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Abstract

This note is concerned with presenting new *dissipativity-based* conditions for the stabilizability of discrete-time systems with time-varying delays by linear static output feedback (SOF). We demonstrate that the feasibility of *nonlinear* matrix inequalities for the design of feedback stabilizing gains derived from the literature is equivalent to the feasibility of a *linear* matrix inequality establishing dissipativity of the system *plus* one matrix inequality constraint. An iterative strategy allowing the computation of stabilizing SOF gains is then developed based on a new relaxed sufficient condition. Compared with other popular approaches in the literature, such as the celebrated cone complementarity linearization (CCL) method, the strategy avoids providing initial "guesses" for the matrix variables, which is especially complicated when dealing with a large number of matrices. Due to being a particular case of SOF with an identity output matrix, static state feedback (SSF) gains can also trivially be computed by exploiting the developed conditions.

1 Introduction

The static output feedback (SOF) stabilization problem has been widely investigated in the control literature (Sadabadi & Peaucelle, 2016). The problem of designing a stabilizing SOF or determining its non-existence remains largely unresolved even for linear time-invariant (LTI) systems, since this problem is generally recognized as NP-hard (Fu, 2004). The reader is encouraged to read the survey papers by Sadabadi & Peaucelle (2016); Syrmos et al. (1997) for a view of several existing strategies for computing SOF gains. Most of the proposed solutions are either iterative (Cao et al., 1998) or impose conservative model transformations and rank constraints on the plant matrices (Crusius & Trofino, 1999).

The study of time delay systems is another important problem since delays can degrade closed-loop stability and performance properties (Fridman, 2014). One strategy to study the stability and stabilization of such systems is by proposing and manipulating so-called Lyapunov-Krasovskii functionals (LKFs) (P. Nam et al., 2015; P. T. Nam et al., 2015; P. T. Nam & Luu, 2020). Even static state feedback (SSF) control design can be challenging for such systems since sufficient and necessary convex design conditions are yet to be found, contrary to the non-delayed LTI case. Some solutions to the state feedback design for time-delay systems can be found in Suplin et al. (2006, 2007); Wu et al. (2010); Fridman et al. (2004); Fridman & Shaked (2002a); Zeng et al. (2015); Zhang et al. (2005); Chen & Zheng (2006). Other control strategies for time-delay

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systems include, for example, predictor, observer-based, and backstepping techniques (Krstic, 2010; Krstic & Smyshlyaev, 2008; González & García, 2021; Castillo & García, 2021).

Although some successful control strategies already exist for output feedback of time-delay systems (TDS), such as the predictor and observer-based ones, the static output feedback stabilization (SOF) of TDS remains largely an open problem. In the continuous-time setting, we can cite the approaches explored in Barreau et al. (2018); Du et al. (2010). Motivated by this fact, in this work, we investigate the problem of *SOF stabilizability* of a specific class of discrete-time delay systems. In particular, we concentrate on the case of systems with *input time-varying* delays. The goal is to examine the relations between the SOF stabilizability of these systems and their dissipativity properties.

Recently, dissipativity theory has been investigated for its relations with the stabilizability of continuous-time nonlinear systems (D. de S. Madeira, 2022), where necessary and sufficient conditions for the exponential stabilizability of nonlinear non-delayed systems by linear SOF were presented. An extension to the case of input-saturated nonlinear systems in differential-algebraic representation (DAR) was also recently studied (Alves Lima et al., 2022). Dissipativity theory has also been recently studied for the stability analysis of aperiodically sampled nonlinear systems with time-varying delay (Thomas et al., 2021).

In this note, we study the relationship between the SOF stabilization of linear discrete-time systems affected by time-varying input delays and dissipativity theory. The main theoretical results are then leveraged for the development of an iterative design algorithm that has, as main advantage over the current literature, the fact that it avoids the cumbersome step of providing initial "guesses" for the Lyapunov matrices, as is the case when using the celebrated cone complementarity linearization (CCL) method. A very preliminary version of this work is Alves Lima & de S. Madeira (2021), which, however, presents neither the main theoretical results of the current work (in Theorem 1 and its proof), nor the new iterative algorithm strategy (given in Section 4.1). In Alves Lima & de S. Madeira (2021), only a sufficient condition that depends on the choice of a tuning matrix that needs to be chosen *a priori* by the designer is provided.

This work is organized as follows: Section 2 reviews preliminaries on dissipativity theory for discrete-time systems. Section 3 details the system of interest, its stability and stabilization, and Section 4 presents the main results on stabilizability and dissipativity, along with a controller design algorithm and numerical example. Finally, Section 5 discusses the main results and future research.

Notation. For a matrix Y in $\mathbb{R}^{n\times m}$, $Y^\top$ means its transpose. For matrices $W = W^\top$ and $Z = Z^\top$ in $\mathbb{R}^{n\times n}$, $W \succ Z$ ($W \succeq Z$) means that $W - Z$ is positive (semi-)definite. $diag(W, Z)$ corresponds to the block-diagonal matrix, the operator $\mathrm{He}\{W\}$ denotes $\mathrm{He}\{W\} = W + W^\top$. $\mathbb{S}_n^+$ stands for the set of symmetric positive definite matrices. I and 0 denote identity and null matrices. The $\star$ in the expression of a matrix denotes symmetric blocks. For integers $a < b$, we use $[\![a, b]\!]$ to denote the set $\{a, a+1, \dots, b-1, b\}$. Finally, $\mathbb{R}^+$ denotes the set $\{\beta \in \mathbb{R} | \beta \geq 0\}$.

2 Dissipativity of discrete-time systems

Concerning the analysis of dynamical systems, a very important property, introduced in the seminal work by Willems (1972), is the energy-based concept of dissipativity. A system is said to be dissipative if the variation of its internal energy (a function of the system states) is not greater than the rate at which energy is supplied to the system by the environment (from external signals) at any given moment. The internal energy of the system is represented by a so-called "storage" function $V(x)$, while the supply rate is modelled by a function $w(u, y)$ of input u and output y signals.

A discrete-time dynamical system given by $x(k+1) = f(x(k), u(k))$ and $y(k) = h(x(k))$, respectively, with state $x(k) \in \mathbb{R}^n$, input $u(k) \in \mathbb{R}^m$, and output $y(k) \in \mathbb{R}^p$ is said to be dissipative with respect to a supply rate function $w(k) = w(u(k), y(k))$ if there exists a continuous nonnegative storage function $V : \mathbb{R}^n \to \mathbb{R}^+$ such that the relation

$$V(x(k+1)) - V(x(k)) \leq w(u(k), y(k)) \tag{1}$$

holds for all initial states, for all admissible $u \in \mathbb{R}^m$, and all $k \in \mathbb{Z}^+$ (Byrnes & Lin, 1994).

The choice of supply rate plays an essential role in dissipativity theory analysis as different supply rate functions can be used to determine different important characteristics of a dynamical system. For example, a system that is dissipative with respect to a supply $w(u(k), y(k)) = y^\top u$ is called passive (valid only for square systems $m = p$), while being dissipative with respect to a supply $w(u(k), y(k)) = \gamma^2 u^\top u - y^\top y$ implies finite induced l_2 gain γ. More details on dissipativity theory can be found in Brogliato et al. (2020). A stronger notion of dissipativity is the so-called "strict" dissipativity. In this case, one demands the existence of a storage function $V(x(k))$ such that $V(x(k+1)) - V(x(k)) < w(u(k), y(k))$, $\forall x(k) \neq 0$, $\forall k$, $\forall u(k)$. In this work, we employ the definition of strict QSR-dissipativity.

Definition 1. *A system is strictly QSR-dissipative if it is strictly dissipative with respect to the quadratic supply rate*

$$w(u(k), y(k)) = y(k)^\top Q y(k) + 2y(k)^\top S u(k) + u(k)^\top R u(k), \tag{2}$$

where $S \in \mathbb{R}^{p\times m}, Q = Q^\top \in \mathbb{R}^{p\times p}$, and $R = R^\top \in \mathbb{R}^{m\times m}$.

3 Systems with time-varying input delays

Consider the system described by the following equations

$$\begin{cases} x(k+1) = Ax(k) + Bu(k - d(k)) \\ y(k) = Cx(k) \\ u(k) = \phi_u(k) = 0,\ k \in [\![-d_M, -1]\!] \end{cases} \tag{3}$$

where $x(k) \in \mathbb{R}^n$ is the system state vector, $y(k) \in \mathbb{R}^p$ is the output, $u(k) \in \mathbb{R}^m$ is the control input, and $x(0)$, $\phi_u(k)$ are initial conditions for the state and control input. The system input delay $d(k) \in \mathbb{N}$ is bounded and time-varying such as $1 \leq d_m \leq d(k) \leq d_M$, and can arbitrarily vary within such limits. Furthermore, integers d_m and d_M are known, whereas the value of $d(k)$ at each sampling time is unknown.

To control (3), consider the following static output feedback control law

$$u(k) = Ky(k), \tag{4}$$

where $K \in \mathbb{R}^{m\times p}$ is a stabilizing gain to be designed. The interconnection (3)-(4) generates the following state-delayed closed-loop system

$$\begin{cases} x(k+1) = Ax(k) + BKCx(k - d(k)) \\ x(k) = \phi(k),\ k \in [\![-d_M, 0]\!] \end{cases} \tag{5}$$

where $\phi(k)$, given by $\phi(0) = x(0)$ and $\phi(k) = 0,\ k \leq -1$, is the initial condition at the interval $[\![-d_M, 0]\!]$.

3.1 Stability of time-delayed systems

In general, stability of time-delayed systems can be tackled by using either delay-independent or delay-dependent conditions. The latter case (in which bounds on the delay are explicitly considered) is preferred in this work. Let us consider discrete-time linear system (5) with time-varying delay $d(k)$ and define its concatenated state as $\bar{x}(k) \triangleq \begin{bmatrix} x^\top(k) & \cdots & x^\top(k-d_M) \end{bmatrix}^\top \in \mathbb{R}^{(d_M+1)n}$. In the Lyapunov-Krasovskii framework, system (5) is asymptotically stable if there exists $V : \mathbb{Z}^+ \times \mathbb{R}^{(d_M+1)n} \to \mathbb{R}^+$, with the shortcut $V(k) = V(k, \bar{x}(k))$, such that $V(k) > 0$, for all $k \in \mathbb{Z}^+$ and $\bar{x}(k) \neq 0$, that is V is positive definite and the forward difference of V with respect to (5) is negative, i.e., $\Delta V(k) := V(k+1) - V(k) = V(k+1, \bar{x}(k+1)) - V(k, \bar{x}(k)) < 0$, for all $k \in \mathbb{Z}^+$. In the last ten years, many different LKF structures and inequalities for their manipulation have been proposed in the literature. Those strategies lead to Linear Matrix Inequalities (LMIs) for the stability analysis of system (5). However, as in the delay-free case, the design of a static output feedback gain is cumbersome due arising nonlinearities between the gain K and the LKF matrices. Here, we aim at giving a different point of view for this problem, by first analysing the relation between the stabilizability of system (3) by SOF and its dissipativity properties. Then, we intend to develop an iterative strategy for the design of stabilizing linear static output feedback gains.

In order to establish stability of (5) for any time-varying delay $d_m \leq d(k) \leq d_M$, we need to find a Lyapunov-Krasovskii functional $V(k)$ satisfying $V(k+1)-V(k) < 0$. Consider the LKF borrowed from Seuret et al. (2015), given by

$$V(k) = V_1(k) + V_2(k) + V_3(k), \tag{6}$$

$$\begin{aligned}
V_1(k) &= \sigma^\top(k) P \sigma(k),\\
V_2(k) &= \sum_{l=k-d_m}^{k-1} x^\top(l) W_1 x(l) + \sum_{l=k-d_M}^{k-d_m-1} x^\top(l) W_2 x(l),\\
V_3(k) &= d_m \sum_{l=-d_m+1}^{0} \sum_{i=k+l}^{k} \zeta^\top(i) Z_1 \zeta(i) + d_\Delta \sum_{l=-d_M+1}^{-d_m} \sum_{i=k+l}^{k} \zeta^\top(i) Z_2 \zeta(i),
\end{aligned}$$

with $\sigma(k) = \begin{bmatrix} x^\top(k) & \sum_{l=k-d_m}^{k-1} x^\top(l) & \sum_{l=k-d_M}^{k-d_m-1} x^\top(l) \end{bmatrix}^\top \in \mathbb{R}^{3n}$ a vector that concatenates the current state with terms having the sum of past states, $d_\Delta = d_M - d_m$, and $\zeta(i) = x(i) - x(i-1)$. Matrices $P \in \mathbb{S}^+_{3n}$, W_1, W_2, Z_1, and $Z_2 \in \mathbb{S}^+_n$ guarantee that the functional is positive definite. For more discussion on this functional, please refer to Seuret et al. (2015).

Consider the partition $P = \begin{bmatrix} P_1 & P_2 & P_3 \\ \star & P_4 & P_5 \\ \star & \star & P_6 \end{bmatrix}$, with P_1, P_4, $P_6 \in \mathbb{S}^+_n$, and $P_2, P_3, P_5 \in \mathbb{R}^{n\times n}$, and auxiliary matrices

$$\begin{aligned}
\overline{B} &= \begin{bmatrix} B^\top & B^\top & B^\top & B^\top & B^\top & B^\top & B^\top \end{bmatrix}^\top \in \mathbb{R}^{7n\times m},\\
\overline{C} &= \begin{bmatrix} 0 & 0 & C & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{p\times 7n}.
\end{aligned}$$

Then, the following lemma, derived from Seuret et al. (2015), with $\Phi(d)$, Ψ_z given in Appendix A, provides sufficient conditions for the stabilizability of (3) by linear SOF.

Lemma 1. *Suppose there exist $K \in \mathbb{R}^{m\times p}$, $P \in \mathbb{S}^+_{3n}$, W_1, W_2, Z_1, Z_2, $\in \mathbb{S}^+_n$, and $X \in \mathbb{R}^{2n\times 2n}$ such that $\Psi_z \succ 0$ (with Ψ_z given in Appendix A) and*

$$J(d) + \mathrm{He}\{G(d)\overline{B}K\overline{C}\} + \overline{C}^\top K^\top B^\top P_{\mathrm{del}} B K \overline{C} \prec 0 \tag{7}$$

hold for $d = d_m$ and $d = d_M$, with $J(d) = \Gamma^\top \Phi(d) \Gamma$, $\Phi(d)$ given in Appendix A, $\Gamma = \left[\begin{array}{cc} A & 0_{n\times 6n} \\ \hdashline \multicolumn{2}{c}{I_{7n}} \end{array}\right] \in \mathbb{R}^{8n\times 7n}$, $P_{\mathrm{del}} = P_1 + Z_1 d_m^2 + Z_2 d_\Delta^2$, and

$$\begin{aligned}
G(d) = diag\big(&A^\top P_{\mathrm{del}} - \left(Z_1 d_m^2 + Z_2 d_\Delta^2\right), -P_2, -P_3, -P_3,\\
&P_2(d_m+1), dP_3 - (d_m-2)P_3^\top, (d_M+1)P_3^\top - dP_3\big).
\end{aligned}$$

Then, the gain K asymptotically stabilizes the closed-loop system (5) for any time-varying delay $d_m \leq d(k) \leq d_M$ and $V(k)$ in (6) is a LKF for (5).

Proof. Lemma 1 is a direct consequence of Seuret et al. (2015)[Thm. 5] by replacing the matrix A_d in Seuret et al. (2015)[Thm. 5] by BKC, and therefore the steps of the proof are not repeated here. ☐

As expected, the main condition (7) in Lemma 1 contains nonlinear terms in the matrix variables. Our goal is to show that there exists a LKF of the form (6) ensuring the stability of the closed-loop system (5) if and only if the open-loop system (3) is dissipative with respect to a quadratic supply rate (QSR) function while an inequality constraint containing the matrix variables hold. Then, an iterative design strategy can be developed to ensure the negativity of the supply rate function subject to the control law (4), and allowing the computation of the stabilizing gain K.

4 Dissipativity-based stabilization

In this section, we establish the main results for the dissipativity-based stabilization of system (3). First, consider the following definition of delay-dependent strict QSR-dissipativity for system (3).

Definition 2. *If there exists a nonnegative storage function $V : \mathbb{Z}^+ \times \mathbb{R}^{(d_M+1)n} \to \mathbb{R}^+$, with the shortcut $V(k) = V(k,\bar{x}(k))$ such that*

$$\Delta V(k) := V(k+1,\bar{x}(k+1)) - V(k,\bar{x}(k)) < w(u(k-d(k)), y(k-d(k))) \tag{8}$$

holds for $\forall k \in \mathbb{Z}^+$, $\forall \bar{x}(k) \neq 0$, and any time-varying delay $d_m \leq d(k) \leq d_M$, where $w(u(k-d(k)), y(k-d(k)))$ is the delayed version of the QSR-supply (2), then $V(k)$ is a storage function for the system (3), and the system (3) is delay-dependent strictly QSR-dissipative with respect to the time-varying "delayed" supply rate function $w(u(k-d(k)), y(k-d(k)))$.

Note that the definition above extends the idea of delay-dependent stability to delay-dependent dissipativity, i.e., a weaker notion of dissipativity for TDS where one needs to search for storage functions V satisfying the dissipation inequality only for delays $d(k)$ belonging to some interval. Other definitions of delay-dependent dissipativity for other classes of TDS in the literature can be found, for example, in Fridman & Shaked (2002b); Feng et al. (2011); Mahmoud (2011). In the sequel, we present a sufficient condition for the existence of a storage function satisfying the dissipation inequality (8).

Lemma 2. *Assume that there exist matrices $P \in \mathbb{S}^+_{3n}$, $N \in \mathbb{S}^+_{7n}$, W_1, W_2, Z_1, $Z_2 \in \mathbb{S}^+_n$, $R \in \mathbb{S}^+_m$, symmetric matrix $Q \in \mathbb{R}^{p\times p}$, matrices $S \in \mathbb{R}^{p\times m}$ and $X \in \mathbb{R}^{2n\times 2n}$ such that $\Psi_z \succ 0$ (with Ψ_z given in Appendix A) and*

$$\begin{bmatrix} J(d) + N - \overline{C}^\top Q\overline{C} & G(d)\overline{B} - \overline{C}^\top S \\ \star & B^\top P_{\mathrm{del}} B - R \end{bmatrix} \prec 0 \tag{9}$$

hold for $d = d_m$ and $d = d_M$, with $J(d)$, $G(d)$,Ψ_z, $\overline{B}$, $\overline{C}$ previously defined. Then, system (3) is strictly dissipative with respect to the quadratic supply rate function $w(u(k-d(k)), y(k-d(k)))$ for any time-varying delay $d_m \leq d(k) \leq d_M$, and the function $V(k)$ in (6) is a storage function for (3) satisfying the dissipation inequality (8).

Proof. First of all, since $N \succ 0$, one can show that (9) implies the satisfaction of

$$\begin{bmatrix} J(d) - \overline{C}^\top Q\overline{C} & G(d)\overline{B} - \overline{C}^\top S \\ \star & B^\top P_{\mathrm{del}} B - R \end{bmatrix} \prec 0. \tag{10}$$

Consider $\kappa(k) = \begin{bmatrix} \xi^\top(k) & u^\top(k-d(k)) \end{bmatrix}^\top$, where

$$\xi(k) := \begin{bmatrix} x^\top(k) & x^\top(k-d_m) & x^\top(k-d(k)) & x^\top(k-d_M) & v^\top \end{bmatrix}^\top$$

$v = \begin{bmatrix} v_1^\top & v_2^\top & v_3^\top \end{bmatrix}^\top$, $v_1 = \frac{1}{d_m+1}\sum_{l=k-d_m}^{k} x(l)$, $v_2 = \frac{1}{d(k)-d_m+1}\sum_{l=k-d(k)}^{k-d_m} x(l)$, $v_3 = \frac{1}{d_M-d(k)+1}\sum_{l=k-d_M}^{k-d(k)} x(l)$. Next, left and right multiply (9) by $\kappa^\top(k)$ and $\kappa(k)$, respectively. Then, considering the structure of the matrix $\overline{C}$ and the fact that $y(k-d(k)) = Cx(k-d(k))$, we can write the obtained expression as

$$\kappa^\top(k) \begin{bmatrix} J(d) & G(d)\overline{B} \\ \star & B^\top P_{\mathrm{del}} B \end{bmatrix} \kappa(k) < w(u(k-d(k)), y(k-d(k))). \tag{11}$$

The next steps of the proof are summarized due to space constraints. Using a summation version of Wirtinger's integral inequality from Seuret & Gouaisbaut (2013) and the reciprocally convex Lemma (Park et al., 2011), one can use similar steps to the proof of Seuret et al. (2015)[Thm. 5] to show that (11) implies (8) with storage function given by the LKF (6), where $J(d)$, defined in Lemma 1 and Appendix A, contains the matrix Ψ_z, which has to be symmetric positive definite and is composed by the LKF matrix Z_2 and by the slack decision variable X in $\mathbb{R}^{2n\times 2n}$. $\square$

By Schur complement, (9) holds with $R_{\rm del} := R - B^\top P_{\rm del} B \succ 0$ if and only if

$$J(d) + N - \overline{C}^\top Q\overline{C} + \left(G(d)\overline{B} - \overline{C}^\top S\right) R_{\rm del}^{-1} (\star) \prec 0. \tag{12}$$

Next, we introduce Theorem 1, a new contribution establishing equivalence between conditions in Lemmas 1 and 2.

Theorem 1. *The three statements are equivalent:*

a) There exist K and $(P, W_1, W_2, Z_1, Z_2, X)$ such that conditions $\Psi_z \succ 0$ and (7) hold;

b) There exist (Q, S, R) and $(P, W_1, W_2, Z_1, Z_2, X)$ such that conditions $\Psi_z \succ 0$, (12), and

$$\Delta_{\rm del} = SR_{\rm del}^{-1} B^\top P_{\rm del} B R_{\rm del}^{-1} S^\top$$

hold, where $\Delta_{\rm del} := SR_{\rm del}^{-1}S^\top - Q$. The induced gain $K = -R_{\rm del}^{-1}S^\top$ is a stabilizing one.

c) There exist (Q', S', R') and $(P, W_1, W_2, Z_1, Z_2, X)$ such that conditions $\Psi_z \succ 0$, (12), and

$$\Delta'_{\rm del} \succeq S'(R'_{\rm del})^{-1} B^\top (P'_{\rm del}) B (R'_{\rm del})^{-1} (S')^\top$$

hold, where $\Delta'_{\rm del} := S'(R'_{\rm del})^{-1}(S')^\top - Q'$. The induced gain $K = -(R'_{\rm del})^{-1}(S')^\top$ is a stabilizing one.

Proof. First of all, the variables $(P, W_1, W_2, Z_1, Z_2, X)$ in one statement are kept the same for the other statement. In addition, Ψ_z depending only on a part of these variables, $\Psi_z \succ 0$ does not need to be proven. Let us be focused on the other variables and conditions.

In the sequence, the proof is done circularly.

• *a) $\Rightarrow$ b):* Inequalities (7) are strict and in finite number. It implies that there exists $N \in \mathbb{S}^+_{7n}$ such that

$$N + J(d) + \mathrm{He}\{G(d)\overline{B}K\overline{C}\} + \overline{C}^\top K^\top B^\top P_{\rm del} BK\overline{C} \prec 0$$

holds for $d \in \{d_m, d_M\}$. For the same reason, there exists a scalar $\epsilon > 0$, large enough, such that:

$$N + J(d) + \mathrm{He}\{G(d)\overline{B}K\overline{C}\} + \overline{C}^\top K^\top B^\top P_{\rm del} BK\overline{C} + \frac{1}{\epsilon} G(d)\overline{B}\,\overline{B}^\top G(d)^\top \prec 0.$$

Let us now introduce $R = B^\top P_{\rm del} B + \epsilon I_m$. It is clear that $R \in \mathbb{S}^+_m$ and that $R_{\rm del} := R - B^\top P_{\rm del} B = \epsilon I_m \succ 0_m$ is invertible. We select also $S = -\epsilon K^\top = -K^\top R_{\rm del} \in \mathbb{R}^{p\times m}$. We have such that $K = -R_{\rm del}^{-1} S^\top$. The previous inequality rewrites

$$N + J(d) - \mathrm{He}\{G(d)\overline{B}\, R_{\rm del}^{-1} S^\top \overline{C}\} + \overline{C}^\top S R_{\rm del}^{-1} B^\top P_{\rm del} B R_{\rm del}^{-1} S^\top \overline{C} + G(d)\overline{B}\, R_{\rm del}^{-1} \overline{B}^\top G(d)^\top \prec 0,$$

leading to

$$N + J(d) - \overline{C}^\top \left(SR_{\rm del}^{-1}S^\top - SR_{\rm del}^{-1}B^\top P_{\rm del} BR_{\rm del}^{-1}S^\top\right)\overline{C} + \left(G(d)\overline{B} - \overline{C}^\top S\right) R_{\rm del}^{-1} \left(G(d)\overline{B} - \overline{C}^\top S\right)^\top \prec 0.$$

Finally setting $Q = SR_{\rm del}^{-1}S^\top - R_{\rm del}^{-1}B^\top P_{\rm del} BR_{\rm del}^{-1}S^\top$ allows to satisfy conditions (12) and $\Delta_{\rm del} = SR_{\rm del}^{-1}B^\top P_{\rm del} BR_{\rm del}^{-1}S^\top$, where $\Delta_{\rm del} := SR_{\rm del}^{-1}S^\top - Q$.

• *b) $\Rightarrow$ c)*: Let us choose $(Q', R', S') = (Q, R, S)$. The equation

$$SR_{\rm del}^{-1}B^\top P_{\rm del} BR_{\rm del}^{-1}S^\top = SR_{\rm del}^{-1}S^\top - Q$$

is a particular case of the non-strict inequality.

• *c) $\Rightarrow$ a)*: Suppose that there exist (Q', S', R') and $(P, W_1, W_2, Z_1, Z_2, X)$ such that $\Psi_z \succ 0$, Inequalities (12) and $\Delta'_{\rm del} \succeq S'(R'_{\rm del})^{-1}B^\top P'_{\rm del} B(R'_{\rm del})^{-1}(S')^\top$ hold, where $\Delta'_{\rm del} := S'(R'_{\rm del})^{-1}(S')^\top - Q'$. Then we have:

$$J(d) + N - \mathrm{He}\{G(d)\overline{B}(R'_{\rm del})^{-1}S^\top\overline{C}\} + G(d)\overline{B}(R'_{\rm del})^{-1}\overline{B}^\top G(d)^\top + \overline{C}^\top S'(R'_{\rm del})^{-1}B^\top P_{\rm del} B(R'_{\rm del})^{-1}(S')^\top\overline{C} \prec 0.$$

By choosing $K = -(R'_{\text{del}})^{-1}(S')^\top$, it yields:

$$J(d) + \text{He}\{G(d)\overline{B}K\overline{C}\} + \overline{C}^\top K^\top B^\top P_{\text{del}} B K \overline{C} \prec -N - G(d)\overline{B}(R'_{\text{del}})^{-1}\overline{B}^\top G(d)^\top \prec 0,$$

which implies feasibility of (7) and completes the proof. $\square$

Remark 1. *Statements b) and c) exhibit similarity, with the primary distinction being the presence of an equality in one and a non-strict inequality in the other. From a numerical standpoint, c) encompasses a larger number of systems and offers greater numerical robustness, making it more amenable to manipulation using computational tools.*

For better readability of the conditions, the developed design theorem was presented by using the LKF strategy from Seuret et al. (2015), which applies Wirtinger's summation inequalities and the reciprocally convex lemma for the LKF manipulation. Nonetheless, in general, newer LKFs (which also serve as storage functions) and strategies for their manipulation (such as the ones in P. T. Nam & Luu (2020); Lee et al. (2019)) could have been cast in the framework of the present work, potentially leading to less conservatism.

4.1 Controller design algorithm

The problem of designing a stabilizing SOF is generally formulated as determining a gain K satisfying item *a*) in Theorem 1. Condition (7) contains many nonlinearities between the gain K and the matrices of the LKF (6). Those nonlinearities pose a significant challenge to computational efficiency in solving them, and current linearization methods for addressing this type of inequality often introduce conservatism. Another option is to apply a Schur complement and use the well-known cone complementarity linearization (CCL) method (El Ghaoui et al., 1997). One difficulty with this method is the need to initialize the Lyapunov matrices, which can be non-intuitive.

Here, we reformulate the problem of designing a stabilizing SOF into the following problem, thanks to item *c*) in Theorem 1, using the dissipativity approach.

Problem 1. *Determine (Q', S', R') and $(P, W_1, W_2, Z_1, Z_2, X)$ such that conditions $\Psi_z \succ 0$, (12), and $\Delta'_{\text{del}} \succeq S'(R'_{\text{del}})^{-1}B^\top(P'_{\text{del}})B(R'_{\text{del}})^{-1}(S')^\top$ hold, where $\Delta'_{\text{del}} := S'(R'_{\text{del}})^{-1}(S')^\top - Q'$.*

Here, we develop an iterative strategy for solving Problem 1. The main potential advantage with the strategy in comparison with the CCL algorithm is that no initial guess on the matrix variables are needed to be provided by the designer, as explained in the sequel. In order to develop such a strategy, consider the following two propositions.

Proposition 1. *Assume that there exist matrices (Q, S, R), an invertible matrix R_{del}, a matrix $L_{\text{del}} \in \mathbb{R}^{(p+m)\times m}$ and a scalar $\lambda \in \mathbb{R}$ such that*

$$\begin{bmatrix} Q & S \\ \star & R \end{bmatrix} + \text{He}\left\{L_{\text{del}} \begin{bmatrix} S^\top & R_{\text{del}} \end{bmatrix}\right\} \preceq \lambda I_{p+m}, \tag{13}$$

then

$$\Delta_{\text{del}} \succeq S R_{\text{del}}^{-1} B^\top P_{\text{del}} B R_{\text{del}}^{-1} S^\top - \lambda\left(I + S R_{\text{del}}^{-2} S^\top\right). \tag{14}$$

Proof. Assume that inequality (13) holds, and multiply this inequality at left by $\begin{bmatrix} I_p & -SR_{\text{del}}^{-1} \end{bmatrix}$ and at right by its transpose to obtain:

$$Q - 2SR_{\text{del}}^{-1}S^\top + SR_{\text{del}}^{-1}RR_{\text{del}}^{-1}S^\top \leq \lambda\left(I_p + SR_{\text{del}}^{-1}R_{\text{del}}^{-1}S^\top\right).$$

By using $R = R_{\text{del}} + B^\top P_{\text{del}} B$ and the definition of Δ_{del}, it leads to (14). $\square$

We can deduce from Proposition 1 that if $\lambda \leq 0$ in (13), then $\Delta_{\text{del}} \succeq S R_{\text{del}}^{-1} B^\top P_{\text{del}} B R_{\text{del}}^{-1} S^\top$, which is a condition in item *c*) in Theorem 1. Inequality (13) is a Bilinear Matrix Inequality (BMI) when L_{del} is a variable. However it is a LMI when L_{del} is fixed. In the sequel, we propose an iterative algorithm that alternatively fix L_{del} and update it thanks to the induced variables. The objective of the algorithm is to look for $\lambda \leq 0$. The following Proposition 2 proposes a generic structure of L_{del} to update it at each iteration with the values of S, R, and R_{del} obtained in the previous iteration.

Proposition 2. *There is no conservatism in constraining the multiplier in* (13) *to be of the form* $L_{\text{del}} = \begin{bmatrix} -R_{\text{del}}^{-1} S^\top & -R_{\text{del}}^{-1} R \end{bmatrix}^\top$.

Proof. Omitted due to space constraints. $\square$

The use of slack variables through the Finsler's Lemma combined with the use of relaxation parameters has been shown useful in recent strategies for iteratively computing stabilizing SOF gains for linear and nonlinear uncertain systems (Felipe & Oliveira, 2021; Alves Lima et al., 2022).

The strategy is summarized by Algorithm 1: the scalar λ should be minimized at each iteration in order to try to enforce the satisfaction of (13) with $\lambda \leq 0$, and consequently of the sufficient stability condition $\Delta_{\text{del}} \succeq S R_{\text{del}}^{-1} B^\top P_{\text{del}} B R_{\text{del}}^{-1} S^\top$.

Algorithm 1: Control design algorithm.

input : i_{max}
output : K
1 $i \leftarrow 0$, $S_0 \leftarrow 0$, $R_0 \leftarrow I$, and $R_{\text{del}_0} \leftarrow I$;
2 **while** $i < i_{max}$ **do**
3 $\quad L_{\text{del}} \leftarrow \begin{bmatrix} -R_{\text{del}_0}^{-1} S_0^\top & -R_{\text{del}_0}^{-1} R_0 \end{bmatrix}^\top$;
4 $\quad \min\limits_{P,N,W_1,W_2,Z_1,Z_2,X,R,Q,S} \lambda$ s.t. (9) and (13);
5 $\quad$ **if** $\lambda \leq 0$ **then**
6 $\quad\quad$ **return** $K = -R_{\text{del}}^{-1} S^\top$;
7 $\quad$ **end**
8 $\quad i \leftarrow i+1$, $S_0 \leftarrow S$, $R_0 \leftarrow R$, $R_{\text{del}_0} \leftarrow R - B^\top P_{\text{del}} B$;
9 **end**

Theorem 2. *The inequality* (13) *is always feasible at the first iteration with the initializing choices provided in Algorithm 1. Furthermore, at each following iteration in the while loop, the objective λ is nonincreasing.*

Proof. Suppose that at the first iteration, LMI (9) holds for a solution set of matrices

$$\mathcal{Y}_1 = \{P_1, N_1, W_{1_1}, W_{2_1}, Z_{1_1}, Z_{2_1}, X_1, R_1, Q_1, S_1\}.$$

Inequality (13) is also feasible since $L_{\text{del}} = \begin{bmatrix} 0 & -I_m \end{bmatrix}^\top$ leads to

$$\operatorname{diag}\left(Q - \lambda_1 I_p, R + 2B^\top P_{\text{del}} B - \lambda_1 I_m\right) \preceq 0,$$

which can always be satisfied with large enough λ_1. Then, due to the structure of inequality (13) derived from Finsler's lemma, at the next iteration there exists large enough λ_2 and a set of matrices $\mathcal{Y}_2$ satisfying the problem in line 4 since this can be achieved at least with the trivial solution $\mathcal{Y}_2 = \mathcal{Y}_1$, $\lambda_2 = \lambda_1$. For each following iteration, the same logic applies, where there exists at least the trivial solution $\lambda_{i+1} = \lambda_i$, $\mathcal{Y}_{i+1} = \mathcal{Y}_i$, meaning that the new λ is at least as good as the one from the previous iteration. $\square$

4.2 Numerical example

Consider the network control system example from Hu et al. (2007)

$$\dot{x} = \begin{bmatrix} -0.80 & -0.01 \\ 1.00 & 0.10 \end{bmatrix} x + \begin{bmatrix} 0.4 \\ 0.1 \end{bmatrix} u.$$

By considering a sampling time of 0.5 seconds and a network induced delay $d(k)$, model (3) is obtained with

$$A = \begin{bmatrix} 0.6693 & -0.0042 \\ 0.04231 & 1.0501 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1647 \\ 0.0960 \end{bmatrix}.$$

In Hu et al. (2007), a static state feedback (SSF) gain was designed that guaranteed stability for a maximum network delay $d_M = 2$. Here, we use the developed strategy for SOF design, considering an output matrix $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$. By running design Algorithm 1 with $d_m = 1$ and $d_M = 19$, after 178 iterations, we find SOF gain $K = -0.1488$ that guarantees stability for any time-varying delay such that $1 \leq d(k) \leq 19$, a much larger bound than the one from Hu et al. (2007).

5 Conclusion

We presented new delay-dependent conditions and an associated iterative algorithm for the SOF stabilization of linear discrete-time systems with input time-varying delays. The conditions were developed using dissipativity theory and employing Lyapunov-Krasovskii functionals as candidate storage functions. A numerical example shows that the algorithm successfully stabilizes an example from the literature. Future work will focus on generalizing the links between the stabilizability of time-delay systems and their dissipation properties by using converse Lyapunov theorems for time-delay systems recently proposed in the literature. Furthermore, the convergence rate of Algorithm 1, which is a nontrivial topic, will be an object of investigation.

A Auxiliary matrices for Lemma 1

All null and identity matrices below are of dimension $n \times n$, unless explicitly indicated in the matrix.

$$\begin{aligned} \Phi(d) = {} & F_2^\top P F_2 - F_1^\top P F_1 + \mathrm{He}\{F^\top(d) P (F_2 - F_1)\} + W \\ & + F_3^\top (d_m^2 Z_1 + d_\Delta^2 Z_2) F_3 - F_s^\top \mathscr{Z}_1(d_m) F_s - F_\Psi^\top \Psi_z F_\Psi, \end{aligned}$$

where $W = \mathrm{diag}(0, W_1, W_2 - W_1, 0, -W_2, 0, 0, 0)$,

$$\mathscr{Z}_1(d_m) = \mathrm{diag}\left(Z_1, 3\alpha(d_m) Z_1\right), \mathscr{Z}_2 = \mathrm{diag}\left(Z_2, 3Z_2\right),$$

$$F_\Psi = \left[\begin{array}{c|cc} 0_{2n\times n} & 0_{2n\times n} & M \\ \hline 0_{2n\times n} & M & 0_{2n\times n} \end{array}\right], M = \begin{bmatrix} 0 & I & -I & 0 & 0 & 0 \\ 0 & I & I & 0 & 0 & -2I \end{bmatrix},$$

$$F_s = \begin{bmatrix} M & 0_{2n\times 2n} \end{bmatrix}, F_3 = \begin{bmatrix} I & -I & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 & 0 & (d_m+1)I & 0 & 0 \\ 0 & 0 & -I & -I & 0 & 0 & (1-d_m)I & (d_M+1)I \end{bmatrix},$$

$$F_2 = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 & (d_m+1)I & 0 & 0 \\ 0 & 0 & 0 & -I & -I & 0 & (1-d_m)I & (d_M+1)I \end{bmatrix},$$

$$F(d) = \begin{bmatrix} 0_{2n\times 6n} & 0_{2n\times n} & 0_{2n\times n} \\ 0_{n\times 6n} & dI_n & -dI_n \end{bmatrix}, \Psi_z = \begin{bmatrix} \mathscr{Z}_2 & X \\ \star & \mathscr{Z}_2 \end{bmatrix}.$$

The function $\alpha(d)$ is defined by $\alpha(d) = 1$, if $d = 1$ and $\alpha(d) = (d+1)/(d-1)$ if $d > 1$.

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