Variations on the theme of slacks-based measure of efficiency in DEA♣

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Abstract: In DEA, there are typically two schemes for measuring efficiency of DMUs; radial and non-radial. Radial models assume proportional change of inputs/outputs and usually remaining slacks are not directly accounted for inefficiency. On the other hand, non-radial models deal with slacks of each input/output individually and independently, and integrate them into an efficiency measure, called slacks-based measure (SBM). In this paper, we point out shortcomings of the SBM and propose 4 variants of the SBM model. The original SBM model evaluates efficiency of DMUs referring to the furthest frontier point within a range. This results in the hardest score for the objective DMU and the projection may go to a remote point on the efficient frontier which may be inappropriate as the reference. In an effort to overcome this shortcoming, we first investigate frontier (facet) structure of the production possibility set. Then we propose Variation I that evaluates each DMU by the nearest point on the same frontier as the SBM found. However, there exist other potential facets for evaluating DMUs. Therefore we propose Variation II that evaluates each DMU from all facets. We then employ clustering methods to classify DMUs into several groups, and apply Variation II within each cluster. This Variation III gives more reasonable efficiency scores with less effort. Lastly we propose a random search method (Variation IV) for reducing the burden of enumeration of facets. The results are approximate but practical in usage.

Keywords: DEA, SBM, facets, enumeration, clustering, random search

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1. Introduction

In most DEA models, the production possibility set is a polyhedral convex set whose vertices correspond to the efficient DMUs in the model. A polyhedral convex set can be defined by its vertices or by its supporting hyperplanes (Simonnard [4]). In DEA literature, main focus is directed to vertices while comparatively few researches are concerned with the supporting hyperplanes.

One of the purposes of this paper is to fill the gap between the two approaches: vertex and hyperplane. We firstly discuss the characteristics of the supporting hyperplanes to the production possibility set in DEA. Then, based on this hyperplanes, we propose several variants of the slacks-based measure of efficiency.

Roughly speaking, we have two types of measure in DEA; radial and non-radial. Radial measures are represented by CCR [2] and BCC [1] models. Their drawbacks exist in that inputs/outputs are assumed to undergo proportional changes and remaining slacks are not accounted for in the efficiency scores.

Non-radial models are represented by the slacks-based measure (SBM) [5]. The SBM evaluates efficiency based on the slacks-based measure to the efficient frontier. However, since its objective is to minimize this measure, the referent point is apt to be far from the objective DMU.

However, there exists other approach; to find the nearest point on the frontier. For this purpose we first modify the SBM to catch the minimum slacks-based measure point on the facet that the SBM found for the DMU. We call this Variation I. Then, after investigation of supporting hyperplanes (facets) to the production possibility set, we extend this approach to consider all facets, resulting in Variation II. Since the enumeration of facets needs massive computation, we propose two more convenient variations; one clustering (Variation III) and the other random search (Variation IV).

This paper unfolds as follows. We introduce the SBM and several properties of facets (hyperplanes) in Section 2. Then we modify the SBM in such a way that instead of minimization of the objective function we maximize it on the facet explored by the SBM (Variation I) in Section 3. We propose a method for finding all facets of the production possibility set in Section 4. Using this result we extend Variation I to employ all facets (Variation II) in Section 5. Then we simplify Variation II to two schemes; one clustering (Variation III) and the other random search (Variation IV) in Section 6. We modify our results to cope with the variable returns-to-scale (VRS) environment in Section 7. We compare our variation with the radial (CCR) model in Section 8. Some concluding remarks follow in the last section.

2. Preliminaries

In this section we introduce the SBM and discuss several properties of the facets of production possibility set.

2.1 Notation and Production Possibility Set

We deal with n DMUs (j=1,…,n) each having m inputs $\left\{x_{ij}\right\}(i=1,\dots,m)$ and s outputs $\left\{y_{ij}\right\}(i=1,\dots,s)$. We denote the DMU j by $(\mathbf{x}_j,\mathbf{y}_j)\ (j=1,\dots,n)$ and the input/output data matrices by $\mathbf{X}=\left(x_{ij}\right)\in R^{m\times n}$ and $\mathbf{Y}=\left(y_{ij}\right)\in R^{s\times n}$, respectively. We assume $\mathbf{X}>\mathbf{0}$ and $\mathbf{Y}>\mathbf{0}$. Under the constant returns-to-scale (CRS) assumption the production possibility set is defined by

$$P=\left\{(\mathbf{x},\mathbf{y})\middle|\mathbf{x}\geq\mathbf{X\lambda},\mathbf{0}\leq\mathbf{y}\leq\mathbf{Y\lambda},\boldsymbol{\lambda}\geq\mathbf{0}\right\} \tag{1}$$

where $\boldsymbol{\lambda}\in R^n$ is the intensity vector. We introduce non-negative input and output slacks $\mathbf{s}^-\in R^m$ and $\mathbf{s}^+\in R^s$ to express

$$\mathbf{x}=\mathbf{X\lambda}+\mathbf{s}^- \text{ and } \mathbf{y}=\mathbf{Y\lambda}-\mathbf{s}^+. \tag{2}$$

2.2 Efficiency and SBM

[Definition 1] (Efficient DMU)

A DMU $(\mathbf{x}_o,\mathbf{y}_o)\in P$ is called CRS-efficient if any solution of the system

$$\mathbf{x}_o=\mathbf{X\lambda}+\mathbf{s}^-,\mathbf{y}_o=\mathbf{Y\lambda}-\mathbf{s}^+,\boldsymbol{\lambda}\geq\mathbf{0},\mathbf{s}^-\geq\mathbf{0},\mathbf{s}^+\geq\mathbf{0}$$

has $\mathbf{s}^-=\mathbf{0}$ and $\mathbf{s}^+=\mathbf{0}$. Otherwise $(\mathbf{x}_o,\mathbf{y}_o)$ is called CRS-inefficient, i.e. there exist non-negative but non-zero (semi-positive) slacks for the above system. .

This definition corresponds to the Pareto-Koopmans definition of efficiency: *A DMU is fully efficient if and only if it is not possible to improve any input or output without worsening some other input or output.* (Cooper et al. [3], p. 45.)

The SBM ([5]) solves the following program for DMU $(\mathbf{x}_o, \mathbf{y}_o)$ $(o = 1, \ldots, n)$.

[Theme -- Original SBM]

$$\rho_o^{\min} = \min \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}} \tag{3}$$

subject to

$$\begin{aligned} &\sum_{j=1}^{n} \mathbf{x}_j \lambda_j + \mathbf{s}^- = \mathbf{x}_o \\ &\sum_{j=1}^{n} \mathbf{y}_j \lambda_j - \mathbf{s}^+ = \mathbf{y}_o \\ &\lambda_j \geq 0 (\forall j) \\ &\mathbf{s}^- \geq \mathbf{0} \\ &\mathbf{s}^+ \geq \mathbf{0}. \end{aligned} \tag{4}$$

This fractional program can be solved by transforming into an equivalent linear program (see [5]). Let an optimal solution of the SBM be $(\boldsymbol{\lambda}^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$.

[Definition 2](Reference set)

The reference set for DMU $(\mathbf{x}_o, \mathbf{y}_o)$ is defined by

$$R = \left\{ j \,\middle|\, \lambda_j^* > 0,\ j = 1, \ldots, n \right\}. \tag{5}$$

[Definition 3](Projection)

The projection of DMU $(\mathbf{x}_o, \mathbf{y}_o)$ is defined by

$$\begin{aligned} \overline{\mathbf{x}}_o &= \mathbf{x}_o - \mathbf{s}^{-*} = \sum_{j \in R} \mathbf{x}_j \lambda_j^* \\ \overline{\mathbf{y}}_o &= \mathbf{y}_o + \mathbf{s}^{+*} = \sum_{j \in R} \mathbf{y}_j \lambda_j^* \end{aligned} \tag{6}$$

[Theorem 1]

The projected DMU $(\overline{\mathbf{x}}_o, \overline{\mathbf{y}}_o)$ is CRS-efficient.

(See Appendix A for a proof.)

As the objective function (3) suggests, the original SBM aims to find the minimum (the worst) score associated with the relatively maximum slacks under the constraint (4). This might project the DMU onto a very remote point on the frontier (facet) and sometimes it is hard to interpret. On the other hand, there is the opposite approach, i.e., to look for the nearest point on the facet, by minimizing the slacks-based measure from the frontiers. For this purpose, we need to investigate the facets of the production possibility set, as we show in the following section.

2.3 Facets of Production Possibility Set

Let $(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)(j = 1, \ldots, k)$ be k DMUs in P. We make a linear combination of these k DMUs with positive coefficients as

$$\begin{aligned}\xi_o &= w_1\xi_1 + \cdots + w_k\xi_k \\ \boldsymbol{\eta}_o &= w_1\boldsymbol{\eta}_1 + \cdots + w_k\boldsymbol{\eta}_k,\end{aligned} \tag{7}$$

where $w_j > 0\ (j = 1,\ldots,k)$.

[Lemma 1]

If any member of $(\xi_j, \boldsymbol{\eta}_j)\ (j = 1,\ldots,k)$ is CRS-inefficient, then $(\xi_o, \boldsymbol{\eta}_o)$ is CRS-inefficient.

Proof: Without losing generality, we assume that $(\xi_1, \boldsymbol{\eta}_1)$ is CRS-inefficient. Then, the system

$$\xi_1 = \mathbf{X}\boldsymbol{\lambda} + \mathbf{s}^-,\ \boldsymbol{\eta}_1 = \mathbf{Y}\boldsymbol{\lambda} - \mathbf{s}^+,\ \boldsymbol{\lambda} \geq \mathbf{0},\ \mathbf{s}^- \geq \mathbf{0},\ \mathbf{s}^+ \geq \mathbf{0} \tag{8}$$

has a solution $(\boldsymbol{\lambda}^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$ with $(\mathbf{s}^{-*}, \mathbf{s}^{+*}) \geq (\mathbf{0},\mathbf{0})$ and $(\mathbf{s}^{-*}, \mathbf{s}^{+*}) \neq (\mathbf{0},\mathbf{0})$. We set

$$\bar{\xi}_1 = \mathbf{X}\boldsymbol{\lambda}^* \text{ and } \bar{\boldsymbol{\eta}}_1 = \mathbf{Y}\boldsymbol{\lambda}^* \tag{9}$$

By inserting (9) into (7), we have

$$\begin{aligned}\xi_o &= w_1\bar{\xi}_1 + \sum\nolimits_{j=2}^{k} w_j\xi_j + w_1\mathbf{s}^{-*} \\ \boldsymbol{\eta}_o &= w_1\bar{\boldsymbol{\eta}}_1 + \sum\nolimits_{j=2}^{k} w_j\boldsymbol{\eta}_j - w_1\mathbf{s}^{+*}.\end{aligned} \tag{10}$$

Let us define

$$\begin{aligned}\bar{\xi}_o &= w_1\bar{\xi}_1 + \sum\nolimits_{j=2}^{k} w_j\xi_j \\ \bar{\boldsymbol{\eta}}_o &= w_1\bar{\boldsymbol{\eta}}_1 + \sum\nolimits_{j=2}^{k} w_j\boldsymbol{\eta}_j.\end{aligned} \tag{11}$$

Since $(\bar{\xi}_1, \bar{\boldsymbol{\eta}}_1) \in P$, and $w_j > 0\ (j = 1,\ldots,k)$, we have

$$(\bar{\xi}_o, \bar{\boldsymbol{\eta}}_o) \in P. \tag{12}$$

Hence, we have

$$\begin{aligned}\xi_o &= \bar{\xi}_o + w_1\mathbf{s}^{-*} \\ \boldsymbol{\eta}_o &= \bar{\boldsymbol{\eta}}_o - w_1\mathbf{s}^{+*}.\end{aligned}$$

Thus, $(\xi_o, \boldsymbol{\eta}_o)$ has non-negative and non-zero slacks $(\mathbf{s}^{-*}, \mathbf{s}^{+*})$ against $(\bar{\xi}_o, \bar{\boldsymbol{\eta}}_o)$. Hence it is CRS-inefficient.

Q.E.D.

As a contraposition of Lemma 1, we have

[Theorem 2]

If $(\xi_o, \boldsymbol{\eta}_o)$ defined by (7) is CRS-efficient, then $(\xi_j, \boldsymbol{\eta}_j)\ (j = 1,\ldots,k)$ must be CRS-efficient.

We notice that the reverse of this theorem is not always true. Now, we assume $(\xi_o, \boldsymbol{\eta}_o)$ in (7) is CRS-efficient and we demonstrate the following theorem.

[Theorem 3]

If $(\xi_o, \boldsymbol{\eta}_o)$ defined by (7) is CRS-efficient, then there exists a supporting hyperplane to P at $(\xi_o, \boldsymbol{\eta}_o)$ which also supports P at $(\xi_j, \boldsymbol{\eta}_j)\ (j = 1,\ldots,k)$.

Proof: By the strong theorem of complementarity, there exist dual variables $\mathbf{v}^* \in R^m, \mathbf{u}^* \in R^s$ with $\mathbf{v}^* > \mathbf{0}, \mathbf{u}^* > \mathbf{0}$ such that[1]

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[1] We can obtain such a strong complementary solution by using the additive model or the non-oriented slacks-based measure (SBM) model [3, 5].

$$\begin{aligned}
&\mathbf{v}^*\boldsymbol{\xi}_o - \mathbf{u}^*\boldsymbol{\eta}_o = 0 \\
&\mathbf{v}^*\boldsymbol{\xi}_j - \mathbf{u}^*\boldsymbol{\eta}_j \geq 0 \ (j = 1, \ldots, k) \\
&\mathbf{v}^*\mathbf{X} - \mathbf{u}^*\mathbf{Y} \geq \mathbf{0}.
\end{aligned} \tag{13}$$

Inserting the definition of $(\boldsymbol{\xi}_o, \boldsymbol{\eta}_o)$ in (7) into the first equality in (13), we have

$$w_1(\mathbf{v}^*\boldsymbol{\xi}_1 - \mathbf{u}^*\boldsymbol{\eta}_1) + \cdots + w_k(\mathbf{v}^*\boldsymbol{\xi}_k - \mathbf{u}^*\boldsymbol{\eta}_k) = 0\,. \tag{14}$$

Taking note of $w_j > 0 \ (j = 1, \ldots, k)$ and the second inequality in (13), the equality (14) holds if and only if

$$\mathbf{v}^*\boldsymbol{\xi}_j - \mathbf{u}^*\boldsymbol{\eta}_j = \mathbf{0} \ (j = 1, \ldots, k). \tag{15}$$

Hence, the hyperplane $\mathbf{v}^*\mathbf{x} - \mathbf{u}^*\mathbf{y} = 0$ passes through $(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j) \ (j = 1, \ldots, k)$ and supports *P*. Q.E.D.

This theorem is helpful in identifying the facets of *P*. Since the system of equations (15) is homogenous, if $(\mathbf{v}^*, \mathbf{u}^*)$ is a solution to (15), then $t(\mathbf{v}^*, \mathbf{u}^*) \ (t > 0)$ is also a solution.

If the rank of the matrix

$$\begin{pmatrix} \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k \\ \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_k \end{pmatrix} \in R^{(m+s)\times k} \tag{16}$$

is not less than $m + s - 1$, then the coefficient $(\mathbf{v}^*, \mathbf{u}^*)$ is uniquely determined except for the scalar multiplier *t*, since the hyperplane $\mathbf{v}^*\mathbf{x} - \mathbf{u}^*\mathbf{y} = 0$ passes through the origin $(\mathbf{x} = \mathbf{0}, \mathbf{y} = \mathbf{0})$ and remaining $m + s - 1$ linearly independent $(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$ determine the hyperplane. This means that the direction $(\mathbf{v}^*, -\mathbf{u}^*)$ is the unique normal to the supporting hyperplane.

If the rank of (16) is less than $m + s - 1$, then there exist multiple $(\mathbf{v}^*, \mathbf{u}^*)$ for the system (13).

[Definition 3] (Facet)

We call the supporting hyperplane $\mathbf{v}^*\mathbf{x} - \mathbf{u}^*\mathbf{y} \leq 0$ defined in Theorem 3 a facet of *P*.[2]

3. Variation I – Minimizing slacks-based measure from the facet

The first variation is a simple modification of the basic SBM in the preceding section. We maximize the objective function rather than minimization.

For each DMU $(\mathbf{x}_o, \mathbf{y}_o) \ (o = 1, \ldots, n)$, we solve the SBM model in (3-4). If it is inefficient, we have its reference set *R* defined by (5). The projected DMU is efficient by Theorem 1 and hence the DMUs in the reference set are efficient by Theorem 2. Furthermore, by Theorem 3, they form a facet of *P*. We evaluate the minimum slacks-based measure and hence the maximum score on the facet as follows.

$$\rho_o^{\max} = \max \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}} \tag{17}$$

subject to

[2] Simonnard (1966) called such hyperplane an *extremal supporting ray*.

$$\begin{aligned}
&\sum_{j\in R}\mathbf{x}_j\lambda_j+\mathbf{s}^-=\mathbf{x}_o\\
&\sum_{j\in R}\mathbf{y}_j\lambda_j-\mathbf{s}^+=\mathbf{y}_o\\
&\lambda_j\ge 0\ (\forall j)\\
&\mathbf{s}^-\ge\mathbf{0}\\
&\mathbf{s}^+\ge\mathbf{0}.
\end{aligned}\tag{18}$$

Since we deal with the same facet as the basic model, we have the relationship:

$$\rho_o^{\max}\ge\rho_o^{\min}.\tag{19}$$

This variation demands one additional LP solution for each inefficient DMU and is computationally rather easy. However, since the facet defined by *R* is an instance of facets and there may be other facets of *P* to be considered in evaluating the maximum efficiency of DMU$(\mathbf{x}_o,\mathbf{y}_o)$, we need to know all facets of *P*. We discuss this subject in the next section. Now we show an example of the SBM and Variation I.

[Example 1]
Table 1 exhibits data for 12 hospitals having two inputs and two outputs.
Inputs: Numbers of doctors and nurses
Outputs: Numbers of outpatients and inpatients

<< Insert Table 1 here **Table 1: Data of 12 hospitals>>**

First, we solved this case by the SBM in (3-4). Then, knowing the reference set and hence a facet of inefficient DMUs, we solved the Variation I in (17-18). The results are displayed in Table 2. Every inefficient DMUs improved their efficiency except H. For example, Hospital C is inefficient $\rho_C^{\min}=0.8265$ by the SBM and its references are B and L. We solved the maximum problem (Variation I) on the facet spanned by B and L, and obtained $\rho_C^{\max}=0.8550$ with the reference B. The difference is the gap between the max and the min objective values measured by (17).

<< Insert Table 2 here. **Table 2: Results of SBM and Variation I>>**

4. Enumeration of facets

In this section, we propose a method for enumerating all facets of *P*.
Let $P_j=(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\ (j=1,\ldots,K)$ be the CRS-efficient DMUs in *P*.

[Definition 3] (friends)
A subset $\{P_{j_1},\ldots,P_{j_k}\}$ of $\{P_j\}=\{(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)\}\ (j=1,\ldots,K)$ is called *friends* if a linear combination with positive coefficients of $\{P_{j_1},\ldots,P_{j_k}\}$ is CRS-efficient.

[Definition 4] (maximal friends)
A *friends* is called *maximal* if any addition of P_j (not in the *friends*) to the *friends* is no more *friends*.

[Definition 5] (dominated friends)
A *friends* is dominated by other *friends* if the set of DMUs is a subset of other's.

We propose an algorithm for finding the *maximal friends* of $P_j = (\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)\ (j = 1, \ldots, K)$.

[Algorithm A]

Begin
  For k = 1 to K
    Find_Maximal_Friends of P_k
  Next k
  Delete *dominated friends* from the set of *friends*
  Obtain the set of facets from the final set of *friends*
End

Subroutine *Find_Maximal_Friends of* P_k
  Exclude $P_1, \ldots, P_{k-1}$ from the candidates of *friends*
  Enumerate all *friends* of P_k
  Remove *dominated friends* from the set of *friends*
Exit sub

Let the number of facets thus generated be H. We have H facets to P:

$$\text{Facet}(h): \mathbf{v}^{(h)*}\mathbf{x} - \mathbf{u}^{(h)*}\mathbf{y} \le 0.\ (h = 1, \ldots, H) \tag{20}$$

Facet(h) passes through its *friends* and supports *P.*

The above facets consist of genuine efficient frontiers of the production possibility set *P*. However, *P* has non-efficient boundaries as we see in Figure 1 as example. In the figure line segments AB and BC are efficient facets, while AD and CE are non-efficient boundaries of *P*. WE notice that, in this paper, we observe and deal only with the efficient portion of the boundary.

<< Insert Figure 1. **Figure 1: Efficient and non-efficient frontiers>>**

[Theorem 4]

For every efficient frontier point of *P*, there exists a Facet(h) that touches the efficient point.

Proof: Every efficient frontier point can be expressed by a positive linear combination of a set of efficient vertices of *P*. By construction of the *maximal_friends* in the Algorithm A, the set as well as the efficient point is on some Facet(h).

Q.E.D.

5. Variation II – Minimizing the SBM from all facets

We deal with a set of DMUs defined in Section 2.

Step 1. Finding Efficient DMUs

Solve the non-oriented SBM model or the additive model and find the set of efficient DMUs.
Let the set be

$$\left\{ \left(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j \right) \middle| j = 1, \ldots, K) \right\}$$

where K is the number of efficient DMUs.

Step 2. Enumeration of Facets

Enumerate all facets applying Algorithm A in Section 4. Let the number of facets thus obtained be *H*. We deal with only facets in the *maximal friends*.

Step 3. Evaluation of Inefficient DMUs

For an inefficient DMU $(\mathbf{x}_o, \mathbf{y}_o)$ we evaluate its efficiency score as follows.

For each $\mathrm{Facet}(h)\ \ (h = 1,\ldots,H)$, we solve the following fractional program:

$$\rho_o^{(h)} = \max \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}}}{1 + \sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}} \tag{21}$$

subject to

$$\begin{aligned} &\sum_{j\in R(h)} \boldsymbol{\xi}_j \lambda_j + \mathbf{s}^- = \mathbf{x}_o \\ &\sum_{j\in R(h)} \boldsymbol{\eta}_j \lambda_j - \mathbf{s}^+ = \mathbf{y}_o \\ &\lambda_j \ge 0 (\forall j), \end{aligned} \tag{22}$$

where $R(h)$ is the set of efficient DMUs that span Facet(h).

We obtain the efficiency score of DMU $(\mathbf{x}_o, \mathbf{y}_o)$ as

$$\rho_o^{all} = \max_h \left\{ \rho_o^{(h)} \right\}. \tag{23}$$

We have the following inequalities among the three scores:

$$\rho_o^{all} \ge \rho_o^{\max} \ge \rho_o^{\min} \tag{24}$$

[Example 2]

In the above example, the set of *friends* composed of two DMUs are found to be AD, BD, AL, BL and DL. The set of *friends* composed of three DMUs are ADL and BDL. The set ABDL cannot be a *friends* (facet). Hence the *maximal friends* are ADL and BDL. Using ADL and BDL as reference respectively, we solved the program (21-22), and obtained the efficiency score for inefficient DMUs as exhibited in Table 3. For example, for DMU E, we have $\rho_E^{ADL} = 0.7682031$ (with reference A) and $\rho_E^{BDL} = 0.7523161$ (with reference D, L). Thus $\rho_E^{all} = 0.7682031$.

<<Insert Table 3 here. **Table 3: Results of SBM and Variation II**>>

Comparisons with Table 2 reveal several interesting features of Variation II. As demonstrated in (24), the efficiency score of Variation II is not less than those of the SBM and Variation I for each DMU.

6. How to reduce a massive enumeration

In Variation II, the enumeration of facets needs an enormous computation time and space for large scale problems, even though advances in recent IT technologies are amazing in both aspects. If we have m=6 (# of inputs), s=5 (# of outputs) and k=20 (# of efficient DMUs), then in the worst case we might enumerate about ${}_{20}C_{10}$=184,756 cases. Of course, most of them would be found to be an inefficient combination.

In this section we propose two modified versions of Variation II which are less time and space consuming.

6.1 Variation III – Clustering

Step 1. Clustering DMUs

Using some clustering method, we classify all DMUs in clusters, say, Cluster 1 to Cluster *L*.

Step 2. Finding efficient DMUs

This step is the same as the Step 1 of the Variation II.

Step 3. Evaluating efficiency score for an inefficient DMU

If the inefficient DMU $(\mathbf{x}_o, \mathbf{y}_o)$ belongs to Cluster *h*, pick up the efficient DMUs in Cluster *h*. If none of DMUs in Cluster *h* is efficient, we pick up the efficient DMUs in the adjacent clusters.

Let the subset of efficient DMUs corresponding to Cluster h be

$$E(h)=\{(\xi_1,\eta_1),\ldots,(\xi_J,\eta_J)\}.$$

We create the facets composed of the efficient DMUs in $E(h)$ using the same procedure as described in Step 2 of the preceding section. We evaluate the efficiency of DMU $(\mathbf{x}_o,\mathbf{y}_o)$ in reference to the facets thus obtained in the same way as the Step 3 of the preceding section. If the program (21-22) has no feasible solution, DMU $(\mathbf{x}_o,\mathbf{y}_o)$ is judged to be efficient in this cluster, i.e., it is globally inefficient but locally efficient.

The merits of this modification are as follows:

(1) By introducing a considerable number of clusters, we can reduce the number of the candidate combinations.

(2) For inefficient DMUs, the efficiency score is obtained in reference to the efficient DMUs in the same cluster. Thus, the results are more acceptable and understandable.

[Example 3]

We classified 12 hospitals in Table 1 into two clusters depending on their size (numbers of doctor and inpatient) as described in the column "Cluster" of Table 4. We solved non-oriented SBM model and found 4 efficient DMUs (A,, B, D, L) and 8 inefficient DMUs with their references as exhibited in the left side of Table 4 where we found several inappropriate references. For example, C has references B and L, whereas L is not in the same cluster as C. In the cluster 1, the maximal friends are AD and BD, and in the cluster 2 we have only one facet L. Finally, we solved the efficiency of inefficient DMUs referring to the facets in the same cluster and found the results recorded in the right half of Table 4. DMU C has its reference D and efficiency score 0.875069 which was upgraded from the SBM score 0.826. DMUs in the cluster 2 were all evaluated their efficiency against L. We found infeasibility for G and J. Hence, we judged them efficient in this cluster. They are globally inefficient but locally efficient.

<< Insert Table 4 here. **Table 4: SBM and Clustering results (Variation III)>>**

6.2 Variation IV – Random Search

In this section we propose an approximate method for finding facets.

Step 1. Finding center of gravity of efficient DMUs.

Let the set of efficient DMUs be $P_j=(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)$ $(j=1,\ldots,K)$. We calculate their center of gravity G as

$$\begin{aligned}\mathbf{x}_G&=(\boldsymbol{\xi}_1+\cdots+\boldsymbol{\xi}_K)/K\\ \mathbf{y}_G&=(\boldsymbol{\eta}_1+\cdots+\boldsymbol{\eta}_K)/K\end{aligned}\tag{25}$$

Figure 2 illustrate an example. We note that we can utilize any positive linear combination of efficient DMUs instead of the center of gravity for our purpose.

Step 2. Creating random directions around efficient DMUs

For each efficient DMU $P_j=(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)$ we compute the direction from G to $P_j=(\boldsymbol{\xi}_j,\boldsymbol{\eta}_j)$ as $(\boldsymbol{\xi}_j-\mathbf{x}_G,\boldsymbol{\eta}_j-\mathbf{y}_G)$ and then perturb the direction slightly using random numbers. Let the direction thus perturbed be $(\mathbf{d}_x,\mathbf{d}_y)$.

Step 3. Finding a facet

We solve the following linear program in $t\in R$ and $\boldsymbol{\lambda}\in R^K$:

$$
\begin{aligned}
&\max t \\
&\text{subjec to} \\
&\mathbf{x}_G + \mathbf{d}_x t \geq \boldsymbol{\xi}_1 \lambda_1 + \cdots + \boldsymbol{\xi}_K \lambda_K \\
&\mathbf{y}_G + \mathbf{d}_y t \leq \boldsymbol{\eta}_1 \lambda_1 + \cdots + \boldsymbol{\eta}_K \lambda_K \\
&t \geq 0, \boldsymbol{\lambda} \geq \mathbf{0}.
\end{aligned} \tag{26}
$$

Let an optimal solution be $(t^*, \boldsymbol{\lambda}^*)$.

If $t^* = 0$, then the center G is efficient and all $P_j = (\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)$ $(j = 1, \ldots, K)$ are *friends*. This case has only one efficient facet by Theorem 3. If $t^* > 0$, then the reference DMUs corresponding to positive λ_j^* s form a facet of *P*, since the optimal solution is obtained on a boundary of *P*.

Step 4. Repeating the random search

We repeat the random search around the *K* efficient DMUs until a sufficient number of facets is found.

Step 5. Evaluating inefficient DMUs

We evaluate the efficiency score of inefficient DMUs using the facets thus found in the same manner as the Variation II.

<<Insert Figure 2 here. **Figure 2: Random search around efficient DMUs**>>

[Example 4]

In the hospital example, DMUs A, B, D and L are efficient. Table 5 denotes their center of gravity and direction vectors from the center to A, B, D and L. We disturb these vectors randomly and, for example for D, we have, dx1=0.7, dx2=-13, dy1=8, dy2=-13. Using this direction we solved the program (26) and obtained $\lambda_A^* = 0.03822, \lambda_D^* = 0.41055, \lambda_L^* = 0.37683$. Thus, ADL spans a facet of *P*. In this way we can find facets of *P* approximately. Table 6 exhibits results of random searches. We tried two random searches (perturbed directions) for each efficient DMU as displayed in the table. Eventually we found the two maximal friends (facets); ADL and BDL.

The reason why we perturb the direction around vertices is that several facets are connected at a vertex and we can find facets with high probability.

<<Insert Table 5 here. **Table 5: Center and directions**>>

＜＜Insert Table 6 here. **Table 6: Results of random search**>>

7. Variable returns-to-scale (VRS) case

So far we have discussed the constant returns to scale case. We need some alternations in the variable returns-to-scale (VRS) case, which requires the convexity condition on the intensity vector $\boldsymbol{\lambda} \in R^n$:

$$\lambda_1 + \cdots + \lambda_n = 1. \tag{27}$$

In this section, we present only important addenda to the preceding sections.

1. The production possibility set (1) and the SBM model (4) have the additional constraint (27).

2. Equation (7) is modified to:

$$\begin{aligned} \boldsymbol{\xi}_o &= w_1\boldsymbol{\xi}_1 + \cdots + w_k\boldsymbol{\xi}_k \\ \boldsymbol{\eta}_o &= w_1\boldsymbol{\eta}_1 + \cdots + w_k\boldsymbol{\eta}_k \\ w_1 + \cdots + w_k &= 1,\ w_j > 0\ (\forall j). \end{aligned} \tag{7A}$$

3. Lemma 1 turns out to:

[Lemma 1A]

If any member of $(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)\ (j = 1, \ldots, k)$ is VRS-inefficient, then $(\boldsymbol{\xi}_o, \boldsymbol{\eta}_o)$ is VRS-inefficient.

Proof: Without losing generality, we assume that $(\boldsymbol{\xi}_1, \boldsymbol{\eta}_1)$ is VRS-inefficient. Then, the system

$$\begin{aligned} \boldsymbol{\xi}_1 &= \mathbf{X}\boldsymbol{\lambda} + \mathbf{s}^-, \\ \boldsymbol{\eta}_1 &= \mathbf{Y}\boldsymbol{\lambda} - \mathbf{s}^+ \\ \mathbf{e}\boldsymbol{\lambda} &= 1,\ \boldsymbol{\lambda} \geq \mathbf{0},\ \mathbf{s}^- \geq \mathbf{0},\ \mathbf{s}^+ \geq \mathbf{0} \end{aligned}$$

has a solution $(\boldsymbol{\lambda}^*, \mathbf{s}^{-*}, \mathbf{s}^{+*})$ with $(\mathbf{s}^{-*}, \mathbf{s}^{+*}) \geq (\mathbf{0}, \mathbf{0})$ and $(\mathbf{s}^{-*}, \mathbf{s}^{+*}) \neq (\mathbf{0}, \mathbf{0})$, where e is the row vector with all elements equal to 1. We set

$$\bar{\boldsymbol{\xi}}_1 = \mathbf{X}\boldsymbol{\lambda}^* \text{ and } \bar{\boldsymbol{\eta}}_1 = \mathbf{Y}\boldsymbol{\lambda}^* \tag{9A}$$

By inserting (9) into (7), we have

$$\begin{aligned} \boldsymbol{\xi}_o &= w_1\bar{\boldsymbol{\xi}}_1 + \sum\nolimits_{j=2}^{k} w_j\boldsymbol{\xi}_j + w_1\mathbf{s}^{-*} \\ \boldsymbol{\eta}_o &= w_1\bar{\boldsymbol{\eta}}_1 + \sum\nolimits_{j=2}^{k} w_j\boldsymbol{\eta}_j - w_1\mathbf{s}^{+*}. \end{aligned} \tag{10A}$$

Let us define

$$\begin{aligned} \bar{\boldsymbol{\xi}}_o &= w_1\bar{\boldsymbol{\xi}}_1 + \sum\nolimits_{j=2}^{k} w_j\boldsymbol{\xi}_j \\ \bar{\boldsymbol{\eta}}_o &= w_1\bar{\boldsymbol{\eta}}_1 + \sum\nolimits_{j=2}^{k} w_j\boldsymbol{\eta}_j. \end{aligned} \tag{11A}$$

Since $(\bar{\boldsymbol{\xi}}_1, \bar{\boldsymbol{\eta}}_1) \in P$ and $\sum_{j=1}^{k} w_j = 1,\ w_j > 0\ (j = 1, \ldots, k)$, we have

$$(\bar{\boldsymbol{\xi}}_o, \bar{\boldsymbol{\eta}}_o) \in P. \tag{12A}$$

Hence, we have

$$\begin{aligned} \boldsymbol{\xi}_o &= \bar{\boldsymbol{\xi}}_o + w_1\mathbf{s}^{-*} \\ \boldsymbol{\eta}_o &= \bar{\boldsymbol{\eta}}_o - w_1\mathbf{s}^{+*}. \end{aligned}$$

Thus, $(\boldsymbol{\xi}_o, \boldsymbol{\eta}_o)$ has non-negative and non-zero slacks $(\mathbf{s}^{-*}, \mathbf{s}^{+*})$ against $(\bar{\boldsymbol{\xi}}_o, \bar{\boldsymbol{\eta}}_o)$. Hence it is VRS-inefficient.

Q.E.D.

4. Theorem 3 changes to:

[Theorem 3A]

If $(\boldsymbol{\xi}_o, \boldsymbol{\eta}_o)$ defined by (7A) is VRS-efficient, then there exists a supporting hyperplane to P at $(\boldsymbol{\xi}_o, \boldsymbol{\eta}_o)$ which also supports P at $(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)\ (j = 1, \ldots, k)$.

Proof: By the strong theorem of complementarity, there exist dual variables $\mathbf{v}^* \in R^m, \mathbf{u}^* \in R^s, u_0^* \in R$ with $\mathbf{v}^* > \mathbf{0}, \mathbf{u}^* > \mathbf{0}$ such that

$$\begin{aligned} &\mathbf{v}^*\boldsymbol{\xi}_o - \mathbf{u}^*\boldsymbol{\eta}_o - u_0^* = 0 \\ &\mathbf{v}^*\boldsymbol{\xi}_j - \mathbf{u}^*\boldsymbol{\eta}_j - u_0^* \geq 0\ (j = 1, \ldots, k) \\ &\mathbf{v}^*\mathbf{X} - \mathbf{u}^*\mathbf{Y} - u_0^*\mathbf{e} \geq \mathbf{0}. \end{aligned} \tag{13A}$$

Inserting the definition of $(\boldsymbol{\xi}_o, \boldsymbol{\eta}_o)$ in (7A) into the first equality in (13A) and noting $\sum_{j=1}^{k} w_j = 1$, we have

$$w_1(\mathbf{v}^*\boldsymbol{\xi}_1 - \mathbf{u}^*\boldsymbol{\eta}_1 - u_0^*) + \cdots + w_k(\mathbf{v}^*\boldsymbol{\xi}_k - \mathbf{u}^*\boldsymbol{\eta}_k - u_0^*) = 0 . \tag{14A}$$

Taking note of $w_j > 0\ (j = 1, \ldots, k)$ and the second inequality in (13A), the equality (14A) holds if and only if

$$\mathbf{v}^*\boldsymbol{\xi}_j - \mathbf{u}^*\boldsymbol{\eta}_j = \mathbf{0}\ (j = 1, \ldots, k). \tag{15A}$$

Hence the hyperplane $\mathbf{v}^*\mathbf{x} - \mathbf{u}^*\mathbf{y} - u_0^* = 0$ passes through $(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)\ (j = 1, \ldots, k)$ and supports *P*. Q.E.D.

Since the system of equations (15A) is homogenous, if $(\mathbf{v}^*, \mathbf{u}^*, u_0^*)$ is a solution to (15A), then $t(\mathbf{v}^*, \mathbf{u}^*, u_0^*)\ (t > 0)$ is also a solution.

If the rank of the matrix

$$\begin{pmatrix} \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k \\ \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_k \end{pmatrix} \in R^{(m+s)\times k} \tag{16A}$$

is not less than $m + s$, then the coefficient $(\mathbf{v}^*, \mathbf{u}^*, u_0^*)$ is uniquely determined except for the scalar multiplier *t*. This means that the direction $(\mathbf{v}^*, -\mathbf{u}^*)$ is the unique normal to the supporting hyperplane.

If the rank of (16A) is less than $m + s$, then there exist multiple $(\mathbf{v}^*, \mathbf{u}^*, u_0^*)$ for the system (13A).

Definition 1 (Facet)

We call the supporting hyperplane $\mathbf{v}^*\mathbf{x} - \mathbf{u}^*\mathbf{y} - u_0^* \leq 0$ defined in Theorem 1A a facet of *P*.

In what follows, we choose the center of gravity of $(\boldsymbol{\xi}_j, \boldsymbol{\eta}_j)(j = 1, \ldots, k)$ as $(\boldsymbol{\xi}_o, \boldsymbol{\eta}_o)$, i.e. $w_j = 1/k\ (\forall j)$.

5. We add the convexity condition $\mathbf{e}\boldsymbol{\lambda} = 1$ to the linear program (26)

8. Comparisons with the radial model

We compared the scores obtained by the SBM, Variation II and the radial CCR models as displayed in Table 7. The CCR score is not less than that of the SBM ([3, p. 111]). However, Variation II and the CCR are mixed. We have no theoretical evidence between the two. The results indicate volatility of score and rank depending on the models, and connote the importance of model selection as is always the case in DEA applications.

<<Insert Table 7 here. **Table 7: Comparisons of SBM, Variation II and CCR>>**

9. Concluding remarks

In this paper, we have proposed 4 variants of the SBM. They have common characteristics as follows:

1. They are units-invariant, i.e. the scores are independent of the units in which the inputs and outputs are measured provided these units are the same for every DMU.
2. We can impose weights exogenously to each input/output depending on their importance, e.g. cost share. Refer to Cooper et al. [4, p.105] and Tsutsui and Goto [7].
3. Although we have developed our model in the so-called non-oriented version, i.e. both input and output inefficiencies are accounted in the efficiency evaluation, we can deal with the input (output) oriented models by taking the numerator (denominator) of the objective function (3, 17, 21) as the target.
4. Future research subjects include (a) experiments on real-world large scale problems and (b) extension to the super-SBM model [6].

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Appendix A

Proof of Theorem 1

Suppose that $(\bar{\mathbf{x}}_o, \bar{\mathbf{y}}_o)$ is CRS-inefficient. Then there exists an optimal solution $(\bar{\rho}_o, \bar{\boldsymbol{\lambda}}^*, \bar{\mathbf{s}}^{-*}, \bar{\mathbf{s}}^{+*})$ with non-zero and non-negative slacks $(\bar{\mathbf{s}}^{-*}, \bar{\mathbf{s}}^{+*})$ for the program:

$$\rho_o = \min \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{\bar{s}_i^-}{\bar{x}_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{\bar{s}_r^+}{\bar{y}_{ro}}} \tag{3B}$$

subject to

$$\begin{aligned} &\sum_{j=1}^{n} \mathbf{x}_j \lambda_j + \bar{\mathbf{s}}^- = \bar{\mathbf{x}}_o \\ &\sum_{j=1}^{n} \mathbf{y}_j \lambda_j - \bar{\mathbf{s}}^+ = \bar{\mathbf{y}}_o \\ &\lambda_j \geq 0 (\forall j) \\ &\bar{\mathbf{s}}^- \geq \mathbf{0} \\ &\bar{\mathbf{s}}^+ \geq \mathbf{0}. \end{aligned} \tag{4B}$$

Inserting (6) to (4B) we have:

$$\begin{aligned} &\sum_{j=1}^{n} \mathbf{x}_j \bar{\lambda}_j^* + \bar{\mathbf{s}}^{-*} + \mathbf{s}^{-*} = \mathbf{x}_o \\ &\sum_{j=1}^{n} \mathbf{y}_j \bar{\lambda}_j^* - \bar{\mathbf{s}}^{+*} - \mathbf{s}^{+*} = \mathbf{y}_o. \end{aligned} \tag{5B}$$

For this manipulation, we have the objective function value for $(\mathbf{x}_o, \mathbf{y}_o)$,

$$\bar{\rho}_o = \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{\bar{s}_i^{-*} + s_i^{-*}}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{\bar{s}_r^{+*} + s_r^{+*}}{y_{ro}}} \tag{6B}$$

Since $(\bar{\mathbf{s}}^{-*}, \bar{\mathbf{s}}^{+*})$ is semi-positive, we have:

$$\bar{\rho}_o < \rho_o^{\min}.$$

This contradicts the optimality of $\rho_o^{\min}$. Q.E.D.

Table 1: Data of 12 hospitals

	Inputs		Outputs	
DMU	Doctor	Nurse	Outpatient	Inpatient
A	20	151	100	90
B	19	131	150	50
C	25	160	160	55
D	27	168	180	72
E	22	158	94	66
F	55	255	230	90
G	33	235	220	88
H	31	206	152	80
I	30	244	190	100
J	50	268	250	100
K	53	306	260	147
L	38	273	250	133

Table 2: Results of SBM and Variation I

DMU	SBM	Ref.	Variation I	Ref.
A	1	A	1	A
B	1	B	1	B
C	0.8264712	B,L	0.8549538	B
D	1	D	1	D
E	0.7276716	B,L	0.7391066	L
F	0.685679	A,L	0.6868147	L
G	0.8765484	B,L	0.9051589	B,L
H	0.7713536	L	0.7713536	L
I	0.9015742	A,L	0.9016285	L
J	0.7653135	B,L	0.7898236	B
K	0.8619133	B,L	0.8622074	L
L	1	L	1	L

Table 3: Results of SBM and Variation II

DMU	SBM	Ref.	Variation II	Ref.
A	1	A	1	A
B	1	B	1	B
C	0.8264712	B,L	0.8750692	D
D	1	D	1	D
E	0.7276716	B,L	0.7682031	A
F	0.685679	A,L	0.7264794	D
G	0.8765484	B,L	0.9368794	D
H	0.7713536	L	0.8091801	D
I	0.9015742	A,L	0.9211676	A,D,L
J	0.7653135	B,L	0.8103234	D
K	0.8619133	B,L	0.8889356	A,D
L	1	L	1	L

Table 4: SBM and Clustering results (Variation III)

DMU	SBM	Ref.	Cluster	Variation III	Ref.	Remark
A	1	A	1	1	A	
B	1	B	1	1	B	
C	0.826	B,L	1	0.875069	D	
D	1	D	1	1	D	
E	0.728	B,L	1	0.768203	A	
F	0.686	A,L	2	0.686815	L	
G	0.877	B,L	2	1	G	locally eff.
H	0.771	L	1	0.80918	D	
I	0.902	A,L	2	0.901629	L	
J	0.765	B,L	2	1	J	locally eff.
K	0.862	B,L	2	0.862207	L	
L	1	L	2	1	L	

Table 5: Center and directions

DMU	(I)Doctor	(I)Nurse	(O)Outpatient	(O)Inpatient
A	20	151	100	90
B	19	131	150	50
D	27	168	180	72
L	38	273	250	133
Center	26	180.75	170	86.25

direction	dx1	dx2	dy1	dy2
A	-6	-29.75	-70	3.75
B	-7	-49.75	-20	-36.25
D	1	-12.75	10	-14.25
L	12	92.25	80	46.75

Table 6: Results of random search

DMU	dx1	dx2	dy1	dy2	Facet found
A	-5.2	-30.3	-75.9	4.8	A
A	-8.5	-25.4	-65.6	2.8	AL
B	-8.2	-45.5	-30.6	-30.9	BL
B	-6.3	-55.5	-10.1	-40.5	BDL
D	0.7	-13.0	8.0	-13.0	ADL
D	1.2	-11.3	12.8	-15.6	BD
L	11.2	100.2	90.4	47.2	BL
L	13.5	80.2	85.2	44.3	ADL

Table 7: Comparisons of SBM, Variation II and CCR

DMU	SBM	Ref.	Rank	Variation II	Ref.	Rank	CCR	Ref.	Rank
A	1	A	1	1	A	1	1	A	1
B	1	B	1	1	B	1	1	B	1
C	0.8264712	B,L	8	0.8750692	D	8	0.8826993	B,D	8
D	1	D	1	1	D	1	1	D	1
E	0.7276716	B,L	11	0.7682031	A	11	0.7631233	A,D,L	12
F	0.685679	A,L	12	0.7264794	D	12	0.8347628	B,D	10
G	0.8765484	B,L	6	0.9368794	D	5	0.9011094	B,L	7
H	0.7713536	L	9	0.8091801	D	10	0.7962596	A,D,L	11
I	0.9015742	A,L	5	0.9211676	A,D,L	6	0.9580663	B,L	5
J	0.7653135	B,L	10	0.8103234	D	9	0.8706379	D	9
K	0.8619133	B,L	7	0.8889356	A,D	7	0.9550884	A,D	6
L	1	L	1	1	L	1	1	L	1

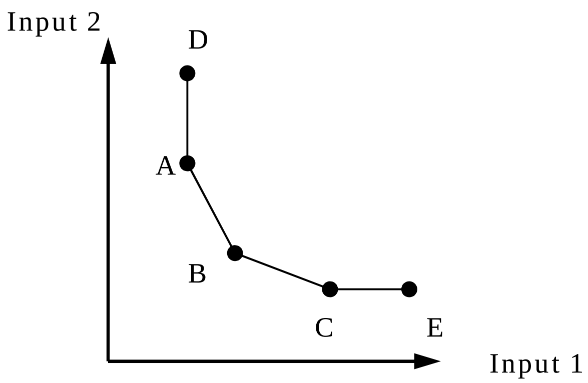

Figure 1: Efficient and non-efficient frontiers

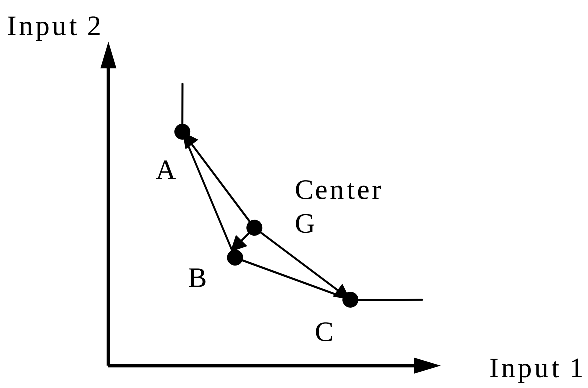

Figure 2: Random search around efficient DMUs