Krivonozhko, Vladimir E.; Førsund, Finn R.; Lychev, Andrey V.

Working Paper

Measurement of returns to scale using non-radial DEA models

Memorandum, No. 29/2012

Provided in Cooperation with:
Department of Economics, University of Oslo

MEMORANDUM

No 29/2012

Measurement of Returns to Scale Using Non-Radial DEA Models

Vladimir E. Krivonozhko, Finn R. Førsund and Andrey V. Lychev

ISSN: 0809-8786

Department of Economics
University of Oslo

This series is published by the
University of Oslo
Department of Economics

P. O.Box 1095 Blindern
N-0317 OSLO Norway
Telephone: + 47 22855127
Fax: + 47 22855035
Internet: http://www.sv.uio.no/econ
e-mail: econdep@econ.uio.no

In co-operation with
The Frisch Centre for Economic Research

Gaustadalleén 21
N-0371 OSLO Norway
Telephone: +47 22 95 88 20
Fax: +47 22 95 88 25
Internet: http://www.frisch.uio.no
e-mail: frisch@frisch.uio.no

Last 10 Memoranda

No 28/12	Derek J. Clark, Tore Nilssen and Jan Yngve Sand *Motivating over Time: Dynamic Win Effects in Sequential Contests*
No 27/12	Erik Biørn and Xuehui Han *Panel Data Dynamics and Measurement Errors: GMM Bias, IV Validity and Model Fit – A Monte Carlo Study*
No 26/12	Michael Hoel, Bjart Holtsmark and Katinka Holtsmark *Faustmann and the Climate*
No 25/12	Steinar Holden *Implication of Insights from Behavioral Economics for Macroeconomic Models*
No 24/12	Eric Nævdal and Jon Vislie *Resource Depletion and Capital Accumulation under Catastrophic Risk: The Role of Stochastic Thresholds and Stock Pollution*
No 23/12	Geir B. Asheim and Stéphane Zuber *Escaping the Repugnant Conclusion: Rank-discounted Utilitarianism with Variable Population*
No 22/12	Erik Biørn *Estimating SUR Systems with Random Coefficients: The Unbalanced Panel Data Case*
No 21/12	Olav Bjerkholt *Ragnar Frisch's Axiomatic Approach to Econometrics*
No 20/12	Ragnar Nymoen and Victoria Sparrman *Panel Data Evidence on the Role of Institutions and Shocks for Unemployment Dynamics and Equilibrium*
No 19/12	Erik Hernaes, Simen Markussen, John Piggott and Ola Vestad *Does Retirement Age Impact Mortality?*

Previous issues of the memo-series are available in a PDF® format at:
http://www.sv.uio.no/econ/english/research/memorandum/

MEASUREMENT OF RETURNS TO SCALE USING NON-RADIAL DEA MODELS

by

Vladimir E. Krivonozhko,

National University of Science and Technology «MISiS», Moscow

Finn R. Førsund[1],

Department of Economics, University of Oslo

Andrey V. Lychev

National University of Science and Technology «MISiS», Moscow

Abstract: There are some specific features of the non-radial DEA (data envelopment analysis) models which cause some problems under the returns to scale measurement. In the scientific literature on DEA, some methods were suggested to deal with the returns to scale measurement in the non-radial DEA models. These methods are based on using Strong Complementary Slackness Conditions in the optimization theory. However, our investigation and computational experiments show that such methods increase computational complexity significantly and may generate "strange" results. In this paper, we propose and substantiate a direct method for the returns to scale measurement in the non-radial DEA models. Our computational experiments documented that the proposed method works reliably and efficiently on the real-life data sets.

Keywords: Data envelopment analysis; Returns to scale; Non-radial models; Efficiency; Strong Complementary Slackness Conditions

JEL classifications: C44, C61, C67, D24

[1] Corresponding author. Telephone +4722855132, fax +4722855030, email: f.r.forsund@econ.uio.no, mail address: Department of Economics, Box 1095 University of Oslo, 0317 Blindern Norway

1. Introduction

The measurement of scale properties of frontier functions estimated using DEA models may in some cases be problematic. In particular the non-radial DEA models (Banker et al., 2004) can possess some specific features that give rise to estimation problems. First, multiple reference sets may exist for a production unit. Second, multiple supporting hyperplanes may occur on optimal units of the frontier. Third, multiple projections (a projection set) may occur in the space of input and output variables. All these features cause certain difficulties under measurement of returns to scale of production units.

Banker et al. (2004) proposed a two-stage approach to determine returns to scale in the non-radial models. Sueyoshi and Sekitani (2007) showed that this approach may generate incorrect results in some cases. An interesting approach was proposed for measurement of returns to scale based on using strong complementary slackness conditions (SCSC) in the non-radial DEA models (Sueyoshi and Sekitani, 2007).

However, our theoretical consideration and computational experiments show that the SCSC non-radial model may not be efficient from the computational point of view. The SCSC non-radial model generates ill-conditioned basic matrices during the solution process, which results in "strange results" that do not coincide with the optimal solution of the corresponding non-radial DEA model[2]. This naturally contradicts the optimization theory.

In our work we propose a two-stage approach to measure returns to scale in the non-radial DEA models. At first stage, an interior point, belonging to the optimal face, is found using a special elaborated method. In Krivonozhko et al. (2012b) it was proved that any interior point of a face has the same returns to scale as any other interior point of this face. At the second stage, we propose to determine the returns to scale at the interior point found in the first stage with the help of Banker and Thrall's (1992) method or using the direct method of Førsund et al. (2007).

Our computational experiments documented that the proposed approach is reliable and efficient for solving real-life DEA problems.

The plan of the paper is to state the problem to be investigated in Section 2, and to develop a direct method for discovering all units belonging to the minimum face in Section 3.

[2] In Krivonozhko et al. (2012a) this was demonstrated for the radial Banker et al. (1984) (BCC hereafter) DEA model. Because the frontier function is identical for the radial and non-radial models the computational problems encountered with the radial model will also occur for the non-radial model.

Section 4 reports some of the numerical experiments with the non-radial model specification based on data for Russian banks. Section 5 concludes.

2. Problem statement

The non-radial DEA model can be written in the following form (Banker et al., 2004; Sueyoshi and Sekitani, 2007)

$$\max h = (C^{+T} S^{+} + C^{-T} S^{-})$$

subject to

$$\begin{aligned}
&\sum_{j=1}^{n} X_j \lambda_j + S^{-} = X_o, \\
&\sum_{j=1}^{n} Y_j \lambda_j - S^{+} = Y_o, \\
&\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \ge 0,\ j = 1,\ldots,n, \\
&S^{+} \ge 0,\ S^{-} \ge 0,
\end{aligned} \tag{1}$$

where $X_j = (x_{1j},\ldots,x_{mj})$ and $Y_j = (y_{1j},\ldots,y_{rj})$ represent the observed inputs and outputs of production units (X_j, Y_j), $j = 1,\ldots,n$, $S^{-} = (s_1^{-},\ldots,s_m^{-})$ and $S^{+} = (s_1^{+},\ldots,s_r^{+})$ are vectors of slack variables. The superscript «T» indicates a vector transpose. The components of the objective-function vectors C^{+} and C^{-} are specified as follows:

$$\begin{aligned}
c_k^{-} &= (m+r)^{-1}\left(\max\{x_{kj} \mid j = 1,\ldots,n\} - \min\{x_{kj} \mid j = 1,\ldots,n\}\right)^{-1}, k = 1,\ldots,m, \\
c_i^{+} &= (m+r)^{-1}\left(\max\{y_{ij} \mid j = 1,\ldots,n\} - \min\{y_{ij} \mid j = 1,\ldots,n\}\right)^{-1}, i = 1,\ldots,r.
\end{aligned}$$

The model (1) is also called range-adjusted model (RAM) (Cooper et al., 2000).

In the model (1), an efficiency score for unit (X_o, Y_o) is evaluated, where (X_o, Y_o) is any production unit from the set (X_j, Y_j), $j = 1,\ldots,n$. If the optimal value h^* of the model is equal to zero, then unit (X_o, Y_o) is considered efficient, if $h^* > 0$, then the unit is inefficient (Banker et al., 2004).

The dual problem to the model (1) is written in the form:

$$\min\left\{v^T X_o - u^T Y_o + u_0\right\}$$

subject to

$$
\begin{aligned}
& v^T X_j - u^T Y_o + u_0 \geq 0,\ j = 1,\ldots,n, \\
& v \geq C^-, u \geq C^+,
\end{aligned}
\tag{2}
$$

where $v = (v_1,\ldots v_m)$ and $u = (u_1,\ldots,u_r)$ are vectors of dual variables associated with the first and the second group of constraints of problem (1), u_0 is a free variable associated with the convex constraint.

In the papers (Sueyoshi and Sekitani, 2007, 2009), it was proposed to use strong complementary slackness conditions from the optimization theory in order to find the set of optimal solutions in the primal and dual space. The SCSC non-radial model (Sueyoshi and Sekitani, 2007) is written in the following form

$$
\max \left\{ \eta \;\middle|\; \begin{aligned}
& \sum_{j=1}^{n} \lambda_j X_j + S^- = X_o, \sum_{j=1}^{n} \lambda_j Y_j - S^+ = Y_o, \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \geq 0,\ j = 1,\ldots,n,\ S^+ \geq 0, S^- \geq 0, \\
& v^T X_j - u^T Y_j + u_0 \geq 0,\ j = 1,\ldots,n, \\
& v \geq C^-, u \geq C^+, C^{-T} S^- + C^{+T} S^+ = v^T X_o - u^T Y_o + u_0, \\
& \lambda_j + v^T X_j - u^T Y_j + u_0 \geq \eta,\ j = 1,\ldots,n, \\
& s_k^- + v_k - c_k^- \geq \eta, k = 1,\ldots,m, \\
& s_i^+ + u_i - c_i^+ \geq \eta, i = 1,\ldots,r
\end{aligned} \right\}. \tag{3}
$$

The first six conditions are from the primal model (1), the next three conditions are from the dual problem (2), and the tenth condition provides the equality of the objective functions of the primal and dual problems. The last three conditions express the SCSC constraints. In order to secure that strong complementarity is obtained the variable η is entered as the objective function in (3).

In paper (Sueyoshi and Sekitani, 2007), the problem (3) is used in order to find the minimum face that contains the set of optimal solutions (a projection set) on the efficient hypersurface of set T in the space of input and output variables. Next, two additional fractional-linear optimization problems are determined for measurement of returns to scale.

The SCSC non-radial model is very interesting as a theoretical idea. However, our computational experiments show that the model (3) may generate strange results even for medium-size problems using well-reputed optimization software. The size of the model (3) increases significantly in comparison with the model (1). Indeed, the size of the model (3) is equal to $(3m+3r+2n+2)\times(2m+2r+n+2)$, where m is the number of inputs, r is the number of outputs, and n is the number of production units. Remember that the size of the model (1) is equal to $(m+r+1)\times(m+r+n)$, and n is usually much greater than $(m+r)$ in real-life models. Moreover, economic interpretation of some constraints of the model (3) does not make sense because in model (3) one has to add variables measured in quite different units. The reasons mentioned above may result in ill-conditioned basic matrices (Wilkinson, 1965) during the solution process. This may generate “strange results” in the optimal solution.

In order to investigate our suspicions about what will happen when using larger real datasets, we conducted computational experiments using two middle-sized models[3]. For the first model, call it Model 1, we took the data for electricity utilities in Sweden 1987; see Førsund et al. (2007). We used the total amount of low- and high-voltage electricity in MWh delivered to the customers and the number of low- and high-voltage customers served as the four outputs. On the input side we use kilometers of low- and high-voltage power lines and total transformer capacity in kVA as the capital variables. Labour is measured in full time equivalent employees. The number of production units in this model is 163.

For Model 2 we took the data from 920 Russia bank’s financial accounts for January 2009, where we used the following variables as inputs: working assets, time liabilities, and demand liabilities. As output variables we took: equity capital, liquid assets, fixed assets.

In the computational experiments we used the well-reputed optimization software CPLEX, the software Mathematica that is very popular among mathematicians, and the software FrontierVision, a specially elaborated program for DEA models that enables one to visualize the multidimensional frontier with the help of constructing two- and three-dimensional sections of the frontier.

It is interesting to note that the three software programs generated identical results when solving model (1) on data sets of Model 1 and Model 2. However, the software CPLEX produced significant discrepancies between the solution of model (1) and model (3), which are described in detail in (Krivonozhko et al., 2012a).

Observe that the software Mathematica is not intended for solution of large-scale LP problems, for this reason this program was not used for the solution of model (3). The software FrontierVision was not elaborated for the solution of the model type (3).

The main discrepancies in solving model (1) and model (3) with the help of the program CPLEX (though theoretically optimal solutions of these problems have to coincide) are as follows: a) efficiency scores of model (1) and model (3) may differ significantly; b) reference sets obtained in the solution of model (3) may contain inefficient units.

[3] The same data as utilised in Krivonozhko (2012a).

3. Direct method for discovering all units belonging to the minimum face

In the linear programming problem (1), the set of optimal points Λ^* (the set of projections of unit (X_o, Y_o) on the frontier) are situated on the boundary (frontier) of the production possibility set *T*. The boundary consists of a number of faces.

Remember that in the DEA models the dimension of face may vary from 0 up to $(m+r-1)$, the maximal dimension. Faces of maximal dimension are called facets. Faces of 0-dimension are known as vertices, 1-dimension as edges. In our exposition, $ri\Gamma$ stands for the relative interior of face Γ.

The solution set Λ^* cannot belong to different faces that do not have common points, otherwise interior points of T would belong to the solution set. Thus, optimal solutions of set Λ^* can belong only to the intersection of some faces of the set T.

Lemma 1. *Let two different faces Γ_1 and Γ_2 of the set T intersect. Then faces Γ_1 and Γ_2 do not have common interior points, i.e.* $\mathrm{ri}\,\Gamma_1 \cap \mathrm{ri}\,\Gamma_2 = \varnothing$.

The proof of Lemma 1 is given in Appendix.

Corollary 1. *Let two different faces Γ_1 and Γ_2 of the set T intersect. Then only one from the following cases occurs:*

(i) One face belongs to the other face entirely, to be precise let $\Gamma_1 \subset \Gamma_2$, and set $\Gamma_1 \cap \Gamma_2$ is a part of the boundary of Γ_2;

(ii) Set $\Gamma_1 \cap \Gamma_2$ is a part of the boundary of the face Γ_1 and face Γ_2, moreover the set $\Gamma_1 \cap \Gamma_2$ is also a face and its dimension is less than the dimensions of the face Γ_1 or the face Γ_2.

It was proven in Lemma 1 that an interior point Z_o of face Γ_1 cannot belong to intersections of $\mathrm{ri}\,\Gamma_1$ and $\mathrm{ri}\,\Gamma_2$. However, an interior point Z_o of Γ_1 may belong to Γ_2. In this case face Γ_1 belongs to face Γ_2 entirely and set $\Gamma_1 \cap \Gamma_2$ is a part of the boundary of Γ_2. This can be shown by the same way as in Lemma 1.

However, if point Z_o belongs to $\Gamma_1 \cap \Gamma_2$ and $Z_o \notin \mathrm{ri}\,\Gamma_1 \cup \mathrm{ri}\,\Gamma_2$, then set $\Gamma_1 \cap \Gamma_2$ is a part of the boundary of the face Γ_1 and the face Γ_2.

From the assertions written above, it follows that faces can intersect only along the boundaries of these faces. Taking into account also that the number of faces of the set T is finite, we obtain that there exists a face of minimum dimension Γ_{min} containing set Λ^*. By virtue of the non-negativity constraints on slack variables and the specific objective function associated with all slack variables in problem (1), set Λ^* may constitute only some part of face Γ_{min}.

Corollary 2. Face Γ_{min} is a bounded polyhedral set (polyhedron).

This assertion follows from the constraints structure of problem (1).

Now, we proceed to the construction of the procedure that finds all production units belonging to the minimum face Γ_{min} and to the set Λ^*.

Let problem (1), respectively problem (2), be solved by the simplex-method (Dantzig and Thapa, 2003) and optimal primal variables $\{\lambda_j^*, j=1,\dots,n;\ s_k^{-*}, k=1,\dots,m;\ s_i^{+*}, i=1,\dots,r\}$ and dual variables $\{v_k^*, k=1,\dots,m;\ u_i^*, i=1,\dots,r;\ u_0^*\}$ be obtained.

Determine the following index sets for primal variables

$$\begin{aligned} I^* &= \{j \mid \lambda_j^* > 0,\ j=1,\dots,n\}, \\ I_x^- &= \{k \mid s_k^{-*} > 0,\ k=1,\dots,m\}, \\ I_y^+ &= \{i \mid s_i^{+*} > 0,\ i=1,\dots,r\}. \end{aligned} \tag{4}$$

Introduce the index sets associated with the dual variables

$$\begin{aligned} J^* &= \{j \mid v^{*T}X_j - u^{*T}Y_j + u_0^* = 0,\ j=1,\dots,n\}, \\ J_v &= \{k \mid v_k^* = c_k^-,\ k=1,\dots,m\}, \\ J_u &= \{i \mid u_i^* = c_i^+,\ i=1,\dots,r\}. \end{aligned} \tag{5}$$

Since the optimal solution is obtained with the help of the simplex-method every non-basic variable is equal to zero. However, some basic variables may be equal to zero also, then the optimal solution is considered degenerate.

For the dual problem (2) all indices belonging to the set $(J^* \cup J_v \cup J_u)$ contain a basic set of indices. However, the set $(J^* \cup J_v \cup J_u)$ may also contain non-basic indices; in this case the dual problem (2) is considered degenerate. Thus, the following relations hold

$$(I^* \cup I_x^- \cup I_y^+) \subseteq J_B \subseteq (J^* \cup J_v \cup J_u),$$

where J_B is a set of optimal basic variables of the problem (1).

Variables λ_j^*, $j \in I^*$ determine only one point on the minimum face. To find all points belonging to the face Γ_{min} it is necessary to solve additional problems.

Problem Q_{ol} ($l \in J^*$):

$$\max f_l = \lambda_l$$

subject to

$$\begin{aligned}
&\sum_{j \in J^*} X_j \lambda_j + \sum_{k \in J_v} e_k s_k^- = X_o, \\
&\sum_{j \in J^*} Y_j \lambda_j - \sum_{i \in J_u} e_i s_i^+ = Y_o, \\
&\sum_{j \in J^*} \lambda_j = 1, \lambda_j \ge 0, \ j \in J^*, \\
&s_k^- \ge 0, k \in J_v, s_i^+ \ge 0, i \in J_u,
\end{aligned} \tag{6}$$

where $e_k^- \in E^m$ and $e_i^+ \in E^r$ are identity vectors associated with variables s_k^- and s_i^+, respectively.

Notice that problem (6) includes only those variables for which the corresponding dual constraints hold strictly equations for optimal dual variables (5). According to the dual theorems of linear programming, this means that optimal variables of problem (6) will also be optimal variables of problem (1).

The Procedure that finds all production units belonging to the minimum face Γ_{min} and to the set Λ^* is described as follows:

1. Initialize sets $J_o = \varnothing$, $JH = J^*$, $J_1 = \varnothing$. If the set JH is not empty, then go to the next step. If set JH is empty then go to step 3.
2. Choose index $l \in JH$, if the set JH is empty, then go to step 3. Solve the problem (6).

   If $f_l^* > 0$, then determine $J_o = J_o \cup l$. If $f_l^* = 1$, then $J_1 = J_1 \cup l$. Delete index l from the set $JH = JH \setminus l$. Go to the beginning of the step.

   If $f_l^* = 0$, then delete index l from the set $JH = JH \setminus l$. Go to the beginning of step 2.
3. Set J_o determines the set of units belonging to the face Γ_{min}. Set J_1 determines the set of units belonging to the set Λ^*.

   The Procedure is completed.

The standard present-day optimization software generates only one point in the multidimensional space as an optimal solution. However, this may be not sufficient in order to determine returns to scale on the whole minimum face, since different vertices of the face may display different returns to scale. Any unit from set J^* may belong to the minimum face. The standard software generates set J^* as a by-product. So, the Procedure enables one to check whether some unit from set J^* belongs to the minimum face or not. The validity of this assertion is based on the theorems given below.

After running the Procedure, the minimum face Γ_{min}, containing the optimal set Λ^*, can be written in the form:

$$\Gamma_{min} = \left\{ (X,Y) \,\middle|\, X = \sum_{j \in J_o} X_j \lambda_j ,\, Y = \sum_{j \in J_o} Y_j \lambda_j ,\, \sum_{j \in J_o} \lambda_j = 1,\, \lambda_j \geq 0,\, j \in J_o \right\}. \tag{7}$$

However, some points of the set Γ_{min} may not belong to the set Λ^* on the frontier. The set Λ^* is written as:

$$\begin{aligned}\Lambda^* = \{ (X,Y) \mid (X,Y) \in \Gamma_{min},\ X \leq X_o,\ Y \geq Y_o, & \\ C^{+T}Y - C^{-T}X = (v^* - C^-)^T X_o - (u^* - C^+)^T Y_o + u_0^* \}. &\end{aligned} \tag{8}$$

The Procedure enables one to find all units belonging to the face Γ_{min} and to the set Λ^*. The validity of this assertion is based on the following theorems.

Theorem 1. *Let unit* $Z_t \in E^{m+r}$ *be an interior point of polyhedron* $\Gamma \subset E^{m+r}$*, let also unit* $Z_p \in E^{m+r}$ *be any point of this polyhedron, which is distinct from point* Z_t*. Then unit* Z_t *can be represented as a convex combination of* $(m+r+1)$ *units of set* Γ *and unit* Z_p *enters this combination with a nonzero coefficient.*

Theorem 2. *The optimal value of problem (6) is strictly positive* $f_l^* > 0$ *if and only if unit* (X_l, Y_l) *belongs to the minimum face* Γ_{min} *that contains the set* Λ^*.

The Proof of Theorems 1 and 2 is given in the Appendix.

In essence, Theorem 1 says that, if some unit Z_p is a vertex of face Γ_{min} or belongs to the face, then it is necessary that there exists such solution that variable λ_p^* enters this solution with a nonzero coefficient.

Corollary 3. *If the optimal value of problem (6)* $f_l^* = 1$*, then unit* (X_l, Y_l) *belongs to the set* Λ^*.

If $f_l^* = 1$, then $\lambda_l^* = 1$, this means that λ_l^* is the only non-negative λ-variable in the optimal basis, hence unit (X_l, Y_l) belongs to the set Λ^*.

It was proved in (Krivonozhko et al., 2012b) that interior points of a face have the same returns to scale, so it is sufficient to determine returns to scale at any interior point of this face. An interior point $(\overline{X}, \overline{Y})$ of the face Γ_{min} can be chosen as a strong convex combination of units from the set J_o, that is

$$\overline{X} = \sum_{j \in J_o} X_j \lambda_j, \ \overline{Y} = \sum_{j \in J_o} Y_j \lambda_j, \ \sum_{j \in J_o} \lambda_j = 1, \ \lambda_j > 0, \ j \in J_o.$$

Returns to scale of unit $(\overline{X}, \overline{Y})$ can be measured by two methods at least. In the first (indirect) method (Banker et al., 2004) the BCC model is solved at the first step, the dimension of this problem is equal to $(m + r + 1) \times (m + r + n)$, then at the second step two additional problems are solved, the dimension of these two problems coincides with the dimension of the BCC problem. Returns to scale is determined with the help of dual variables.

In the second (direct) method (Førsund et al., 2009), an intersection of the set T and two-dimensional plane is constructed with the help of some algorithms at the first step. At the second step returns to scale of any point on the graph is measured by using derivatives of this graph.

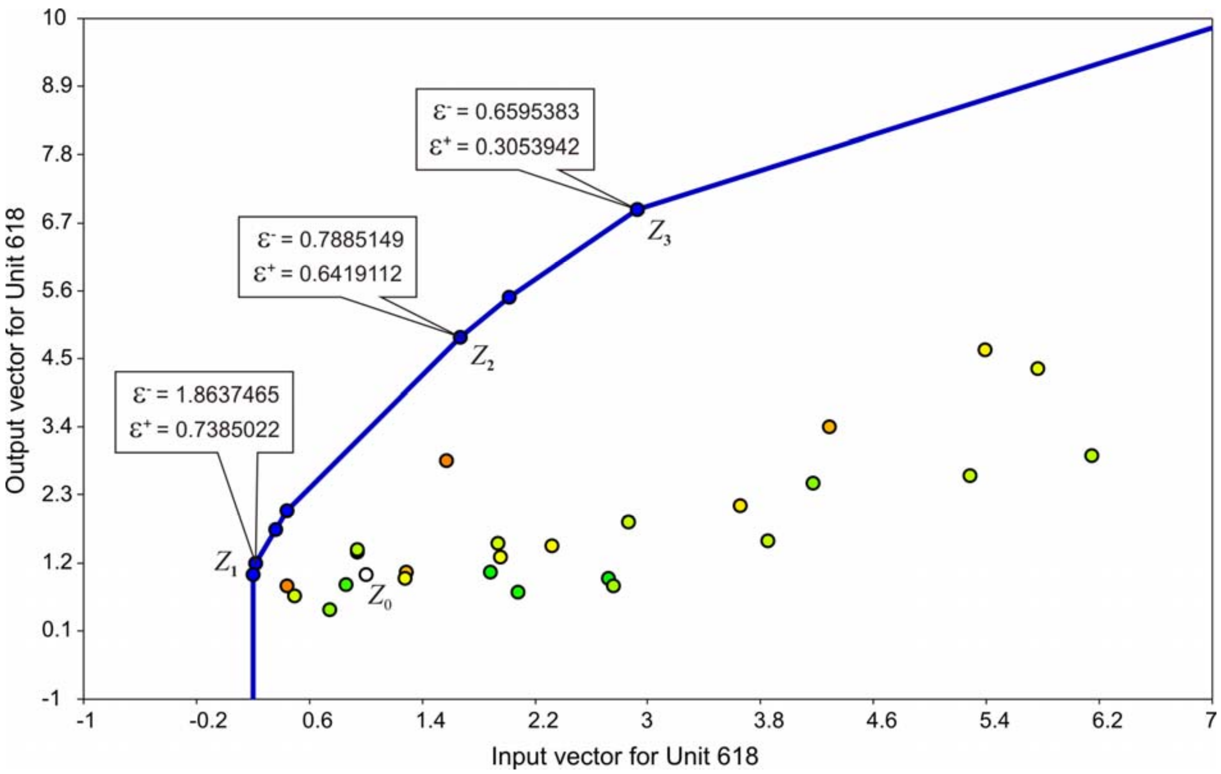

Fig. 1. Production function for Unit 618

In Figure 1, the production function for unit 618 is depicted, which is constructed as an intersection of the two-dimensional plane and six-dimensional production possibility set T in Model 2 for banks. In Figure 1 point Z_o is the bank that has number 618 in the data set of banks. Small circles denote the projections of the actual observations of banks onto the two-dimensional plane. In the figure, scale elasticity values (from the left and from the right) are shown for three points of the graph Z_1, Z_2, Z_3, respectively. The software FrontierVision constructs intersections for any unit, spending less than one second per section, and enables one to calculate scale elasticity instantly, hence to measure returns to scale of any points on the graph.

4. Computational experiments

Consider computational experiments with the data set of Model 1 in detail. The problem (1) was solved for every unit (X_j, Y_j), $j = 1,\dots,n$ in Model 1, where every (X_j, Y_j) was inserted in problem (1) instead of (X_o, Y_o). Next, problem of the form (6) was determined on the basis of optimal solution of problem (1) and solved for every $j \in J^*$ using the software CPLEX. Thus, all production units belonging to Γ_{min} or to Λ^* were found for every solution of problem (1).

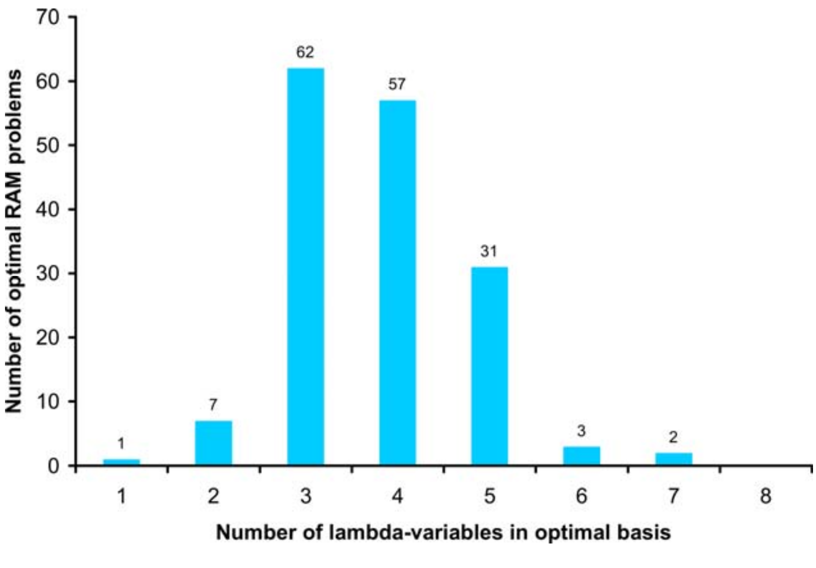

Fig. 2. Distribution of λ-variables on optimal solutions of problems (1)

Figure 2 represents the dependence of the number of problems of the form (1) on the number of λ-variables in the optimal basis of problem (1). Remember that the number of λ-variables in the optimal basis determines the dimension of the face associated with this solution for the bounded face. The dimension of space in Model 1 is equal to 8, so the maximal dimension of face is equal to 7; the optimal basis associated with a face of maximal dimension has to have eight λ-variables in the basis.

Figure 2 shows that faces of maximal dimension did not appear in optimal solutions of problems (1). This fact can be easily explained. The number of faces of the production possibility set is a finite number. However, the number of possible directions of vector (C^+, C^-) in problem (1) is a non-countable infinite number. So, the possibility of cases, where vector (C^+, C^-) is perpendicular to the face of maximal dimension, is very low.

As one can see from the figure, the maximal number of faces (62) turned out to have dimension 2.

Computational experiments showed that there are only two cases (for units 75 and 254) when a vertex belongs to set Γ_{min} and Λ^* simultaneously. Consider the first case in detail.

Solving problem (1) by CPLEX program for unit 75, we obtain the following reference set:

$$\lambda_{139} = 0.403,\ \lambda_{241} = 0.023,\ \lambda_{246} = 0.574,$$

all other λ-variables in the optimal solution are equal to zero.

Units 139 and 246 display constant returns to scale, unit 241 has increasing returns to scale.

After getting the solution of problem (6) for unit 75, we obtain that only for unit 139 the objective function $f^*_{139} = 1$. This means that unit 139 belongs to Γ_{min} and Λ^* simultaneously.

Define the convex combination of units 139, 241 and 246 as artificial unit 1075

$$Z_{1075} = \tfrac{1}{3}(Z_{139} + Z_{241} + Z_{246}).$$

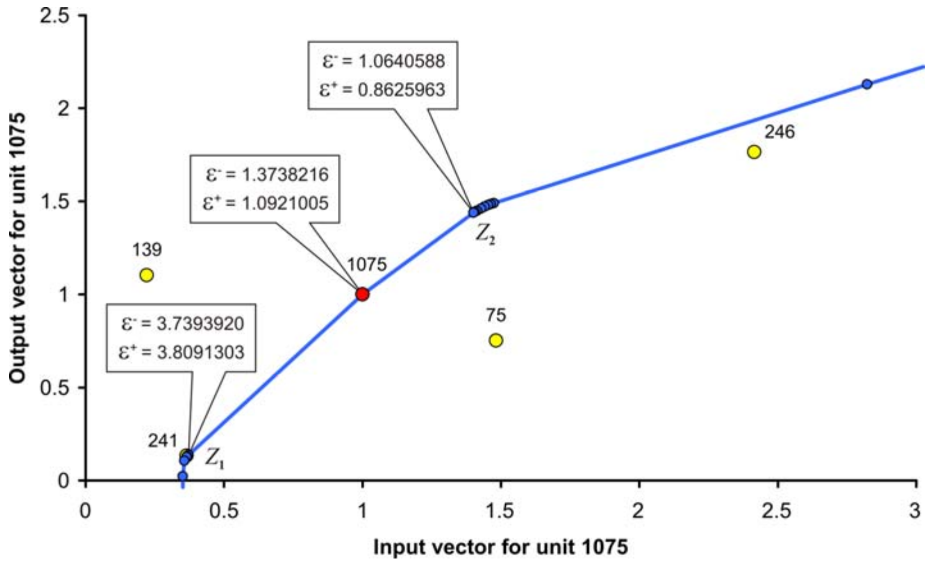

Fig. 3. Production function for unit 1075

Figure 3 represents an intersection of the eight-dimensional production possibility set with a two-dimensional plane for unit 1075, where the directions of the plane are determined by the input and output vectors of unit 1075. The light and dark points in the figure denote projections of units from the reference set onto the two-dimensional plane.

Figure 3 shows also scale elasticities $(\varepsilon^{-}, \varepsilon^{+})$ for corner points Z_1, Z_2 and artificial unit 1075. Projection of unit 241 onto the plane almost coincides with corner point Z_1. It follows from the figure that unit 1075 has increasing returns to scale. Since unit 1075 is an interior unit of the face formed by units 139, 241 and 246, so points lying on this face display increasing returns to scale, see (Krivonozhko et al., 2012b). However, the software could generate units 139 plus slacks as optimal solution. If one determined returns to scale only on the basis of this information, then all units of the face would display constant returns to scale.

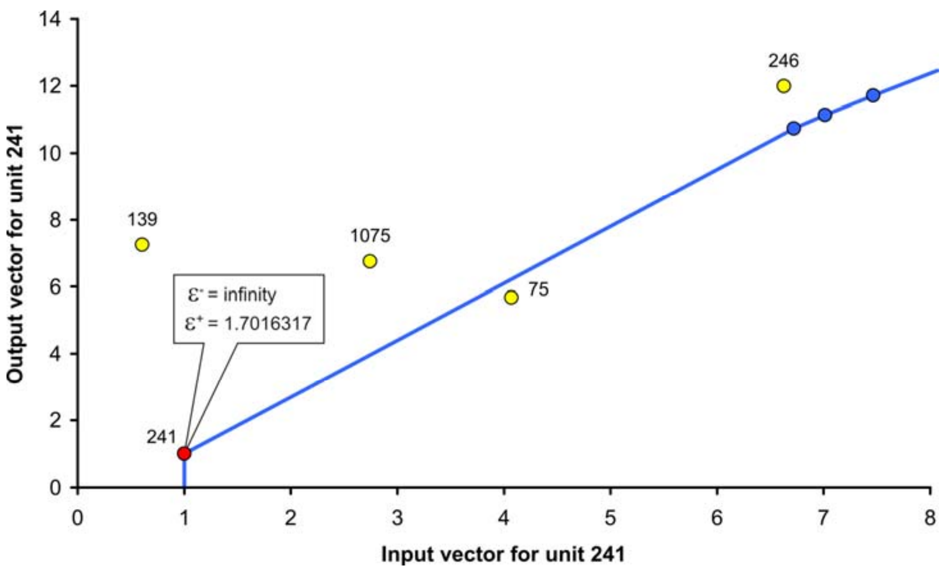

Fig. 4. Production function for unit 241

Production function for unit 241 is depicted in Figure 4, where the vectors of the two-dimensional plane are determined by input and output vectors of unit 241. Again, the dark and light points in the figure denote projections of units from the reference set onto the two-dimensional plane. Left and right scale elasticities (ε^{-} and ε^{+}) are shown also in the figure for unit 241, according to their values unit 241 displays increasing returns to scale.

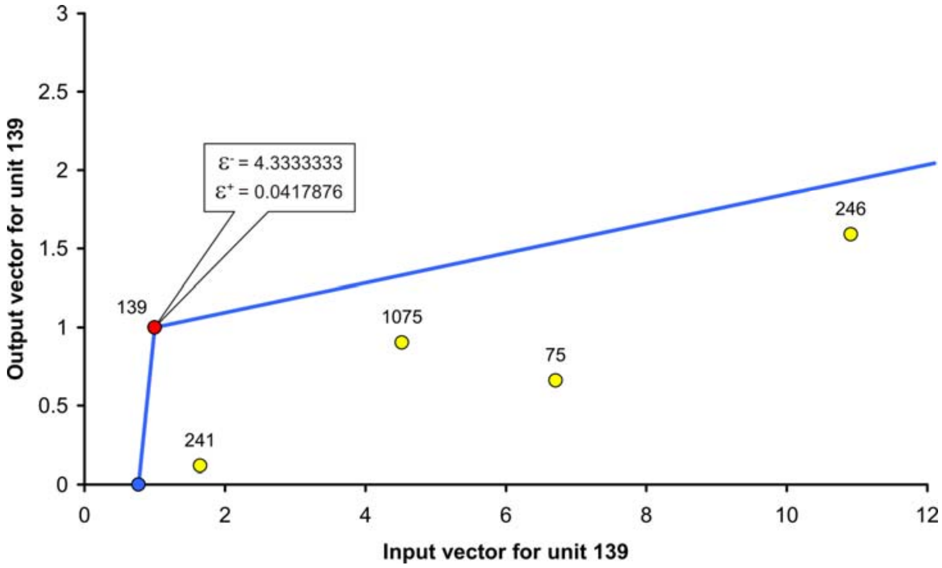

Fig. 5. Production function for unit 139

Figure 5 represents production function for unit 139. Left and right scale elasticities shown in the figure indicate that this unit displays constant returns to scale.

Hence, this example show that only an interior point of a face can determine return to scale of points lying on this face.

However, the question arises: What is the 'payment' for discovering all vertices of the face Γ_{min}?

We calculated the number of iterations accomplished by CPLEX software in order to solve problem (1) for all units (X_j, Y_j), $j = 1, \ldots, n$. This number is equal to $\alpha = 2876$ iterations. Next, we calculated the number of iterations accomplished by CPLEX in order to solve problems (6) for all units (X_j, Y_j), $j = 1, \ldots, n$. This number makes up $\beta = 1782 = 0.62\,\alpha$ iterations in the problems of the type (6). That is really not a heavy burden, even for an ordinary notebook.

It is worth noting that interior unit 1075 of face Γ_{min} is depicted as a corner point in Figure 3. Moreover, the intersection of the two-dimensional face Γ_{min} and the two-dimensional plane spanned by input and output vectors of unit 1075 generates just one point (unit 1075). This situation is impossible in three-dimensional space. However, such cases do exist in the multidimensional case. Let us dwell on this more detail.

Designate *(m+r)* by *p*. Every face of the production possibility set in Euclidean space E^p belongs uniquely to some minimum affine set.

Remember that the affine set Aff (Rockafellar, 1970) generated by units $Z_l = (X_l, Y_l) \in E^p$, $l \in I_A$, where I_A is a subset of index set $j = 1, \ldots, n$, can be written in the following form

$$Aff = \left\{ Z \middle| Z = \sum_{l \in I_A} Z_l \mu_l, \sum_{l \in I_A} \mu_l = 1, where\ \mu_l\ are\ any\ real\ numbers \right\}. \qquad (9)$$

The affine set Aff is parallel to the subspace L (Rockafellar, 1970), passing through the origin, which can be written in the form:

$$L = \left\{ Z \middle| Z = \sum_{l \in I_A} (Z_l - Z_p)\gamma_l, where\ Z_p\ is\ any\ unit\ from\ the\ set\ I_A, \gamma_l\ are\ any\ real\ numbers \right\}.$$

The dimension of the affine set Aff is equal to the dimension of the subspace L.

Now, we are ready to state the following result.

Theorem 3. The intersection of the 2-dimensional plane and the k-dimensional affine set, $k \geq 2$, result in a point in the multidimensional space E^p only if $p \geq k + 2$.

The proof of the theorem is given in the Appendix.

It follows from Theorem 3 that in the 3-dimensional space the intersection of two 2-dimensional planes cannot result in a point. In fact, Figure 3 shows that interior artificial unit 1075 represents in the 8-dimensional space the intersection of the 2-dimensional face Γ_{min} and the 2-dimensional plane spanned by input and output vectors of the unit 1075.

Finding. The non-radial DEA models possess some specific features. However this is a not a problem in order for find returns to scale of the set of optimal points. For this purpose it is sufficient to find an interior point of the minimum face that contains the set of optimal solutions of problem (1) and to determine returns to scale of this interior point. Such solution requires much less computations than to solve problem (1) for the specific unit.

5. Conclusion

In Sueyoshi and Sekitani (2007; 2009), a method was proposed in order to measure returns to scale in the non-radial DEA models using strong complementary slackness conditions. However, the size of the SCSC non-radial model (3) increases significantly in comparison with the model (1). In particular, for the banks data set the size of basic matrices during the solution process becomes (1840×1840) instead of (7×7) in the model (1). In addition, some constraints in model (3) do not make sense from an economic point of view. Unreliable solutions may follow due to these reasons.

In our method, it is sufficient to solve several problems of the form of model (1), however such problems have much less variables than problem (1). Thus, the proposed approach is reliable and efficient for solutions of real-life problems. Moreover, it was stressed in Sueyoshi and Sekitani (2007) that the method of Banker et al. (2004) cannot always generate reliable results because of difficulties described above in the non-radial DEA models. However, the method of Banker et al. (2004) can also be used to measure returns to scale from our point of view. For this purpose, it is sufficient to take an interior point of the minimum face, which can be found by the method proposed in this paper, after this one can use the method proposed in Banker et al. (2004).

Acknowledgement

The research is carried out with financial support of the Programme of Creation and Development of the National University of Science and Technology «MISiS». Financial support from the Russian Foundation for Basic Research (RFBR) is also gratefully acknowledged (grants 11-07-00698-a and 12-07-31136-mol_a).

Appendix

Proof of Lemma 1. Assume that point $Z_o \in \mathrm{ri}\,\Gamma_1 \cap \mathrm{ri}\,\Gamma_2$. Since faces Γ_1 and Γ_2 are different, then there exists a direction that belongs to face Γ_1 and does not belong to Γ_2 or, on the contrary, belongs to Γ_2 and does not belong to Γ_1. Hence there exist such points Z_1 and Z_2 that $Z_o = \lambda_1 Z_1 + \lambda_2 Z_2$, $\lambda_1 + \lambda_2 = 1$, $\lambda_1, \lambda_2 \geq 0$, Z_1 and Z_2 belong to $\mathrm{ri}\,\Gamma_1$ and Z_1 and Z_2 do not belong to $\mathrm{ri}\,\Gamma_2$ or, on the contrary, Z_1 and Z_2 belong to $\mathrm{ri}\,\Gamma_2$ and Z_1 and Z_2 do not belong to $\mathrm{ri}\,\Gamma_1$. Without loss of generality, let $Z_1, Z_2 \in \mathrm{ri}\,\Gamma_1$ and $Z_1, Z_2 \notin \mathrm{ri}\,\Gamma_2$. Therefore point Z_o of face Γ_2 can be represented as a convex combination of points Z_1 and Z_2 that do not belong to face Γ_2. This contradiction completes the proof.

Proof of Theorem 1. According to convex analysis (Nikaido, 1968), there exists a unique point Z_q, that belongs to some face of the polyhedron Γ, and point Z_t can be represented in the form

$$Z_t = (1-\rho)Z_q + \rho Z_p,\ 0 < \rho < 1. \tag{A.1}$$

Polyhedron Γ is a bounded set; hence the face is also bounded.

Since Z_q belongs to some bounded face of polyhedron Γ, then according to the Carathéodory's Theorem (Rockafellar, 1970), point Z_q can be written in the form

$$Z_q = \sum_{j \in J_1} Z_j \lambda_j^{'} ,$$

$$\sum_{j \in J_1} \lambda_j^{'} = 1, \ \lambda_j^{'} \geq 0, \ j \in J_1, \tag{A.2}$$

where set J_1 is a subset of all vertices of the face, and the number of elements in J_1 is less or equal to $(m+r)$.

It follows from (A.1) and (A.2) that

$$Z_t = (1-\rho)\sum_{j \in J_1} Z_j \lambda_j^{'} + \rho Z_p, \tag{A.3}$$

where $0 < \rho < 1$.

From (A.3), we can obtain

$$Z_t = \sum_{j \in J_1} Z_j \lambda_j^{''} + \rho Z_p, \ \lambda_j^{''} = (1-\rho)\lambda_j^{'}, \ j \in J_1 \tag{A.4}$$

$$\sum_{j \in J_1} \lambda_j^{''} + \rho = 1 - \rho + \rho = 1.$$

Hence unit Z_t can be represented as a convex combination (A.4) of units that belong to the polyhedron Γ and unit Z_p enters this combination with a positive coefficient.

This completes the proof.

Proof of Theorem 2. Let be $f_l^* = \lambda_l^* > 0$. This means that unit (X_l, Y_l) enters a convex combination with a nonzero coefficient that determines some optimal point belonging to set Λ^*. Hence $(X_l, Y_l) \in \Gamma_{min}$.

Conversely. Take any unit $(X_l, Y_l) \in \Gamma_{min}$, where $l \in J^*$. Set J^* includes indices of vertices that form face $\Gamma_{\min}$. Consider some point $(X_k, Y_k) \in ri\Lambda^*$. Point (X_k, Y_k) can be represented as a convex combination of units from set Γ_{min}. This convex combination is associated with a feasible solution of problem (6), we call it Combination 1. By virtue of Theorem 1 there exists such convex combination of points from set Γ_{min}, which includes point (X_l, Y_l) with a nonzero coefficient $\lambda_l > 0$, we call it Combination 2. Substitute this Combination 2 into the problem (6) instead of Combination 1, we obtain a feasible solution of (6) containing $\lambda_l > 0$. By optimizing problem (6), we obtain $\lambda_l^* \geq \lambda_l > 0$.

This completes the proof.

Proof of Theorem 3. The 2-dimensional plane in the p-dimensional space E^p is determined by $(p-2)$ linear independent equations. The k-dimensional affine set in the p-dimensional space E^p is determined by $(p-k)$ linear independent equations. So, the intersection of 2-dimensional plane and k-dimensional affine space is determined by the system of $(p-k+p-2)$ linear equations.

This system determines a point in the multidimensional space E^p only if the following conditions hold:

a) The number of equations of the system $(p-k+p+2) \geq p$ or $p \geq k+2$;

b) The rank of this system of linear equations is equal to *p*.

Hence the intersection of the 2-dimensional plane and the k-dimensional affine set may result in a point in the multidimensional space.

This completes the proof.

References

Banker, R. D., Charnes, A., & Cooper, W. W. (1984). Some models for estimating technical and scale inefficiency in data envelopment analysis. *Management Science* 30(9): 1078–1092.

Banker, R. D., Cooper, W. W., Seiford, L. M., Thrall, R. M., & Zhu, J. (2004). Returns to scale in different DEA models. *European Journal of Operational Research* 154(2), 345–362.

Cooper, W. W., Park, K. S., & Pastor, J. T. (2000). RAM: A range adjusted measure of efficiency. *Journal of Productivity Analysis* 11: 5–42.

Dantzig, G. B. & Thapa, M. N. *Linear programming 2: Theory and Extensions.* New York: Springer-Verlag, 2003.

Førsund, F. R., Hjalmarsson, L., Krivonozhko, V. E., & Utkin, O. B. (2007). Calculation of scale elasticities in DEA models: direct and indirect approaches. *Journal of Productivity Analysis* 28: 45–56.

Førsund, F. R., Kittelsen, S. A. C., & Krivonozhko, V. E. (2009). Farrell revisited – Visualizing properties of DEA production frontiers. *Journal of the Operational Research Society* 60: 1535–1545.

Krivonozhko, V. E., Førsund, F. R., & Lychev, A. V. (2012a). A note on imposing strong complementary slackness conditions in DEA. *European Journal of Operational Research* 220(3): 716–721.

Krivonozhko, V. E., Førsund, F. R., & Lychev, A. V. (2012b). Returns-to-scale properties in DEA models: the fundamental role of interior points. *Journal of Productivity Analysis* 38: 121-130.

Nikaido, H. (1968). *Convex structures and economic theory*. New York: Academic Press.

Rockafellar, R. T. (1970). *Convex analysis*. Princeton University Press, New Jersey.

Sueyoshi, T. & Sekitani, K. (2007). Measurement of returns to scale using a non-radial DEA model: A range-adjusted measure approach. *European Journal of Operational Research* 176: 1918–1946.

Sueyoshi, T. & Sekitani, K. (2009). An occurrence of multiple projections in DEA-based measurement of technical efficiency: theoretical comparison among DEA models from desirable properties. *European Journal of Operational Research* 196: 764–794.

Wilkinson, J. H. (1965). *The algebraic eigenvalue problem*. Oxford: Oxford University Press.