial and final time) over a fixed time window $[T_0, T_f]$, we design a smooth trajectory that

accelerates and decelerates symmetrically. This is done using the fixed end state and fixed end time boundary conditions C2 of the calculus of variations problem discussed in Lemma 1. Thus the parameter trajectory $g_s(t)$ and its derivatives are as follows:

$$g_s(t) = \Delta g \left(10T_p^2 - 15T_p\Delta t + 6\Delta t^2\right) \frac{\Delta t^3}{T_p^5} + g_0, \quad (8)$$

$$\dot{g}_s(t) = \frac{30\Delta g}{T_p^5} \Delta t^2 (T_p - \Delta t)^2, \quad (9)$$

$$\ddot{g}_s(t) = \frac{60\Delta g}{T_p^5} \Delta t (T_p^2 - 3T_p\Delta t + 2\Delta t^2), \quad (10)$$

$$\ddot{\ddot{g}}_s(t) = \frac{60\Delta g}{T_p^5} (T_p^2 - 6T_p\Delta t + 6\Delta t^2). \quad (11)$$

Some properties of the $g_s(t)$ and its derivatives are summarised as follows:

1. $\dot{g}_s(T_0) = 0$, $\ddot{g}_s(T_0) = 0$, $\dot{g}_s(T_f) = 0$ and $\ddot{g}_s(T_f) = 0$.
2. From (10), $t = T_0, T_0 + \frac{T_p}{2}$ and T_f are the solutions of $\ddot{g}_s(t) = 0$, and are the only extremizers of $\dot{g}_s(t)$. For $g_0 < g_f$, from (11), $\ddot{\ddot{g}}_s(T_0) = \ddot{\ddot{g}}_s(T_f) = \frac{60\Delta g}{T_p^5} > 0$ and $\ddot{\ddot{g}}_s(T_0 + \frac{T_p}{2}) = -\frac{30\Delta g}{T_p^5} < 0$ which implies $\dot{g}_s(t)$ attains maximum value at $t = T_0 + \frac{T_p}{2}$, and minimum value at $t = T_0$ and $t = T_f$. By similar arguments, for $g_f < g_0$, $\dot{g}_s(t)$ attains its minimum value at $t = T_0 + \frac{T_p}{2}$ and maximum value at $t = T_0$ and $t = T_f$.
3. For $\dot{g}_s(t)$, the maximum value \dot{g}_{max} for the case $g_0 < g_f$ and the minimum value \dot{g}_{min} for case $g_f < g_0$ are both attained at $t = T_0 + \frac{T_p}{2}$ and

$$\dot{g}_{max} = \frac{15|g_f - g_0|}{8T_p}, \quad \dot{g}_{min} = -\frac{15|g_f - g_0|}{8T_p}. \quad (12)$$

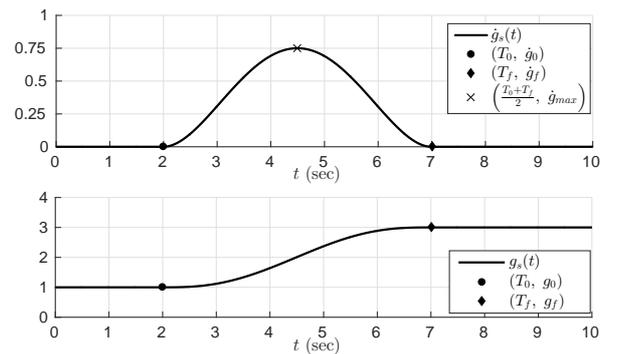


Figure 3: A symmetric transition trajectory example where: $T_0 = 2$ sec, $T_p = 5$ sec, $g_0 = 1$, $g_f = 3$. Thus $T_f = 7$ sec and $\dot{g}_{max} = 0.75/\text{sec}$ at $t = \frac{T_0 + T_f}{2} = 4.5$ sec

An example of this class of trajectories is shown in Fig. 3. The constants which completely specify this trajectory are:

$$T_0, T_f, g_0 \text{ and } g_f. \quad (13)$$

The monotone transition trajectory is used to decelerate and accelerate the multi-agent formation along the Lissajous curve.

The symmetric transition trajectory is used to design collision-free trajectories between Lissajous curves. The next section summarises and builds upon the prior work in [2].

4. Proposed surveillance strategy

The proposed surveillance strategy discussed in this section meets the objectives O1, O2 and O3 listed in Section 1 (as shown in [2]). To develop the theory for the proposed trajectory plans, we make the following assumptions:

1. Agents are helicopter or quadrotor like agents capable of hovering.
2. The search area is an obstacle-free rectangle of dimensions $L \times H$ and all agents are homogeneous and identical.
3. Each agent has a circular noise-free sensor footprint of radius $r_s < \frac{1}{2} \sqrt{L^2 + H^2}$ (i.e., half the diagonal length).
4. Position and timing information for each agent is available from an external source (e.g., visual feedback using cameras, GPS, etc.).
5. We assume ideal communication links without any delays or packet losses.

Given any non-degenerate Lissajous curve, the proposed multi-agent surveillance strategy defines collision-free trajectories for multiple agents on this curve, while ensuring that the agent formation lies on an elliptical locus centered around the origin at any instant of time. This is done by initially placing $N = a + b$ agents at equi-parametric separations on the Lissajous curve with constants a, b in (1), and moving them along the curve at equal parametric speed.

We briefly recall the results of the surveillance strategy from [2] in the following subsections. Later in the paper, we extend this strategy to a reconfigurable formation of agents on Lissajous curves.

4.1. Multi-Agent formation

For the proposed placement of agents on the Lissajous curve, the initial parameter value of the agent i' is $s^{i'}(0) = s_0^{i'} = \frac{2\pi(i'-1)}{a+b}$ where $i' \in \{1, 2, \dots, a+b\}$. For the surveillance strategy, since all agents move at the same parametric rate \dot{s} on the Lissajous curve, the parameter value of the agent at any given point in time is given by $s^{i'}(t) = s_0^{i'} + s(t)$, where $s(t) = \int_0^t \dot{s} dt$. The position of the agent i' is a function of $s(t)$. From (1), $y_{i'}(s(t)) = B \sin\left(\frac{2(i'-1)b\pi}{a+b} + bs(t)\right)$ and $x_{i'}(s(t)) = A \cos\left(\frac{2(i'-1)a\pi}{a+b} + as(t)\right)$. Using the identity $\cos(\theta) = \cos(2\pi(i' - 1) - \theta)$, $x_{i'}(s(t)) = A \cos\left(\frac{2(i'-1)b\pi}{a+b} - as(t)\right)$. With a little abuse of notation, we denote the running parameter $s(t)$ by s for brevity. In Claim 2 given in the online supplement¹, we have shown that by appropriate renumbering of the index i' by i , the agents can be numbered along the elliptical locus rather than the Lissajous curve, and the resulting position coordinates of the agent i with this new renumbering is given by

$$x_i(s) = A \cos(\tilde{\psi}^i - as), \quad y_i(s) = B \sin(\tilde{\psi}^i + bs), \quad (14)$$

where $\tilde{\psi}^i = \frac{2\pi(i-1)}{a+b}$ for $i \in \{1, 2, \dots, a+b\}$.

There can be many pairs of mutually co-prime integers satisfying the relation $N = a + b$. For example, with $N = 7$ the mutually co-prime (a, b) pairs satisfying $N = a + b$ are $(1, 6)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 2)$ and $(6, 1)$. In [2] an algorithm was proposed for choosing the optimal (a, b) pair that maximises the size bound on the agent and minimises the area coverage time (discussed in subsection 4.6). This algorithm could select a degenerate (a, b) pair with even a and odd b , for example $(a, b) = (2, 5)$ for $N = 7$. In [2], this issue was addressed by swapping the value of a with b and A with B , so as to get a non-degenerate Lissajous curve. This is equivalent to a rotation of reference frame.

In this paper, we propose an online formation reconfiguration strategy that switches between Lissajous curves. Hence it is convenient to maintain the same reference frame across the selected Lissajous curves. Therefore, we represent the swapping of a with b and A with B , by an equivalent phase shift in the original frame as follows:

After the swap, the position coordinates of agent i on the resulting non-degenerate Lissajous curve are

$$(\hat{x}_i(s'), \hat{y}_i(s')) = (B \cos(\tilde{\psi}^i - bs'), A \sin(\tilde{\psi}^i + as')).$$

These coordinates can be expressed in the original reference frame by a rotation of $-\frac{\pi}{2}$, as given below:

$$\begin{bmatrix} x_i(s') \\ y_i(s') \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_i(s') \\ \hat{y}_i(s') \end{bmatrix} = \begin{bmatrix} A \cos(\tilde{\psi}^i + \frac{\pi}{2} + as') \\ B \sin(\tilde{\psi}^i + \frac{\pi}{2} - bs') \end{bmatrix}.$$

By substituting a negative parameter $s = -s'$ (reversing the direction of traversal), we get $(x_i(s), y_i(s)) = (A \cos(\tilde{\psi}^i + \frac{\pi}{2} - as), B \sin(\tilde{\psi}^i + \frac{\pi}{2} + bs))$. Thus the general representation of the agent positions on the Lissajous curve for the proposed strategy is

$$\begin{aligned} x_i(s, \psi)|_{\psi=\psi^i} &= A \cos(\psi - as), \\ y_i(s, \psi)|_{\psi=\psi^i} &= B \sin(\psi + bs), \end{aligned} \quad (15)$$

where $\psi^i = \frac{2\pi(i-1)}{a+b} + \frac{o\pi}{2}$ and the offset o is

$$o = 1 - (a \bmod 2). \quad (16)$$

4.2. Elliptical formation locus

In [2], it has been shown that the agent positions given by (15) lie on a conic curve given by

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} - \frac{2xy \sin((a+b)s)}{AB} = \cos^2((a+b)s), \quad (17)$$

which represents an elliptical locus that is always centered about the origin $(x_o, y_o) = (0, 0)$.

For different values of parameter s , different elliptical loci are obtained as shown in Fig. 4. For all $k \in \mathbb{N}$, by (17) the parameter values $s = \frac{k\pi}{(a+b)}$ result in the ellipse $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$. Similarly parameter values $s = \frac{(2k-1)\pi}{2(a+b)}$ result in a degenerate ellipse

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + (-1)^k \frac{2xy}{AB} = 0,$$

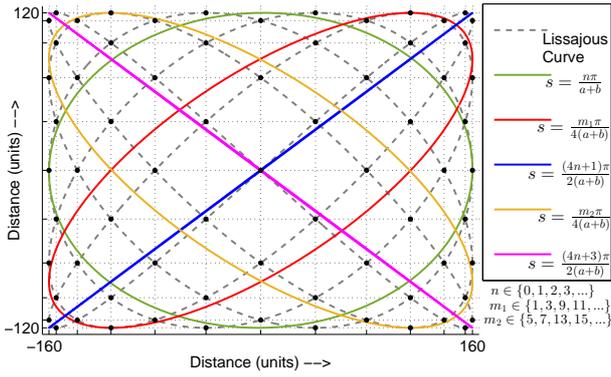


Figure 4: Loci of the agent positions with placements given by (14) at different values of parameter s for Lissajous curve having $a = 5$, $b = 6$, $A = 160$ and $B = 120$.

which is the straight line $y = \frac{(-1)^{k+1}B}{A}x$ along the diagonals of the rectangular region defined by $[-A, A] \times [-B, B]$. Notice that the equation of the ellipse given by (17), which is the locus for the multi-agent formation, is only dependent on the parameter s and not ψ^i . Now in (15), for an agent i , if we make s constant (fixing the ellipse) and vary the parameter ψ , then we can achieve a parametric motion along the formation ellipse. Note that the formation ellipse has a parametric length 2π in terms of the parameter ψ , thus the proposed formation places the agents at equi-parametric intervals ($\tilde{\psi}^i = \frac{2\pi(i-1)}{N}$ for $i \in \{1, 2, \dots, N\}$) along the parametric length of the formation ellipse. We use this idea later to move the agents from one Lissajous curve to another for formation reconfiguration.

4.3. Speed profile of the agents

Let V_{max} be the largest permissible linear speed for each agent. Differentiating (15) and assuming the running parameters ψ and s to be functions of time, the components of velocity along the x -direction and the y -direction are

$$\dot{x} = -A \sin(\psi - as)(\dot{\psi} - a\dot{s}), \quad (18)$$

$$\dot{y} = B \cos(\psi + bs)(\dot{\psi} + b\dot{s}), \quad (19)$$

and the resultant speed V is given by

$$V(t) = \sqrt{\dot{y}^2 + \dot{x}^2}. \quad (20)$$

It is desirable to have a continuous velocity profile of the agents in order to facilitate a practical implementation of the proposed surveillance and formation reconfiguration strategy. Furthermore, the resultant speed must be maintained below V_{max} . For the surveillance mission, it was proposed in [2] that the parameter $\psi = \psi^i = \frac{2\pi(i-1)}{N}$ be a constant and the agents move on the Lissajous curve at a constant non-zero parametric speed \dot{s}_{nom} given by

$$\dot{s}_{nom} = \frac{V_{max}}{\sqrt{A^2 a^2 + B^2 b^2}}. \quad (21)$$

Thus $\dot{\psi} = 0$ and it was shown in [2] that selecting the nominal value of $\dot{s} = \dot{s}_{nom}$ guarantees $V(t) \leq V_{max}$.

4.4. Sensing and communication range of agents

From (15), the x and y coordinate separation between any two agents i, j in the proposed formation, is given by

$$|x_i - x_j| = 2A \left| \sin(\Psi_p - as) \sin(\Psi_m) \right| \text{ and} \\ |y_i - y_j| = 2B \left| \cos(\Psi_p + bs) \sin(\Psi_m) \right|,$$

where $\Psi_p = \frac{\psi^i + \psi^j}{2}$, and $\Psi_m = \frac{\psi^i - \psi^j}{2}$. As a result the Euclidean distance between the agents i and j is

$$D_{ij} = 2 \sin(\Psi_m) \sqrt{A^2 \sin^2(\Psi_p - as) + B^2 \cos^2(\Psi_p + bs)}. \quad (22)$$

For adjacent agents along the elliptical locus, $\psi^j = \psi^i \pm \frac{2\pi}{N}$, because for the proposed strategy, the agents are equi-parametrically distributed along the elliptical locus for all time. Hence the Euclidean distance between adjacent agents is given by

$$D_{ad} = 2 \sin\left(\frac{\pi}{N}\right) \sqrt{A^2 \sin^2(\Psi_p - as) + B^2 \cos^2(\Psi_p + bs)}. \quad (23)$$

D_{ad} is bounded above by $D_M = 2 \sin\left(\frac{\pi}{N}\right) \sqrt{A^2 + B^2}$. As a result, if each agent has a circular sensor footprint of radius r_s , then overlapping sensor footprints of parametrically adjacent agents along the elliptical locus can be guaranteed at any value of the parameter s by ensuring $r_s \geq \frac{D_M}{2}$. Similarly, by ensuring that each agent's spherical communication range has radius $r_{com} > D_M$, we guarantee that the adjacent agents can communicate for cooperation during formation reconfiguration manoeuvres. In practice, considering the presence of curve tracing errors in the implementation using real robotic platforms such as quadrotors, the lower bounds on sensor footprint radius r_s and communication range radius r_{com} are chosen as

$$r_{sm} = \eta \sin\left(\frac{\pi}{N}\right) \sqrt{A^2 + B^2} \text{ and} \quad (24)$$

$$r_{cm} = 2\eta \sin\left(\frac{\pi}{N}\right) \sqrt{A^2 + B^2} \quad (25)$$

respectively, where $\eta \geq 1$ is a safety factor to ensure sufficient sensor footprint overlap and communication range.

4.5. Agents with non-zero size and coverage time

In practice, real agents such as ground robots or quadrotors have a non-zero size and are not point agents. Thus in [2], an upper bound r_{du} on the radius of the circular hull was derived, which contains the physical dimensions of the agent. This upper bound is given by:

$$r_{du} = \sin\left(\frac{\pi}{N}\right) \frac{AB}{\sqrt{A^2 a^2 + B^2 b^2}}. \quad (26)$$

Agents having sizes smaller than this bound are guaranteed to have collision-free trajectories for the proposed surveillance strategy.

Since N agents (where $N = a + b$) are initially placed along the curve with a parametric separation of $\frac{2\pi}{a+b}$ and all agents

move with equal parametric speed \dot{s}_{nom} along the curve, a parametric displacement of $\frac{2\pi}{a+b}$ for all the agents guarantees that the entire Lissajous curve is collectively traversed by all the agents. As a consequence, an upper bound on the time taken to collectively cover the entire rectangular area by the multi-agent formation is

$$T_{cov} = \frac{2\pi}{(a+b)\dot{s}_{nom}} = \frac{2\pi\sqrt{A^2a^2 + B^2b^2}}{NV_{max}}. \quad (27)$$

The results proved in [2] can be summarised by the following theorem:

Theorem 1. *Given a non-degenerate Lissajous curve described by (1) having parameters A, B, a, b , the multi-agent formation of parametric point agents on the curve given by (14), equipped with a circular sensor footprint of radius r_s given by (24) and moving along the curve with an equal parametric speed \dot{s}_{nom} given by (21) guarantees the fulfilment of objectives:*

- 1) *Collision-free paths for the agents in the formation, with agent speed bounded above by V_{max} .*
- 2) *Complete and repeated coverage of the rectangular area by the multi-agent formation.*
- 3) *Finite time detection of a rogue element trying to escape from the region starting from the center of the region.*

4.6. Selection of Lissajous curve parameters

Suppose the number of agents to be used for the proposed surveillance strategy is N , and they monitor a rectangular area of dimensions $L \times H$. Then we choose the Lissajous curve constants $A = \frac{L}{2}$ and $B = \frac{H}{2}$ in (1). Recall that to achieve the proposed multi-agent formation the agent positions are defined by (15) where $N = a + b$ for a pair of co-prime positive integers (a, b) . For the proposed strategy we select the (a, b) pair considering the following claim.

Claim 1. *([2]) For a given N , the value $a^* = \frac{B^2N}{A^2+B^2}$ with $b = N - a^*$ is the minimizer of T_{cov} and maximizer of r_{du} , where T_{cov} and r_{du} are given by (27) and (26) respectively.*

In [2], it has been argued that for a $\delta > 0$, $T_{cov}(a^* + \delta) = T_{cov}(a^* - \delta)$. Hence, though a^* may not be an integer, we find the positive integer k_c such that it is the nearest integer to a^* that yields a co-prime $(k_c, N - k_c)$ pair. Then we select $(a, b) = (k_c, N - k_c)$. This choice is the coprime integer pair that minimises the value of T_{cov} and maximises r_{du} while satisfying $a + b = N$. We search for this mutually co-prime pair iteratively using Algorithm 1. The completeness of this algorithm has been proved in [2]. In the worst case, Algorithm 1 selects the following:

$$(a, b) = \begin{cases} (1, N - 1), & \text{if } A \geq B \\ (N - 1, 1), & \text{if } A < B. \end{cases} \quad (28)$$

In case the selected (a, b) pair corresponds to a degenerate Lissajous curve, then instead of the swapping a with b and A with B , as done in [2], the Algorithm 1 computes the phase offset o given by (16).

Algorithm 1 CURVE_SELECT

Inputs: A, B, N

Functions: GCD

Outputs: a, b, o

```

1:  $a^* = \frac{B^2N}{A^2+B^2}$ ,  $d_u = \lceil a^* \rceil - a^*$ ,  $d_l = a^* - \lfloor a^* \rfloor$ 
2:  $c = 1$ 
3: if  $d_u \leq d_l$  or  $a^* < 1$  then
4:    $k_c = \lceil a^* \rceil$ ,  $m = 0$ 
5: if  $d_u > d_l$  or  $a^* > N - 1$  then
6:    $k_c = \lfloor a^* \rfloor$ ,  $m = 1$ 
7: while  $GCD(k_c, N - k_c) \neq 1$  do
8:    $k_c = k_c + (-1)^{c+m}c$ 
9:    $c = c + 1$ 
10:  $a = k_c$ ,  $b = N - k_c$ 
11:  $o = 0$ 
12: if  $GCD(k_c, 2) = 2$  then
13:    $o = 1$ 

```

4.7. Number of agents in the formation

To practically implement the proposed surveillance strategy we need to ensure the following:

1. Sufficient sensor footprint radius to ensure overlapping sensing ring formation (i.e., $r_s \geq r_{sm}$ in (24)).
2. Sufficient communication range between agents for co-operation (i.e., $r_{com} \geq r_{cm}$ in (25)).

To ensure this for a given sensing capability r_s and communication range r_{com} of a single agent, we need to compute the minimum number of agents N_{min} for which the formation is defined. Thus from (24), defining $\mathcal{R} = \eta\sqrt{A^2 + B^2}$, the minimum number of agents necessary to ensure $r_s \geq r_{sm}$ is

$$N_s = \left\lceil \pi \left| \sin^{-1} \left(\frac{r_1}{\mathcal{R}} \right) \right|^{-1} \right\rceil \text{ with } r_1 = \begin{cases} r_s, & \text{if } r_s < \mathcal{R} \\ \mathcal{R}, & \text{otherwise.} \end{cases}$$

Similarly, from (25), the minimum number of agents required to ensure $r_{com} \geq r_{cm}$ are

$$N_c = \left\lceil \pi \left| \sin^{-1} \left(\frac{r_2}{2\mathcal{R}} \right) \right|^{-1} \right\rceil \text{ with } r_2 = \begin{cases} r_{com}, & \text{if } r_{com} < 2\mathcal{R} \\ 2\mathcal{R}, & \text{otherwise.} \end{cases}$$

Thus the minimum number of agents in the formation that are necessary to guarantee both $r_s \geq r_{sm}$ and $r_{com} \geq r_{cm}$, is given by

$$N_{min} = \max\{N_s, N_c\} \geq 2. \quad (29)$$

For the reconfiguration strategy, we design trajectories for addition, removal and replacement of agents in the subsequent section. We assume that the maximum number of extra agents $N_{extra} > 0$ to be used in addition to the N_{min} agents is pre-defined, and the maximum number of agents in the formation is thus calculated as

$$N_{max} = N_{min} + N_{extra}. \quad (30)$$

For normal operation we use N agents where $N_{min} < N \leq N_{max}$.

4.8. Bound on agent size

Our objective is to design smooth trajectories for reconfiguring the multi-agent formation from one Lissajous curve to another depending on the number of agents being used. Given a number of agents N_j and the corresponding Lissajous curve (a_j, b_j) selected using Algorithm 1, from (26) the upper bound on the circular hull radius encompassing the dimensions of the agent for the reconfigurable formation is selected as

$$r_{dm} = \min_{j \in S_N} \left\{ \frac{AB}{\sqrt{A^2 a_j^2 + B^2 b_j^2}} \middle| N_j = N_{min} + j \right\} \times \sin\left(\frac{\pi}{N_{max}}\right), \quad (31)$$

where $S_N := \{0, 1, 2, \dots, N_{extra}\}$. Thus $r_{dm} \leq r_{du_j} = \frac{AB}{\sqrt{A^2 a_j^2 + B^2 b_j^2}} \sin\left(\frac{\pi}{N_j}\right), \forall j \in S_N$.

Thus we have Algorithm 2, which initialises all the parameters discussed above for all the agents.

Algorithm 2 INITIALISATION

Inputs: $L, H, r_s, r_{com}, V_{max}, N_{extra}, \eta$
Functions: CURVE_SELECT
Outputs: $A, B, a, b, o, N, N_{min}, N_{max}, \dot{s}_{nom}, r_{dm}$ and $(x_i(0), y_i(0)) \forall i \in \{1, \dots, N\}$

- 1: $A = \frac{L}{2}, B = \frac{H}{2}$
- 2: **if** $r_s < \eta \sqrt{A^2 + B^2}$ **then**
- 3: $r_1 = r_s$
- 4: **else** $r_1 = \eta \sqrt{A^2 + B^2}$
- 5: **if** $r_{com} < 2\eta \sqrt{A^2 + B^2}$ **then**
- 6: $r_2 = r_{com}$
- 7: **else** $r_2 = 2\eta \sqrt{A^2 + B^2}$
- 8: $N_s = \left\lceil \pi \left| \sin^{-1} \left(\frac{r_1}{\eta \sqrt{A^2 + B^2}} \right) \right|^{-1} \right\rceil$
- 9: $N_c = \left\lceil \pi \left| \sin^{-1} \left(\frac{r_2}{2\eta \sqrt{A^2 + B^2}} \right) \right|^{-1} \right\rceil$
- 10: $N_{min} = \max\{N_s, N_c\}, N_{max} = N_{min} + N_{extra}$
- 11: **for** $j = 0, 1, \dots, N_{extra}$ **do**
- 12: $N_j = N_{min} + j$
- 13: $[a_j, b_j, o_j] = \text{CURVE_SELECT}(A, B, N_j)$
- 14: $r_{d_j} = \frac{AB}{\sqrt{A^2 a_j^2 + B^2 b_j^2}}$
- 15: $r_{dm} = \min \{r_{d_j} \mid j \in \{0, \dots, N_{extra}\}\} \times \sin\left(\frac{\pi}{N_{max}}\right)$
- 16: $N = N_1, a = a_1, b = b_1$ and $o = o_1$
- 17: $\dot{s}_{nom} = \frac{V_{max}}{\sqrt{A^2 a_1^2 + B^2 b_1^2}}$
- 18: **for** $i = 1, \dots, N$ **do**
- 19: $\begin{bmatrix} x_i(0) \\ y_i(0) \end{bmatrix} = \begin{bmatrix} A \cos\left(\frac{2\pi(i-1)}{N} + \frac{o_1\pi}{2}\right) \\ B \sin\left(\frac{2\pi(i-1)}{N} + \frac{o_1\pi}{2}\right) \end{bmatrix}$

5. Formation reconfiguration strategy

In this section, we extend the proposed surveillance strategy (discussed in Section 4) for a more practical setting where agents may need to be added, removed or replaced from the

formation on the go. This is useful for applications where the surveillance task might last for long durations. Furthermore, the trajectories designed for these tasks must guarantee collision-free motion of the agents. For this an appropriate selection of trajectories is done by the agents via cooperation. This is achieved by information exchange on a communication channel and this channel is established between adjacent agents of the formation by ensuring that the communication range $r_{com} > r_{cm}$ (given by (25)).

Suppose the formation initially consists of N agents and our objective is to switch to a formation of $N - 1$ or $N + 1$ agents. For a reconfiguration, the trajectories are planned to move these agents from the current Lissajous curve corresponding to N agents to the Lissajous curve corresponding to $N - 1$ or $N + 1$ agents (selected in either case by Algorithm 1). For the replacement operation, we design a simple exchange step where the agent to be replaced is removed and a new agent takes its place on the same Lissajous curve. The proposed surveillance and reconfiguration strategy has been designed considering aerial agents such as helicopters and quadrotors, which are capable of safely decelerating to a zero speed in flight (hover) during operation.

Using the parametric representation in (15), we design collision-free parametric trajectories based on cooperation for removal, addition and replacement of a single agent. For the discussions in subsequent subsections, we use the notations given in Table 1.

Table 1: Notations

N_c	Number of agents before reconfiguration
(a_c, b_c, o_c)	Lissajous curve constants for N_c agents given by Algorithm 1 with $N_c = a_c + b_c$
$s_c^i(t), \psi_c^i(t)$	Curve parameters of agent i in terms of Lissajous curve for N_c agents at time t
N_d	Number of agents after reconfiguration
(a_d, b_d, o_d)	Lissajous curve constants for N_d agents given by Algorithm 1 with $N_d = a_d + b_d$
$s_d^i(t), \psi_d^i(t)$	Curve parameters of agent i in terms of Lissajous curve for N_d agents at time t
ψ_D^i	ψ_d parameter value corresponding to formation positions on Lissajous curve for N_d agents assigned to agent i for transition
Δ_ψ^i	Displacement in parameter ψ_d^i for agent i for reconfiguration to ψ_D^i
$\Delta^{ij}\psi(t)$	ψ parameter separation between agent i and j at time t

Prior to any reconfiguration, the N_c agents lie on an elliptical locus defined by the value of the parameter $s_c \in [0, 2\pi)$ according to (17). Also for N_c agents on Lissajous curve described by (a_c, b_c, o_c) , the agents are equi-parametrically spaced along an ellipse in terms of parameter $\psi_c \in [0, 2\pi)$ (refer (15)). Thus parametric separation between the adjacent agents in terms of parameter ψ_c is $\frac{2\pi}{N_c}$. This fact is used by each agent to identify its adjacent agents' parameters through communication.

5.1. Steps for formation reconfiguration

A brief outline of the steps involved in the reconfiguration operation are as follows:

- The multi-agent formation of N_c agents decelerates to a stop on the Lissajous curve described by constants (a_c, b_c, o_c) . This is done by using the monotone deceleration trajectory (2) for the parameter s_c , where the value of $\dot{s}_c = \dot{s}_{nom}$ is decelerated smoothly to $\dot{s}_c = 0$
- For addition and removal operations, where the number of agents after reconfiguration $N_d \neq N_c$, a motion of the agents along the formation ellipse is necessary to reconfigure to the destination Lissajous curve for N_d agents (described by constants (a_d, b_d, o_d)). Thus the parameter transformation is done to express the curve parameters (ψ_c, s_c) in terms of the destination Lissajous curve as (ψ_d, s_d) , for all agents.
- The agents are then assigned a destination position on the Lissajous curve for N_d agents by a cooperative assignment scheme, and this is followed by a motion of the agents along the formation ellipse to these assigned positions by variation of parameter ψ_d . Since the agents require to accelerate from rest from the Lissajous curve for N_c agents and decelerate back to rest on their assigned positions on the Lissajous curve for N_d agents, the parameter ψ_d is varied using the symmetric transition trajectory (8).
- Upon reaching the Lissajous curve for N_d agents, the agents now accelerate along this new Lissajous curve to $\dot{s}_d = \dot{s}_{nom}$ using the monotone trajectory in (2) for parameter s_d to resume normal surveillance operation.

The common steps involved in a reconfiguration operation are discussed in further detail below:

5.1.1. Monotone deceleration of s_c

When any one of the three reconfiguration operations is initiated at time T_R , all the formation agents are brought to a halt on the Lissajous curve for N_c agents. This is done by decelerating the \dot{s}_c to 0. One of the formation agents chosen as the reconfiguration initiator agent i_l , computes $\tilde{s}_f = s_c^i(T_R) + \frac{\pi}{8N_c}$. The choice of i_l for each reconfiguration operation is operation specific and will be discussed later. If \tilde{s} is used as the stopping s_c parameter value for all active formation agents the formation locus lies on the elliptical locus given by (17) for $s_c = \tilde{s}_f$.

For the replacement operation the final stopping value for agent i in the formation is selected as

$$s_f^i = s_{cf}^i = \tilde{s}_f, \quad (32)$$

and this value is communicated to all formation agents by agent i_l via the communication links. For the addition and removal applications, we intend to reconfigure the formation by moving along this ellipse. For agents having non-zero dimensions, collisions can occur for a narrow elliptical locus as shown in Fig. 5. Thus it is necessary to derive the range of parameter s_c for which motion along the ellipse should be prohibited, and

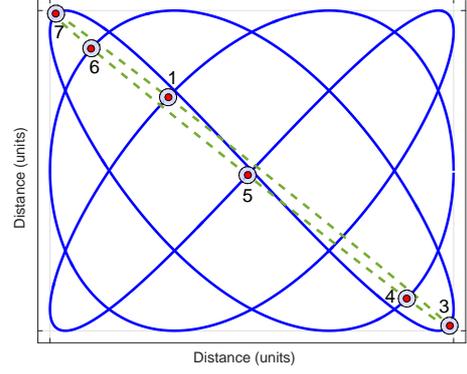


Figure 5: An agent removal example where agent 2 is removed and the remaining agents cannot have a collision-free transition along the dotted green elliptical locus due to their non-zero size

accordingly select an appropriate stopping parameter value \tilde{s}_f . To do this, we have the following proposition:

Proposition 1. For adjacent agents, i and j of an N_c agent formation having dimensions within a circular hull bound r_{dm} , and minimum ψ parameter separation $\Delta_{min}^{ij} = \min_{t \in \mathbb{R}^+} |\psi_c^j(t) - \psi_c^i(t)| > 0$, all elliptical loci corresponding to parameter

$$s_c \notin S_{avoid} \quad (33)$$

are feasible for collision-free agent transitions along these loci, where $S_{avoid} = \cup_{k \in \mathbb{N}} (s_{diag}(k) - \delta_s, s_{diag}(k) + \delta_s) \pmod{2\pi}$, with $s_{diag}(k) = \frac{(2k-1)\pi}{2N_c}$, $\delta_s = \frac{\pi}{2N_c} \frac{r_{dm} \sqrt{A^2+B^2}}{AB} \left| \sin\left(\frac{\Delta_{min}^{ij}}{2}\right) \right|^{-1}$.

The proof of Proposition 1 is given in the appendix.

Remark 1. In Proposition 1 if the value of Δ_{min}^{ij} decreases, the δ_s increases as it is proportional to $\left| \sin\left(\frac{\Delta_{min}^{ij}}{2}\right) \right|^{-1}$. If $\delta_s \geq \frac{\pi}{2N_c}$ then no ellipse is feasible for collision-free transitions as $S_{avoid} = [0, 2\pi)$. Thus to have a feasible elliptical locus for transitions we must ensure $\delta_s < \frac{\pi}{2N_c}$.

It will be shown in later sections that for addition $\Delta_{min}^{ij} = \frac{2\pi}{N_d}$ and for removal $\Delta_{min}^{ij} = \frac{2\pi}{N_c}$. Thus the stopping parameter value for agent i_l is chosen as:

$$s_{cf}^{i_l} = \begin{cases} s_{diag}(k') + \delta_s, & \text{if } \tilde{s}_f \in S_{avoid} \text{ for } k = k' \\ \tilde{s}_f, & \text{otherwise,} \end{cases} \quad (34)$$

where s_{diag} and S_{avoid} are as defined in Proposition 1. The value of $s_{cf}^{i_l}$ is communicated to the remaining agents in the formation using the communication links and

$$s_{cf}^i = s_{cf}^{i_l}. \quad (35)$$

We assume that the transition is initiated for all agents at the same time, i.e., $T_R^i \approx T_R^j$. The formation agents use the monotone transition trajectory given by (2) (discussed in Section 3) to smoothly decelerate to rest at $s_c^i = s_{cf}^i$. Thus from (7), the constants that characterize this trajectory for agent i are:

$$T_0 = T_R^i, \quad \dot{g}_0 = \dot{s}_{nom}, \quad \dot{g}_f = 0, \quad g_0 = s_c^i(T_0) \text{ and } g_f = s_{cf}^i.$$

5.1.2. Parameter transformation

This step is carried out by each formation agent after completion of the monotone deceleration of \dot{s}_c to 0 (discussed in Section 5.1.1). For brevity of notation, parameters $s_c(t)$ and $\psi_c(t)$, which are functions of time, are written as s_c and ψ_c respectively. This step is common to the agent addition and removal operations as both involve agent motion from Lissajous curve for N_c agents to the Lissajous curve for N_d agents along the formation ellipse, and is not necessary for agent replacement as $N_c = N_d$. For such a motion to be possible, the agent positions on the current Lissajous curve and the assigned positions on the Lissajous curve for N_d agents must lie on the same ellipse at any given instant. In other words, for (a_c, b_c, s_c) and (a_d, b_d, s_d) , (17) must result in the same ellipse. This implies that $(a_c + b_c)s_c = (a_d + b_d)s_d$, or,

$$s_d = \frac{a_c + b_c}{a_d + b_d} s_c. \quad (36)$$

From (15), the position of agent i on the Lissajous curve (a_c, b_c) in terms of parameters ψ_c^i and s_c^i is:

$$(x_i, y_i) = (A \cos(\psi_c^i - a_c s_c^i), B \sin(\psi_c^i + b_c s_c^i)),$$

where ψ_c^i is a constant. We can rewrite $\psi_c^i - a_c s_c^i = \psi_c^i + a_d s_d^i - a_c s_c^i - a_d s_d^i$ and $\psi_c^i + b_c s_c^i = \psi_c^i + b_c s_c^i - b_d s_d^i + b_d s_d^i$. From (36), substituting $s_c^i = \frac{a_d + b_d}{a_c + b_c} s_d^i$ leads to

$$\psi_c^i - a_c s_c^i = \psi_c^i - a_d s_d^i \text{ and } \psi_c^i + b_c s_c^i = \psi_c^i + b_d s_d^i,$$

where

$$\psi_d^i = \psi_c^i + \frac{a_d b_c - a_c b_d}{a_c + b_c} s_d^i \pmod{2\pi}. \quad (37)$$

Thus $(x_i, y_i) = (A \cos(\psi_d^i - a_d s_d^i), B \sin(\psi_d^i + b_d s_d^i))$ is unchanged under this transformation. Note that the number pairs (a_c, b_c) and (a_d, b_d) are co-prime and both $a_c = a_d$ and $b_c = b_d$ cannot hold simultaneously as $(N_c \neq N_d)$. This implies that $\frac{a_c}{b_c} \neq \frac{a_d}{b_d}$ which means $\frac{a_d b_c - a_c b_d}{a_c + b_c} \neq 0$. Thus we see that (36) and (37) transform the parameters s_c^i, ψ_c^i of the Lissajous curve with (a_c, b_c, o_c) to the parameters s_d^i, ψ_d^i of the Lissajous curve with (a_d, b_d, o_d) without affecting the position coordinates of the agents. Since agents are at rest, from (36), $\dot{s}_d^i = \dot{s}_c^i = 0$ and $\dot{\psi}_d^i = \dot{\psi}_c^i = 0$ for all $i \in \{1, \dots, N_c\}$.

Remark 2. The ψ parameter separation between agents i and j along the ellipse is remains unchanged under the parameter transformation in (37), i.e.,

$$\Delta^{ij}\psi(t) = \psi_c^j(t) - \psi_c^i(t) = \psi_d^j(t) - \psi_d^i(t). \quad (38)$$

5.1.3. Symmetric transition of ψ_d

For both agent addition and removal operations the formation agents move from the Lissajous curve for N_c agents to the Lissajous curve for N_d agents. Separate cooperative leader selection and transition assignment schemes are proposed for either operation in later sections that guarantee collision-free transition trajectories. These schemes assign a destination parameter

value ψ_D^i (on the Lissajous curve for N_d agents) to each agent i , from the set

$$\Psi_D = \left\{ \frac{2\pi(p-1)}{N_d} + \frac{o_d\pi}{2} \pmod{2\pi} : p \in \{1, \dots, N_d\} \right\}. \quad (39)$$

These correspond to the agent formation positions on the Lissajous curve for N_d agents from (15). Suppose the formation agent i is assigned destination parameter values at time $t = T_\Psi^i$. Each formation agent i must travel the parametric displacement to reach the assigned ψ_D^i value equal to

$$\Delta_\psi^i = \psi_D^i - \psi_d^i(T_\Psi), \quad (40)$$

For both addition and removal, the cooperative leader selection and transition assignment schemes also communicate necessary information to the formation agents that allows them to compute

$$\Delta_{max} = \max_{i \in \{1, \dots, N_d\}} |\Delta_\psi^i|. \quad (41)$$

The agents then use the symmetric transition trajectory given by (8) to move to the positions corresponding to the assigned ψ_D^i values. For this transition, $\dot{s}_d^i = 0$ and as a result the agent speed depends only on $|\dot{\psi}_d^i|$. From (20), $V^i < \sqrt{A^2 + B^2} |\dot{\psi}_d^i|$. Therefore, $V^i < V_{max}$, if $|\dot{\psi}_d^i| \leq \dot{\psi}_{max}$, where $\dot{\psi}_{max} = \frac{V_{max}}{\sqrt{A^2 + B^2}}$. Thus from (12), considering peak parametric speed as $\dot{g}_{max} = \dot{\psi}_{max}$ for parametric interval $|g_f - g_0| = \Delta_{max}$, rearrangement of the equation gives

$$T_p = \frac{15\Delta_{max}}{8V_{max}} \sqrt{A^2 + B^2}.$$

Using T_p as transition time period in (8) for the agent $i_m = \arg \max_{i \in \{1, \dots, N_d\}} |\Delta_\psi^i|$ guarantees that its maximum parametric speed is $\dot{\psi}_{max} = \frac{15\Delta_{max}}{8T_p}$ (hence limiting physical speed below V_{max}). We also use T_p as the transition time period for the remaining formation agents. Thus from (12), for all $i \neq i_m$, $|\dot{\psi}_{d_{max}}^i| = |\dot{\psi}_{d_{min}}^i| = \frac{15\Delta_\psi^i}{8T_p} \leq \frac{15\Delta_{max}}{8T_p} = \dot{\psi}_{max}$. Hence speeds of all the agents are bounded above by V_{max} .

From (13), the constants that characterise the symmetric trajectory in (8) for agent i are

$$T_0 = T_\Psi^i, T_p = \frac{15\Delta_{max} \sqrt{A^2 + B^2}}{8V_{max}}, g_0 = \psi_d^i(T_0), g_f = \psi_D^i. \quad (42)$$

For both addition and removal the destination parameter values in Ψ_D are assigned such that the symmetric transitions along the ellipse are in the same direction. Assuming negligible communication delays, the transition start times for formation agents i, j ($i \neq j$) satisfy $T_\Psi^i \approx T_\Psi^j = T_0$. As a consequence we have the following result for the relative parametric displacement $\Delta^{ij}\psi$:

Proposition 2. For formation agents i, j moving along the symmetric transition trajectories for parameter ψ_d characterised by (42) for the time window $t \in [T_0, T_f]$, $\Delta^{ij}\psi(t)$ given by (38) is monotone in nature and achieves its extremal values at $t = T_0, T_f$.

Proof: Since symmetric transition trajectory for reconfiguration is initiated at $T_0 = T_\psi^i \approx T_\psi^j$ and ends at $T_f = T_0 + T_p$, from (9), the relative parametric speed between agents i and j , is given by

$$\Delta^{ij}\dot{\psi}(t) = \dot{\psi}_d^j(t) - \dot{\psi}_d^i(t) = \frac{30(\Delta_\psi^j - \Delta_\psi^i)}{T_p^5} \Delta t^2 (T_p - \Delta t)^2, \quad (43)$$

where $\Delta t = t - T_0$, Δ_ψ^j and Δ_ψ^i are given by (40). Thus from (43), for $t \in (T_0, T_f)$, $\Delta^{ij}\dot{\psi}(t) > 0$, if $\Delta_\psi^j - \Delta_\psi^i > 0$ and $\Delta^{ij}\dot{\psi}(t) < 0$, if $\Delta_\psi^j - \Delta_\psi^i < 0$. This proves that $\Delta^{ij}\psi(t) = \psi_d^j(t) - \psi_d^i(t)$ is monotone in nature and its extremal values are attained at $t = T_0, T_f$. \square

5.1.4. Monotone acceleration of s_d

This step is common to all three reconfiguration operations. After completing the symmetric transition trajectory along the elliptical locus for agent addition and removal operations, and agent exchange for the replacement operation, all the N_d agents have reached their destination ψ_d^i values on the Lissajous curve selected for N_d agents, and are at rest. Now the agents accelerate along this curve using the monotone transition trajectory for parameter s_d given by (2) to resume performing the proposed surveillance strategy with N_d agents with parametric speed $\dot{s}_d = \dot{s}_{nom}$ according to (21) for the new Lissajous curve with (a_d, b_d, o_d) . This is initiated by an initiator agent i_l (operation specific) via the communication links. We assume that the transition is initiated by the leader i_L at $T_0 = T_a^i$ for agent i (where $T_a^i \approx T_a^j$ for $i \neq j$ and $i, j \in \{1, \dots, N_d\}$). From (7) the constants that specify this trajectory in (2) for each agent i are

$$T_0 = T_a^i, \quad g_0 = \dot{s}_d^i(T_0), \quad g_f = g_0 + \frac{\pi}{8N_d}, \quad \dot{g}_0 = 0, \quad \dot{g}_f = \dot{s}_{nom}. \quad (44)$$

5.1.5. Symmetric transition to way-point

Here an agent moves from its initial position P_0 with coordinates (x_0, y_0) , to a final way-point position P_f with coordinates (x_f, y_f) by performing a symmetric transition trajectory (given by (8)) for the displacement along the vector $\overrightarrow{P_0 P_f}$ (having length $d_f = \sqrt{(x_f - x_0)^2 + (y_f - y_0)^2}$). Assuming this transition is done over a time window $[T_0, T_f]$ with $T_f = T_0 + T_p$ where $T_p \geq \frac{15d_f}{8V_{max}}$, then from (12) selecting the constants

$$T_0 = T_0^i, \quad T_p = T_p^i, \quad g_0 = 0 \quad \text{and} \quad g_f = d_f$$

in (13) for the trajectory given by (8), ensures that the agent speed is bounded above by V_{max} .

We now discuss the reconfiguration steps involved in each of the three operations separately.

5.2. Agent removal

The removed agent i_r has two adjacent neighbours i_p and i_n on the formation ellipse with parameter values $\psi_c^{i_n} = \psi_c^{i_r} + \frac{2\pi}{N_c} \bmod 2\pi$ and $\psi_c^{i_p} = \psi_c^{i_r} - \frac{2\pi}{N_c} \bmod 2\pi$ respectively. The algorithmic sketch of the steps for the agent removal operation are as follows:

Initial condition: Proposed formations of N_c agents moving on the Lissajous curve for surveillance at altitude h_F .

- 1: **Removal initialisation:** Agent i_r stops communication lowers altitude at $t = T_R$ to $h_L < h_F$, and returns to base to land. Number of formation agents remaining $N_d = N_c - 1$.
- 2: **Monotone deceleration of parameter s_c :** initiated by agent i_n for N_d formation agents.
- 3: **Parameter transformation:** from (s_c, ψ_c) to (s_d, ψ_d) , done when $\dot{s}_c^i = 0$ for all formation agents i .
- 4: **Leader selection:** Leader agent i_L selected from $\{i_n, i_p\}$
- 5: **Transition assignment:** Destination positions on the Lissajous curve for N_d agents are assigned to formation agents by leader i_L
- 6: **Symmetric transition trajectory of parameter ψ_d :** Agents move along formation ellipse to reach assigned destination positions on the Lissajous curve for N_d agents.
- 7: **Monotone acceleration of parameter s_d :** initiated by agent i_n after the previous step, N_d formation agents accelerate along the Lissajous curve for N_d agents to resume area surveillance.

The reconfiguration steps for the formation agents unique to the agent removal operation are as follows:

5.2.1. Removal initialisation

When agent i_r is removed from the formation of N_c agents at time $t = T_R$, its next agent i_n alerts the remaining agents about the removal of agent i_r via the communication links. We assume that the remaining $N_d = N_c - 1$ formation agents (having indices $j \in \{1, \dots, N_c\} \setminus \{i_r\}$) are updated about the removal at time $t = T_R^j \approx T_R$. This is followed by the monotone deceleration of the parameter s_c for which agent i_n is the initiator agent i_l (refer Section 5.1.4).

5.2.2. Leader selection and transition assignment

For the agent removal operation $N_d = N_c - 1$, assuming agent i_r is removed, the leader agent i_L is selected from the agents i_n and i_p (neighbours of i_r in the formation with parameter values $\psi_c^{i_n} = \psi_c^{i_r} + \frac{2\pi}{N_c} \bmod 2\pi$ and $\psi_c^{i_p} = \psi_c^{i_r} - \frac{2\pi}{N_c} \bmod 2\pi$, respectively). If agent i_n is selected as the leader (as shown in Fig 6) then after the parameter transformation, the direction of transition along the ellipse is selected in the direction of the increasing ψ_d parameter (i.e., $\dot{\psi}_d > 0$) as shown in the Fig. 6, as it is guaranteed to have an adjacent agent within its communication range of r_{com} (at parameter $\psi_c = \psi_c^{i_n} + \frac{2\pi}{N_c} \bmod 2\pi$ before the parameter transformation). Similarly, if agent i_p is selected as the leader then the direction of transition along the ellipse is selected in the direction of the decreasing ψ_d parameter (i.e., $\dot{\psi}_d < 0$). The destination parameter values on the Lissajous curve for N_d agents lie in the set Ψ_D given by (39). The element in Ψ_D which is nearest to i_L along the chosen direction of transition is selected as the destination parameter value for the leader i_L . For agent i_n (shown in Fig. 6), the nearest element of Ψ_D encountered along the $\dot{\psi}_d > 0$ direction is at $\psi_{cl}^n = \left[\left(\psi_d^{i_n} - \frac{o_d\pi}{2} \right) \frac{N_d}{2\pi} \right] \frac{2\pi}{N_d} + \frac{o_d\pi}{2}$, and parametric interval $\delta_\psi^{i_n} = \psi_{cl}^n - \psi_d^{i_n}$ as shown in Fig. 6. Similarly for agent i_p , the nearest element of Ψ_D encountered along the

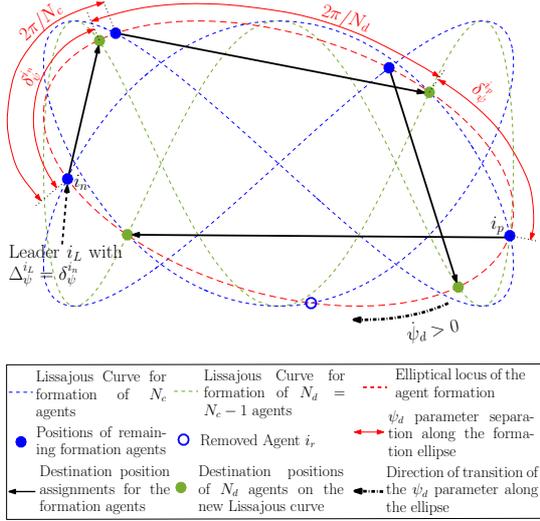


Figure 6: Transition assignment example for the removal of an agent from a $N_c = 5$ agent formation for leader $i_L = i_n$.

$\dot{\psi}_d < 0$ direction is at $\psi_{cl}^p = \left\lfloor \left(\psi_d^{i_p} - \frac{\alpha_d \pi}{2} \right) \frac{N_d}{2\pi} \right\rfloor \frac{2\pi}{N_d} + \frac{\alpha_d \pi}{2}$, and parametric interval $\delta_{\psi}^{i_p} = \psi_d^{i_p} - \psi_{cl}^p$ as shown in Fig. 6. Both agents i_n and i_p compute the values of $\delta_{\psi}^{i_p}$ and $\delta_{\psi}^{i_n}$, and the leader agent i_L is selected as follows:

$$i_L = \begin{cases} i_n, & \text{if } \delta_{\psi}^{i_n} \leq \delta_{\psi}^{i_p}, \\ i_p, & \text{if } \delta_{\psi}^{i_n} > \delta_{\psi}^{i_p}. \end{cases} \quad (45)$$

The destination parameter $\psi_D^{i_L}$ and the corresponding parametric distance $|\Delta_{\psi}^{i_L}|$ (given by (40)) for the leader i_L is selected as

$$(\psi_D^{i_L}, |\Delta_{\psi}^{i_L}|) = \begin{cases} (\psi_{cl}^{i_n}, \delta_{\psi}^{i_n}), & \text{if } i_L = i_n, \\ (\psi_{cl}^{i_p}, \delta_{\psi}^{i_p}), & \text{if } i_L = i_p. \end{cases} \quad (46)$$

As a consequence of choices in (45) and (46), the selected leader has smaller value of the parametric transition distance $|\Delta_{\psi}^{i_L}|$ assigned for reconfiguration. The leader communicates $\psi_D^{i_L}$ and $\psi_D^{i_L}$ values to the remaining formation agents using the communication links between adjacent agents. The subsequent values in Ψ_D are assigned as destination parameter values to the subsequent agents, in the sequence in which they are encountered on the formation ellipse along the $\dot{\psi}_d < 0$ direction for $i_L = i_p$ and $\dot{\psi}_d > 0$ direction for $i_L = i_n$ (shown in Fig. 6). Mathematically this can be written as

$$\psi_D^i = \begin{cases} \psi_D^{i_L} + \frac{2\pi n_i}{N_d} \bmod 2\pi, & \text{if } i_L = i_n, \\ \psi_D^{i_L} - \frac{2\pi n_i}{N_d} \bmod 2\pi, & \text{if } i_L = i_p, \end{cases} \quad (47)$$

where n_i is the count of agent i relative to i_L along the transition direction, and is computed as

$$n_i = \begin{cases} (\psi_D^{i_L} - \psi_D^i \bmod 2\pi) \frac{N_c}{2\pi}, & \text{if } i_L = i_n, \\ (\psi_D^i - \psi_D^{i_L} \bmod 2\pi) \frac{N_c}{2\pi}, & \text{if } i_L = i_p. \end{cases} \quad (48)$$

Here $n_i \leq N_d - 1$ is an integer because prior to any transition along the ellipse, $|\psi^i - \psi^{i_L}|$ is an integer multiple of $\frac{2\pi}{N_c}$ for the agents $i \in \{1, \dots, N_c\} \setminus \{i_r, i_L\}$.

For the destination parameter assignment given by (46) and (47), $|\Delta_{\psi}^i|$ (given by (40)) is the parametric transition distance that each agent must move along the formation ellipse to reach the assigned destination value ψ_D^i . Suppose the leader agent $i_L = i_n$. Then as illustrated in Fig. 6, the values of $|\Delta_{\psi}^i|$ for agents $i \in \{1, \dots, N_c\} \setminus \{i_r\}$ are given by

$$|\Delta_{\psi}^i| = |\Delta_{\psi}^{i_L}| + n_i \left(\frac{2\pi}{N_d} - \frac{2\pi}{N_c} \right). \quad (49)$$

Equation (49) also holds for the case $i_L = i_p$. From (49), for the agent removal case, the longest parametric transition distance Δ_{max} in (41) is obtained for $n_i = N_d - 1$ and $N_d = N_c - 1$, and is given by

$$\Delta_{max} = |\Delta_{\psi}^{i_L}| + 2\pi \frac{N_c - 2}{N_c(N_c - 1)}.$$

This is followed by the symmetric transition in the ψ_d parameter for the N_d formation agents as discussed in Section 5.1.3.

Claim 2. The destination parameter assignment scheme given by (47) guarantees collision-free symmetric transition trajectories in parameter ψ_d for the N_d formation agents.

Proof: Prior to the agent removal, $|\Delta^{ij}\psi(t)| = |\psi_c^j(t) - \psi_c^i(t)| = \frac{2\pi}{N_c}$ for adjacent agents i and j . We assume that agent j succeeds agent i along the selected direction of transition, i.e., i, j satisfy $n_j = n_i + 1$ in (48). Thus in the time interval $[T_0, T_f]$ for the symmetric transition of formation agents, with $T_0 = T_{\psi}^i \approx T_{\psi}^j$ and $T_f = T_0 + T_p$, the initial and final parametric displacement (in parameter ψ_d) from agent i to the adjacent agent on the formation ellipse along the direction of transition is given by:

$$\Delta^{ij}\psi(T_0) = \begin{cases} \frac{4\pi}{N_c} & \text{if } i = i_p, \\ \frac{2\pi}{N_c} & \text{if } i \neq i_p, \end{cases} \Delta^{ij}\psi(T_f) = \frac{2\pi}{N_d} \text{ for } i_L = i_n, \quad (50)$$

$$\Delta^{ij}\psi(T_0) = \begin{cases} -\frac{4\pi}{N_c} & \text{if } i = i_n, \\ -\frac{2\pi}{N_c} & \text{if } i \neq i_n, \end{cases} \Delta^{ij}\psi(T_f) = \frac{-2\pi}{N_d} \text{ for } i_L = i_p. \quad (51)$$

From Proposition 2, we know that $\Delta^{ij}\psi(t)$ is monotone for $t \in [T_0, T_f]$ and attains its maximum and minimum values at T_0 or T_f . Thus from (50) and (51), the minimum value of $|\Delta^{ij}\psi(t)| = \Delta_{min}^{ij} = \frac{2\pi}{N_c}$ for the agent removal operation. From Proposition 1 and (31), $\delta_s = \frac{\pi}{2N_c} \frac{\sqrt{A^2+B^2} \sin(\frac{\pi}{N_{max}})}{\sqrt{A^2a_j^2+B^2b_j^2} \left| \sin(\frac{\pi}{N_c}) \right|}$.

Since $N_c \leq N_{max}$ and (a_j, b_j) are co-prime positive integers, $\delta_s < \frac{\pi}{2N_c}$. Moreover, as the formation agents stop on a feasible ellipse with $s_c \neq S_{avoid}$ (which is ensured as discussed in Section 5.1.1), the symmetric transition trajectories in the parameter ψ_d for the proposed transition assignment scheme are collision-free. \square

5.3. Agent addition

For the agent addition operation, i_p and i_n are the formation agents parametrically preceding and succeeding the assigned formation position for added agent i_a respectively. The algorithmic sketch of the steps for adding agent i_a to a formation of N_c agents to get a formation of $N_d = N_c + 1$ agents are as follows:

Initial condition: Proposed formations of N_c agents moving on the Lissajous curve for surveillance at altitude h_F .

- 1: **Addition Initialisation:** Agent i_a waiting at height h_L , alerts the closest formation agent i_c to initiate the agent addition operation when in communication range.
- 2: **Monotone deceleration of parameter s_c :** initiated by agent i_c at $t = T_R$ for all formation agents.
- 3: **Parameter update for added agent:** The formation agents cooperatively calculate the s_d^i, ψ_d^i parameters for the formation position of agent i_a , and communicate the same to i_a . i_a moves to its assigned formation position at height h_L .
- 4: **Parameter transformation:** From (s_c, ψ_c) to (s_d, ψ_d) , done for formation agents i at height h_F when $s_c^i = 0$.
- 5: **Leader selection:** Leader agent i_L is selected from $\{i_n, i_p\}$.
- 6: **Transition assignment:** Destination positions on the Lissajous curve for N_d agents are assigned to formation agents by i_L .
- 7: **Symmetric transition trajectory of parameter ψ_d :** Agents move along formation ellipse to reach assigned destination positions on new Lissajous curve.
- 8: **Agent i_a enters formation:** During the symmetric transition trajectory, i_a rises to height h_F when the ascent is collision-free.
- 9: **Monotone acceleration of parameter s_d :** initiated by agent i_a after the previous two steps. The N_d formation agents accelerate along the Lissajous curve for N_d agents to resume area surveillance.

We now discuss the reconfiguration steps specific to the addition operation for the added agent i_a and the formation agents.

5.3.1. Addition initialisation

The agent to be added to the formation is launched from the home base and it hovers at altitude $h_L < h_F$. When formation agent i_c is in communication range, it is alerted by i_a to initiate the agent addition operation. Agent i_c then initiates the monotone deceleration of the parameter s_c for the N_c formation agents.

5.3.2. Parameter update for added agent

The formation agents calculate the position of entry on the elliptical locus for the added new agent and communicate the corresponding ψ_d and s_d values to the added agent. The calculation of these values is done considering Lemma 2 (proof given in the Appendix).

Lemma 2. Suppose P_1, P_2, \dots, P_N are N parametrically equi-spaced points on a closed curve C with the convention $P_{N+1} = P_1$, and Q_1, Q_2, \dots, Q_{N+1} are $N + 1$ parametrically equi-spaced points on the same curve C with convention $Q_{N+2} = Q_1$. Then there is exactly one pair of adjacent Q points contained in the interval $[P_i, P_{i+1})$ of adjacent P points for some $i \in \{1, \dots, N\}$.

From (15), we know that for the proposed formation, the agent positions on the Lissajous curve partition the elliptical locus in parametrically equal parts in terms of parameter ψ_c . Thus from Lemma 2, at any point in time exactly two adjacent agent positions on the Lissajous curve for $N_d = N_c + 1$

agents (shown as green spots in Fig. 7) must lie between two adjacent agent positions on the Lissajous curve for N_c agents (shown as blue spots in Fig. 7) along the elliptical locus. Hence

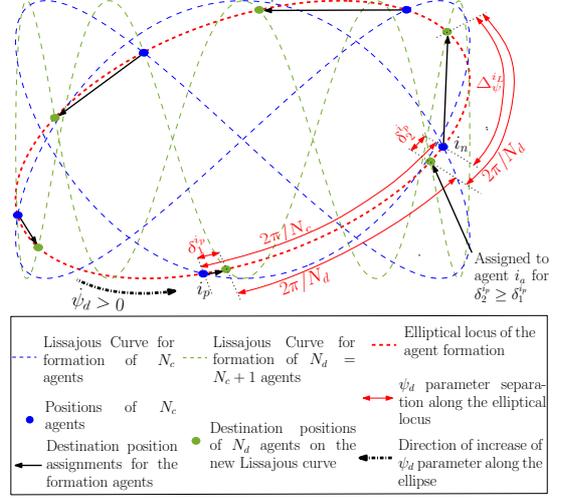


Figure 7: Transition assignment example for the addition of an agent from a $N_c = 5$ agent formation for leader $i_L = i_n$.

after the calculation of the stopping parameter value s_f^i as discussed in Section 5.1.1, each of the formation agents computes the transformed values s_d^i and ψ_d^i of the stopping parameter value $s_f^i = s_c^i$ and ψ_c^i using (36) and (37) respectively. Defining Ψ_D as in (39), each formation agent then computes the following terms:

$$\delta_1^i = \left[\left(\psi_d^i - \frac{o_d \pi}{2} \right) \frac{N_d}{2\pi} \right] \frac{2\pi}{N_d} + \frac{o_d \pi}{2} - \psi_d^i, \quad (52)$$

$$\delta_2^i = \frac{2\pi}{N_c} - \left(\delta_1^i + \frac{2\pi}{N_d} \right), \quad (53)$$

where δ_1^i is the parameter separation between ψ_d^i and the closest destination position on the Lissajous curve for N_d agents (from the set Ψ_D) along the $\psi_d > 0$ direction. As a consequence of Lemma 2, the value of $\delta_2^i > 0$ for exactly one of the formation agents (as shown in Fig. 7). We call this agent as agent i_p . Agent i_p then selects the value of the destination parameter ψ_D^a (from the set Ψ_D) for the entry position of the added agent i_a as follows:

$$\psi_D^a = \begin{cases} \psi_d^{i_p} + \delta_1^{i_p}, & \text{if } \delta_1^{i_p} > \delta_2^{i_p} \\ \psi_d^{i_p} + \frac{2\pi}{N_c} - \delta_2^{i_p}, & \text{if } \delta_1^{i_p} \leq \delta_2^{i_p}. \end{cases} \quad (54)$$

It will be shown later in the transition assignment step that this choice ensures a shorter transition along the elliptical locus for the formation agents to reconfigure to the Lissajous curve for N_d agents. Assuming that the reconfiguration is initiated at time T_R , the values of N_c , ψ_D^a , $s_c^i(T_R)$ and $s_{cf}^a = s_{cf}^i$ are communicated to agent i_a by the formation agents. Agent i_a transforms s_{cf}^a to s_{df}^a using (36), and also calculates coordinates of the entry point into the formation using (15) as

$$(x_E^a, y_E^a) = \left(A \cos(\psi_D^a - a_d s_{df}^a), B \sin(\psi_D^a + b_d s_{df}^a) \right). \quad (55)$$

It then computes the time period for monotone deceleration of $s_c^{i_c}$ for agent i_c using (6) as $T_p^{i_c} = \frac{2(s_{cf}^{i_c} - s_c^{i_c}(T_R^i))}{\delta_{nom}}$. Agent i_a then moves to position given by (55) using the symmetric transition for a way-point (discussed in Section 5.1.5) with transition period $T_p = \max\left(T_p^{i_c}, \frac{15d_f}{8v_{max}}\right)$. This ensures that the added agent i_a does not reach its formation position before the formation agents decelerate to a halt with $\dot{s}_c = 0$.

5.3.3. Leader selection and transition assignment

After the parameter transformation step is completed for the formation agents at height h_F , and the agent i_a has reached formation position given by (55) at height $h_L < h_F$, the formation agents must be assigned a destination ψ_D^j parameter values on the Lissajous curve for $N_d = N_c + 1$ agents, from the set Ψ_D given by (39), so that they can transition along the formation ellipse t these locations for reconfiguration. This can be achieved by motion along the $\dot{\psi}_d > 0$ or the $\dot{\psi}_d < 0$ direction.

The parameter value ψ_d^i of the formation agent i on the Lissajous curve for N_c agents (corresponding to the blue spots in Fig. 7) and elements of set Ψ_D , corresponding to the location of the formation agents on the Lissajous curve for $N_d = N_c + 1$ agents (green spots in Fig. 7), are equi-spaced on the formation ellipse, with a parametric separation of $\frac{2\pi}{N_c}$ and $\frac{2\pi}{N_d}$ respectively. In Section 5.3.2 from Lemma 2, agent i_p was identified as the only formation agent having exactly one pair of destination values from Ψ_D in the interval $[\psi_d^{i_p}, \psi_d^{i_p} + \frac{2\pi}{N_c})$ (i.e., with $\delta_2^{i_p} > 0$ in (53)). We call the adjacent agent parametrically succeeding i_p as i_n , having parameter value $\psi_d^{i_n} = \psi_d^{i_p} + \frac{2\pi}{N_c} \bmod 2\pi$. From (54), the formation position of agent i_a lies between i_p and i_n on the formation ellipse. The leader agent to initialise the direction of transition assignment is chosen as

$$i_L = \begin{cases} i_p, & \text{if } \delta_1^{i_p} > \delta_2^{i_p} \\ i_n, & \text{if } \delta_1^{i_p} \leq \delta_2^{i_p} \end{cases} . \quad (56)$$

Thus the destination values from Ψ_D are assigned to the formation agents as the closest value along $\dot{\psi}_d > 0$ for $i_L = i_n$, and along $\dot{\psi}_d < 0$ for $i_L = i_p$. This is mathematically written as

$$\psi_D^i = \begin{cases} \left(\psi_d^i - \frac{o_d\pi}{2}\right) \frac{N_d}{2\pi} \left\lfloor \frac{2\pi}{N_d} + \frac{o_d\pi}{2} \text{ if } i_L = i_p, \\ \left(\psi_d^i - \frac{o_d\pi}{2}\right) \frac{N_d}{2\pi} \left\lceil \frac{2\pi}{N_d} + \frac{o_d\pi}{2} \text{ if } i_L = i_n. \end{cases} \quad (57)$$

As a consequence of Lemma 2, outside the segment $[\psi_d^{i_p}, \psi_d^{i_n})$, the positions corresponding to values of ψ_d^i and elements of Ψ_D alternate along the elliptical locus as shown in Fig. 7. Thus this assignment yields parametrically non-overlapping transition intervals as shown in Fig. 7. The destination parameter values from Ψ_D are assigned according to (47) and (48) (as in the removal case).

Also, the magnitude of the parametric transition distance $|\Delta_\psi^i|$ (given by (40)) for agent i can be expressed in terms of the parametric transition distance of the leader $|\Delta_\psi^{i_L}|$ using (49), where n_i is calculated according to (48). If $i_L = i_n$ as shown in Fig. 7, then $|\Delta_\psi^{i_L}| = |\Delta_\psi^{i_n}| = \frac{2\pi}{N_d} - \delta_2^{i_p}$. Similarly if $i_L = i_p$, then $|\Delta_\psi^{i_L}| = |\Delta_\psi^{i_p}| = \frac{2\pi}{N_d} - \delta_1^{i_p}$. Thus the leader selection in (56) ensures

that the transition direction corresponding to the shorter parametric transition distance is selected. Since $N_d = N_c + 1$ from (49), the longest transition interval for any formation agent for the agent addition case is given by

$$\Delta_{max} = \Delta_\psi^{i_L}.$$

This is followed by the symmetric transition in the ψ_d parameter for the N_c formation agents at formation height h_F (as discussed in Section 5.1.3).

Claim 3. *The destination parameter assignment scheme given by (47) guarantees collision-free symmetric transition trajectories in parameter ψ_d for the N_c formation agents at formation height h_F .*

Proof: We discuss collision-free transition between i_a (waiting at formation height h_L) and leader i_L separately in the subsection 5.3.4. The remaining N_c agents are at formation altitude h_F and equi-parametrically spaced along the elliptical locus prior to any transitions in parameter ψ_d at time $t = T_0$. We assume agent j succeeds agent i along the selected direction of transition, i.e., i, j satisfy $n_j = n_i + 1$ in (48). Thus the initial value of $\Delta^{ij}\psi$ (given in (38)) for adjacent formation agents i and j prior to reconfiguration is

$$\Delta^{ij}\psi(T_0) = \begin{cases} \frac{2\pi}{N_c} \text{ if } i_L = i_n, \\ -\frac{2\pi}{N_c} \text{ if } i_L = i_p. \end{cases} \quad (58)$$

After the symmetric transition is over, i_a is at its formation position at height h_F and we have N_d agents at their assigned parameter values given by (57). As a consequence of Lemma 2, outside the segment $[\psi_d^{i_p}, \psi_d^{i_n})$, the positions corresponding to values of ψ_d^i and elements of Ψ_D assigned by (57) alternate along the elliptical locus as shown in Fig. 7. Thus after the completion of the symmetric transition,

$$\Delta^{ij}\psi(T_f) = \psi_D^j - \psi_D^i = \begin{cases} \frac{2\pi}{N_d} \text{ if } i_L = i_n, \\ -\frac{2\pi}{N_d} \text{ if } i_L = i_p. \end{cases} \quad (59)$$

As a consequence of Proposition 2, from (58) and (59), the minimum value of $|\Delta^{ij}\psi(t)| = \Delta_{min}^{ij} = \frac{2\pi}{N_d}$ for the agent addition operation. From Proposition 1 and (31), $\delta_s = \frac{\pi}{2N_c} \frac{\sqrt{A^2+B^2}}{\sqrt{A^2a_j^2+B^2b_j^2}} \frac{\sin(\frac{\pi}{N_{max}})}{\left|\sin(\frac{\pi}{N_d})\right|}$.

Since $N_d \leq N_{max}$ and (a_j, b_j) are co-prime positive integers, $\delta_s < \frac{\pi}{2N_c}$. Moreover, as the formation agents stop on a feasible ellipse with $s_c \neq S_{avoid}$ (which is ensured as discussed in Section 5.1.1), the symmetric transition trajectories in the parameter ψ_d for the proposed transition assignment scheme are collision-free. \square

5.3.4. Entry of agent i_a in the formation

Recall that agent i_a is still waiting at its formation position at $h_L < h_F$. As a consequence of leader selection in (56) and parameter assignment in (54), agent i_L is the nearest formation agent to agent i_a in terms of the parametric separation in ψ_d

before the commencement of the transition along the formation ellipse. During the symmetric transition in parameter ψ_d , agent i_a rises to formation altitude h_F when its Euclidean distance from agent i_L is greater than $2r_{dm}$. This guarantees the collision-free rise of agent i_a to result in a formation of $N_d = N_c + 1$ agents at height h_F at the end of the symmetric transition trajectory discussed in Section 5.1.3.

5.4. Agent replacement

The agent replacement operation replaces formation agent i_r with a new agent i_R . The algorithmic sketch of the steps for replacing an agent in the formation of N_c agents are as follows:

Initial condition: Proposed formations of N_c agents moving on the Lissajous curve for surveillance.

- 1: **Replacement Initialisation:** Agent i_R initialised with id of agent i_r , and waiting at height $h_L < h_F$, alerts the closest formation agent i_c to initiate the agent replacement operation.
- 2: **Monotone deceleration of parameter s_c :** initiated by agent i_c for all formation agents.
- 3: **Parameter update for agent i_R :** Formation agent i_r (at height h_F) communicates the $s_c^{i_r}, \psi_c^{i_r}$ parameters to agent i_R . i_R moves to the position of i_r at height $h_L < h_F$.
- 4: **Position exchange of i_r and i_R :** This is done after the previous two steps are complete. Agent i_r returns to base and lands.
- 5: **Monotone acceleration of parameter $s_d = s_c$:** initiated by agent i_R after the previous step, N_c formation agents accelerate along the Lissajous curve for N_c agents to resume area surveillance.

We discuss the reconfiguration steps unique to the replacement of the formation agent i_r by agent i_R below.

5.4.1. Replacement initialisation

The agent i_R is initialised with the id i_r of the formation agent that it is meant to replace. Agent i_R takes off from the base location and waits at height $h_L < h_F$ for the formation agents (at height h_F) to approach. When the closest formation agent i_c is within communication range, agent i_R alerts agent i_c to initiate the agent replacement operation. Also, agent i_R communicates the id of the formation agent i_r to the formation agents via agent i_c . Agent i_c then initiates the monotone deceleration of the parameter s_c (discussed in Section 5.1.1) for the N_c formation agents.

5.4.2. Parameter update of agent i_R

With the initiation of the monotone deceleration trajectory for s_c , the agent i_r sends its parameter $\psi_c^{i_r}, s_c^{i_r}(T_R)$ and parameter $s_{cf}^{i_r}$ corresponding to the stopping formation ellipse, to agent i_R using the communication links via agent i_c . Using (15), the agent i_R at height h_L calculates coordinates directly below agent i_r at height h_F as

$$(x_E^{i_R}, y_E^{i_R}) = (A \cos(\psi_c^{i_r} - a_c s_c^{i_r}), B \sin(\psi_c^{i_r} + b_c s_c^{i_r})). \quad (60)$$

Similar to the addition case (in Section 5.3.2) the agent i_R computes time period for monotone deceleration of $s_c^{i_r}$ using

(6) as $T_p^{i_r} = \frac{2(s_c^{i_r} - s_c^{i_r}(T_R))}{s_{nom}}$, and moves to the position given by (60) using the symmetric transition for a way-point (discussed in Section 5.1.5) with transition period $T_p = \max\left(T_p^{i_r}, \frac{15d_f}{8V_{max}}\right)$.

5.4.3. Position exchange of i_r and i_R

After the formation agents decelerate to rest at height h_F , and agent i_R reaches the position coordinates (60) at height $h_L < h_F$ (say at time $t = T_0^s$), then the agents i_R and i_r are both at rest at the same position coordinates, and are separated in altitude by distance $h_F - h_L$. In Section 4.2, we have seen that for a fixed value of $s = s_o$, $\mathcal{E}(\psi) = [A \cos(\psi - a_s o) B \sin(\psi + b_s o)]^T$ is the parametric equation of an ellipse with parameter ψ . Thus the tangent vector is given by $\mathcal{T}(\psi) = [-A \sin(\psi - a_c s_o) B \cos(\psi + b_c s_o)]^T$. Consider the vector $\mathcal{N}(\psi) = \pm[B \cos(\psi + b_c s_o) A \sin(\psi - a_c s_o)]^T$. Then the inner product $\langle \mathcal{N}(\psi), \mathcal{T}(\psi) \rangle = 0$ for all ψ , which implies that it gives the direction of the local normal to the ellipse. If $\mathcal{N}(\psi)$ is an outward normal to the elliptical locus at the point $\mathcal{E}(\psi)$, then $\langle \mathcal{N}(\psi), \mathcal{E}(\psi) \rangle = AB \cos(N_c s_o) \geq 0$, because the elliptical locus is always centered at the origin, and is a convex curve. Thus to ensure the selection of the outward normal, $\mathcal{N}(\psi)$ is chosen as

$$\mathcal{N}(\psi) = \text{sign}(\cos(N_c s_o)) \begin{bmatrix} B \cos(\psi + b_s o) \\ A \sin(\psi - a_s o) \end{bmatrix} \quad (61)$$

where $\text{sign}(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0 \end{cases}$.

The agent i_r computes a way-point on unit outward normal direction $\hat{\mathcal{N}}(\psi) = \frac{\mathcal{N}(\psi)}{\|\mathcal{N}(\psi)\|}$ at distance $d_f^{i_r} = 3r_{dm}$, and moves to this point using the symmetric trajectory to a waypoint (discussed in Section 5.1.5) with $T_p = \frac{15d_f}{4V_{max}}$. Upon completion of this motion, the agent i_r alerts agent i_R , which rises to the formation height h_F and agent i_r simultaneously reduces its altitude to h_L . The formation containing agent i_R , then performs monotone acceleration of parameter $s_d = s_c$ along the Lissajous curve for N_c agents (discussed in Section 5.1.4) and agent i_r returns to base to land.

Remark 3. In both reconfiguration operations of removal and replacement, the removed or replaced agent i_r is at a height $h_L < h_F$ at the end of the reconfiguration and is made to return to the base and land, using the symmetric trajectory to a way-point (discussed Section 5.1.5) with $T_p = \frac{15d_f}{8V_{max}}$.

6. Simulation and experimental validation

The proposed surveillance strategy of using an elliptical formation of multiple agents on a Lissajous curve discussed in Section 4 was validated through simulation and experiments with differential drive robots in prior work in [2]. The video of these experiments and simulation can be found at the web-link: <https://youtu.be/rhygE32UD08>

The reconfiguration scheme for the formation discussed in Section 5 is validated here by simulation in MATLAB® for parametric agents (agents whose positions are defined by (15)) having finite non-zero sizes. In order to achieve a decentralized implementation of the reconfiguration strategy on actual quadrotors, we first develop and test the on-board software for path planning and inter-agent communication

for the quadrotors as agents, using a Software-In-The-Loop (SITL) simulator in a ROS-Gazebo environment. The same software is then used on the actual quadrotors (developed in-house) to experimentally validate the reconfiguration strategy in a motion capture environment.

Table 2: Inputs to Algorithm 2. (Distances in meters)

	L	H	r_s	r_{com}	V_{max}	N_{extra}	η
MATLAB Simulation 1	10	7	4.7	9.5	0.5	2	1.05
MATLAB Simulation 2	10	7	1.5	3.2	1	2	1.05
SITL Simulation	25	16	7	11	0.3	1	1.05
Experiment	5	5	2.7	5.5	0.2	1	1.05

Table 3: Output of Algorithm 2. (Distances in meters)

	A	B	a	b	ϕ	N	N_{min}	N_{max}	\dot{s}_{nom}	r_{dm}
MATLAB Simulation 1	5	3.5	2	3	$\frac{\pi}{2}$	5	4	6	0.0345	0.481
MATLAB Simulation 2	5	3.5	4	11	$\frac{\pi}{2}$	15	14	16	0.023	0.074
SITL Simulation	12.5	8	3	7	0	10	9	10	0.0045	0.459
Experiment	2.5	2.5	3	2	0	5	4	5	0.0222	0.407

6.1. MATLAB® simulations

To validate the theory of the proposed surveillance (discussed in Section 4) and reconfiguration strategy (discussed in Section 5), we present two MATLAB® simulations: The first to illustrate the nature of the agent trajectories and the second to demonstrate scalability. The inputs to Algorithm 2 for both simulations are given in Table 2 and the corresponding outputs are given in Table 3. The addition of agent 6 to a 5 agent formation at time $t = 12 \text{ sec}$ is considered for discussion here, and the simulated trajectories shown in the Fig. 8 include the speeds of the agents, the parameter rates $\dot{\psi}$, \dot{s} and parameter values ψ , s of all agents. As illustrated by the Fig. 8, the monotone parametric trajectory (2) smoothly accelerates and decelerates the agents along the Lissajous curve and the symmetric trajectory (8) smoothly transitions the agents from one Lissajous curve to the other. Furthermore, as discussed in the theory, the speeds of the agents are always maintained below $V_{max} = 50 \text{ cm/s}$ in the X-Y plane. (For altitude changes we use step change commands when X-Y plane velocities are zero). The parameter trajectories are smooth except for the step just before the ΔT_3 interval. This jump in the parameter value corresponds to the parameter transformation step where the agent parameters ψ and s for the current Lissajous curve are expressed in terms of the destination Lissajous curve (ψ_d, s_d) while conserving position coordinates (as discussed in Section 5.1.2). The s parameter trajectories of agents 1-5 overlap as they all lie on the same formation ellipse, both before and after the reconfiguration. The s parameter value of the agent 6 is initialised on the destination Lissajous curve as it reaches its assigned formation position directly after the ΔT_1 transition. The parameter values ψ prior to the addition at $t = 12 \text{ sec}$ and after reconfiguration $t > 60 \text{ sec}$ are equispaced, indicating equi-parametric formation along the elliptical locus on both the Lissajous curves, before and after reconfiguration.

In order to demonstrate the scalability for the second simulation, we assume a smaller sensor footprint radius and communication range (refer to Table 2). Thus the number of agents required for this case is larger. For both simulations, the circular hull radius of the agents is selected as the sufficient bound r_{dm} given by Algorithm 2. From the simulations, we observe that the motion of the agents both during

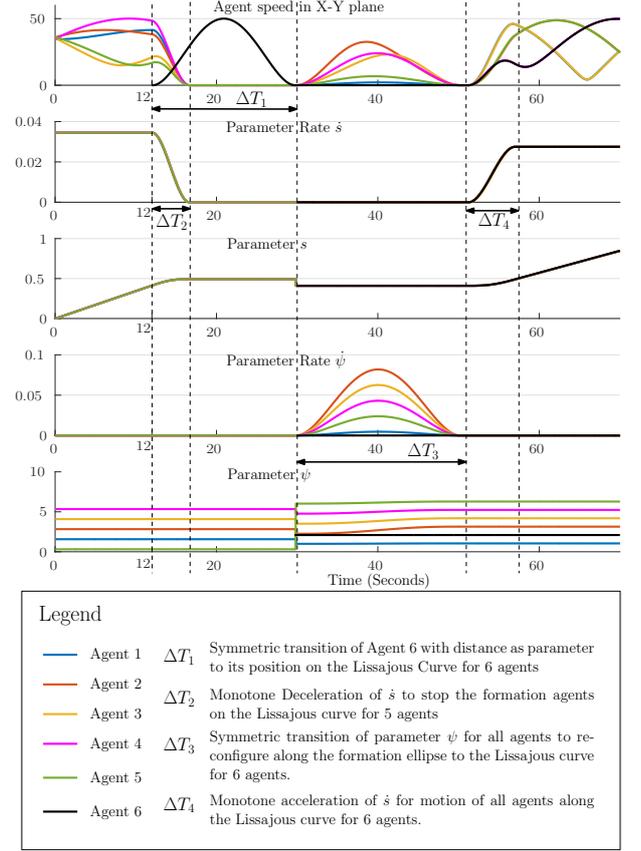


Figure 8: Trajectories for agent addition case for MATLAB simulation 1

surveillance along the Lissajous curve and during reconfiguration are collision-free, thus validating the strategies for surveillance and reconfiguration designed in Sections 4 and 5.

Remark 4. The videos of both the MATLAB® simulations for all three reconfiguration operations can be found at the web-link: <https://youtu.be/HEg5XfbBusY>

6.2. Software-In-The-Loop simulation

Since the proposed surveillance and reconfiguration strategies are developed for aerial agents such as quadrotors or helicopters, our final objective is to implement it using programmable autonomous quadrotors.

The quadrotors built for the experiment use the Pixhawk v1² flight controller running the PX4 flight stack³ for stabilisation of the drone. Reference commands (such as commanded position) can be sent to the Pixhawk using the MAVLink communication packets⁴ on a serial channel.

To simulate this setup, we use the Software-In-The-Loop (SITL) simulator⁵ for the PX4 flight stack that uses the Robot Operating System (ROS) (Version: Kinetic Kame⁶) along with the physics simulator

²https://docs.px4.io/en/flight_controller/pixhawk.html

³<https://dev.px4.io/en/>

⁴<https://mavlink.io/en/>

⁵<https://dev.px4.io/en/simulation/gazebo.html>

⁶<http://wiki.ros.org/kinetic>

Gazebo⁷ (Version 8). This simulation is done on a computer system equipped with a NVIDIA Geforce GTX 1060 graphics card and running the Ubuntu 16.04 Xenial LTS operating system. For each quadrotor, an instance of the PX4 flight stack is simulated, and the proposed strategy for surveillance (Section 4) and reconfiguration (Section 5) is implemented as a C++ script, which is written adhering to the node-topic structure of ROS (called *UAV_i_ctrl* node for agent *i*). The *MAVROS*⁸ package is used to translate between the ROS interface and the MAVLink packets, which are sent via a UDP port to the PX4 flight stack simulation (SITL component) of the corresponding quadrotor. In this manner the SITL simulation environment allows simulation of the physics for multiple quadrotors along with a simulated instance of the PX4 flight stack for each agent. The SITL component acquires the quadrotor states as feedback and sends actuator command values to the Gazebo simulator as shown in the Fig. 9.

Each quadrotor receives communication data from all other quadrotors in the simulation. This is done for ease of implementing the multi-agent network in ROS, and the limited communication range r_{com} is simulated in the navigation code for each agent by ignoring received data from agents outside a sphere of radius r_{com} centered around the agent. We use the *joy*⁹ package to issue basic commands to the multi-agent formation such as take-off, mission start, land, and reconfiguration commands, namely:

- 1) removal command with agent ID
- 2) replacement command with agent ID
- 3) addition command to initiate formation reconfiguration.

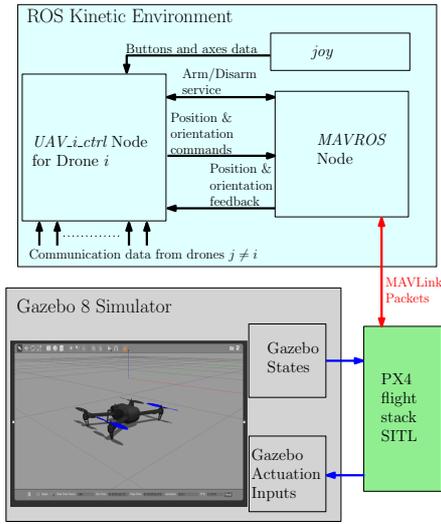


Figure 9: SITL simulation framework for each 3DR Iris quadrotor in ROS-Gazebo environment, using the PX4 flight stack simulated in Loop

The inputs and outputs of the Algorithm 2 for the SITL simulation are given in Tables 2 and 3 respectively. From Table 3, we see that the upper bound on circular hull radius of the agents is $r_{dm} = 0.459 m$. For the SITL simulator, we have selected the 3DR Iris quadrotor¹⁰ model (shown in Fig. 9) from the available models. This quadrotor has a motor to motor length of $55 cm$ and a propeller diameter of $10 inch$ (or $25.4 cm$). Thus it has a circular hull of radius $r_d = 0.402 m < r_{dm}$

(calculated as $\frac{55+24.5}{2} cm$). Gazebo simulates the complete physics of all the 3DR Iris quadrotors, along with an instance of the simulated PX4 flight stack for each quadrotor, which tracks the reference position commands issued using its internal PID control loops.

Remark 5. The video of the SITL multi-quadrotor simulation can be found at the web-link:

<https://youtu.be/XKXlvEDB-Qo>

The top-left window is a video recording of the Gazebo SITL simulation, and the top-right window shows the video of the recorded simulation coordinates of the quadrotors, plotted in MATLAB®.

6.3. Implementation

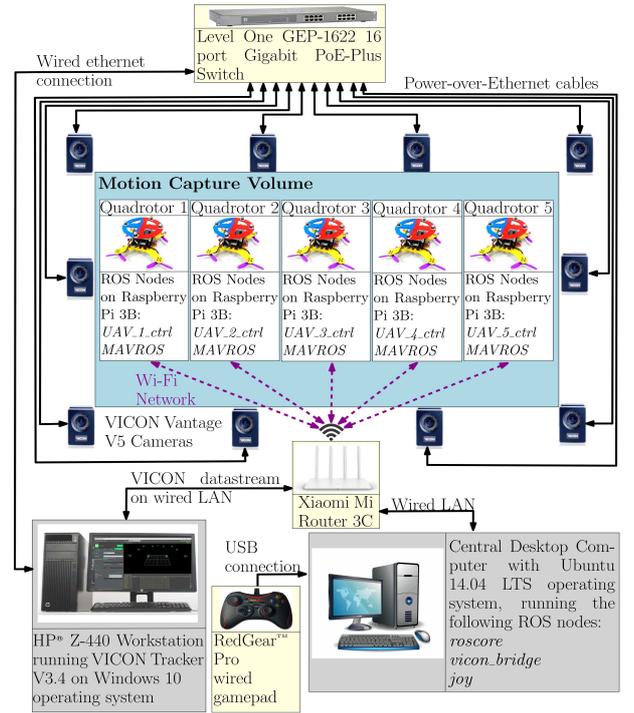


Figure 10: Block diagram of the experimental setup

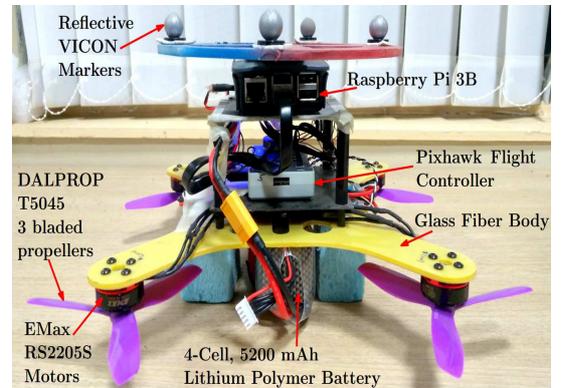


Figure 11: Quadrotor used for experiment

The experiment discussed here has been performed in an indoor laboratory space using a motion capture system for localisation. This

⁷<http://gazebosim.org/>

⁸<http://wiki.ros.org/mavros>

⁹<http://wiki.ros.org/joy>

¹⁰<http://www.arducopter.co.uk/iris-quadcopter-uav.html>

system comprises of ten Vantage V5¹¹ cameras as shown in Fig. 10. The quadrotors used for the experiment were made in-house (shown in Fig. 11). The design specifications of the quadrotors are given in Table 4. The sensor calibration and inner PID control-loops tuning for these quadrotors for the PX4 flight stack running on the Pixhawk v1 flight controller was done using the QGroundControl¹² software.

Table 4: Quadrotor Specifications

Structure	X-type frame 20 cm × 20 cm 3 mm thick glass fiber sheet
Motors	EMax RS2205S 2300kV
Propellers	DALPROP T5045 3 bladed 5 inch diameter, 4.5 inch pitch
Speed Controllers	BLHeli_S DShot600 30A Cicada ESC
Flight Controller	Pixhawk v1 with Firmware: px4fmu-v2.1pe v1.6.0 release candidate 4
Companion Computer	Raspberry Pi 3B with Ubuntu Mate 16.04 Operating System and ROS Kinetic Kame installed
Battery	4-Cell(4S), 25C, 5200mAh Lithium-Polymer with 16.8V peak voltage
Total Weight	1.18 kg
Endurance	12-14 minutes

The inputs and outputs of the Algorithm 2 for the experiment are given in Tables 2 and 3 respectively. From Table 3 we see that the upper bound on circular hull radius of the agents is $r_{dm} = 0.407 m$. The quadrotor we have constructed has a motor to motor length of 28.3 cm and a propeller diameter of 5 inch (or 12.7 cm). Thus it has a circular hull of radius $r_d = 0.205 m < r_{dm}$ (calculated as $\frac{28.3+12.7}{2} cm$). The same C++ script *UAV_j_ctrl* developed for the SITL simulation in Section 6.2 is used for implementing the proposed surveillance and reconfiguration strategy with the quadrotors shown in Fig. 11. Each quadrotor has a Raspberry Pi 3B¹³ companion computer onboard which runs the *UAV_j_ctrl* and *MAVROS* nodes for the corresponding quadrotor. As a result, the proposed strategy is implemented in a decentralized manner. A block diagram of the experimental setup is shown in Fig. 10. The quadrotors are fitted with reflective markers for operation in the VICON motion capture space. The VICON cameras detect these markers and the HP workstation processes the camera data using the VICON Tracker V3.4 software¹⁴ and broadcasts its data stream on a Local Area Network. The *roscore* node¹⁵ which is the master node for handling the complete ROS network runs on a central computer running ROS Indigo Igloo¹⁶ on the Ubuntu 14.04 LTS. This computer processes the VICON data stream and converts it to a ROS compatible format using the *vicon_bridge* node¹⁷ in ROS Indigo. It also runs the *joy* node to read joystick commands for initiating take off, land, agent removal with ID, agent replacement with ID and agent addition. The *roscore* node running on the central computer interacts with the Raspberry Pi's via a Wi-Fi network setup using a wireless router. For this we use the concept of a multicomputer ROS network¹⁸. Thus each

Raspberry Pi receives localisation as well as inter-agent communication data via Wi-Fi within this multi-computer ROS network, and in turn commands the Pixhawk flight controller via a local instance of the *MAVROS* Node, using a USB wired link. The Raspberry Pi issues (X, Y, Z) position commands corresponding to the s^i, ψ^i parameters of agent i , and the flight altitude ($h_F = 1.5 m$ or $h_L = 0.5 m$). It also issues a constant heading command of $0 rad$. The Pixhawk v1 flight controller tracks these commands using the PID loops in the PX4 firmware. These loops in turn use the on-board sensor data and localisation data available from the motion capture system as feedback.

Remark 6. *The video of the multi-quadrotor experiment discussed above can be found at the web-link:*

<https://youtu.be/DUNR0-T9zTA>

The top-left window is a video recording of the quadrotors in flight, and the top-right window shows the video of recorded position coordinates of the quadrotors as captured by the motion capture system (orthographic top view), plotted in MATLAB®. The bottom left video shows the motion capture markers on the quadrotors seen by the VICON cameras on the VICON Tracker V3.4 software.

From the videos in Remark 5 and Remark 6, we see that for the SITL simulation and the experiment in a VICON environment, all three reconfiguration operations are performed with smooth collision-free trajectories of the quadrotors in the same simulation/flight, hence validating our proposed multi-agent surveillance and formation reconfiguration strategy. The step commands are only used for initialisation of the quadrotor positions in the formation, and for changing altitude from h_F to h_L and vice-versa.

7. Conclusion

We have proposed in [2], a multi-agent formation on Lissajous curves which performs collision-free surveillance of a rectangular area. We have proposed here a reconfiguration strategy whereby a quadrotor can be added, removed or replaced using a decentralized cooperating scheme.

We have validated our results through MATLAB® simulations for agents having a non-zero size satisfying a theoretically derived size bound. To demonstrate the practical applicability of the proposed surveillance and reconfiguration strategies, we have also presented simulations, for quadrotors in a ROS-Gazebo based Software-In-The-Loop simulator and have implemented the same with a team of five quadrotors in a motion capture environment. This work has potential applications in security, asset protection, agricultural monitoring, distributed sensing, etc.

Acknowledgements

We thank Vraj Parikh and Shoeb Ahmed Adeel for their help with the design and construction of the quadrotors. We also thank Gaurav Gardi for assistance with setting up the SITL simulator, and Mr. Kumar Khot for help with logistics and setting up of the VICON motion capture system. The quadrotors were constructed and flight tested in the Miniature Aerial Vehicle (MAV) Laboratory at the Department of Aerospace Engineering, IIT Bombay, and the experiments were conducted in the Autonomous Robots and Multi-agent Systems (ARMS) Laboratory at the Interdisciplinary Programme in Systems and Control Engineering, IIT Bombay.

¹¹<https://www.vicon.com/products/camerasystems/vantage>

¹²<http://qgroundcontrol.com/>

¹³<https://www.raspberrypi.org/products/raspberry-pi-3-model-b/>

¹⁴<https://docs.vicon.com/display/Tracker34/Vicon+Tracker+User+Guide>

¹⁵<http://wiki.ros.org/roscore>

¹⁶<http://wiki.ros.org/indigo>

¹⁷http://wiki.ros.org/vicon_bridge

¹⁸<http://wiki.ros.org/ROS/Tutorials/MultipleMachines>

Appendix

Proof of Lemma 1: Let $V(\ddot{g}(t), t) = \frac{\dot{g}^2(t)}{2}$, then cost $J = \int_0^{T_f} V(\ddot{g}(t), t) dt$. For convenience of notation denoting k^{th} derivative of $g(t)$ as $g^{(k)}(t)$, the first variation ∂J is computed as:

$$\partial J = \int_0^{T_f} \left(V(\ddot{g}(t) + \delta_{g^{(3)}}(t), t) - V(\ddot{g}(t), t) \right) dt \quad (62)$$

and by the necessary condition for optimality, at the optimal trajectory $g^*(t)$ (denoted as g_t^* for short), $\partial J = 0$. Thus using Taylor series expansion in (62) about $g(t) = g_t^*$ yields $\partial J = \int_0^{T_f} \frac{\partial V(g^{(3)}(t), t)}{\partial g^{(3)}(t)} \Big|_{g_t^*} \delta_{g^{(3)}}(t) dt$. By repeated application of integration by parts,

$$\begin{aligned} \partial J &= \frac{\partial V(g^{(3)}(t), t)}{\partial g^{(3)}(t)} \Big|_{g_t^*} \delta_{g^{(2)}}(t) \Big|_0^{T_f} - \frac{d}{dt} \left(\frac{\partial V(g^{(3)}(t), t)}{\partial g^{(3)}(t)} \Big|_{g_t^*} \right) \delta_{g^{(1)}}(t) \Big|_0^{T_f} \\ &+ \frac{d^2}{dt^2} \left(\frac{\partial V(g^{(3)}(t), t)}{\partial g^{(3)}(t)} \Big|_{g_t^*} \right) \delta_g(t) \Big|_0^{T_f} - \int_0^{T_f} \frac{d^3}{dt^3} \left(\frac{\partial V(g^{(3)}(t), t)}{\partial g^{(3)}(t)} \Big|_{g_t^*} \right) \delta_g(t) dt. \end{aligned}$$

From the fixed boundary conditions, $\delta_{g^{(2)}}(T_f) = \delta_{g^{(2)}}(0) = 0$, $\delta_{g^{(1)}}(T_f) = \delta_{g^{(1)}}(0) = 0$, $\delta_g(0) = 0$. Thus, the first variation simplifies to

$$\partial J = \frac{d^2}{dt^2} \left(\frac{\partial V(g^{(3)}(t), t)}{\partial g^{(3)}(t)} \Big|_{g_t^*} \right) \delta_g(T_f) - \int_0^{T_f} \frac{d^3}{dt^3} \left(\frac{\partial V(g^{(3)}(t), t)}{\partial g^{(3)}(t)} \Big|_{g_t^*} \right) \delta_g(t) dt. \quad (63)$$

By substituting $V(\ddot{g}(t), t) = \frac{\dot{g}^2(t)}{2}$, the Euler-Lagrange equation (63) simplifies to $g^{(6)}(t) = 0$. Then from the Euler-Lagrange equation, for some constants c_1, c_2, c_3, c_4, c_5 and c_6 ,

$$g^*(t) = c_1 t^5 + c_2 t^4 + c_3 t^3 + c_4 t^2 + c_5 t + c_6, \quad (64)$$

$$g^{*(1)}(t) = 5c_1 t^4 + 4c_2 t^3 + 3c_3 t^2 + 2c_4 t + c_5, \quad (65)$$

$$g^{*(2)}(t) = 20c_1 t^3 + 12c_2 t^2 + 6c_3 t + 2c_4, \quad (66)$$

$$g^{*(3)}(t) = 60c_1 t^2 + 24c_2 t + 6c_3, \quad (67)$$

$$g^{*(4)}(t) = 120c_1 t + 24c_2, \quad (68)$$

$$g^{*(5)}(t) = 120c_1. \quad (69)$$

The solutions for both the boundary value conditions C1 and C2 are as follows:

1. Since $\partial J = 0$ and $\delta_g(T_f)$ need not be zero, $\frac{d^2}{dt^2} \left(\frac{\partial V(g^{(3)}(t), t)}{\partial g^{(3)}(t)} \Big|_{g_t^*} \right) = 0$ in addition to the Euler-Lagrange equation. This simplifies to $g^{*(5)}(t) = 0$ and from (69), $c_1 = 0$. From the given boundary conditions at $t = 0$ and (64), (65), (66) we get $c_6 = g_0$, $c_5 = \dot{g}_0$ and $c_4 = 0$ respectively. The boundary conditions at $t = T_f$ gives $0 = 12c_2 T_f^2 + 6c_3 T_f$ and $\dot{g}_f = 4c_2 T_f^3 + 3c_3 T_f^2 + \dot{g}_0$ which gives rise to this linear system of equations

$$\begin{bmatrix} 2T_f & 1 \\ 4T_f^3 & 3T_f^2 \end{bmatrix} \begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{g}_f - \dot{g}_0 \end{bmatrix}. \quad (70)$$

Solving this system of linear equations, $c_2 = -\frac{\dot{g}_f - \dot{g}_0}{2T_f^2}$, $c_3 = \frac{\dot{g}_f - \dot{g}_0}{T_f^2}$.

Thus, $g^*(t) = (\dot{g}_f - \dot{g}_0) \left(-\frac{t^4}{2T_f^3} + \frac{t^3}{T_f^2} \right) + \dot{g}_0 t + g_0$.

2. The second set of boundary conditions C2 represent a fixed end-time T_f and fixed end state $g(T_f) = g_f$ problem. Thus at $t = 0$ and $t = T_f$, (64)- (69) result in the following Linear system of equations.

$$\begin{bmatrix} T_f^5 & T_f^4 & T_f^3 & T_f^2 & T_f & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 5T_f^4 & 4T_f^3 & 3T_f^2 & 2T_f & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 20T_f^3 & 12T_f^2 & 6T_f & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} g_f \\ g_0 \\ \dot{g}_f \\ \dot{g}_0 \\ \ddot{g}_f \\ \ddot{g}_0 \end{bmatrix} = \begin{bmatrix} g_f \\ g_0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (71)$$

Solving (71), gives

$$c_1 = \frac{6(g_f - g_0)}{T_f^5}, \quad c_2 = \frac{15(g_0 - g_f)}{T_f^4}, \quad c_3 = \frac{10(g_f - g_0)}{T_f^3}, \quad c_4 = c_5 = 0, \quad c_6 = g_0$$

$$\text{Thus, } g^*(t) = (g_f - g_0) \left(10T_f^2 - 15T_f t + 6t^2 \right) \frac{t^3}{T_f^3} + g_0. \quad \square$$

Lemma 3. For $u \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$, $\sin^2 u \geq \frac{4}{\pi^2} u^2$

Proof: Consider the following cases:

Case 1: For $u = 0, \pm \frac{\pi}{2}$, $\sin^2 u = \frac{4}{\pi^2} u^2$. Thus the claim holds true at these values.

Case 2: For $u \in \left(-\frac{\pi}{2}, 0 \right) \cup \left(0, \frac{\pi}{2} \right)$, let $f(u) = \frac{\sin^2 u}{u^2}$. By differentiating and simplifying the resultant expression, $f'(u) = \frac{\sin 2u}{u^2} (u - \tan u)$. Using the infinite Taylor series for $\tan u = u + \frac{u^3}{3} + \frac{2u^5}{15} + \dots$ for $|u| < \frac{\pi}{2}$, $f'(u) = -\sin 2u \left(\frac{1}{3} + \frac{2u^2}{15} + \dots \right)$ for $|u| < \frac{\pi}{2}$.

Thus for $u \in \left(-\frac{\pi}{2}, 0 \right)$, $f'(u) > 0$ which implies that $f(u)$ is monotone increasing and thus $f(u) > f\left(-\frac{\pi}{2}\right)$ which implies $\frac{\sin^2 u}{u^2} > \frac{4}{\pi^2}$ or $\sin^2 u > \frac{4}{\pi^2} u^2$. Similarly for $u \in \left(0, \frac{\pi}{2} \right)$, $f'(u) < 0$ which implies that $f(u)$ is monotone decreasing and thus $f(u) > f\left(\frac{\pi}{2}\right)$ which implies $\frac{\sin^2 u}{u^2} > \frac{4}{\pi^2}$ or $\sin^2 u > \frac{4}{\pi^2} u^2$. Hence, *Case 1* and *Case 2* considered together prove the claim. \square

Proof of Proposition 1: For any two agents i, j in the formation, $\psi_c^j = \psi_c^i + \Delta^{ij} \psi \bmod 2\pi$. From (22), the Euclidean distance between agents i and j is given by $D_{ij} = 2 \sin\left(\frac{\Delta^{ij} \psi}{2}\right) \sqrt{A^2 \sin^2(\Psi_p - a_c s_c) + B^2 \cos^2(\Psi_p + b_c s_c)}$ and $\Psi_p = \psi_c^i + \frac{\Delta^{ij} \psi}{2}$. From Lemma 3 in the appendix, we get a parabolic lower bound for $A^2 \sin^2\left(\psi_c^i + \frac{\Delta^{ij} \psi}{2} - a_c s_c\right)$ and $B^2 \cos^2\left(\psi_c^i + \frac{\Delta^{ij} \psi}{2} + b_c s_c\right)$ as $\frac{4}{\pi^2} A^2 u^2$ and $\frac{4}{\pi^2} B^2 \left(u + (a_c + b_c) s_c - \frac{(2k-1)\pi}{2} \right)^2$ respectively, where $u = \psi_c^i + \frac{\Delta^{ij} \psi}{2} - a_c s_c$ for some $k \in \mathbb{N}$. Thus $D_{ij}^2 > 4 \sin^2\left(\frac{\Delta^{ij} \psi}{2}\right) f(u)$, where

$$f(u) = \frac{4}{\pi^2} A^2 u^2 + \frac{4}{\pi^2} B^2 \left(u + N_c s_c - \frac{(2k-1)\pi}{2} \right)^2.$$

Solving $\frac{df(u)}{du} = 0$ we get $u^* = -\frac{B^2}{A^2 + B^2} \left(N_c s_c - \frac{(2k-1)\pi}{2} \right)$ which is a minimizer as $f(u)$ is a sum of parabolic functions having positive coefficients. Thus if $D_{ij \min}$ is the minimum distance between the agents i and j , then $D_{ij \min}^2 \geq 4 \sin^2\left(\frac{\Delta^{ij} \psi}{2}\right) f(u^*)$. Substituting value of u^* in this inequality and simplifying the resultant expression leads to

$$D_{ij \min} \geq \frac{4}{\pi} \frac{ABN_c}{\sqrt{A^2 + B^2}} \left| \sin\left(\frac{\Delta^{ij} \psi}{2}\right) \right| \Delta_s, \quad (72)$$

where $\Delta_s = \left| s_c - \frac{(2k-1)\pi}{2N_c} \right|$. Hence if we guarantee

$$\frac{4}{\pi} \frac{ABN_c}{\sqrt{A^2 + B^2}} \left| \sin\left(\frac{\Delta^{ij} \psi}{2}\right) \right| \Delta_s > 2r_{dm} \quad (73)$$

where $\Delta_{min}^{ij} = \min_{t \in \mathbb{R}^+} |\Delta^{ij}\psi(t)|$, then we can ensure $D_{ijmin} > 2r_{dm}$ and the transition along the ellipse is guaranteed to be collision-free. Rearranging (73) results in the following inequalities: $s_c > s_{diag}(k) + \delta_s$ and $s_c < s_{diag}(k) - \delta_s$, where $s_{diag}(k) = \frac{(2k-1)\pi}{2N_c}$ for $k \in \mathbb{N}$ is the parameter value for which the agents lie on the the degenerate ellipse (or the diagonal line) as shown in Fig. 4, and $\delta_s = \frac{\pi}{2N_c} \frac{r_{dm} \sqrt{A^2+B^2}}{AB} \left| \sin\left(\frac{\Delta_{min}^{ij}}{2}\right) \right|^{-1}$.

Thus all ellipses corresponding to all the s values satisfying

$$s_c \notin \cup_{k \in \mathbb{N}} (s_{diag}(k) - \delta_s, s_{diag}(k) + \delta_s) \pmod{2\pi} \quad (74)$$

are feasible for agent transitions. \square

Proof of Lemma 2: If Q_1, \dots, Q_{N+1} are $N + 1$ equi-spaced points on C , by pigeonhole principle, at least one of the segments $[P_i, P_{i+1})$ contains two Q_j 's. This proves the existence of at least one such Q_j, Q_{j+1} pair within a $[P_i, P_{i+1})$ segment (with $j \in \{1, \dots, N + 1\}$).

There are N equi-spaced segments of the form $[P_i, P_{i+1})$ which are disjoint and cover C , so if two of them contain a pair of Q_j 's each, the remaining $N - 3$ of Q_j 's are distributed among the remaining $N - 2$ segments, thus at least one of them does not contain any Q_j points. Suppose the segment $[P_{i'} P_{i'+1})$ for some $i' \in \{1, \dots, N\}$, doesn't contain a Q_j , then this segment in turn is contained within a $[Q_{j'}, Q_{j'+1})$ segment for some $j' \in \{1, \dots, N + 1\}$. But if the parametric curve length of C is L , then $[P_{i'} P_{i'+1})$ is of length $\frac{L}{N}$ and $[Q_{j'}, Q_{j'+1})$ is of length $\frac{L}{N+1}$ and $[P_{i'} P_{i'+1}) \not\subset [Q_{j'}, Q_{j'+1})$, which is a contradiction. Thus there is exactly one Q_j pair contained in a $[P_i, P_{i+1})$ segment. \square

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