Lyapunov-based nonlinear boundary control design with predefined convergence for a class of 1D linear reaction-diffusion equations

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Salim Zekraoui,[1] Nicolas Espitia[1] and Wilfrid Perruquetti[2]

Abstract— **In this paper, we treat the problem of Lyapunov-based nonlinear boundary stabilization of a class of one-dimensional reaction-diffusion systems with any predefined convergence (asymptotic or non-asymptotic). As an application, we focus on the non-asymptotic notions (finite-time and fixed-time) for which we give some particular explicit control designs followed by some numerical simulations. The key idea of our approach is to use a "spatially weighted L^2-norm" as a Lyapunov functional to design a nonlinear controller and to ensure stability with any desired convergence.**

I. Introduction

In recent years, increasing attention has been paid to the problem of stabilization of Partial Differential Equations (in short PDEs) since they model the evolution in time and space of complex systems such that the heat transfer, traffic flow, fluids flow, chemical reactor processes, string vibration, the behavior of electromagnetic phenomena, and many other systems. Unlike Ordinary Differential Equations (ODEs), PDEs are of infinite-dimensional nature. This nature makes it difficult to adapt the existing methods used to stabilize ODEs. Therefore, it is of great significance to investigate the problems of control for PDEs. Among these ones, the problem of boundary control for PDEs is more challenging and important.

In the framework of first and second-order PDEs, we can distinguish three major classes: elliptic equations (e.g. Poisson equation), hyperbolic equations (e.g. wave propagation equation, transport equation), and parabolic equations (e.g. heat conduction equation, reaction-diffusion equation). For these last two classes, most of the existing stabilization results ensure asymptotic (exponential) convergences. However, in many applications where strict time performances are required, the notion of non-asymptotic stability/stabilization (i.e. stability/stabilization in a finite time) is strongly needed, especially in certain applications where the transient process must occur within a given time (e.g., rendezvousing of multi-agents, ABS (anti-lock braking system), missile tactical guidance). The non-asymptotic convergence notion can be classified as finite-time convergence which refers to a convergence in a finite time that depends on the initial conditions, fixed-time convergence which refers to a convergence in a finite time uniformly bounded by a constant independent of the initial conditions, and prescribed–time convergence which refers to a convergence in a finite time prescribed independently of initial conditions. Despite all the benefits of non-asymptotic stabilization, its design is still a challenging topic for PDEs.

For parabolic PDEs, the backstepping approach has contributed significantly to solving the problem of prescribed-time stabilization in [18], [4] (respectively finite-time stabilization in [15]) for a scalar reaction-diffusion using (respectively switching) time-varying feedback laws equation with boundary control. Moreover, the approach has helped in solving the problems of null controllability in [3] and finite-time stabilization using periodic time-varying feedback laws for a class of reaction-diffusion equations. The backstepping approach consists in transforming the studied parabolic system, using an invertible Volterra and/or Fredholm type transformation, into another system of the same type, called the target system, satisfying the desired non-asymptotic stability. Then, using the inverse transformation the desired stability property is transferred back to the original system. Finding a suitable target system for a general class of complex systems is sometimes not straightforward and studying its stability may be complicated, especially when dealing with non-asymptotic stabilization. To avoid this problem, one can use the notion of generalized homogeneity (introduced in [13]) as in [16], [14] or Lyapunov Based techniques as in [19] to achieve finite-time stabilization.

In this paper, we revisit the problem of boundary control design for a class of one-dimensional linear reaction-diffusion equations. The main idea of our approach is to choose a Lyapunov function that helps directly in designing a simple nonlinear boundary controller which guarantees the desired stability for the studied system. Unlike the Backstepping-based approach, our control design will not use boundary time-varying feedbacks and will be easily modifiable to guarantee any desired stability (non-asymptotic or asymptotic stability). Our approach is similar to the Control Lyapunov function (CLF) approach, which has been investigated for parabolic PDEs in [5], [6], in the sense of using the Lyapunov function directly to design the boundary control. As an application, we focus on the notions of finite-time/fixed-time stability, where we give some particular explicit control designs followed by some numerical simulations.

This paper is organized as follows. In Section II, we introduce the one-dimensional reaction-diffusion system with Dirichlet actuation. In Section III, we introduce and give some properties of the "spatially weighted L^2-norm" which is chosen as a Lyapunov functional (inspired from results in [1] and [7, Chapter 11, page 178] for hyperbolic sys-

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tems). Next, we establish the Lyapunov stability analysis where we design a nonlinear controller that will ensure stability with a predefined convergence (asymptotic or non-asymptotic). In Section IV, we provide some explicit control designs ensuring non-asymptotic stability (finite-time and fixed-time). In Section V, to illustrate the results we give some numerical simulations for both cases: finite-time/fixed-time stabilization. Finally, conclusions and perspectives are given in Section VI.

Notations:

$\mathbb{R}_+$ denotes the set of non negative real numbers. For all $a \in \mathbb{R}_+$ and all $x \in \mathbb{R}$ we define the signed power a of x by $\{x\}^a = \mathrm{sign}(x)|x|^a$. We denote by $\mathbb{1}_{\{x>0\}} : \mathbb{R} \to \{0,1\}$ the function defined by $\mathbb{1}_{\{x>0\}}(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$.

$L^2((0,1),\mathbb{R})$ denotes the set $\{f : [0,1] \to \mathbb{R} : \int_0^1 |f(x)|^2 dx < \infty\}$ with the scalar product $\langle f,\ g\rangle_{L^2} := \int_0^1 f(x)g(x)dx$, and the norm $\|f\|_{L^2} := \left(\int_0^1 f(x)^2 dx\right)^{\frac{1}{2}}$. We denote by $H^1((0,1),\mathbb{R})$ the set $\{f \in L_2; f' \in L_2\}$, with the scalar product $\langle f,\ g\rangle_{H^1} := \langle f,\ g\rangle_{L^2} + \langle f',\ g'\rangle_{L^2}$ and with the norm $\|f\|_{H^1} := \left(\|f\|^2_{L^2} + \|f'\|^2_{L^2}\right)^{\frac{1}{2}}$. For simplicity, we will use the notation L^2 (resp. H^1) instead of $L^2((0,1),\mathbb{R})$ (resp. $H^1((0,1),\mathbb{R})$).

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a class-$\mathcal{K}$ function if it is continuous, zero at zero, and strictly increasing. If in addition, α is unbounded with its argument then α is said to be a class-$\mathcal{K}_\infty$.

II. PROBLEM STATEMENT, CONCEPTS AND PRELIMINARY RESULTS

A. Preliminaries on non-asymptotic concepts for PDEs

In this section, we recall some definitions of asymptotic and non-asymptotic concepts (finite-time, and fixed-time stability) in the framework of infinite dimensional systems.

Let us consider the following evolution system described by:

$$z_t(t,\cdot) = Az(t,\cdot), \tag{1}$$

with $t \geq t_0 \geq 0$, where $A : \mathcal{D}(A) \subset L^2 \to L^2$ is a (possibly unbounded) linear operator, t_0 is the initial time, and z_0 will denote the initial condition.

Definition 1: The origin of system (1) is said to be

- **stable**[2, Definition 1.1.] if for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $t_0 \geq 0$ and $z_0 \in L^2$,
$$(\|z_0\|_{L^2} \leq \delta) \Longrightarrow (\|z(t,\cdot)\|_{L^2} \leq \varepsilon, \quad \forall t \geq t_0),$$
- **asymptotically stable (AS)** [1] if it is stable and $\lim_{t\to+\infty} \|z(t,\cdot)\|_{L^2} = 0$ for any $z_0 \in L^2$,
- **finite-time stable (FTS)** if it is stable and for any $z_0 \in L^2$ there exists $0 \leq T^{z_0} < +\infty$ such that $\|z(t,\cdot)\|_{L^2} = 0$ for all $t \geq T^{z_0}$. The functional $T(z_0) = \inf\{T^{z_0} \geq 0 : \|z(t,\cdot)\|_{L^2} = 0, \forall t \geq T^{z_0}\}$ defines the settling time of the system (1),
- **fixed-time stable (FxTS)** if it is FTS and $\sup_{z_0 \in L^2} T(z_0) < +\infty$,

In the above-given definitions uniformly with respect to initial time t_0 has been omitted for sake of brevity.

Based on Definition 1, let us give some sufficient conditions for the previous stability notions.

Proposition 1: Let $V : \Omega \subset \mathcal{D}(A) \to \mathbb{R}_+$ be a functional continuous on Ω, continuously differentiable on $\Omega \setminus \{0\}$, and satisfying the coercivity condition (i.e. there exist two class-$\mathcal{K}_\infty$ functions φ_1 and φ_2 such that $\varphi_1(\|z(t,\cdot)\|_{L^2}) \leq V(z(t,\cdot)) \leq \varphi_2(\|z(t,\cdot)\|_{L^2})$ for all $t \geq t_0$). Then,

- if the time derivative of V along the solutions of (1) (denoted in all the rest of the paper by $\frac{d}{dt}V(z(t,\cdot))$) satisfies $\frac{d}{dt}V(z(t,\cdot)) \leq 0$ in Ω for all $t \geq t_0$, then the origin of system (1) is **stable**.
- Furthermore, if there exists a class-$\mathcal{K}_\infty$ function such that $\frac{d}{dt}V(z(t,\cdot)) \leq -\varphi_3(\|z(t,\cdot)\|_{L^2})$ in Ω for any $t \geq t_0$, then the origin of system (1) is **AS** (see [8, Proposition 3.2]).
- or if there exists $0 \leq T^{V(z_0)} < +\infty$ such that $V(z(t,\cdot)) = 0$ for all $t \geq T^{V(z_0)}$, then the origin of system (1) is **FTS** with the settling time $T(V(z_0))$ defined similarly as in Definition 1. In particular, if $\sup_{z_0\in\Omega} T(V(z_0)) < +\infty$, then the origin of system (1) is **FxTS**.

Remark 1: Note that if V is continuously differentiable on $\Omega \setminus \{0\}$, then $\frac{d}{dt}V(z(t,\cdot)) = \left\langle \frac{\partial V(z(t,\cdot))}{\partial z},\ Az(t,\cdot)\right\rangle_{L^2}$. Note that if one can find a suitable coercive Lyapunov function then, using the comparison Lemma, one may reduce the complexity of the stability analysis to the study of the following simple scalar ordinary differential equation:

$$\dot{x} = -\mathcal{K}(x),\ x \in \mathbb{R}. \tag{2}$$

as detailed in what follows.

The set of $L^1_{loc}(\mathbb{R})$ functions $\mathcal{K}$ such that the origin is (globally uniformly) asymptotically stable can be specified as follows: $\mathcal{S}$ is the set of $L^1_{loc}(\mathbb{R})$ functions $\mathcal{K} : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{K}(x) = 0 \Leftrightarrow x = 0$ (the origin is the unique equilibrium point) and $x\mathcal{K}(x) > 0, \forall x \in \mathbb{R} \setminus \{0\}$. Next, using Landau notations[2], let us introduce $\mathcal{E}_{k_0,a_0} = \{\mathcal{K} \in \mathcal{S} : \mathcal{K}(x) \underset{x\to 0}{\sim} k_0\{x\}^{a_0}, \lim_{|x|\to\infty} \mathcal{K}(x) \neq 0\}$ (**FTS**) and $\mathcal{E}^{k_\infty,a_\infty}_{k_0,a_0} = \{\mathcal{K} \in \mathcal{E}_{k_0,a_0} : \mathcal{K}(x) \underset{|x|\to\infty}{\sim} k_\infty\{x\}^{a_\infty}\}$ (**FxTS**).

Examples 1: Let $a_0 \in [0,1), a_\infty > 1, k_0 > 0, k_\infty > 0$ and ψ be any continuous positive function which is zero at 0 and at ∞. Let

$$\mathcal{K}_1(x) = k_0\{x\}^{a_0}(1 + \psi(x)), \tag{3}$$
$$\mathcal{K}_2(x) = (k_0\{x\}^{a_0} + k_\infty\{x\}^{a_\infty})(1 + \psi(x)), \tag{4}$$

then $\mathcal{K}_1 \in \mathcal{E}_{k_0,a_0}$, $\mathcal{K}_2 \in \mathcal{E}^{k_\infty,a_\infty}_{k_0,a_0}$.

[1] one can use $\mathcal{KL}-$function (see [8, Definition 2.8.])

[2] $f(x) \underset{x\to a}{\sim} g(x)$ if and only if $\frac{f(x)-g(x)}{g(x)} \underset{x\to a}{\longrightarrow} 0$.

Corollary 1: Let $V : \Omega \subset \mathcal{D}(A) \to \mathbb{R}_+$ be a continuous function on Ω, continuously differentiable on $\Omega \setminus \{0\}$, and satisfying the coercivity condition. If there exists a continuous function $\mathcal{K} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that

$$\frac{d}{dt}V(z(t,\cdot)) \leq -\mathcal{K}(V(z(t,\cdot))), \tag{5}$$

and $\mathcal{K} \in \mathcal{E}_{k_0,a_0}$ (resp. $\mathcal{K} \in \mathcal{E}_{k_0,a_0}^{k_\infty,a_\infty}$) with $a_0 \in [0,1)$ (resp. $a_0 \in [0,1), a_\infty > 1$), then the origin of (1) is **FTS** (resp. **FxTS**).

Proof: To prove that the origin of (1) is **FTS**, it is sufficient to notice that from the corollary conditions, there exists $\mathcal{O}$ a neighborhood of 0, where equation (5) is equivalent to $\dot{V}(z(t,\cdot)) \leq -k_0 V(z(t,\cdot))^{\alpha_0}$ which guarantees the stability of (1) and also the finite-time convergence to the origin by integrating with respect to t, i.e. $V(z(t,\cdot)) \leq \left[V(z_0)^{1-\alpha_0} - k_0(1-\alpha_0)(t-t_0)\right]^{\frac{1}{1-\alpha_0}}$. A similar proof can be provided for the **FxTS** case. ∎

Remark 2: Note that inequality (5), for particular cases of $\mathcal{K}$, has been proved in the framework of non-asymptotic stabilization for a class of parabolic PDEs with distributed control (see e.g. [12], [11], [16], [10]). In particular, in [16], the problem of finite-time stabilization of the following reaction-diffusion PDE with distributed control:

$$\begin{aligned} z_t(t,x) &= z_{xx}(t,x) - \frac{cz(t,x)}{\|z(t,\cdot)\|_{L^2}^{2-2\alpha}}, \\ z(t,0) &= z(t,1) = 0, \end{aligned}$$

has been solved. Furthermore, in [16], inequality (5) with $\mathcal{K}(V) = cV^\alpha$ has been obtained using the following Lyapunov function: $V(z(t,\cdot)) = \|z(t,\cdot)\|_{L^2}^2$ for any $t \geq 0$, $c > 0$ and $\alpha \in (0,1)$. Note that $\mathcal{K}$ is in the set $\mathcal{E}_{c,\alpha}$ and satisfies the conditions of Proposition 1.

The goal now is to obtain inequality (5) in the case of boundary actuation. This motivates the following statement.

B. Problem statement:

We consider the following reaction-diffusion equation with constant reaction term and Dirichlet actuation:

$$\begin{aligned} z_t(t,x) &= z_{xx}(t,x) + \lambda z(t,x), \\ z(t,0) &= 0, \\ z(t,1) &= U(t), \\ z(t_0,x) &= z_0(x), \end{aligned} \tag{6}$$

where $t \geq t_0 \geq 0$, $x \in [0,1]$, the reaction term $\lambda \in \mathbb{R}$, the state $z(t,\cdot) \in \mathcal{D}(A) := \{z \in H^1 : \frac{\partial^2 z}{\partial x^2} \in L^2,\ z(0) = 0,\ z(1) = U(t)\}$ with the operator $A = \frac{\partial^2}{\partial x^2}$ is the second-order partial derivative with respect to space, the control $U(t) \in \mathbb{R}$, and the initial condition $z_0 \in \mathcal{D}(A)$.

Our goal is to design a nonlinear control $U(t)$ and a Lyapunov functional $V(z(t,\cdot))$ such that the time derivative of V along the solutions of (6) satisfies (5) for any continuous function $\mathcal{K} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\mathcal{K}(0) = 0$. As an application, we choose the function $\mathcal{K}$ such that the closed-loop system (6) with the control $U(t)$ is finite-time stable or fixed-time stable in light of the notions presented in the previous section.

III. Stability analysis

In this Section, we first introduce the Lyapunov functional candidate and we give some of its properties. Then, by computing its time derivative along the solutions of (6), we design a nonlinear control $U(t)$ that will ensure inequality (5) for all $t \geq t_0$ and all continuous function $\mathcal{K} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that $\mathcal{K}(0) = 0$.

Let us consider the following spatially weighted L^2-norm [3] as a Lyapunov function candidate:

$$V(z) = \int_0^1 e^{\sigma x}|z(x)|^2 dx,\ \sigma > 0. \tag{7}$$

We can clearly see that V satisfies for any $\sigma > 0$ the following property:

$$\|z(t,\cdot)\|_{L^2}^2 \leq V(z(t,\cdot)) \leq e^\sigma \|z(t,\cdot)\|_{L^2}^2. \tag{8}$$

Moreover, by computing the time derivative of V along the solutions (6), we can establish the following proposition

Proposition 2: Let $\mathcal{K} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a continuous function such that $\mathcal{K}(0) = 0$. Then, the functional V given in (7) satisfies the following inequality for every $t \geq t_0 \geq 0$ and every $\sigma > 0$:

$$\begin{aligned} \frac{d}{dt}V(z(t,\cdot)) \leq -2e^\sigma \Big[&\frac{\sigma}{2}U(t)^2 - z_x(t,1)U(t) \\ &-\mathcal{B}(V(z(t,\cdot))) + \frac{e^{-\sigma}}{2}\mathcal{K}(V(z(t,\cdot)))\Big], \end{aligned} \tag{9}$$

where $\mathcal{B}(\cdot)$ is given by

$$\begin{aligned} \mathcal{B}(V(z(t,\cdot))) =& e^{-\sigma}\left(\lambda + \frac{\sigma^2}{2}\right)\mathbb{1}_{\{x>0\}}\left(\lambda + \frac{\sigma^2}{2}\right)V(z(t,\cdot)) \\ &+ \frac{e^{-\sigma}}{2}\mathcal{K}(V(z(t,\cdot))) \geq 0. \end{aligned} \tag{10}$$

Proof: Let us start by computing the time derivative of V in (7) along the solutions of (6),

$$\begin{aligned} \frac{d}{dt}V(z(t,\cdot)) &= 2\int_0^1 e^{\sigma x} z(t,x) z_t(t,x) dx, \\ &= 2\lambda \int_0^1 e^{\sigma x}|z(t,x)|^2 dx \\ &\quad + 2\int_0^1 e^{\sigma x} z(t,x) z_{xx}(t,x) dx. \end{aligned}$$

Next, by integration by parts on the last term, we get,

$$\begin{aligned} \frac{d}{dt}V(z(t,\cdot)) = 2\lambda V(z(t,\cdot)) &- 2\int_0^1 e^{\sigma x}|z_x(t,x)|^2 dx \\ &- 2\sigma \int_0^1 e^{\sigma x} z(t,x) z_x(t,x) dx \\ &+ 2\left[e^{\sigma x} z(t,x) z_x(t,x)\right]_0^1, \end{aligned}$$

[3] similar functionals have been used in the framework of exponential stabilization (e.g. for linear conservation laws in [1] or for a transport PDE with a zero input at the boundary in [7, Chapter 11, page 178]).

Then, we get

$$\begin{aligned}\frac{d}{dt}V(z(t,\cdot)) &\leq 2\lambda V(z(t,\cdot)) - \sigma\int_0^1 e^{\sigma x}\frac{\partial|z(t,x)|^2}{\partial x}dx \\ &\quad + 2e^{\sigma}z(t,1)z_x(t,1) - 2z(t,0)z_x(t,0), \\ &= 2\lambda V(z(t,\cdot)) + 2e^{\sigma}z_x(t,1)U(t) \\ &\quad - \sigma\int_0^1 e^{\sigma x}\frac{\partial|z(t,x)|^2}{\partial x}dx.\end{aligned}$$

Now, by a second integration by parts on the last term, we obtain,

$$\begin{aligned}\frac{d}{dt}V(z(t,\cdot)) &\leq 2\lambda V(z(t,\cdot)) + 2e^{\sigma}z_x(t,1)U(t) \\ &\quad - \sigma\left[e^{\sigma x}|z(t,x)|^2\right]_0^1 \\ &\quad + \sigma^2\int_0^1 e^{\sigma x}|z(t,x)|^2dx, \\ &= 2\left(\lambda+\frac{\sigma^2}{2}\right)V(z(t,\cdot)) + 2e^{\sigma}z_x(t,1)U(t) \\ &\quad - \sigma e^{\sigma}U(t)^2, \\ &= -2e^{\sigma}\left[\frac{\sigma}{2}U(t)^2 - z_x(t,1)U(t)\right. \\ &\qquad \left. -e^{-\sigma}\left(\lambda+\frac{\sigma^2}{2}\right)V(z(t,\cdot))\right].\end{aligned}$$

Then, using the fact that $a \leq a\mathbb{1}_{\{x>0\}}(a)$ for any $a\in\mathbb{R}$, we get,

$$\begin{aligned}\frac{d}{dt}V(z(t,\cdot)) &\leq -2e^{\sigma}\left[\frac{\sigma}{2}U(t)^2 - z_x(t,1)U(t)\right. \\ &\qquad \left. -e^{-\sigma}\left(\lambda+\frac{\sigma^2}{2}\right)\mathbb{1}_{\{x>0\}}\left(\lambda+\frac{\sigma^2}{2}\right)V(z(t,\cdot))\right], \\ &= -2e^{\sigma}\left[\frac{\sigma}{2}U(t)^2 - z_x(t,1)U(t)\right. \\ &\qquad \left. -\mathcal{B}(V(z(t,\cdot))) + \frac{e^{-\sigma}}{2}\mathcal{K}(V(z(t,\cdot)))\right],\end{aligned}$$

with $\mathcal{B}(\cdot)$ being given in (10). ■

Let us now give the first main result of our paper,

Theorem 1: Let $t_0 \geq 0$, $\sigma > 0$. Let $\mathcal{K}:\mathbb{R}_+\mapsto\mathbb{R}_+$ be a continuous function such that $\mathcal{K}(0)=0$. Let $\mathcal{B}(\cdot)$ be given as in (10). Then, under the following control:

$$U(t) = \frac{z_x(t,1) - \sqrt{z_x(t,1)^2 + 2\sigma\mathcal{B}(V(z(t,\cdot)))}}{\sigma}, \quad (11)$$

or

$$U(t) = \frac{z_x(t,1) + \sqrt{z_x(t,1)^2 + 2\sigma\mathcal{B}(V(z(t,\cdot)))}}{\sigma}, \quad (12)$$

the inequality (5) is satisfied for all $t \geq t_0$.

Proof: The proof of Theorem 1 is a direct application of the quadratic formula on the inequality (9), where we chose $U(t)$ to be the solution of the following second-degree equation:

$$\frac{\sigma}{2}U(t)^2 - z_x(t,1)U(t) - \mathcal{B}(V(z(t,\cdot))) = 0.$$

■

IV. Application to finite-time, fixed-time stabilization

In this section, we use Theorem 1 to establish the second main result of our paper which proves the finite-time stability (resp. fixed-time stability) of the closed-loop system (6) with the nonlinear control (11) (or (12)).

Theorem 2: Let $t_0 \geq 0$, $\sigma > 0$. Let $\mathcal{K}:\mathbb{R}_+\mapsto\mathbb{R}_+$ be a continuous function such that $\mathcal{K}(0)=0$. Let $\mathcal{B}(\cdot)$ be given as in (10). Then, if $\mathcal{K}$ is in the set $\mathcal{E}_{k_0,a_0}$ (resp. in the set $\mathcal{E}_{k_0,a_0}^{k_\infty,a_\infty}$), then the closed-loop system (6) with the nonlinear control (11) (or (12)) is finite-time stable (resp. fixed-time stable).

Moreover, there exists a settling time $T_{max}(V(z_0))$ (upper bounded by a constant when FxTS) such that $V(z(t,\cdot))=0$ when $t \geq t_0 + T_{max}(V(z_0))$. By the coercivity condition $\|z(t,\cdot)\|_{L^2}^2 = 0$ when $t \geq t_0 + T_{max}(V(z_0))$.

In particular, if the control (11) is replaced by

$$U(t) = \frac{z_x(t,1) - \text{sign}(z_x(t,1))\sqrt{z_x(t,1)^2 + 2\sigma\mathcal{B}(V(z(t,\cdot)))}}{\sigma}, \quad (13)$$

we have in addition, $|U(t)| \to 0$ when $t \to t_0 + T_{max}(V(z_0))$ and $|U(t)| = 0$ for any $t \geq t_0 + T_{max}(V(z_0))$.

Proof: The proof of Theorem 2 is a straightforward application of Corollary 1 and Theorem 1. In fact, from Theorem 2, we have that (5) is satisfied for all $t \geq t_0$ and any continuous function $\mathcal{K}:\mathbb{R}_+\mapsto\mathbb{R}_+$ such that $\mathcal{K}(0)=0$. In particular, for $\mathcal{K}\in\mathcal{E}_{k_0,a_0}$ (resp. $\mathcal{K}\in\mathcal{E}_{k_0,a_0}^{k_\infty,a_\infty}$). Then from Corollary 1, we conclude that the closed-loop system (6) with (11) (or (12)) is **FTS** (resp. **FxTS**).

Furthermore, if (11) is replaced by (13), we can prove that:

$$\begin{aligned}|U(t)|^2 &= \frac{1}{\sigma^2}\left[\sqrt{z_x(t,1)^2} - \sqrt{z_x(t,1)^2 + 2\sigma\mathcal{B}(V(z(t,\cdot)))}\right]^2, \\ &\leq \frac{1}{\sigma^2}\left[\sqrt{z_x(t,1)^2 + 2\sigma\mathcal{B}(V(z(t,\cdot))) - z_x(t,1)^2}\right]^2, \\ &= \frac{2}{\sigma}\mathcal{B}(V(z(t,\cdot))),\end{aligned}$$

where we have used the fact that $|\sqrt{a_1} - \sqrt{a_2}| \leq \sqrt{|a_1 - a_2|}$, for any $a_1, a_2 \geq 0$. Using in addition the fact that $V(z(t,\cdot)) = 0 \implies \mathcal{K}(V(z(t,\cdot))) = 0 \implies \mathcal{B}(V(z(t,\cdot))) = 0$, we conclude that $|U(t)| \to 0$ for any $t \to t_0 + T_{max}(V(z_0))$ and $|U(t)| = 0$ for any $t \geq t_0 + T_{max}(V(z_0))$. ■

V. Simulations

In this section, we give numerical simulations for the closed-loop system (6) for three different initial conditions $z_0 = x - x^2$, $100z_0$, and $1000z_0$ with the following reaction coefficient $\lambda = 20$, the initial time $t_0 = 0$, and with the nonlinear control $U(t)$ defined as in (11).

A. The case of finite-time stabilization

Let us take the parameters $c = 0.5$, $\alpha = 0.5$, $\sigma = 2$. Figure 1 shows the evolution of the state $z(t,x)$ of the closed-loop system (6) with the control $U(t)$ in (11) (whose time evolution for $\mathcal{K}(V(z(t,\cdot))) = \frac{c}{2}V(z(t,\cdot))^{\alpha}$ is described in Figure 2 for the initial condition $z_0 = x - x^2$) with $\mathcal{K}(V(z(t,\cdot))) = cV(z(t,\cdot))^\alpha$, for the initial condition $z_0 = x - x^2$. Finally, Figure 3 shows in a logarithmic scale the evolution of the norm $\|z(t,\cdot)\|^2_{L^2}$ of the closed-loop system (6) with the nonlinear control $U(t)$ in (11) with $\mathcal{K}(V(z(t,\cdot))) = cV(z(t,\cdot))^\alpha$ in solid lines and with $\mathcal{K}(V(z(t,\cdot))) = cV(z(t,\cdot))$ in dashed lines, and for three different initial conditions: $z_0 = x - x^2$ in blue lines, $100z_0$ in red lines, and $1000z_0$ in black lines. Hence, we can observe from the solid lines that the larger the initial condition, the larger the settling time (i.e. the time of convergence depends on the initial condition).

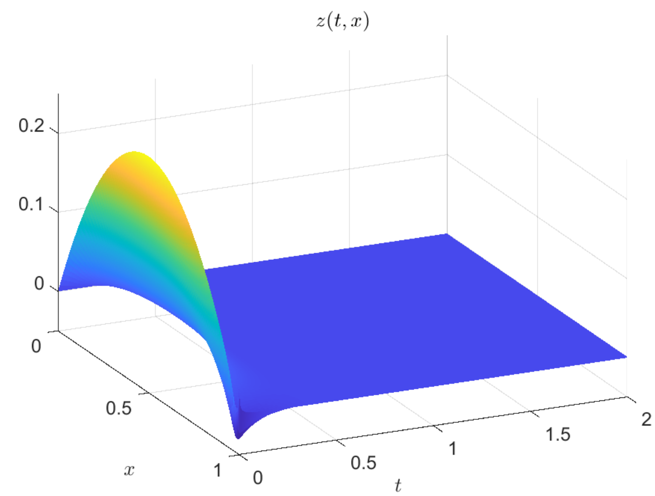

Fig. 1. The evolution of the state $z(t,x)$ of the closed-loop system (6) with the control $U(t)$ in (11) with $\mathcal{K}(V(z(t,\cdot))) = cV(z(t,\cdot))^\alpha$, for the initial condition $z(t_0,x) = x - x^2$,

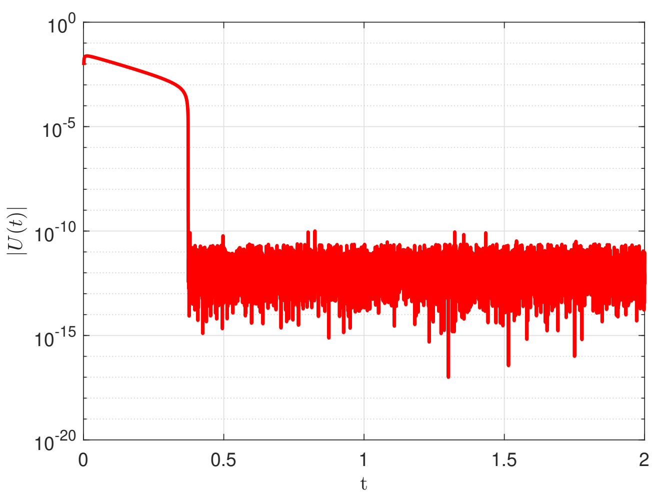

Fig. 2. The evolution of the nonlinear control $U(t)$ given in (11) with $\mathcal{K}(V(z(t,\cdot))) = cV(z(t,\cdot))^\alpha$ for the initial condition $z(t_0,x) = x - x^2$.

B. *The case of fixed-time stabilization*

Let us take the parameters $c_1 = 0.5$, $c_2 = 1$, $\alpha = 0.5$, $\beta = 2$, $\sigma = 3$. Figure 4 shows the evolution of the state $z(t,x)$ of the closed-loop system (6) with the control $U(t)$

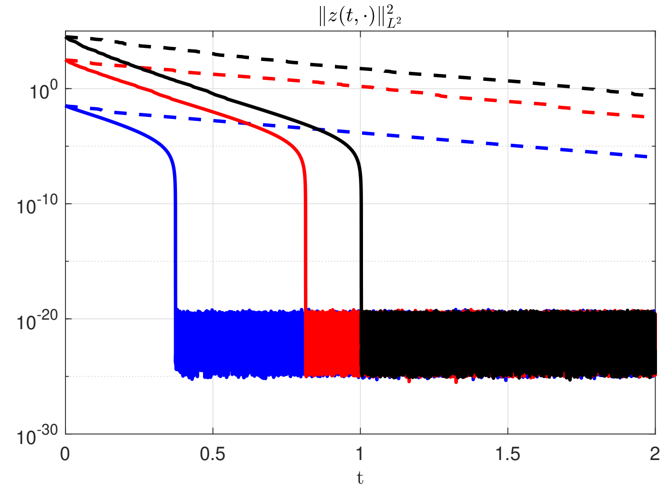

Fig. 3. The evolution of the norm $\|z(t,\cdot)\|^2_{L^2}$ of the closed-loop system (6) in a logarithmic scale in a blue line for the initial condition $z(t_0,x) = x - x^2$, in a red line for $z(t_0,x) = 100(x - x^2)$, and in a black line for $z(t_0,x) = 1000(x - x^2)$, where we used the nonlinear control $U(t)$ given in (11) with $\mathcal{K}(V(z(t,\cdot))) = cV(z(t,\cdot))^\alpha$ to get **FTS** as shown in solid lines, and with $\mathcal{K}(V(z(t,\cdot))) = cV(z(t,\cdot))$ to get exponentially stability shown in dashed lines.

in (11) with $\mathcal{K}(V(z(t,\cdot))) = c_1V(z(t,\cdot))^\alpha + c_2V(z(t,\cdot))^\beta$, for the initial condition $z_0 = x - x^2$. Next, in Figure 5, we present the time evolution of the control $U(t)$ in (11) with $\mathcal{K}(V(z(t,\cdot))) = c_1V(z(t,\cdot))^\alpha + c_2V(z(t,\cdot))^\beta$ and for the initial condition $z_0 = x - x^2$. Finally, Figure 6 shows in a logarithmic scale the evolution of the norm $\|z(t,\cdot)\|^2_{L^2}$ of the closed-loop system (6) with the nonlinear control $U(t)$ in (11) with $\mathcal{K}(V(z(t,\cdot))) = c_1V(z(t,\cdot))^\alpha + c_2V(z(t,\cdot))^\beta$ in solid lines and with $\mathcal{K}(V(z(t,\cdot))) = cV(z(t,\cdot))$ in dashed lines, and for three different initial conditions: $z_0 = x - x^2$ in blue solid lines, $100z_0$ in red solid lines, and $1000z_0$ in black solid lines. Hence, we can observe that the settling time is upper bounded by a constant that does not depend on the initial conditions (i.e. the time of the convergence does not depend on the initial conditions).

VI. Conclusion

This paper deals with the problem of nonlinear boundary stabilization with a predefined type of convergence for a class of reaction-diffusion systems with a constant reaction term using a Lyapunov-based approach. The key idea is to use the "spatially weighted L^2-norm" as a Lyapunov functional V to design a nonlinear control $U(t)$ and to obtain a generalized Lyapunov inequality of the type $\frac{d}{dt}V(z(t,\cdot)) \leq -\mathcal{K}(V(z(t,\cdot)))$ for any continuous function $\mathcal{K}$ such that $\mathcal{K}(0) = 0$. As an application, we deal with the problem of finite-time, and fixed-time stabilization of a class of reaction-diffusion systems with a constant reaction term and we give some numerical simulations to illustrate the results.

The present paper did not study the existence/uniqueness issue for the solutions of the closed-loop system. This is a completely different topic for future research. To this purpose, ideas contained in [17], [9] can be used. However, the obtained stability estimates will certainly help the analysis.

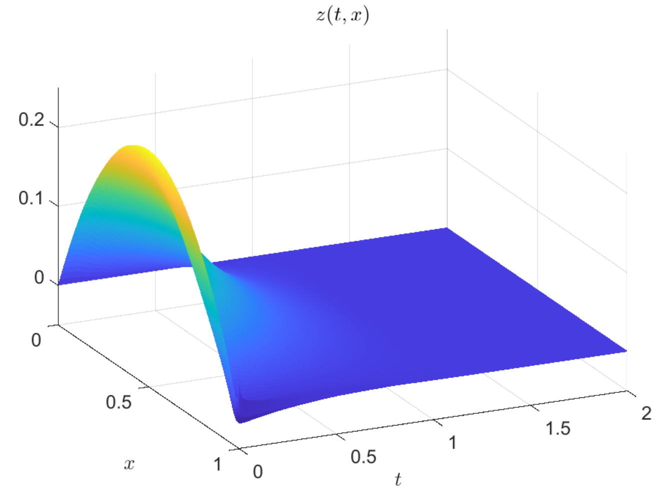

Fig. 4. The evolution of the state $z(t,x)$ of the closed-loop system (6) with the control $U(t)$ in (11) with $\mathcal{K}(V(z(t,\cdot))) = c_1 V(z(t,\cdot))^\alpha + c_2 V(z(t,\cdot))^\beta$, for the initial condition $z(t_0,x) = x - x^2$,

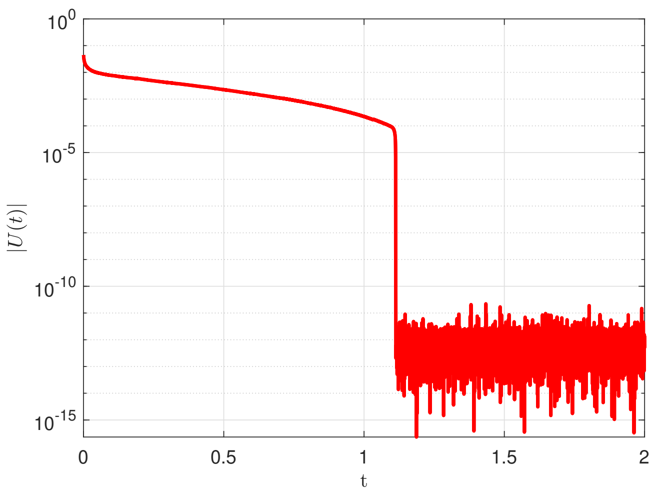

Fig. 5. The evolution of the nonlinear control $U(t)$ given in (11) with $\mathcal{K}(V(z(t,\cdot))) = c_1 V(z(t,\cdot))^\alpha + c_2 V(z(t,\cdot))^\beta$ for the initial condition $z(t_0,x) = x - x^2$.

Future work will extend this result to a class of nonlinear reaction-diffusion-advection systems with a non-delayed/delayed boundary control.

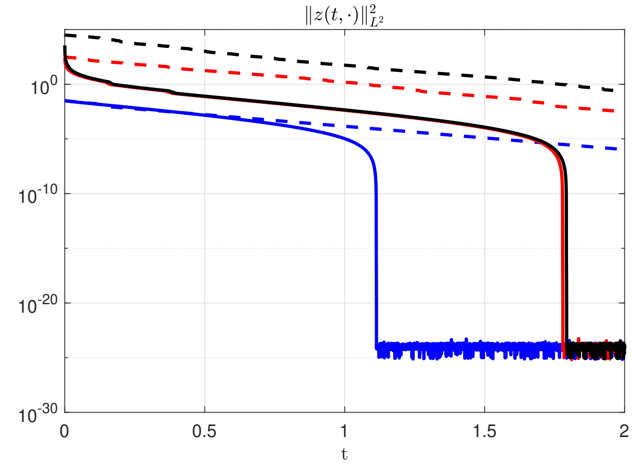

Fig. 6. The evolution of the norm $\|z(t,\cdot)\|^2_{L^2}$ of the closed-loop system (6) in a logarithmic scale in a blue line for the initial condition $z(t_0,x) = x - x^2$, in a red line for $z(t_0) = 100(x - x^2)$, and in a black line for $z(t_0) = 1000(x - x^2)$, where we used the nonlinear control $U(t)$ given in (11) and we took $\mathcal{K}(V(z(t,\cdot))) = c_1 V(z(t,\cdot))^\alpha + c_2 V(z(t,\cdot))^\beta$ to get FxTS as shown in solid lines. Then, we took $\mathcal{K}(V(z(t,\cdot))) = c_1 V(z(t,\cdot))$ to get exponentially stability shown in dashed lines.

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