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Decision Support

Tree, web and average web values for cycle-free directed graph games ☆

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ABSTRACT

On the class of cycle-free directed graph games with transferable utility solution concepts, called web values, are introduced axiomatically, each one with respect to a chosen coalition of players that is assumed to be an anti-chain in the directed graph and is considered as a management team. We provide their explicit formula representation and simple recursive algorithms to calculate them. Additionally the efficiency and stability of web values are studied. Web values may be considered as natural extensions of the tree and sink values as has been defined correspondingly for rooted and sink forest graph games. In case the management team consists of all sources (sinks) in the graph a kind of tree (sink) value is obtained. In general, at a web value each player receives the worth of this player together with his subordinates minus the total worths of these subordinates. It implies that every coalition of players consisting of a player with all his subordinates receives precisely its worth. We also define the average web value as the average of web values over all management teams in the graph. As application the water distribution problem of a river with multiple sources, a delta and possibly islands is considered.

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1. Introduction

In standard cooperative game theory it is assumed that any coalition of players may form. However, in many practical situations the collection of coalitions that can be formed is restricted by some social, economical, hierarchical, communication, or technical structure. The study of games with transferable utility and limited cooperation introduced by means of communication graphs was initiated by Myerson (1977). In this paper we restrict our consideration to the class of cycle-free digraph games in which the players are partially ordered and the communication via bilateral agreements between players is represented by a directed graph without directed cycles. A cycle-free digraph cooperation structure allows modeling of various flow situations when several links may merge at a node, while other links split at a node into several separate ones.

It is assumed that a directed link represents a one-way communication situation. This restricts the set of coalitions that can be formed. There are different scenarios possible for controlling cooperation in case of directed communication. It is possible that players are controlled only by their predecessors. Another scenario assumes that players are controlled only by their successors. But it is also possible that the management team is located neither at the top nor at the bottom of the given directed communication structure but somewhere in between and each manager keeps control over all of his successors and predecessors. Important is that no manager is a subordinate of any other manager. In general, any anti-chain in the digraph can be chosen as a management team.

We introduce web values for cycle-free digraph games axiomatically, each one with respect to a chosen management team, and provide their explicit formula representation. The web value assigns to every player what he contributes when he joins his subordinates in the graph and that the total payoff for any player together with all his subordinates is equal to the worth they can get all together by their own. We also provide simple recursive computational methods for computing web values and study their efficiency and when possible stability. The values are introduced for arbitrary cycle-free digraph games and can be considered as natural extensions of the tree and sink values defined for rooted and sink forest digraph games, respectively (cf. Demange (2004), Khmelnitskaya (2010)). Besides, we define the average web value by taking the average of web values over all management teams of the graph. This value depends only on a given TU game and a given cycle-free directed communication graph and does not depend on the choice among different options for controlling cooperation. Furthermore, we extend the Ambec and Sprumont (2002)

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line-graph river game model of sharing a river to the case of a river with multiple sources, a delta and possibly islands by applying the results obtained to this more general setting of sharing a river among different agents located at different levels along the river bed restated in terms of a cycle-free digraph game.

The study of cooperative games with limited cooperation depending on partial orders of players is not new. Faigle and Kern (1992) initiated the study of cooperative games under precedence constraints that can be reformulated in terms of directed graphs, possibly disconnected. They consider different coalitional structures and allow for certain coalitions to be disconnected. Another extension of the tree value for arbitrary cycle-free digraph games has been recently proposed in Li and Li (2011). In that paper it is assumed that on each edge of a cycle-free digraph that underlies the game there is a flow amount along the edge. As solution the authors consider the value which assigns to every player the worth of this player together with his successors minus a weighted sum of the worths of these successors together with their successors, where the weights are determined by the relative flow amounts. The definition of this value requires additional and in general not easily available information about the flow distribution in the digraph. Besides, in this value the payoffs for the players are not adjusted for the fact that sets of successors of different successors of a same player may overlap each other.

The paper has the following structure. Basic definitions and notation are introduced in Section 2. In Section 3 we discuss different scenarios possible for controlling the situation defined by a digraph communication structure. Section 4 investigates a particular case when the control is going from the top to the bottom, which provides the so-called tree value. In Section 5 the general case of web values is studied. The average web value is introduced in Section 6. In Section 7 the application to the water distribution problem of a river with multiple sources, a delta and possibly islands is considered.

2. Preliminaries

A *cooperative game with transferable utility*, or *TU game*, is a pair $\langle N, v\rangle$, where $N = \{1,\dots,n\}$ is a finite set of $n \geqslant 2$ players and $v := 2^N \to \mathbb{R}$ is a *characteristic function* with $v(\emptyset) = 0$, assigning to any *coalition* $S \subseteq N$ its *worth* $v(S)$. The set of TU games with fixed player set N is denoted by $\mathcal{G}_N$. For simplicity of notation and if no ambiguity appears we write v when we refer to a TU game $\langle N, v\rangle$. The *subgame* of a TU game $v \in \mathcal{G}_N$ with nonempty player set $T \subseteq N$ is the TU game $v|_T \in \mathcal{G}_T$ defined by $v|_T(S) = v(S), S \subseteq T$. A *payoff vector* is a vector $x \in \mathbb{R}^N$ with x_i the payoff to player $i \in N$ and $x(S) = \sum_{i\in S} x_i$ the total payoff to the members of coalition $S \subseteq N$.

The cooperation structure on the player set N is specified by a graph, directed or undirected, on N, determining which coalitions are feasible. A *graph* on N consists of N as the set of nodes and for a *directed graph*, or *digraph*, a collection of ordered pairs $\Gamma \subseteq \{(i,j)|i,j \in N, i \neq j\}$ as the set of directed links from one node to another node in N, and for an *undirected graph* a collection of unordered pairs $\Gamma \subseteq \{\{i,j\}|i,j \in N, i \neq j\}$ as the set of links between two nodes in N. For a subgraph $\Gamma' \subseteq \Gamma$, $N(\Gamma')$ denotes the set of nodes in Γ'. For a digraph Γ on N and a coalition $S \subseteq N$, $\Gamma|_S = \{(i,j) \in \Gamma | i,j \in S\}$ is the *subgraph* of Γ on S.

For a digraph Γ on N, a sequence of different nodes $(i_1,\dots,i_r)$, $r \geqslant 2$, is a *path* in Γ between nodes i_1 and i_r if $\{(i_h, i_{h+1}), (i_{h+1}, i_h)\} \cap \Gamma \neq \emptyset$ for $h = 1,\dots,r-1$, and a *directed path* in Γ from node i_1 to node i_r if $(i_h, i_{h+1}) \in \Gamma$ for $h = 1,\dots,r-1$. A path $(i_1,\dots,i_r)$ in digraph Γ is a *cycle* if $r \geq 3$ and $\{(i_r, i_1), (i_1, i_r)\} \cap \Gamma \neq \emptyset$, and a directed path $(i_1,\dots,i_r)$ in Γ is a *directed cycle* if $(i_r, i_1) \in \Gamma$. Digraph Γ is *cycle-free* if it contains no directed cycles, and Γ is *strongly cycle-free* if it is cycle-free and contains no cycles. Nodes i and j in N are *connected* in Γ if there exists a path in Γ between i and j. Γ is *connected* if any two different nodes in N are connected in Γ. A subset $S \subseteq N$ is *connected* in Γ if the subgraph $\Gamma|_S$ is connected. For $S \subseteq N$, $C^\Gamma(S)$ denotes the collection of subsets of S being connected in Γ, S/Γ is the collection of maximally connected subsets, called *components*, of S in Γ, and $(S/\Gamma)_i$ is the (unique) component of S in Γ containing $i \in S$.

For a cycle-free digraph Γ on N and $i,j \in N$, $\vec{P}^\Gamma(i,j)$ denotes the set of directed paths in Γ from node i to node j. A node i on a (directed) path p we denote as an element of p, i.e., $i \in p$. For a directed path $\vec{p}$ in Γ we write $(i,j) \in \vec{p}$ if i and j are consecutive nodes in $\vec{p}$. For any set P of (directed) paths in Γ, $N(P) = \{i \in p | p \in P\}$. A link $(i,j) \in \Gamma$ is *inessential* if there exists $\vec{p} \in \vec{P}^\Gamma(i,j)$ such that $\vec{p} \neq (i,j)$, otherwise (i,j) is *essential*. A directed path $\vec{p}$ is *proper* if it contains no inessential links. In a strongly cycle-free digraph all links are essential.

For a cycle-free digraph Γ on N and $i,j \in N$, j is a *(proper) successor* of i and i is a *(proper) predecessor* of j if there is a (proper) directed path in Γ from i to j. For an (essential) link $(i,j) \in \Gamma$, i is the *origin* and j is the *terminus*, i is a *(proper) immediate predecessor* of j and j is a *(proper) immediate successor* of i. For $i \in N$, we denote by $P^\Gamma(i)$ $(S^\Gamma(i))$ the set of predecessors (successors) of i in Γ, by $\widehat{P}^\Gamma(i)$ $(\widehat{S}^\Gamma(i))$ the set of immediate predecessors (successors) of i in Γ, and by $\widehat{P}^\Gamma_*(i)$ $\left(\widehat{S}^\Gamma_*(i)\right)$ the set of proper immediate predecessors (successors) of i. For $i \in N$, we define $\overline{P}^\Gamma(i) = P^\Gamma(i) \cup \{i\}$, $\overline{S}^\Gamma(i) = S^\Gamma(i) \cup \{i\}$, and the set $W^\Gamma(i) = S^\Gamma(i) \cup P^\Gamma(i) \cup \{i\}$ as the *web* of node i with i its *hub* and each node $j \in W^\Gamma(i) \backslash \{i\}$ being a *subordinate* of i. For $S \subseteq N$, we define $P^\Gamma(S) = \cup_{i\in S} P^\Gamma(i)$, $S^\Gamma(S) = \cup_{i\in S} S^\Gamma(i)$, $W^\Gamma(S) = \cup_{i\in S} W^\Gamma(i)$, $\overline{P}^\Gamma(S) = P^\Gamma(S) \cup S$, and $\overline{S}^\Gamma(S) = S^\Gamma(S) \cup S$. A coalition $S \subseteq N$ is a *full successors set* (*full predecessors set*) in Γ if $S = \overline{S}^\Gamma(i)$ $(S = \overline{P}^\Gamma(i))$ for some $i \in N$, and S is a *full web set* in Γ if $S = W^\Gamma(i))$ for some $i \in N$. For a node $i \in N$, $d^\Gamma(i) = \left|\widehat{P}^\Gamma_*(i)\right|$ is the *in-degree of* i in Γ and $e^\Gamma(i) = \left|\widehat{S}^\Gamma_*(i)\right|$ is the *out-degree of* i in Γ. Moreover, for $j \in S^\Gamma(i)$, $d^\Gamma_i(j) = \left|\widehat{P}^{\Gamma^i}_*(j)\right|$ is the *in-degree of* j *from* i in Γ, where $\Gamma^i = \Gamma|_{\overline{S}^\Gamma(i)}$, and for $j \in P^\Gamma(i)$, $e^\Gamma_i(j) = \left|\widehat{S}^{\Gamma_i}_*(j)\right|$ is the *out-degree of* j *to* i in Γ, where $\Gamma_i = \Gamma|_{\overline{P}^\Gamma(i)}$.

For a cycle-free digraph Γ on N, a node $i \in N$ having no predecessor (successor) in Γ, i.e., $P^\Gamma(i) = \emptyset$ $(S^\Gamma(i) = \emptyset)$, is a *source* (*sink*) in Γ. For a coalition $S \subseteq N$, $R^\Gamma(S)$ is the set of sources in $\Gamma|_S$ and $L^\Gamma(S)$ is the set of sinks in $\Gamma|_S$. A strongly cycle-free digraph Γ on N is a *(rooted) tree* if it has only one source in Γ, denoted by the *root* $r(\Gamma)$, and Γ is a *sink tree* if it has only one sink in Γ, denoted by $s(\Gamma)$. A (*rooted* or *sink*) *forest* is composed of a finite number of disjoint (rooted or sink) trees. A *line-graph* is a digraph for which each node has at most one immediate successor and at most one immediate predecessor. A subgraph T of a digraph Γ on N is a *subtree* of Γ if T is a tree on $N(T)$. A subtree T of Γ is a *full subtree* of Γ if $N(T) = \overline{S}^\Gamma(r(T))$. A full subtree T of Γ is a *maximal subtree* if $r(T)$ is a source in Γ.

We assume that the cooperation structure on the player set N is specified by a cycle-free directed graph. A pair $\langle v, \Gamma\rangle$ of a TU-game $v \in \mathcal{G}_N$ and a cycle-free directed graph Γ on N constitutes a game with cycle-free digraph communication structure and is called a *cycle-free directed graph game* or *cycle-free digraph game*. The set of all cycle-free digraph games on a fixed player set N is denoted $\mathcal{G}^\Gamma_N$. A *value* on $\mathcal{G}^\Gamma_N$ is a function $\xi : \mathcal{G}^\Gamma_N \to \mathbb{R}^N$ that assigns to every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}^\Gamma_N$ a payoff vector $\xi(v, \Gamma) \in \mathbb{R}^N$. For a game $\langle v, \Gamma\rangle \in \mathcal{G}^\Gamma_N$, a payoff vector $x \in \mathbb{R}^N$ is *component efficient* if for every component $C \in N/\Gamma$ it holds that $x(C) = v(C)$, and x is *component feasible* if for every component $C \in N/\Gamma$ it holds that $x(C) \leqslant v(C)$. A *value* ξ on $\mathcal{G}^\Gamma_N$ satisfies one of these properties on a subset $\mathcal{G}$ of $\mathcal{G}^\Gamma_N$ if for any digraph game $\langle v, \Gamma\rangle \in \mathcal{G}$ it holds that $\xi(v, \Gamma)$ satisfies this property.

3. Web connectedness and management teams

For a directed link in an arbitrary digraph there are two different interpretations possible. One interpretation is that a link is directed to indicate which player has initiated the communication, but at the same time it represents a fully developed communication link. In such a case, following Myerson (1977), it is assumed that cooperation is possible among any set of connected players, i.e., the coalitions in which players are able to cooperate, the *feasible coalitions*, are all the connected coalitions. In this case the focus is on component efficient values, at which all components of the graphs get their worth. Another interpretation of a directed link assumes that a directed link represents the only one-way communication situation. In that case not every connected coalition might be feasible. In this paper we abide by the second interpretation of a directed link and consider different scenarios possible for controlling cooperation and creation of feasible coalitions under the assumption of one-directional communication.

In directed communication structures it is often assumed that management is organized downwards from the top when players are controlled by their predecessors and the main managers are located at the sources of a given digraph, e.g., see Demange (2004) for tree structures and Faigle and Kern (1992) in the case of precedence constraints. However, the opposite direction of management is also possible when main managers are located at the sinks and players are controlled by their successors, see Khmelnitskaya (2010). This, for example, may happen in multistage technological processes when subsequent players determine the amount of production on previous stages that they may handle. In a directed graph each player is in fact a sink for his predecessors and a source for his successors and, therefore, his communication is restricted by these two sets of players with whom he is connected via directed paths, and no communication is possible with other players. In general in a directed graph any player can be chosen as a manager for controlling the situation and he is able to keep control over his full web consisting of all his subordinates. As adjunct manager a successor of a manager is able to control only his own successors set and a predecessor of a manager is able to control only his own predecessors set. The links of the digraph show which sets of players can be controlled by a given management team, not individually, but as coalitions. Talking about control we do not assume the individual control of the players, but we assume that the (local) managers regardless of whether they are sources or not control the cooperation within feasible coalitions of players as is reflected by their worths. For example in case of the river application discussed in Section 7 a manager not being a source may build a dam allowing him to control the total amount of water that he wants to consume for his own purposes plus what he accepts to pass through his territory for extra consumption of downstream users, if they are, and to leave it to the entire coalition of upstream players what to do with the remaining water.

For a coalition of players to create a management team its members cannot be subordinates of each other and together they keep control over the entire society given by N. Therefore, given a cycle-free digraph Γ on N, a coalition $M \subseteq N$ is a *management team* in Γ if

(i) $W^{\Gamma}(M) = N$,
(ii) $i \notin W^{\Gamma}(j)\ \forall i,\ j \in M,\ \ i \neq j$.

For a cycle-free digraph Γ the set of all possible management teams we denote by $\mathcal{M}(\Gamma)$. Notice that a management team is an anti-chain in terms of graph theory. Observe that we prescribe the subordination of players in a given digraph Γ when we choose a management team. It is easy to see that for every player there exists at least one management team containing this player, in particular, some managers might be simply sources or sinks in the digraph. Moreover, there exist two particular management teams – one composed by all sources in the digraph and another one composed by all sinks in the digraph. Besides, as a consequence of condition (ii), we obtain that each management team M in a digraph Γ is minimal since $W^{\Gamma}(M\backslash\{j\}) \neq N$ for any $j \in M$. Furthermore, the set of predecessors $P^{\Gamma}(M)$ and the set of successors $S^{\Gamma}(M)$ of a management team M in Γ are well defined because $P^{\Gamma}(M) \cap S^{\Gamma}(M) = \emptyset$. In fact, $\{P^{\Gamma}(M),\ M,\ S^{\Gamma}(M)\}$ is a partition of the player set N.

For any coalition $S \subseteq N$ to keep the subordination prescribed by a given management team $M \in \mathcal{M}(\Gamma)$ a local management team $M(S) \subseteq S$ in $\Gamma|_S$ needs to consist of the nodes in S that are closest in subordination to the management team M. Besides the managers of M who are already in S, if any, the management team $M(S)$ of S should also contain predecessors (successors) of M who are not in the web of those managers in S and who are either sinks (sources) in $\Gamma|_S$ or whose immediate successors (predecessors) in $\Gamma|_S$ are also successors (predecessors) of the management team M. In this way coalition S inherits the subordination of players induced by M in the sense that for any $i \in S\backslash M(S)$ it holds that $i \in P^{\Gamma|_S}(M(S))$ if $i \in P^{\Gamma}(M)$ and $i \in S^{\Gamma|_S}(M(S))$ if $i \in S^{\Gamma}(M)$. However, when there is a link in $\Gamma|_S$ from one of the predecessors of M to one of the successors of M, then both have equal rights to become a local manager in S but only one can be chosen, i.e., in general $M(S)$ might be not uniquely determined.

To avoid this ambiguity, given a cycle-free digraph Γ on N and a management team $M \in \mathcal{M}(\Gamma)$, we define the *(local) management team* $M(S)$ of a coalition $S \subseteq N$ induced by M as

$$M(S) = M^1(S) \cup M^2(S) \cup M^3(S),$$

where

$$M^1(S) = M \cap S,$$
$$M^2(S) = \{i \in P^{\Gamma}(M) \cap S \mid i \notin W^{\Gamma|_S}(M \cap S) \text{ and } \widehat{S}^{\Gamma|_S}(i) \subseteq S^{\Gamma}(M)\},$$
$$M^3(S) = \{i \in S^{\Gamma}(M) \cap S \mid i \notin W^{\Gamma|_S}(M \cap S) \text{ and } \widehat{P}^{\Gamma|_S}(i) \subseteq P^{\Gamma}(M) \setminus M^2(S)\}.$$

If node $i \in P^{\Gamma}(M) \cap S$ ($i \in S^{\Gamma}(M) \cap S$) and $i \notin W^{\Gamma|_S}(M \cap S)$ is a sink (source) in $\Gamma|_S$, then i has no immediate successors (predecessors) in $\Gamma|_S$, i.e., $\widehat{S}^{\Gamma|_S}(i) = \emptyset$ ($\widehat{P}^{\Gamma|_S}(i) = \emptyset$), and therefore $i \in M^2(S)$ ($i \in M^3(S)$) automatically. When coalition S contains two players i and j with $(i,j) \in \Gamma$ such that $i,j \notin W^{\Gamma|_S}(M \cap S)$, $i \in P^{\Gamma}(M) \cap S$ and $\widehat{S}^{\Gamma|_S}(i) \subseteq S^{\Gamma}(M)$, and $j \in S^{\Gamma}(M) \cap S$ and $\widehat{P}^{\Gamma|_S}(j) \subseteq P^{\Gamma}(M)$, then only one of these players can become a local manager in S. The definition chooses for the predecessor, player i, to become local manager.

When a directed link binding a manager is broken we get the following rule.

Management team development rule: Given a cycle-free digraph Γ on N and management team M in Γ, for an immediate successor $j \in \widehat{S}^{\Gamma}(i)$ of some manager $i \in M$, $M \cup \{j\}$ becomes a management team in $\Gamma\backslash\{(i,\ j)\}$ if $j \notin \widehat{S}^{\Gamma}(h)$ for all $h \in M$, $h \neq i$, and similar, for an immediate predecessor $k \in \widehat{P}^{\Gamma}(i)$ of some $i \in M$, $M \cup \{k\}$ becomes a management team in $\Gamma\backslash\{(k,\ i)\}$ if $k \notin \widehat{P}^{\Gamma}(h)$ for all $h \in M$, $h \neq i$.

Observe that in the first case the adjunct manager j is not necessarily a source in $\Gamma\backslash\{(i,\ j)\}$ because j may have predecessors among players in $P^{\Gamma}(M)$, in particular, j might be a sink in $\Gamma\backslash\{(i,\ j)\}$ (see Example 1). A similar remark concerns the second case when the adjunct manager k is not a sink in $\Gamma\backslash\{(k,\ i)\}$ if k has successors among players in $S^{\Gamma}(M)$.

In real-life situations usually no agent who is (adjunct) manager will accept that one of his subordinates becomes his equal partner if a coalition forms. So, given a cycle-free digraph Γ on N and a management team $M \in \mathcal{M}(\Gamma)$, we assume that the only feasible coalitions are the so-called *M-web connected* coalitions, being the connected coalitions $S \in C^{\Gamma}(N)$ that meet the condition that for every local manager $i \in M(S)$ it holds that $i \notin W^{\Gamma}(j)$ for any other local manager $j \in M(S)$. This means that no local manager can be in the web of another local manager. It guarantees that an

M-web connected coalition inherits the subordination of players prescribed by the management team M in Γ. Obviously, every component $C \in N/\Gamma$ is M-web connected. Also, any full web set in Γ with its hub being a manager in M is M-web connected. An M-web connected coalition is *full M-web connected* if it also contains all subordinates of the local management team. A full M-web connected coalition is the union of one or more full web sets. For a given cycle-free digraph Γ on N, management team $M \in \mathcal{M}(\Gamma)$ and coalition $S \subseteq N$, by $C^{\Gamma}_{M}(S)$ we denote the set of M-web connected subsets of S, by $[S/\Gamma]^M$ the set of maximally M-web connected subsets of S, called the *M-web components of S in* Γ, and by $[S/\Gamma]^M_i$ the M-web component of S containing player $i \in S$.

Example 1. The set of management teams in the cycle-free digraph Γ depicted in Fig. 1 equals to

$$\mathcal{M}(\Gamma) = \{\{1,2\},\{2,3\},\{2,5\},\{3,4,10\},\{4,5,10\},\{6,7\},\{7,9\},\{7,10\},\{8,9\}\}.$$

For management team $M = \{6,7\}$, the local management team in coalition $S = \{3,4,6,8,10\}$ is $M(S) = \{3,6\}$ where $6 \in M^1(S)$ and $3 \in M^2(S)$, and in coalition $S' = \{2,3,4,7,8,9,10\}$ the local management team is $M(S') = \{3,7,9,10\}$ where $7 \in M^1(S')$, $3,10 \in M^2(S')$ and $9 \in M^3(S)$. For management team $M = \{4,5,10\}$ the deletion of link $(5,6)$ does not lead to the change of the management team while in case of management team $M = \{7,9\}$ the deletion of link $(7,8)$ is accompanied by the creation of a new management team $M' = \{7,8,9\}$. In the latter case the adjunct manager 8 is a sink in the digraph $\Gamma\backslash\{(7,8)\}$. For management team $M = \{3,4,10\}$ coalitions $\{5,6,7,8\}$ and $\{6,7,8\}$ are M-web connected, but coalition $S = \{3,6,7,8\}$ is not M-web connected since $M(S) = \{3,6,7\}$ and $6,7 \in W^{\Gamma}(3)$.

For a given cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}^{\Gamma}_N$ the set of triples $\{\langle v,\Gamma,M\rangle\}_{M\in\mathcal{M}(\Gamma)}$ determines the set of different scenarios possible in the TU game v for controlling the cooperation defined by digraph communication structure Γ. In the remaining of this section and in Sections 4 and 5 we assume that for every cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}^{\Gamma}_N$ some management team $M \in \mathcal{M}(\Gamma)$ is a priori fixed. When we consider a particular management team $M \in \mathcal{M}(\Gamma)$, we write $\langle v, \Gamma, M\rangle$ instead of $\langle v, \Gamma\rangle$.

For efficiency of a value we require that every M-web connected coalition composed by one of the managers together with all his subordinates realizes its worth. This gives the first axiom a value must satisfy, called M-web efficiency.

A value ξ on $\mathcal{G}^{\Gamma}_N$ is *M-web efficient* (MWE) if for every cycle-free digraph game $\langle v,\Gamma,M\rangle \in \mathcal{G}^{\Gamma}_N$ it holds that

$$\sum_{j\in W^{\Gamma}(i)} \xi_j(v,\Gamma,M) = v(W^{\Gamma}(i)), \quad \text{for all } i \in M.$$

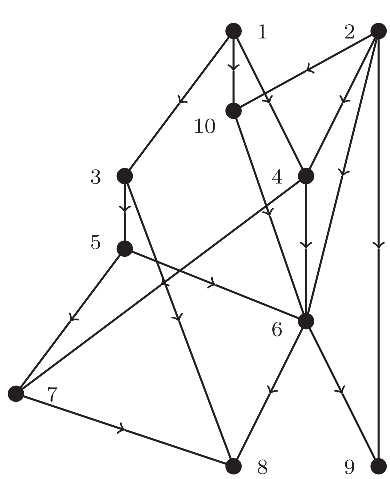

Fig. 1. The digraph in Example 1).

MWE generalizes the usual definition of efficiency for a (rooted/sink) tree. Indeed, in a (rooted) tree when it is assumed that the root is the only manager, M-web efficiency just says that the total payoff should be equal to the worth of the grand coalition N. A similar remark holds true for a sink tree with the sink as only manager. Still, MWE is different from component efficiency. Different from the Myerson (1977) case with undirected communication graph we do not assume that every component is able to realize its exact capacity but only the components having a web structure. For example, if one worker works in two different divisions, the two managers of these divisions and the worker may form a feasible coalition. Yet, it is impossible to guarantee the efficiency of this coalition because there is no communication link between the managers of the two divisions.

The next two axioms reflect the desirable property of stability of the management system – any changes on the upper levels of the management hierarchy should not destroy the stable performance at the lower levels. The first axiom, called M-web successor equivalence, says that if a link with terminus being a successor of the given management team is deleted, then this player and all his successors still receive the same payoff.

A value ξ on $\mathcal{G}^{\Gamma}_N$ is *M-web successor equivalent* (MWSE) if for every cycle-free digraph game $\langle v,\Gamma,M\rangle \in \mathcal{G}^{\Gamma}_N$ it holds that for all $(i,j) \in \Gamma$ such that $i,j \in \bar{S}^{\Gamma}(M)$,

$$\xi_k(v,\Gamma\setminus\{(i,j)\},M) = \xi_k(v,\Gamma,M), \quad \text{for all } k \in \bar{S}^{\Gamma}(j).$$

MWSE means that the payoff to each player in the full successors set of any successor of the given management team does not change if any of the immediate predecessors of that successor breaks his link to him. It implies that for every successors set of a successor of the management team the payoff distribution is completely determined by the players of this set.

The second axiom, called M-web predecessor equivalence, says that if a link with the origin being a predecessor of the given management team is deleted, then this origin and all his predecessors still receive the same payoff.

A value ξ on $\mathcal{G}^{\Gamma}_N$ is *M-web predecessor equivalent* (MWPE) if for every cycle-free digraph game $\langle v,\Gamma,M\rangle \in \mathcal{G}^{\Gamma}_N$ it holds that for all $(i,j) \in \Gamma$ such that $i,j \in \bar{P}^{\Gamma}(M)$,

$$\xi_k(v,\Gamma\setminus\{(i,j)\},M) = \xi_k(v,\Gamma,M), \quad \text{for all } k \in \bar{P}^{\Gamma}(i).$$

MWPE means that the payoff to each player in the full predecessors set of any predecessor of the given management team does not change if any of the immediate successors of that predecessor breaks his link from him. It implies that for every predecessors set of a predecessor of the management team the payoff distribution is fully determined by the players of this set.

Along with MWE we consider also two other efficiency properties requiring that the full sets of subordinates of a player, not only of a manager, are also able to realize their full capacity. M-web full-tree efficiency and M-web full-sink efficiency require correspondingly that every full successors set within the set of successors of a given management team and every full predecessors set within the set of predecessors of a given management team realize their worths.

A value ξ on $\mathcal{G}^{\Gamma}_N$ is *M-web full-tree efficient* (MWFTE) if for every cycle-free digraph game $\langle v,\Gamma,M\rangle \in \mathcal{G}^{\Gamma}_N$ it holds that

$$\sum_{j\in\bar{S}^{\Gamma}(i)} \xi_j(v,\Gamma,M) = v(\bar{S}^{\Gamma}(i)), \quad \text{for all } i \in S^{\Gamma}(M).$$

A value ξ on $\mathcal{G}^{\Gamma}_N$ is *M-web full-sink efficient* (MWFSE) if for every cycle-free digraph game $\langle v,\Gamma,M\rangle \in \mathcal{G}^{\Gamma}_N$ it holds that

$$\sum_{j\in\bar{P}^{\Gamma}(i)} \xi_j(v,\Gamma,M) = v(\bar{P}^{\Gamma}(i)), \quad \text{for all } i \in P^{\Gamma}(M).$$

4. The tree value

In this section we consider the situation when a management team in a digraph is composed by the set of all sources of the graph.

4.1. Axiomatic definition

For a management team that consists of all sources of a given cycle-free digraph M-web connectedness can be restated in terms of tree-connectedness. For a cycle-free digraph Γ on N a connected coalition $S \in C^{\Gamma}(N)$ is *tree-connected*, or simply *t-connected*, if it meets the condition that for every source $i \in R^{\Gamma}(S)$ it holds that $i \notin S^{\Gamma}(j)$ for every other source $j \in R^{\Gamma}(S)$. A t-connected coalition is *full t-connected*, if it contains all successors of its sources.

In what follows, for a cycle-free digraph Γ on N and a coalition $S \subseteq N$, let $C^{\Gamma}_t(S)$ denote the set of t-connected subsets of S, $[S/\Gamma]^t$ the set of maximally t-connected subsets of S, called the *t-connected components of S*, and $[S/\Gamma]^t_i$ the t-connected component of S containing $i \in S$.

In case the management team M in the digraph is the set of sources, M-web efficiency reduces to maximal-tree efficiency, M-web successor equivalence to successor equivalence, and M-web full-tree efficiency to full-tree efficiency, being stronger than maximal-tree efficiency, while M-web predecessor equivalence and M-web full-sink efficiency become redundant. Moreover, $M(S) = R^{\Gamma}(S)$ for all $S \subseteq N$.

A value ξ on $\mathcal{G}^{\Gamma}_N$ is *maximal-tree efficient* (MTE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}^{\Gamma}_N$ it holds that

$$\sum_{j\in\overline{S}^{\Gamma}(i)} \xi_j(v,\Gamma) = v(\overline{S}^{\Gamma}(i)), \quad \text{for all } i \in R^{\Gamma}(N).$$

A value ξ on $\mathcal{G}^{\Gamma}_N$ is *successor equivalent* (SE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}^{\Gamma}_N$ it holds that for all $(i,j) \in \Gamma$

$$\xi_k(v, \Gamma \setminus \{(i,j)\}) = \xi_k(v,\Gamma), \quad \text{for all } k \in \overline{S}^{\Gamma}(j).$$

A value ξ on $\mathcal{G}^{\Gamma}_N$ is *full-tree efficient* (FTE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}^{\Gamma}_N$ it holds that

$$\sum_{j\in\overline{S}^{\Gamma}(i)} \xi_j(v,\Gamma) = v(\overline{S}^{\Gamma}(i)), \quad \text{for all } i \in N. \tag{1}$$

Proposition 1. *On the class of cycle-free digraph games $\mathcal{G}^{\Gamma}_N$, MTE and SE imply FTE.*

Proof. Let ξ be a value on $\mathcal{G}^{\Gamma}_N$ that meets MTE and SE, and let a cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}^{\Gamma}_N$ be arbitrarily chosen. For every given $i \in N$, the subgraph Γ^i is a maximal tree in the subgraph $\Gamma' = \Gamma \setminus \{(k,i) | k \in \widehat{P}^{\Gamma}(i)\}$. Since $\overline{S}^{\Gamma'}(i) = \overline{S}^{\Gamma}(i)$, $i \in R^{\Gamma'}(N)$, and due to MTE,

$$\sum_{j\in\overline{S}^{\Gamma}(i)} \xi_j(v, \Gamma \setminus \{(k,i)|k \in \widehat{P}^{\Gamma}(i)\}) \overset{\text{MTE}}{=} v(\overline{S}^{\Gamma}(i)).$$

By successive application of SE,

$$\xi_j(v, \Gamma \setminus \{(k,i)|k \in \widehat{P}^{\Gamma}(i)\}) \overset{\text{SE}}{=} \xi_j(v,\Gamma), \quad \text{for all } j \in \overline{S}^{\Gamma}(i).$$

Whence,

$$\sum_{j\in\overline{S}^{\Gamma}(i)} \xi_j(v,\Gamma) = v(\overline{S}^{\Gamma}(i)), \quad \text{for all } i \in N,$$

i.e., the value ξ meets FTE. □

Given a cycle-free digraph Γ on N, for $i \in N$ and $j \in S^{\Gamma}(i)$ we define the integer κ^{Γ}_{ij} by

$$\kappa^{\Gamma}_{ij} = \sum_{r=0}^{n-2} (-1)^r \kappa^{\Gamma,r}_{ij}, \tag{2}$$

where, for $r = 0, 1, \ldots, n-2$, $\kappa^{\Gamma,r}_{ij}$ is the number of tuples $(i_0, \ldots, i_{r+1})$ such that $i_0 = i$, $i_{r+1} = j$, $i_h \in S^{\Gamma}(i_{h-1})$, $h = 1, \ldots, r+1$. Since all nodes forming a tuple $(i_0, \ldots, i_{r+1})$ in which $i_0 = i$, $i_{r+1} = j$, $i_h \in S^{\Gamma}(i_{h-1})$, $h = 1, \ldots, r+1$, belong to some directed path $\vec{p}$ in $\vec{P}^{\Gamma}(i,j)$, any κ^{Γ}_{ij} is defined only via tuples of nodes from the set $N(\vec{P}^{\Gamma}(i,j))$.

It turns out that MTE and SE uniquely define a value on the class of cycle-free digraph games.

Theorem 1. *On the class of cycle-free digraph games $\mathcal{G}^{\Gamma}_N$ there is a unique value t that satisfies MTE and SE. For every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}^{\Gamma}_N$, the value $t(v, \Gamma)$ possesses the following properties:*

(i) *it obeys the recursive equality*

$$t_i(v,\Gamma) = v(\overline{S}^{\Gamma}(i)) - \sum_{j\in S^{\Gamma}(i)} t_j(v,\Gamma), \quad \textit{for all } i \in N; \tag{3}$$

(ii) *it admits the explicit representation in the form*

$$t_i(v,\Gamma) = v(\overline{S}^{\Gamma}(i)) - \sum_{j\in S^{\Gamma}(i)} \kappa^{\Gamma}_{ij}\, v(\overline{S}^{\Gamma}(j)), \quad \textit{for all } i \in N. \tag{4}$$

Proof. Due to Proposition 1 the value t on $\mathcal{G}^{\Gamma}_N$ that satisfies MTE and SE meets FTE as well, wherefrom the recursive equality (3) follows straightforwardly. Next, we show that the representation in the form (3) is equivalent to the representation in the form (4). According to (3) it holds for the value t that every player receives what this player together with his successors can get on their own, their worth, minus what all his successors will receive by themselves. Since the same property holds for these successors as well, it is not difficult to see that (4) follows directly from (3) by successive substitution. Indeed, for any $\langle v, \Gamma\rangle \in \mathcal{G}^{\Gamma}_N$ and $i \in N$ it holds that

$$\begin{aligned}
t_i(v,\Gamma) = {} & v(\overline{S}^{\Gamma}(i)) - \sum_{j\in S^{\Gamma}(i)} t_j(v,\Gamma) \overset{(3)}{=} \\
& v(\overline{S}^{\Gamma}(i)) - \sum_{j\in S^{\Gamma}(i)} v(\overline{S}^{\Gamma}(j)) + \sum_{j\in S^{\Gamma}(i)} \sum_{k\in S^{\Gamma}(j)} t_k(v,\Gamma) \overset{(3)}{=} \\
& v(\overline{S}^{\Gamma}(i)) - \sum_{j\in S^{\Gamma}(i)} v(\overline{S}^{\Gamma}(j)) + \sum_{j\in S^{\Gamma}(i)} \sum_{k\in S^{\Gamma}(j)} v(\overline{S}^{\Gamma}(k)) \\
& - \sum_{j\in S^{\Gamma}(i)} \sum_{k\in S^{\Gamma}(j)} \sum_{h\in S^{\Gamma}(k)} t_h(v,\Gamma) \overset{(3)}{=} \ldots = \\
& v(\overline{S}^{\Gamma}(i)) - \sum_{j\in S^{\Gamma}(i)} \sum_{r=0}^{n-2} (-1)^r \kappa^{\Gamma,r}_{ij}\, v(\overline{S}^{\Gamma}(j)) = \\
& v(\overline{S}^{\Gamma}(i)) - \sum_{j\in S^{\Gamma}(i)} \kappa^{\Gamma}_{ij}\, v(\overline{S}^{\Gamma}(j)).
\end{aligned}$$

From (4), we obtain immediately that the value t meets SE, because in any digraph Γ for all $(i,j) \in \Gamma$ and $k \in \overline{S}^{\Gamma}(j)$ the full subtrees Γ^k and $(\Gamma \setminus \{(i,j)\})^k$ coincide. This completes the proof, since MTE follows from FTE automatically. □

According to the recursive formula (3), in a cycle-free digraph game the value t assigns to every player the worth of his full successors set minus the total payoff to his successors. This implies that every player receives as payoff what he contributes when he joins his successors in the digraph. In particular, every player who is a sink receives as payoff just his own worth, every player who has only sinks as successors receives as payoff the worth of him together with his succeeding sinks minus what the sinks already receive, and so on.

Corollary 1. *There exists a simple recursive algorithm for computing the value t going upstream from the sinks of the given digraph.*

The computation of the coefficients κ^{Γ}_{ij}, $i \in N$, $j \in S^{\Gamma}(i)$, defined by (2) in the explicit formula representation (4) requires, in general, the enumeration of quite a lot of possibilities. We show below that in many cases the coefficients κ^{Γ}_{ij} can be more easily computed and the value t can be presented in a computationally more transparent and simpler form. For $i \in N$, $j \in S^{\Gamma}(i)$ and $S \subseteq N(\vec{P}^{\Gamma}(i,j))$ containing nodes i and j, define

$$\kappa_{ij}^{\Gamma}(S) = \sum_{r=0}^{n-2} (-1)^r \kappa_{ij}^{\Gamma,r}(S), \tag{5}$$

where, for $r = 0, 1, \ldots, n-2$, $\kappa_{ij}^{\Gamma,r}(S)$ counts all tuples $(i_0, \ldots, i_{r+1})$ for which $i_0 = i$, $i_{r+1} = j$, and $i_h \in S^{\Gamma}(i_{h-1}) \cap S$, $h = 1, \ldots, r+1$. Remark that $\kappa_{ij}^{\Gamma} = \kappa_{ij}^{\Gamma}(N(\vec{P}^{\Gamma}(i,j)))$ for all $j \in S^{\Gamma}(i)$, $i \in N$. For any cycle-free digraph Γ on N, $i \in N$ and $j \in S^{\Gamma}(i)$, the set $\vec{P}^{\Gamma}(i,j)$ of directed paths in Γ from i to j can be partitioned into a number of separate subsets of paths of two types, possibly only one subset of one of the types, or some of the subsets containing only one path, such that paths from different subsets do not intersect between i and j, in subsets of the first type all paths belonging to the same subset have at least one common node different from i and j, and for the paths in each subset of the second type it holds that every path intersects at least one of the other paths between i and j but all of them together have no other nodes in common than i and j. More exactly, given a cycle-free digraph Γ on N, for all $i \in N$ and $j \in S^{\Gamma}(i)$ there exist two integers $0 \leqslant \tilde{q}_{ij}^{\Gamma} \leqslant q_{ij}^{\Gamma}$ and a partition of $\vec{P}^{\Gamma}(i,j)$ into sets

$$\vec{P}_1(i,j), \ldots, \vec{P}_{\tilde{q}_{ij}^{\Gamma}}(i,j), \vec{P}_{\tilde{q}_{ij}^{\Gamma}+1}(i,j), \ldots, \vec{P}_{q_{ij}^{\Gamma}}(i,j) \tag{6}$$

satisfying

(i) $\vec{p}_1 \cap \vec{p}_2 = \{i,j\}$ for all $\vec{p}_1 \in \vec{P}_h(i,j)$, $\vec{p}_2 \in \vec{P}_l(i,j)$, $h, l = 1, \ldots, q_{ij}^{\Gamma}$, $h \neq l$;
(ii) $(\bigcap_{\vec{p} \in \vec{P}_h(i,j)} \vec{p}) \setminus \{i,j\} \neq \emptyset$ for all $h = 1, \ldots, \tilde{q}_{ij}^{\Gamma}$;
(iii) $\bigcap_{\vec{p} \in \vec{P}_h(i,j)} \vec{p} = \{i,j\}$ and $(\vec{p}_0 \cap (\bigcup_{\vec{p} \in \vec{P}_h(i,j) \setminus \{\vec{p}_0\}} \vec{p})) \setminus \{i,j\} \neq \emptyset$ for all $\vec{p}_0 \in \vec{P}_h(i,j)$, $h = \tilde{q}_{ij}^{\Gamma} + 1, \ldots, q_{ij}^{\Gamma}$.

Example 2. The set of paths from i to j depicted in Fig. 2 is composed by three subsets of paths, two of the first type and one of the second type.

Given a digraph Γ on N and a set of paths $\vec{P} \subseteq \vec{P}^{\Gamma}(i,j)$, $i \in N$, $j \in S^{\Gamma}(i)$, we may consider the subgraph $\Gamma|_{\vec{P}}$ on $N(\vec{P})$ induced by the paths in $\vec{P}$, i.e., $\Gamma|_{\vec{P}} = \{(h,l) \in \vec{p} \mid \vec{p} \in \vec{P}\}$. A node $h \in N(\vec{P})$ which has at least two proper immediate predecessors or at least two proper immediate successors in $\Gamma|_{\vec{P}}$, i.e., if $|\widehat{P}_*^{\Gamma|_{\vec{P}}}(h)| \cdot |\widehat{S}_*^{\Gamma|_{\vec{P}}}(h)| > 1$, is called a *proper intersection point* in $N(\vec{P})$. At a proper intersection point in $N(\vec{P})$ two or more different paths in $\vec{P}$ join, split, or cross each other. As shown below in Lemma 1 only these proper intersection points and the proper immediate successors of i which are also predecessors of j are needed in the computation of κ_{ij}^{Γ}. The subset of $N(\vec{P})$ composed by i, j, all proper immediate successors $h \in \widehat{S}_*^{\Gamma|_{\vec{P}}}(i)$ of i in $\Gamma|_{\vec{P}}$ and all proper intersection points in $N(\vec{P})$ defines the *upper covering set* $\overline{C}^{\Gamma}(\vec{P})$ for $\vec{P}$, and the subset of $N(\vec{P})$ composed by i, j, all proper immediate predecessors $h \in \widehat{P}_*^{\Gamma|_{\vec{P}}}(j)$ of j in $\Gamma|_{\vec{P}}$ and all proper intersection points in $N(\vec{P})$ defines the *lower covering set* $\underline{C}^{\Gamma}(\vec{P})$ for $\vec{P}$.

Theorem 2. *For every cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^{\Gamma}$ the value t given by (4) admits the equivalent representation in the form*

$$t_i(v,\Gamma) = v(\overline{S}^{\Gamma}(i)) - \sum_{j \in \widehat{S}_*^{\Gamma}(i)} v(\overline{S}^{\Gamma}(j)) + \sum_{\substack{j \in S^{\Gamma}(i) \\ d_i^{\Gamma}(j) > 1}} \left(q_{ij}^{\Gamma} - 1 - \sum_{h=\tilde{q}_{ij}^{\Gamma}+1}^{q_{ij}^{\Gamma}} \kappa_{ij}^{\Gamma}(\overline{C}^{\Gamma}(\vec{P}_h(i,j))) \right) v(\overline{S}^{\Gamma}(j)), \quad \text{for all } i \in N, \tag{7}$$

where, for all $i \in N$ and $j \in S^{\Gamma}(i)$, $\vec{P}_h(i,j)$, $h = 1, \ldots, q_{ij}^{\Gamma}$, form the partition of $\vec{P}^{\Gamma}(i,j)$ in (6).

If $\langle v, \Gamma \rangle$ is a strongly cycle-free digraph game, then the above representation reduces to

$$t_i(v,\Gamma) = v(\overline{S}^{\Gamma}(i)) - \sum_{j \in \widehat{S}^{\Gamma}(i)} v(\overline{S}^{\Gamma}(j)), \quad \text{for all } i \in N. \tag{8}$$

For rooted forest digraph games defined by rooted forest digraph structures, which are strongly cycle-free, the value given by (8) coincides with the tree value introduced first under the name of hierarchical outcome in Demange (2004), where it is also shown that under the mild condition of superadditivity it belongs to the core of the restricted game as defined in Myerson (1977). More recently, the tree value for rooted forest games was used as a basic element in the construction of the average tree solution for cycle-free undirected graph games in Herings et al. (2008). In Khmelnitskaya (2010) it is shown that on the class of rooted forest digraph games the tree value can be characterized via component efficiency and successor equivalence; moreover, it is shown that the class of rooted forest digraph games is the maximal subclass in the class of strongly cycle-free digraph games where this axiomatization holds true. Recall that the subgraph of any component in a forest digraph is a rooted tree. Hence, on the class of rooted forest digraph games maximal-tree efficiency coincides with component efficiency.

From now on we refer to the value t for cycle-free digraph games given by (3), or equivalently by (4) or (7) and for strongly cycle-free digraph games by (8), as the *root tree value*, or simply the *tree value*.

The validity of the first statement of Theorem 2 follows from Theorem 1 and Lemma 1 below and Corollary 2 to it. The second statement follows easily from the first one. Indeed, in any strongly cycle-free digraph Γ all links are essential, whence $\widehat{S}_*^{\Gamma}(i) = \widehat{S}^{\Gamma}(i)$, and $d_i^{\Gamma}(j) = 1$ for all $i \in N$ and $j \in S^{\Gamma}(i)$.

Lemma 1. *For any cycle-free digraph Γ on N, the coefficients κ_{ij}^{Γ}, $i \in N$, $j \in S^{\Gamma}(i)$, defined by (2) possess the following properties:*

(i) *if a link $(k, l) \in \Gamma$ is inessential, then $\kappa_{ij}^{\Gamma} = \kappa_{ij}^{\Gamma'}$ for all $i \in N$ and $j \in S^{\Gamma}(i)$, where $\Gamma' = \Gamma \setminus \{(k, l)\}$;*
(ii) *$\kappa_{ij}^{\Gamma} = 1$ for all $i \in N$ and $j \in \widehat{S}_*^{\Gamma}(i)$;*
(iii) *$\kappa_{ij}^{\Gamma} = -q_{ij}^{\Gamma} + 1 + \sum_{h=\tilde{q}_{ij}^{\Gamma}+1}^{q_{ij}^{\Gamma}} \kappa_{ij}^{\Gamma}(\overline{C}^{\Gamma}(\vec{P}_h(i,j)))$ for all $i \in N$ and $j \in S^{\Gamma}(i) \setminus \widehat{S}_*^{\Gamma}(i)$;*
(iv) *$\sum_{h \in N(\vec{P}^{\Gamma}(i,j)) \setminus \{j\}} \kappa_{hj}^{\Gamma} = 1$ and $\sum_{h \in N(\vec{P}^{\Gamma}(i,j)) \setminus \{i\}} \kappa_{ih}^{\Gamma} = 1$ for all $i \in N$ and $j \in S^{\Gamma}(i)$.*

Proof.

(i). It is sufficient to prove the statement only in case when $k \in S^{\Gamma}(i)$ and $j \in S^{\Gamma}(l)$. Let $\vec{p} \in \vec{P}^{\Gamma}(i,j)$ be such that $\vec{p} \ni (k,l)$. By definition of an inessential link there exists $\vec{p}_0 \in \vec{P}^{\Gamma}(k,l)$ such that $\vec{p}_0 \neq (k,l)$. It is not difficult to see that the path $\vec{p}_1 = \vec{p} \setminus \{(k,l)\} \cup \vec{p}_0$ obtained from the path $\vec{p}$ by replacing the link (k, l) by the path $\vec{p}_0$ belongs to $\vec{P}^{\Gamma}(i,j)$, and moreover, all tuples $(i_0, \ldots, i_{r+1})$ in the definition of κ_{ij}^{Γ} that belong to $\vec{p}$ also belong to $\vec{p}_1$. Whence deleting an inessential link does not change the value of κ_{ij}^{Γ}.

In the remaining of the proof without loss of generality we may assume that $\vec{P}^{\Gamma}(i,j)$ consists of proper paths.

(ii). If $j \in \widehat{S}_*^{\Gamma}(i)$ for some $i \in N$, then $\vec{P}^{\Gamma}(i,j)$ contains only the path $\vec{p} = (i,j)$. Wherefrom it follows that $\kappa_{ij}^{\Gamma} = 1$.

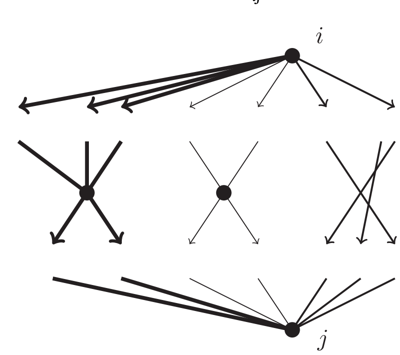

Fig. 2. The set of paths in Example 2.

(iii). Take $i \in N$ and $j \in S^\Gamma(i) \setminus \widehat{S}^\Gamma_*(i)$ and let $\vec{P}_h(i,j)$, $h = 1, \ldots, q^\Gamma_{ij}$, form the partition of $\vec{P}^\Gamma(i,j)$ in (6). Then

$$\kappa^\Gamma_{ij} = \kappa^\Gamma_{ij}(N(\vec{P}_1(i,j))) + \left[\kappa^\Gamma_{ij}(N(\vec{P}_2(i,j))) - \kappa^\Gamma_{ij}(N(\vec{P}_1(i,j) \cap \vec{P}_2(i,j)))\right] + \cdots + \left[\kappa^\Gamma_{ij}\left(N\left(\vec{P}_{q^\Gamma_{ij}}(i,j)\right)\right) - \kappa^\Gamma_{ij}\left(N\left(\bigcap_{h=1}^{q^\Gamma_{ij}} \vec{P}_h(i,j)\right)\right)\right].$$

Since the paths from different $\vec{P}_h(i,j)$ do not intersect between i and j,

$$\kappa^\Gamma_{ij}\left(N\left(\bigcap_{h=1}^{k} \vec{P}_h(i,j)\right)\right) = 1, \quad \text{for } k = 2, \ldots, q^\Gamma_{ij}.$$

Whence it follows that

$$\kappa^\Gamma_{ij} = -q^\Gamma_{ij} + 1 + \sum_{h=1}^{q^\Gamma_{ij}} \kappa^\Gamma_{ij}(N(\vec{P}_h(i,j))).$$

First, consider $h \in \left\{1, \ldots, \tilde{q}^\Gamma_{ij}\right\}$, then there exists $k \in N(\vec{P}_h(i,j))$, $k \neq i,j$, such that $k \in \vec{p}$ for all $\vec{p} \in \vec{P}_h(i,j)$. By definition, $\kappa^{\Gamma,r}_{ij}(N(\vec{P}_h(i,j)))$ is equal to the number of tuples $(i_0, \ldots, i_{r+1})$ such that $i_0 = i$, $i_{r+1} = j$, $i_l \in S^\Gamma(i_{l-1}) \cap N(\vec{P}_h(i,j))$, $l = 1, \ldots, r+1$, or equivalently, $\kappa^{\Gamma,r}_{ij}$ is equal to the number of these tuples $(i_0, \ldots, i_{r+1})$ that do not contain k plus the number of these tuples $(i_0, \ldots, i_{r+1})$ that contain k. Since $k \in \vec{p}$ for all $\vec{p} \in \vec{P}_h(i,j)$, for every $(r+2)$-tuple $(i_0, \ldots, i_{r+1})$ that does not contain k there exists a uniquely defined $(r+3)$-tuple composed by the same nodes plus node k. Wherefrom together with equality (5) it follows that $\kappa^\Gamma_{ij}(N(\vec{P}_h(i,j))) = 0$.

Next, consider $h \in \left\{\tilde{q}^\Gamma_{ij} + 1, \ldots, q^\Gamma_{ij}\right\}$. We show that $\kappa^\Gamma_{ij}(N(\vec{P}_h(i,j))) = \kappa^\Gamma_{ij}(\overline{C}^\Gamma(\vec{P}_h(i,j)))$. Take any $k \in N(\vec{P}_h(i,j)) \setminus \overline{C}^\Gamma(\vec{P}_h(i,j))$. Then

$$\kappa^\Gamma_{ij}(N(\vec{P}_h(i,j))) = \kappa^\Gamma_{ij}(N(\vec{P}_h(i,j)); k) + \kappa^\Gamma_{ij}(N(\vec{P}_h(i,j)) \setminus \{k\}),$$

where $\kappa^\Gamma_{ij}(N(\vec{P}_h(i,j)); k)$ counts all tuples in $N(\vec{P}_h(i,j))$ containing k. By definition of upper covering set, $\overline{C}^\Gamma(\vec{P}_h(i,j))$ contains some predecessor of k, i.e., $\overline{C}^\Gamma(\vec{P}_h(i,j)) \cap P^\Gamma(k) \neq \emptyset$. Moreover, since $k \notin \overline{C}^\Gamma(\vec{P}_h(i,j))$, i.e., k is neither a proper immediate successor of i nor a proper intersection point in the subgraph $\Gamma|_{N(\vec{P}_h(i,j))}$, there exists $l \in \overline{C}^\Gamma(\vec{P}_h(i,j)) \cap P^\Gamma(k)$ that belongs to all paths in $\vec{P}_h(i,j)$ containing k. Applying the same argument as before to $\vec{P}_l(i,j)$ of the first type, we obtain that $\kappa^\Gamma_{ij}(N(\vec{P}_h(i,j)); k) = 0$. Thus $\kappa^\Gamma_{ij}(N(\vec{P}_h(i,j))) = \kappa^\Gamma_{ij}(N(\vec{P}_h(i,j)) \setminus \{k\})$. Repeating the same reasoning successively with respect to all $k' \in N(\vec{P}_h(i,j)) \setminus (\overline{C}^\Gamma(\vec{P}_h(i,j)) \cup \{k\})$ we obtain $\kappa^\Gamma_{ij}(N(\vec{P}_h(i,j))) = \kappa^\Gamma_{ij}(\overline{C}^\Gamma(\vec{P}_h(i,j)))$.

(iv). Take any $i \in N$ and $j \in S^\Gamma(i)$. By definition, $\kappa^{\Gamma,0}_{ij} = 1$ and, for $r \geqslant 1$, $\kappa^{\Gamma,r}_{ij} = \sum_{h \in S^\Gamma(i) \cap P^\Gamma(j)} \kappa^{\Gamma,r-1}_{hj} = \sum_{h \in S^\Gamma(i) \cap P^\Gamma(j)} \kappa^{\Gamma,r-1}_{ih}$. Hence, $\kappa^\Gamma_{ij} = 1 - \sum_{h \in S^\Gamma(i) \cap P^\Gamma(j)} \kappa^\Gamma_{hj}$ and $\kappa^\Gamma_{ij} = 1 - \sum_{h \in S^\Gamma(i) \cap P^\Gamma(j)} \kappa^\Gamma_{ih}$. Since $S^\Gamma(i) \cap P^\Gamma(j) = N(\vec{P}^\Gamma(i,j)) \setminus \{i,j\}$, this implies (iv). $\square$

Remark that the system of equations in (iv) also uniquely determines the coefficients κ^Γ_{ij}, $i \in N$, $j \in S^\Gamma(i)$. From case (iii) of Lemma 1 we obtain the next corollary.

Corollary 2. *For a cycle-free digraph Γ it holds that $\kappa^\Gamma_{ij} = 0$ for all $i \in N$ and $j \in S^\Gamma(i) \setminus \widehat{S}^\Gamma_*(i)$ for which $q^\Gamma_{ij} = \tilde{q}^\Gamma_{ij} = 1$. In particular, $\kappa^\Gamma_{ij} = 0$ for all $i \in N$ and $j \in S^\Gamma(i) \setminus \widehat{S}^\Gamma_*(i)$ with $d^\Gamma_i(j) = 1$.*

The second statement holds because for all $j \in S^\Gamma(i) \setminus \widehat{S}^\Gamma_*(i)$ with $d^\Gamma_i(j) = 1$ there is a unique proper immediate predecessor of j that belongs to all paths in $\vec{P}^\Gamma(i,j)$.

Example 3. Fig. 3 illustrates the situation when $j \in S^\Gamma(i) \setminus \widehat{S}^\Gamma_*(i)$ with $d^\Gamma_i(j) = 1$.

Example 4. The examples of digraphs depicted in Fig. 4 demonstrate the situation when for some $i \in N$ and $j \in S^\Gamma(i)$ the paths in $\vec{P}^\Gamma(i,j)$ constitute one subset of the second type, i.e., paths in $\vec{P}^\Gamma(i,j)$ do intersect but have no other nodes in common than i and j.

For the digraph depicted in Fig. 4(a) it holds that $d^\Gamma_1(7) = 2$ and $\kappa^\Gamma_{17} = 0$, for the one in Fig. 4(b) $d^\Gamma_1(6) = 2$ and $\kappa^\Gamma_{16} = 1$, and for the one in Fig. 4(c) $d^\Gamma_1(8) = 2$ and $\kappa^\Gamma_{18} = -1$.

According to formula (4), or equivalently (7), the tree value assigns to a player the worth of his full successors set minus appropriate positive or negative multiples of the worths of the full successors sets of all his successors such that, in order to correct for overlapping successor sets, as stated in (iv) of Lemma 1, for each successor the sum of the positive and negative multiples, with which the worths of the full successors sets he belongs to are multiplied, is equal to 1. It is worth to note that from this it follows that the right hand side of formula (4), being considered with respect not to coalitional worths but to players in these coalitions, contains only player i when counting the total of all multiple pluses and minuses.

A value ξ on $\mathcal{G}^\Gamma_N$ is *independent of inessential links* if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}^\Gamma_N$ and cycle-free digraph game $\langle v, \Gamma'\rangle \in \mathcal{G}^\Gamma_N$ with Γ' being the subgraph Γ' of Γ composed by all essential links of Γ it holds that $\xi(v, \Gamma) = \xi(v, \Gamma')$.

Corollary 3. *The tree value satisfies independence of inessential links.*

From Theorem 2 the tree value is determined only by the coalitions having the full successors set structure. Deletion of inessential links does not change this set of coalitions. All other coalitions, in particular any t-connected coalition composed of nodes connected by inessential links, even if their worths are very high, are irrelevant. A similar situation occurs in the commonly accepted Myerson (1977) undirected graph game model where every disconnected coalition, even being of very high worth, is irrelevant. The property of independence of inessential links in fact reflects the rigidity of the entire management system in a sense of importance of all lower level managers since an attempt of a higher level manager to control any one of his not immediate subordinates directly, which is represented as an inessential link, does not change the total distribution of payoffs.

Example 5. Fig. 5 provides an example of the tree value for a 10-person game with cycle-free but not strongly cycle-free digraph structure as depicted in Fig. 1. If there is no confusion, a set $\{i_1, \ldots, i_k\}$ is denoted by $i_1 \cdots i_k$.

The tree value can be computed by using the recursive formula (3) or the explicit representation (7). We explain in detail the computation of $t_1(v, \Gamma)$ based on the explicit formula (7):

$\widehat{S}^\Gamma_*(1) = \{3, 4, 10\} \Rightarrow \kappa^\Gamma_{13} = \kappa^\Gamma_{14} = \kappa^\Gamma_{1,10} = 1$;
$S^\Gamma(1) \setminus \widehat{S}^\Gamma_*(1) = \{5, 6, 7, 8, 9\}$:
$d^\Gamma_1(5) = d^\Gamma_1(9) = 1 \Rightarrow \kappa^\Gamma_{15} = \kappa^\Gamma_{19} = 0$;
$\vec{P}^\Gamma(1,6) = \{(1,3,5,6), (1,4,6), (1,10,6)\}$, no intersections $\Rightarrow q^\Gamma_{16} = \tilde{q}^\Gamma_{16} = 3 \Rightarrow \kappa^\Gamma_{16} = -2$;
$\vec{P}^\Gamma(1,7) = \{(1,3,5,7), (1,4,7)\}$, no intersections $\Rightarrow q^\Gamma_{17} = \tilde{q}^\Gamma_{17} = 2 \Rightarrow \kappa^\Gamma_{17} = -1$;
$\vec{P}^\Gamma(1,8)$ is composed by $\vec{p}_1 = (1,3,5,7,8), \vec{p}_2 = (1,3,5,6,8)$, $\vec{p}_3 = (1,10,6,8)$, $\vec{p}_4 = (1,4,7,8)$,
$\vec{p}_5 = (1,4,6,8), \vec{p}_6 = (1,3,8)$; eliminate path $\vec{p}_6$ containing inessential link *(3,8)*;
paths $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$, $\vec{p}_4$ and $\vec{p}_5$ form one subset of the second type $\Rightarrow q^\Gamma_{18} = 1$, $\tilde{q}^\Gamma_{18} = 0$;

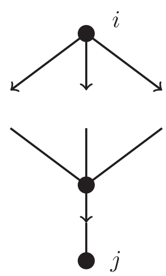

Fig. 3. Illustration of Example 3.

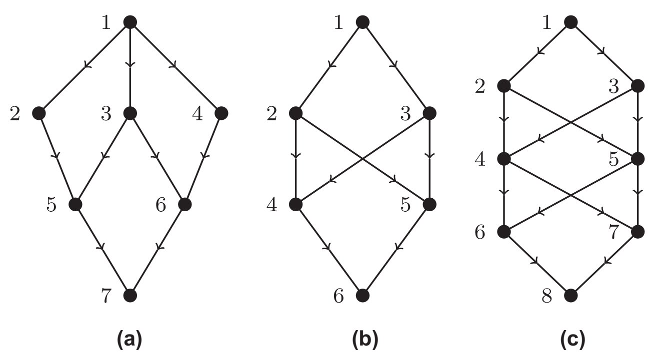

Fig. 4. The digraphs in Example 4.

$\overline{C}^\Gamma(\vec{P}^\Gamma(1,8)) = \{1,4,5,6,7,8,10\}$;
$\kappa^\Gamma_{18}(\vec{p}_1) = 0$;
$\vec{p}_2 \setminus \vec{p}_1$ contains tuples (1,6,8) and $(1,5,6,8) \Rightarrow \kappa^\Gamma_{18}(\vec{p}_2 \setminus \vec{p}_1) = 0$;
$\vec{p}_3 \setminus (\vec{p}_1 \cup \vec{p}_2)$ contains tuples $(1,10,8)$, $(1,10,6,8) \Rightarrow \kappa^\Gamma_{18}(\vec{p}_3 \setminus (\vec{p}_1 \cup \vec{p}_2)) = 0$;
$\vec{p}_4 \setminus (\vec{p}_1 \cup \vec{p}_2 \cup \vec{p}_3)$ contains $(1,4,8)$, $(1,4,7,8) \Rightarrow \kappa^\Gamma_{18}(\vec{p}_4 \setminus (\vec{p}_1 \cup \vec{p}_2 \cup \vec{p}_3)) = 0$;
$\vec{p}_5 \setminus (\vec{p}_1 \cup \vec{p}_2 \cup \vec{p}_3 \cup \vec{p}_4)$ contains $(1,4,6,8) \Rightarrow \kappa^\Gamma_{18}(\vec{p}_5 \setminus (\vec{p}_1 \cup \vec{p}_2 \cup \vec{p}_3 \cup \vec{p}_4)) = 1$;
$\Rightarrow \kappa^\Gamma_{18} = 1$.

Therefore, $t_1(v, \Gamma) = v(13456789,10) - v(356789) - v(46789) - v(689,10) + 2v(689) + v(78) - v(8)$.

Example 6. Fig. 6 gives an example of the tree value for a 10-person game with strongly cycle-free digraph structure.

On the class of cycle-free digraph games the tree value not only meets FTE but FTE alone uniquely defines the tree value.

Theorem 3. *On the class of cycle-free digraph games $\mathcal{G}^\Gamma_N$ the tree value is the unique value that satisfies FTE.*

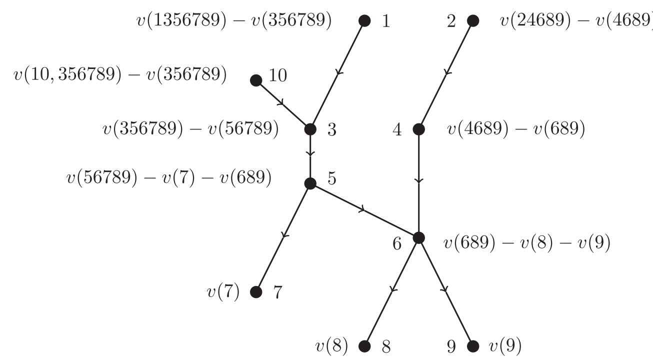

Fig. 6. Illustration of Example 6.

Proof. Since the tree value satisfies FTE, it is enough to show that the tree value is the unique value that meets FTE on $\mathcal{G}^\Gamma_N$. Let a value ξ on $\mathcal{G}^\Gamma_N$ satisfy FTE. Then, (1) holds for every $\langle v, \Gamma \rangle \in \mathcal{G}^\Gamma_N$. Every digraph Γ under consideration is cycle-free, i.e., no player in N appears to be a successor of itself. Hence, due to the arbitrariness of game $\langle v, \Gamma \rangle$, the n equalities in (1) are independent. Thus, we have a system of n independent linear equalities with respect to n variables $\xi_j(v, \Gamma)$ which uniquely determines $\xi(v, \Gamma)$ that in this case coincides with $t(v, \Gamma)$. $\square$

Corollary 4. *On the class of cycle-free digraph games $\mathcal{G}^\Gamma_N$ FTE is equivalent to MTE and SE.*

Remark 1. Observe that the independence of inessential links of the tree value can be also obtained as a corollary to Theorem 3.

4.2. Component efficiency and stability

In this subsection we consider component efficiency and stability of the tree value. First we derive the total payoff given by the tree value to any t-connected coalition.

Theorem 4. *Given a cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}^\Gamma_N$, for any t-connected coalition $S \in C^\Gamma_t(N)$ it holds that*

$$\sum_{i\in S} t_i(v,\Gamma) = \sum_{i\in R^\Gamma(S)} v(\overline{S}^\Gamma(i)) - \sum_{i\in S\setminus R^\Gamma(S)} (\kappa^\Gamma_{S,i} - 1)\, v(\overline{S}^\Gamma(i)) - \sum_{i\in \overline{S}^\Gamma(S)\setminus S} \kappa^\Gamma_{S,i}\, v(\overline{S}^\Gamma(i)), \qquad (9)$$

where $\kappa^\Gamma_{S,j} = \sum_{i\in P^\Gamma(j)\cap S} \kappa^\Gamma_{ij}$ for all $j \in \overline{S}^\Gamma(S)$.

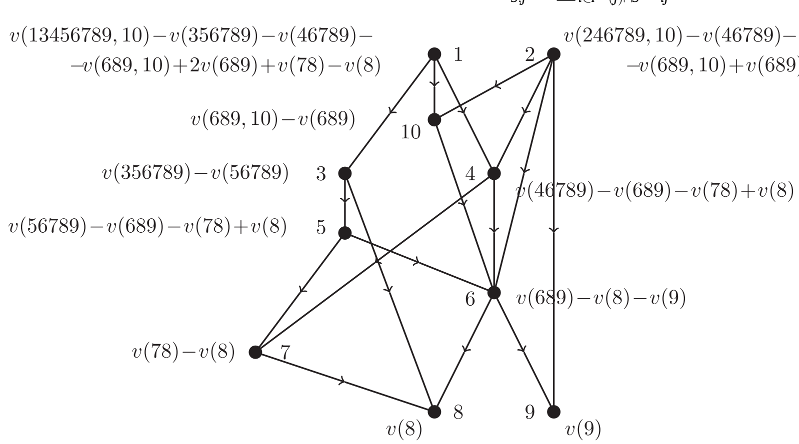

Fig. 5. Illustration of Example 5.

If $\langle v, \Gamma\rangle$ is a strongly cycle-free digraph games, then for any t-connected coalition $S \in C^{\Gamma}_t(N)$ it holds that

$$\sum_{i\in S} t_i(v,\Gamma) = \sum_{i\in R^{\Gamma}(S)} v(\bar{S}^{\Gamma}(i)) - \sum_{i\in S\setminus R^{\Gamma}(S)} (d^{\Gamma}_S(i)-1)\,v(\bar{S}^{\Gamma}(i)) - \sum_{i\in R^{\Gamma}(\bar{S}^{\Gamma}(S)\setminus S)} d^{\Gamma}_S(i)\,v(\bar{S}^{\Gamma}(i)), \tag{10}$$

where $d^{\Gamma}_S(j) = |\widehat{P}^{\Gamma}_*(j) \cap \bar{S}^{\Gamma}(S)|$ for all $j \in \bar{S}^{\Gamma}(S)$.

Proof. For any $S \in C^{\Gamma}_t(N)$ it holds that

$$\sum_{i\in S} t_i(v,\Gamma) \overset{(4)}{=} \sum_{i\in S}\left(v(\bar{S}^{\Gamma}(i)) - \sum_{j\in S^{\Gamma}(i)} \kappa^{\Gamma}_{ij}\, v(\bar{S}^{\Gamma}(j))\right)$$
$$= \sum_{i\in S} v(\bar{S}^{\Gamma}(i)) - \sum_{j\in \bar{S}^{\Gamma}(S)\setminus R^{\Gamma}(S)} \left(\sum_{i\in P^{\Gamma}(j)\cap S} \kappa^{\Gamma}_{ij}\, v(\bar{S}^{\Gamma}(j))\right)$$
$$= \sum_{i\in R^{\Gamma}(S)} v(\bar{S}^{\Gamma}(i)) - \sum_{i\in S\setminus R^{\Gamma}(S)} \left(\kappa^{\Gamma}_{S,i}-1\right) v(\bar{S}^{\Gamma}(i)) - \sum_{i\in \bar{S}^{\Gamma}(S)\setminus S} \kappa^{\Gamma}_{S,i}\, v(\bar{S}^{\Gamma}(i)).$$

In case Γ is a strongly cycle-free digraph, it holds that

$$\sum_{i\in S} t_i(v,\Gamma) \overset{(8)}{=} \sum_{i\in S}\Big(v(\bar{S}^{\Gamma}(i)) - \sum_{j\in \hat{S}^{\Gamma}(i)} v(\bar{S}^{\Gamma}(j))\Big)$$
$$= \sum_{i\in R^{\Gamma}(S)} v(\bar{S}^{\Gamma}(i)) - \sum_{i\in S\setminus R^{\Gamma}(S)} \left(d^{\Gamma}_S(i)-1\right) v(\bar{S}^{\Gamma}(i))$$
$$- \sum_{\substack{j\in \widehat{S}^{\Gamma}(i)\\ i\in S,\ j\notin S}}$$

To complete the proof of (10) it suffices to notice that, since Γ is a strongly cycle-free digraph, every $j \in \widehat{S}^{\Gamma}(i)$ such that $i \in S$ and $j \notin S$ is a source in $\bar{S}^{\Gamma}(S) \setminus S$. □

Observe that for $j \in \bar{S}^{\Gamma}(S)$ and $S \in C^{\Gamma}_t(N)$ the number $d^{\Gamma}_S(j)$ can be interpreted as the *in-degree of j from S*. Remark also that for any connected component $C \in N/\Gamma$ it holds that $d^{\Gamma}_C(i) = d^{\Gamma}(i)$ for all $i \in C$.

From Theorem 4 it follows that for any cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}^{\Gamma}_N$ the total payoff to any component $C \in N/\Gamma$ is given by

$$\sum_{i\in C} t_i(v,\Gamma) = \sum_{i\in R^{\Gamma}(C)} v(\bar{S}^{\Gamma}(i)) - \sum_{i\in C\setminus R^{\Gamma}(C)} \left(\kappa^{\Gamma}_{C,i}-1\right) v(\bar{S}^{\Gamma}(i)), \tag{11}$$

while if $\langle v, \Gamma\rangle$ is a strongly cycle-free digraph game, (11) reduces to

$$\sum_{i\in C} t_i(v,\Gamma) = \sum_{i\in R^{\Gamma}(C)} v(\bar{S}^{\Gamma}(i)) - \sum_{i\in C\setminus R^{\Gamma}(C)} (d^{\Gamma}(i)-1)\, v(\bar{S}^{\Gamma}(i)). \tag{12}$$

To support these expressions we recall the Myerson model in Myerson (1977) of a TU game with undirected cooperation structure, in which the total payoff to each component $C \in N/\Gamma$ equals its worth

$$\sum_{i\in C} \xi_i(v,\Gamma) = v(C). \tag{13}$$

While in the Myerson model the components are the only efficient feasible coalitions, the building bricks in (11) and (12) are the full successors sets which are the efficient feasible coalitions under the assumption of *t*-connectedness. Observe also that for strongly cycle-free rooted forest digraph games (12) reduces to (13),

$$\sum_{i\in C} t_i(v,\Gamma) = v(\bar{S}^{\Gamma}(r(\Gamma|_C))) = v(C).$$

The concept of the *core* of a TU game as the set of efficient payoff vectors that are not dominated by any coalition of players was introduced in Gillies (1953). A solution for a class of TU games is *stable* if it belongs to the core of any game of this class with nonempty core. For the class of cycle-free digraph games $\mathcal{G}^{\Gamma}_N$ we define *t-stability* of a solution by the *t*-core. For a cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}^{\Gamma}_N$ the *t-core* $C^t(v,\Gamma)$ is defined as the set of component efficient payoff vectors that are not dominated by any *t*-connected coalition,

$$C^t(v,\Gamma) = \left\{x \in \mathbb{R}^N | x(C) = v(C),\ \forall C \in N/\Gamma;\ x(S) \geqslant v(S),\ \forall S \in C^{\Gamma}_t(N)\right\}.$$

A game $v \in \mathcal{G}_N$ is *superadditive* if $v(S) + v(T) \leqslant v(S \cup T)$ for all S, $T \subseteq N$, such that $S \cap T = \emptyset$.

Theorem 5. *On the subclass of superadditive rooted forest digraph games the tree value is an element of the t-core.*

Proof. Let $\langle v,\Gamma\rangle \in \mathcal{G}^{\Gamma}_N$ be any superadditive rooted forest digraph game. We show that the tree value $t(v, \Gamma)$ belongs to $C^t(v, \Gamma)$. Consider an arbitrary $C \in N/\Gamma$, then C is a tree. Let $i \in C$ be a source in Γ, then $C = \bar{S}^{\Gamma}(i)$ because of the rooted forest structure of Γ. Due to the full-tree efficiency of the tree value, it holds that

$$\sum_{j\in \bar{S}^{\Gamma}(i)} t_j(v,\Gamma) \overset{FTE}{=} v(\bar{S}^{\Gamma}(i)),$$

wherefrom it follows that

$$\sum_{j\in C} t_j(v,\Gamma) = v(C).$$

Take any $S \in C^{\Gamma}_t(N)$. Because of the rooted forest structure of Γ, it holds that $d^{\Gamma}_N(i) = 1$ for all $i \in N\setminus R^{\Gamma}(N)$, from which it follows that $\Gamma|_S$ contains exactly one source, say, node i, i.e., $\Gamma|_S$ is a subtree, and $S \subseteq \bar{S}^{\Gamma}(i)$. Moreover, since Γ is strongly cycle-free, $\Gamma|_{\bar{S}^{\Gamma}(i)}$ is a full subtree, and because of the tree structure of $\Gamma|_S$, $\Gamma|_{\bar{S}^{\Gamma}(i)\setminus S}$ is a forest of full subtrees on disjoint node sets, say, $T_1,\ldots,T_q$. Hence,

$$\bar{S}^{\Gamma}(i) = S \cup \left(\bigcup_{k=1}^{q} T_k\right).$$

Applying again the full-tree efficiency of the tree value, we obtain that

$$\sum_{j\in \bar{S}^{\Gamma}(i)} t_j(v,\Gamma) \overset{FTE}{=} v(\bar{S}^{\Gamma}(i)) \quad \text{and} \quad \sum_{j\in T_k} t_j(v,\Gamma) \overset{FTE}{=} v(T_k) \quad \text{for } k = 1,\ldots,q.$$

From the superadditivity of v and the last three equalities, it follows that

$$\sum_{j\in S} t_j(v,\Gamma) = v(\bar{S}^{\Gamma}(i)) - \sum_{k=1}^{q} v(T_k) \geqslant v(S). \qquad \square$$

Remark 2. The statement of Theorem 5 can also be obtained as a corollary of the stability result proved in Demange (2004). Indeed, in a rooted forest every component has a tree structure and, therefore, is *t*-connected. Whence, for any rooted forest digraph game the *t*-core coincides with the core of the Myerson restricted game.

The following examples show that for *t*-stability of a superadditive digraph game the requirement on the digraph to be a rooted forest is non-reducible. In Example 7 the tree value of a superadditive cycle-free but not strongly cycle-free digraph game violates individual rationality and therefore does not meet the inequality constraints of the *t*-core, while in Example 8 the tree value of a superadditive strongly cycle-free game in which the graph contains two sources violates feasibility.

Example 7. Consider a 4-person cycle-free superadditive digraph game $\langle v, \Gamma\rangle$ with $v(\{2,4\}) = v(\{3,4\}) = v(\{2,3,4\}) = v(N) = 1$, $v(S) = 0$ otherwise, and Γ depicted in Fig. 7.

Then $t(v, \Gamma) = (-1,1,1,0)$, whence $t_1(v, \Gamma) = -1 < 0 = v(\{1\})$. By definition, every singleton coalition, in particular $S = \{1\}$, is *t*-connected.

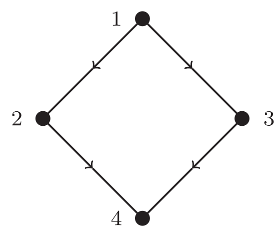

Fig. 7. The digraph in Example 7.

Example 8. Consider a 3-person cycle-free superadditive digraph game $\langle v, \Gamma\rangle$ with $v(\{1,2\}) = v(\{1,3\}) = v(N) = 1$, $v(S) = 0$ otherwise, and Γ depicted in Fig. 8.

Then $t(v,\Gamma) = (1,1,0)$, whence $t_1(v,\Gamma) + t_2(v,\Gamma) + t_3(v,\Gamma) = 2 > 1 = v(N)$.

A game $v \in \mathcal{G}_N$ is *convex* if for all $T, Q \subseteq N$ it holds that

$$v(T) + v(Q) \leqslant v(T \cup Q) + v(T \cap Q). \tag{14}$$

For TU games the notion of convexity was introduced in Shapley (1971), where it is shown that unlike for the superadditive games on the class of convex games the Shapley value is stable. For cycle-free undirected graph games if we choose a node in a given cycle-free undirected graph as a root of the rooted tree and apply the corresponding tree value as a solution of the original undirected graph game, superadditivity guarantees stability of this solution. However, for strongly cycle-free digraph games convexity ensures only component feasibility. In fact, to guarantee component feasibility it suffices that the strongly cycle-free digraph game is t-convex, which is a weaker condition than convexity of the game v.

A cycle-free digraph game $\langle v, \Gamma\rangle$ is *t-convex*, if the inequality (14) holds for all t-connected coalitions $T, Q \subset C_t^\Gamma(N)$ such that T is a full t-connected set, Q is a full successors set, and $T \cup Q \in C_t^\Gamma(N)$.

Theorem 6. *On the subclass of t-convex strongly cycle-free digraph games the tree value is component feasible.*

Proof. Let $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ be any t-convex strongly cycle-free digraph game. Assume that Γ is connected, otherwise we apply the same argument to any component $C \in N/\Gamma$. If there is only one source in Γ, it holds that $\sum_{i=1}^n t_i(v,\Gamma) = v(N)$ and the tree value is even efficient. So, assume that there are q different sources $r_1,\ldots,r_q$ in Γ for some $q \geqslant 2$. Since Γ is connected, the sources in Γ can be ordered in such a way that

$$\left(\bigcup_{h=1}^{j-1} \overline{S}^\Gamma(r_h)\right) \cap \overline{S}^\Gamma(r_j) \neq \emptyset, \quad \text{for } j = 2,\ldots,q.$$

For $j = 1,\ldots,q$, let $T_j = \bigcup_{h=1}^{j} \overline{S}^\Gamma(r_h)$. From the strongly cycle-freeness of Γ it follows that for $j = 2,\ldots,q$ there exists a unique $i_j \in N$ such that

$$T_{j-1} \cap \overline{S}^\Gamma(r_j) = \overline{S}^\Gamma(i_j).$$

By t-convexity of the digraph game $\langle v, \Gamma\rangle$ it holds that

$$v(T_{j-1}) + v(\overline{S}^\Gamma(r_j)) \leqslant v(T_j) + v(\overline{S}^\Gamma(i_j)), \quad \text{for } j = 2,\ldots,q.$$

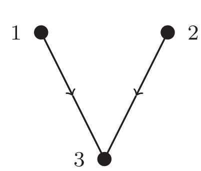

Fig. 8. The digraph in Example 8.

Since $T_1 = \overline{S}^\Gamma(r_1)$ and $T_q = N$ and applying the last inequality successively for $j = 2,\ldots,q$, we obtain that

$$\sum_{j=1}^q v(\overline{S}^\Gamma(r_j)) \leqslant v(N) + \sum_{j=2}^q v(\overline{S}^\Gamma(i_j)).$$

Hence,

$$v(N) \geqslant \sum_{j=1}^q v(\overline{S}^\Gamma(r_j)) - \sum_{j=2}^q v(\overline{S}^\Gamma(i_j)).$$

Since Γ is strongly cycle-free, for any $i \in N\backslash R^\Gamma(N)$, node i has $d^\Gamma(i)$ different sources as predecessors, which implies that the term $v(\overline{S}^\Gamma(i))$ appears precisely $d^\Gamma(i) - 1$ times. Therefore,

$$v(N) \geqslant \sum_{i\in R^\Gamma(N)} v(\overline{S}^\Gamma(i)) - \sum_{i\in N\backslash R^\Gamma(N)} (d^\Gamma(i) - 1)\, v(\overline{S}^\Gamma(i)). \qquad \square$$

The following example shows that under the assumption of convexity, which is stronger than t-convexity, one or more constraints for not being dominated in the definition of the t-core might be violated for the tree value.

Example 9. Consider a 5-person strongly cycle-free convex digraph game $\langle v,\Gamma\rangle$ with $v(N) = 10$, $v(\{1,2,3\}) = v(\{1,2,3,4\}) = v(\{1,2,3,5\}) = 3$, $v(\{1,3,4,5\}) = v(\{2,3,4,5\}) = 1$, $v(S) = 0$ otherwise, and digraph Γ depicted in Fig. 9.

Then $t(v, \Gamma) = (1,1,0,0,0)$, whence the total payoff of the t-connected coalition $S = \{1,2,3\}$ is equal to 2, which is smaller than $v(S)$ equal to 3.

From (11) it follows that for a cycle-free (connected) digraph game $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ a necessary and sufficient condition for the overall feasibility of the tree value is that

$$\sum_{i\in R^\Gamma(N)} v(\overline{S}^\Gamma(i)) \leqslant v(N) + \sum_{i\in N\backslash R^\Gamma(N)} \left(\kappa_{N,i}^\Gamma - 1\right) v(\overline{S}^\Gamma(i)). \tag{15}$$

Since $N = \cup_{i\in R^\Gamma(N)} \overline{S}^\Gamma(i)$, the grand coalition equals the union of the successors sets of all sources in the graph Γ. In case there is only one source in Γ, condition (15) is redundant, because the left side is then equal to $v(N)$. In case there is more than one source in Γ, the different successors sets of the sources of Γ will intersect each other and for any $i \in N\backslash R^\Gamma(N)$ the number $\kappa_{N,i}^\Gamma - 1$ is the number of times that the successors set $\overline{S}^\Gamma(i)$ of node i equals the intersection of successors sets of the sources of Γ. Therefore, condition (15) is a kind of convexity condition for the grand coalition saying that the sum of the worths of the successors sets of all the sources of the graph should be less than or equal to the worth of the grand coalition (their union) plus the total worths of their intersections. In a firm where any full successors set of a source is a division within the firm and subdivisions that are intersections of several divisions are shared by these divisions, in (15) the left-side minus the sum in the right-side can be economically interpreted as the total worths of the divisions when they do not cooperate, while $v(N)$ is the worth of the firm when the divisions do cooperate. To have feasibility the latter value should be at least equal to the former value. Remark that $v(N)$ minus the total payoff at the tree value can be interpreted as the net profit of the firm (or the synergy effect from cooperation) that can be given to its shareholders.

5. Web values

In this section we consider the case of an arbitrary management team in a given cycle-free directed communication graph and assume that for every cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ some management team $M \in \mathcal{M}(\Gamma)$ is a priori fixed.

To a cycle-free digraph Γ on N and management team $M \in \mathcal{M}(\Gamma)$ we associate the digraph

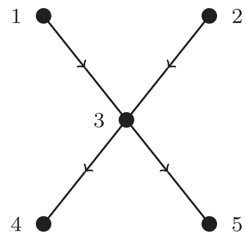

Fig. 9. The digraph in Example 9.

$$\Gamma^M = \{(i,j) \in \Gamma \mid j \in S^\Gamma(M)\} \bigcup \{(j,i) \mid (i,j) \in \Gamma,\ i \in P^\Gamma(M),\ j \notin S^\Gamma(M)\},$$

composed by the same links as Γ but with reversed orientation of any link with origin a predecessor of M and terminus not a successor of M. The set of sources in Γ^M coincides with the management team M in Γ, i.e., $R^{\Gamma^M}(N) = M$.

Example 10. Fig. 10 provides an example of the digraph Γ^M for the cycle-free digraph Γ depicted in Fig. 1 and the management team $M = \{3,4,10\}$.

Due to the management team development rule and the agreements on the asymmetries in the definitions of Γ^M and of the management team of a coalition, the assumption of M-web connectedness with respect to M in Γ is equivalent to the assumption of tree connectedness in digraph Γ^M, and the requirements of axioms MWE, MWSE together with MWPE, and MWE together with MWFTE and MWFSE with respect to game $\langle v, \Gamma, M\rangle$ are equivalent to the requirements of axioms MTE, SE and FTE with respect to game $\langle v, \Gamma^M\rangle$ correspondingly. The latter observations allow to get for the general case of M-web connectedness the following results obtained straightforwardly from the results proved in Section 4 for the case of tree connectedness.

Proposition 2. *On the class of cycle-free digraph games $\mathcal{G}^\Gamma_N$ MWE together with MWSE imply MWFTE, and MWE together with MWPE imply MWFSE with respect to any management team.*

Theorem 7. *On the class of cycle-free digraph games $\mathcal{G}^\Gamma_N$ there is a unique value w that satisfies MWE, MWSE and MWPE with respect to any management team. For every cycle-free digraph game $\langle v, \Gamma, M\rangle \in \mathcal{G}^\Gamma_N$, the payoff vector $w(v, \Gamma, M)$ possesses the following properties:*

(*i*) *it meets the equality*

$$w(v,\Gamma,M) = t(v,\Gamma^M); \tag{16}$$

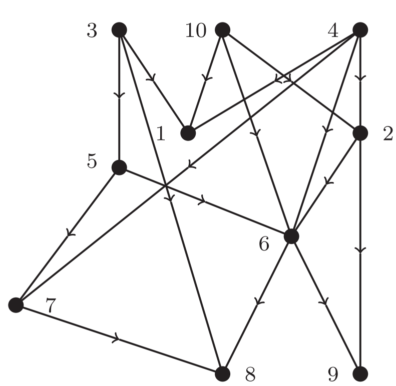

Fig. 10. The digraph in Example 10.

(*ii*) *it obeys the recursive formula*

$$w_i(v,\Gamma,M) = \begin{cases} v(\overline{S}^\Gamma(i)) - \sum\limits_{j\in S^\Gamma(i)} w_j(v,\Gamma,M), & \forall i \in S^\Gamma(M),\\ v(\overline{P}^\Gamma(i)) - \sum\limits_{j\in P^\Gamma(i)} w_j(v,\Gamma,M), & \forall i \in P^\Gamma(M),\\ v(W^\Gamma(i)) - \sum\limits_{j\in W^\Gamma(i)\setminus\{i\}} w_j(v,\Gamma,M), & \forall i \in M; \end{cases} \tag{17}$$

(*iii*) *it admits the explicit representation in the form*

$$w_i(v,\Gamma,M) = \begin{cases} v(\overline{S}^\Gamma(i)) - \sum\limits_{j\in S^\Gamma(i)} \kappa^\Gamma_{ij}\, v(\overline{S}^\Gamma(j)), & \forall i \in S^\Gamma(M),\\ v(\overline{P}^\Gamma(i)) - \sum\limits_{j\in P^\Gamma(i)} \kappa^\Gamma_{ji}\, v(\overline{P}^\Gamma(j)), & \forall i \in P^\Gamma(M),\\ v(W^\Gamma(i)) - \sum\limits_{j\in S^\Gamma(i)} \kappa^\Gamma_{ij}\, v(\overline{S}^\Gamma(j)) - \sum\limits_{j\in P^\Gamma(i)} \kappa^\Gamma_{ji}\, v(\overline{P}^\Gamma(j)), & \forall i \in M, \end{cases} \tag{18}$$

where for all $i \in N$ and $j \in S^\Gamma(i)$, κ^Γ_{ij} is defined by (2).

We refer to the value w as to the *M-web value* or *web value* for cycle-free digraph games with respect to management team M.

According to (17) the web value assigns to every manager the worth of his full web minus the total payoff to all his subordinates and to every successor (predecessor) of the given management team the worth of his full successors (predecessors) set minus the total payoff to his successors (predecessors). Wherefrom we obtain a simple recursive algorithm for computing the web value by going upstream from the sinks and downstream from the sources till the chosen management team is reached.

Remark 3. From (17) it follows that each member of a management team M can be considered as an independent entity in a sense that for every manager $i \in M$ it holds that $w_j(v,\Gamma,M) = w_j(v|_{W^\Gamma(i)}, \Gamma|_{W^\Gamma(i)}, \{i\})$ for all $j \in W^\Gamma(i)$.

The next theorem provides an explicit representation of the M-web value.

Theorem 8. *For any cycle-free digraph game $\langle v, \Gamma, M\rangle \in \mathcal{G}^\Gamma_N$, the M-web value $w(v, \Gamma, M)$ admits the equivalent representation in the form*

$$w_i(v,\Gamma,M) = \begin{cases} v(\overline{S}^\Gamma(i)) - \sum\limits_{j\in \widehat{S}^\Gamma_*(i)} v(\overline{S}^\Gamma(j)) \\ \quad + \sum\limits_{\substack{j\in S^\Gamma(i)\\ d^\Gamma_i(j)>1}} \left(q^\Gamma_{ij} - 1 - \sum\limits_{h=\hat{q}^\Gamma_{ij}+1}^{q^\Gamma_{ij}} \kappa^\Gamma_{ij}(\overline{C}^\Gamma(\vec{P}_h(i,j))) \right) v(\overline{S}^\Gamma(j)), & \forall i \in S^\Gamma(M),\\ v(\overline{P}^\Gamma(i)) - \sum\limits_{j\in \widehat{P}^\Gamma_*(i)} v(\overline{P}^\Gamma(j)) \\ \quad + \sum\limits_{\substack{j\in P^\Gamma(i)\\ e^\Gamma_i(j)>1}} \left(q^\Gamma_{ji} - 1 - \sum\limits_{h=\hat{q}^\Gamma_{ji}+1}^{q^\Gamma_{ji}} \kappa^\Gamma_{ji}(\underline{C}^\Gamma(\vec{P}_h(i,j))) \right) v(\overline{P}^\Gamma(j)), & \forall i \in P^\Gamma(M),\\ v(W^\Gamma(i)) - \sum\limits_{j\in \widehat{S}^\Gamma_*(i)} v(\overline{S}^\Gamma(j)) - \sum\limits_{j\in \widehat{P}^\Gamma_*(i)} v(\overline{P}^\Gamma(j)) \\ \quad + \sum\limits_{\substack{j\in S^\Gamma(i)\\ d^\Gamma_i(j)>1}} \left(q^\Gamma_{ij} - 1 - \sum\limits_{h=\hat{q}^\Gamma_{ij}+1}^{q^\Gamma_{ij}} \kappa^\Gamma_{ij}(\overline{C}^\Gamma(\vec{P}_h(i,j))) \right) v(\overline{S}^\Gamma(j)) \\ \quad + \sum\limits_{\substack{j\in P^\Gamma(i)\\ e^\Gamma_i(j)>1}} \left(q^\Gamma_{ji} - 1 - \sum\limits_{h=\hat{q}^\Gamma_{ji}+1}^{q^\Gamma_{ji}} \kappa^\Gamma_{ji}(\underline{C}^\Gamma(\vec{P}_h(i,j))) \right) v(\overline{P}^\Gamma(j)), & \forall i \in M. \end{cases} \tag{19}$$

If $\langle v, \Gamma, M\rangle$ is a strongly cycle-free digraph games, then the above representation reduces to

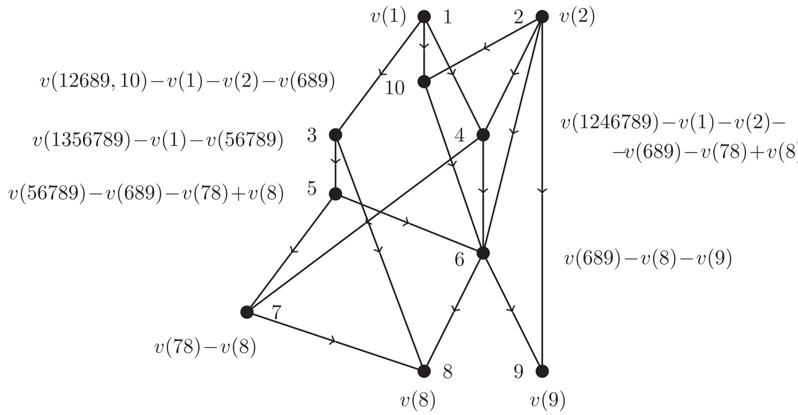

Fig. 11. Illustration of Example 11.

$$w_i(v,\Gamma,M)=\begin{cases} v(\bar{S}^\Gamma(i))-\sum\limits_{j\in\hat{S}^\Gamma(i)} v(\bar{S}^\Gamma(j)), & \forall i\in S^\Gamma(M),\\ v(\bar{P}^\Gamma(i))-\sum\limits_{j\in\hat{P}^\Gamma(i)} v(\bar{P}^\Gamma(j)), & \forall i\in P^\Gamma(M),\\ v(W^\Gamma(i))-\sum\limits_{j\in\hat{S}^\Gamma(i)} v(\bar{S}^\Gamma(j))-\sum\limits_{j\in\hat{P}^\Gamma(i)} v(\bar{P}^\Gamma(j)), & \forall i\in M. \end{cases} \tag{20}$$

The M-web value assigns to every successor (predecessor) of a given management team the payoff equal to the worth of his full successors (predecessors) set minus the worths of all full successors (predecessors) sets of his proper immediate successors (predecessors) plus or minus appropriate multiples of the worths of all full successors (predecessors) sets of any other of his successors (predecessors) to correct for multiple overlaps of these sets. The M-web value assigns to every manager of a given management team the payoff equal to the worth of his full web minus the worths of all full successors sets of his proper immediate successors plus or minus appropriate multiples of the worths of all full successors sets of any other of his successors and minus the worths of all full predecessors sets of his proper immediate predecessors plus or minus appropriate multiples of the worths of all full predecessors sets of any other of his predecessors. In fact, each player receives what he contributes when he joins his subordinates when we count only the efficient feasible coalitions that are full webs for the managers, full successors sets for the successors of the management team, and full predecessors sets for the predecessors of the management team. Again, it is worth to notice that the right hand sides of both formulas (19) and (20) being considered with respect not to coalitional worths but to players in these coalitions contain only player i when taking into account all weighted pluses and minuses.

Example 11. Fig. 11 provides an example of the M-web value $w(v, \Gamma, M)$ for a 10-person game v with cycle-free digraph Γ depicted in Fig. 1 and the management team $M = \{3,4,10\}$.

The M-web value not only meets MWE, MWFTE and MWFSE but also these three efficiency properties alone uniquely define the M-web value on the class $\mathcal{G}_N^\Gamma$.

Theorem 9. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ the M-web value w is the unique value that satisfies MWE, MWFTE and MWFSE.*

Corollary 5. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ MWE, MWFTE and MWFSE together imply MWSE and MWPE.*

Corollary 6. *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ the M-web value meets the independence of inessential links.*

For a cycle-free digraph game $\langle v,\Gamma,M\rangle\in\mathcal{G}_N^\Gamma$, we define the *M-web core* $C^M(v, \Gamma, M)$ as the set of component efficient payoff vectors that are not dominated by any M-web connected coalition,

$$C^M(v,\Gamma,M)=\left\{x\in\mathbb{R}^N | x(C)=v(C),\ \forall C\in N/\Gamma;\ x(S)\geqslant v(S),\ \forall S\in C_M^\Gamma(N)\right\}.$$

Theorem 10. *On the class of superadditive line-graph digraph games the M-web value is an element of the M-web core.*

We remark that for M-web stability of a superadditive digraph game when the management team M is composed neither only by sources nor only by sinks the requirement on the digraph to be a line-graph is non-reducible.

A cycle-free digraph game $\langle v,\Gamma,M\rangle\in\mathcal{G}_N^\Gamma$ is *M-web-convex*, if for all M-web connected coalitions $T,Q\subset C_M^\Gamma(N)$ such that T is a full M-web connected set, Q is a web, and $T\cup Q\in C_M^\Gamma(N)$, it holds that

$$v(T)+v(Q)\leqslant v(T\cup Q)+v(T\cap Q). \tag{21}$$

Theorem 11. *On the subclass of M-web-convex strongly cycle-free digraph games $\mathcal{G}_N^\Gamma$ the M-web value is component feasible.*

If the management team of a cycle-free digraph Γ on N consists of all sinks, web connectedness can be restated in terms of sink connectedness, where a connected coalition $S\in C^\Gamma(N)$ is *sink connected*, or simply *s-connected*, if for every sink $i\in L^\Gamma(S)$ it holds that $i\notin P^\Gamma(j)$ for any other sink $j\in L^\Gamma(S)$. In this case M-web efficiency becomes maximal sink efficiency, M-web predecessor equivalence predecessor equivalence, M-web efficiency together with M-web full-sink efficiency provide full sink efficiency, the axioms of M-web successor equivalence and M-web full-tree efficiency become redundant, and the M-web core reduces to the *s-core* $C^s(v, \Gamma)$ defined as the set of component efficient payoff vectors that are not dominated by any s-connected coalition,

$$C^s(v,\Gamma)=\{x\in\mathbb{R}^N | x(C)=v(C),\ \forall C\in N/\Gamma;\ x(S)\geqslant v(S),\ \forall S \in C_s^\Gamma(N)\},$$

where $C_s^\Gamma(N)$ denotes the set of all s-connected subcoalitions of N. Besides, formulas (18)–(20) that provide representations of the M-web value reduce correspondingly to[1]

[1] In the next formulas we denote the value relevant to the case of sink connectedness by s instead of w used in the general case.

$$s_i(v,\Gamma) = v(\overline{P}^\Gamma(i)) - \sum_{j\in P^\Gamma(i)} \kappa^\Gamma_{ji} v(\overline{P}^\Gamma(j)), \quad \text{for all } i\in N, \tag{22}$$

$$s_i(v,\Gamma) = v(\overline{P}^\Gamma(i)) - \sum_{j\in \hat{P}^\Gamma_*(i)} v(\overline{P}^\Gamma(j)) + \sum_{\substack{j\in P^\Gamma(i)\\ d^\Gamma_i(j)>1}} \left(q^\Gamma_{ji} - 1 - \sum_{h=\bar{q}^\Gamma_{ji}+1}^{q^\Gamma_{ji}} \kappa^\Gamma_{ji}(\underline{C}^\Gamma(\vec{P}_h(j,i))) \right) v(\overline{P}^\Gamma(j)), \quad \text{for all } i\in N, \tag{23}$$

and

$$s_i(v,\Gamma) = v(\overline{P}^\Gamma(i)) - \sum_{j\in \hat{P}^\Gamma(i)} v(\overline{P}^\Gamma(j)), \quad \text{for all } i\in N. \tag{24}$$

For sink forest digraph games defined by sink forest digraph structures that are strongly cycle-free, the value given by (24) coincides with the sink value introduced in Khmelnitskaya (2010). We refer to the value s given by (22), or equivalently by (23), as the *sink tree value*, or simply the *sink value*, for cycle-free digraph games.

Theorem 12. *On the subclass of superadditive sink forest digraph games the sink value belongs to the s-core.*

6. The average web value

In this section we introduce the average web value for cycle-free directed graphs. This value only depends on a given TU game and digraph. By taking the average web value we equalize the players' control assuming that every player may become a manager and that every possible management team is equally likely to occur.

For any cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$, the *average web value* (*AW-value*) is defined as the average of M-web values over the set $\mathcal{M}(\Gamma)$ of all management teams in the digraph Γ,

$$AW(v,\Gamma) = \frac{1}{|\mathcal{M}(\Gamma)|} \sum_{M\in\mathcal{M}(\Gamma)} w(v,\Gamma,M).$$

It is not difficult to see that the AW-value inherits the independence of inessential links property from M-web values. Moreover, since convexity of a digraph game $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$ is stronger than M-web-convexity with respect to any management team $M\in\mathcal{M}(\Gamma)$, we obtain from Theorem 11 the next theorem.

Theorem 13. *On the class of convex strongly cycle-free digraph games $\mathcal{G}^\Gamma_N$ the AW-value is component feasible.*

The *average tree solution* (*AT solution*) for undirected cycle-free graph games, introduced in Herings et al. (2008), assigns to any undirected cycle-free graph game $\langle v, \Gamma\rangle$ to player $i\in N$ the average of his tree value payoffs in all rooted spanning trees[2] in the subgraph $\langle (N/\Gamma)_i, \Gamma|_{(N/\Gamma)_i}\rangle$:

$$AT_i(v,\Gamma) = \frac{1}{|(N/\Gamma)_i|} \sum_{j\in(N/\Gamma)_i} t_i(v|_{(N/\Gamma)_i}, T(j)), \quad \text{for all } i\in N,$$

where, for $j\in(N/\Gamma)_i$, $T(j)$ is the rooted tree on $(N/\Gamma)_i$ with j as root and composed of all links of $\langle (N/\Gamma)_i, \Gamma|_{(N/\Gamma)_i}\rangle$ with orientation directed away from the root and t is the tree value given by (8).

With any digraph Γ on N we associate the undirected graph $\widetilde{\Gamma}$ on N defined as

$$\widetilde{\Gamma} = \{\{i,j\} | i,j\in N, \{(i,j),(j,i)\}\cap\Gamma \neq \emptyset\}.$$

[2] Given an undirected graph Γ on N, a rooted tree Γ' on N is a *spanning tree* of Γ if for every $(i, j)\in\Gamma'$ it holds that $\{i, j\}\in\Gamma$.

Theorem 14. *The AW-value for a strongly cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$ coincides with the AT solution for the corresponding undirected graph game $\langle v,\widetilde{\Gamma}\rangle$, i.e.,*

$$AW(v,\Gamma) = AT(v,\widetilde{\Gamma}), \tag{25}$$

if and only if Γ is a line-graph.

Proof. Let $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$ be a strongly cycle-free digraph game for which equality (25) holds. From Theorem 7 it follows that for any management team $M\in\mathcal{M}(\Gamma)$, $w(v,\Gamma,M) = t(v,\Gamma^M)$. Besides, for any management team $M\in\mathcal{M}(\Gamma)$ the undirected graphs corresponding to Γ and Γ^M coincide, i.e., $\widetilde{\Gamma} = \widetilde{\Gamma^M}$. Whence and due to the arbitrary choice of the game v in $\mathcal{G}_N$ it follows that for equality (25) to hold all management teams $M\in\mathcal{M}(\Gamma)$ need to be singletons. To complete the proof it is enough to notice that the condition of a strongly cycle-free digraph to be a line-graph is a necessary and sufficient condition for all management teams of the digraph to be singletons, and moreover, in a line-graph Γ on N the total number of management teams in Γ is equal to n, i.e., $|\mathcal{M}(\Gamma)| = n$. □

In Herings et al. (2008) it is shown that the AT solution defined on the class of superadditive undirected cycle-free graph games is stable, and that on the entire class of undirected cycle-free graph games the AT solution is characterized via component efficiency and component fairness, see also Béal et al. (2010).

A value ξ on the class of graph games is *component fair* (CF) if, for any cycle-free graph game $\langle v, \Gamma\rangle$, for every link $\{i, j\}\in\Gamma$, it holds that

$$\frac{1}{|(N/\Gamma\setminus\{i,j\})_i|}\sum_{t\in(N/\Gamma\setminus\{i,j\})_i} (\xi_t(v,\Gamma) - \xi_t(v,\Gamma\setminus\{i,j\}) = \frac{1}{|(N/\Gamma\setminus\{i,j\})_j|}\sum_{t\in(N/\Gamma\setminus\{i,j\})_j} (\xi_t(v,\Gamma) - \xi_t(v,\Gamma\setminus\{i,j\}).$$

From Theorem 14 and the axiomatization and properties of the AT solution we obtain the next corollary.

Corollary 7. *On the subclass of line-graph games $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$ the AW-value is characterized by CE and CF and, moreover, on the subclass of superadditive line-graph games $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$ the AW-value belongs to the core of the undirected graph game $\langle v,\widetilde{\Gamma}\rangle$.*

7. Sharing a river with multiple sources, a delta and possible islands

Ambec and Sprumont (2002) approach the problem of optimal water allocation for a given river with certain capacity over the agents (cities, countries) located along the river from the game theoretic point of view. Their model assumes that between each pair of neighboring agents there is an additional inflow of water. Each agent, in principal, can use all the inflow between itself and its upstream neighbor, however, this allocation in general is not optimal in respect to total welfare. To obtain a more profitable allocation it is allowed to allocate more water to downstream agents which in turn can compensate the extra water obtained by side-payments to upstream ones. The problem of optimal water allocation is approached as the problem of optimal welfare distribution. Brink et al. (2007) show that the Ambec-Sprumont river game model can be naturally embedded into the framework of a graph game with line-graph cooperation structure. In Khmelnitskaya (2010) the line-graph river model is extended to the rooted tree and sink tree digraph model of a river with a delta or with multiple sources, respectively. We extend the line-graph, rooted tree or sink tree model of a river to the cycle-free digraph model of a river with

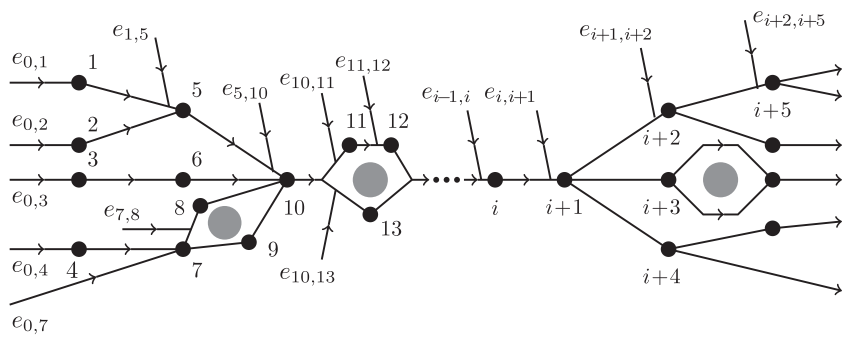

Fig. 12. A river with multiple sources, a delta, and several islands along the river bed.

both multiple sources and a delta, and also possible islands along the river bed as well.

Let N be a set of players (users of water) located along the river from upstream to downstream. Let $e_{ki} \geqslant 0$, $i \in N$, $k \in \widehat{P}^{\Gamma}(i)$, be the inflow of water in front of the most upstream player(s) when $k = 0$, or the inflow of water entering the river between neighboring players when player k is in front of player i. Fig. 12 provides a schematic representation of the model.

Following Ambec and Sprumont (2002) it is assumed that each player $i \in N$ has a quasi-linear utility function given by $u^i(x_i, t_i) = b^i(x_i) + t_i$ where t_i is a monetary compensation to player i, x_i is the amount of water allocated to player i, and $b^i := \mathbb{R}_+ \to \mathbb{R}$ is a continuous nondecreasing function providing benefit $b^i(x_i)$ to player i when he consumes the amount x_i of water. Moreover, it is also assumed that if a splitting of the river into branches happens to occur after a certain player, then this player takes, besides his own quota, also the responsibility to split the rest of the water flow to the branches such to guarantee the realization of the water distribution plan to his successors.

A light modification of the introduced under the same assumptions in Khmelnitskaya (2010) superadditive river game for a river with multiple sources or a delta given by $v \in \mathcal{G}_N$ defined as:

for any $S \in C^{\Gamma}(N)$, $v(S) = \sum_{i \in S} b^i(x_i^S)$, where $x^S \in \mathbb{R}^S$ solves

$$\max_{x \in \mathbb{R}^S_+} \sum_{i \in S} b^i(x_i) \quad \text{s.t.} \quad \sum_{i \in \overline{P}^{\Gamma}(T)} x_i \leqslant \sum_{i \in \overline{P}^{\Gamma}(T)} \sum_{j \in \widehat{P}^{\Gamma|_T}(i)} e_{ji}, \quad \text{for all } T \in C^{\Gamma}(S),$$

and for any other $S \subset N$, $v(S) = \sum_{T \in S/\Gamma} v(T)$, suits to the case of a river with both multiple sources and a delta, and also possible islands.

To solve the digraph river game with the digraph representing the river flow, we may apply one of the values for cycle-free digraph games developed in the paper. In some situations a subcoalition of users of the river water can be chosen to be responsible for the regulation of the total water distribution. This, for example, might happen due to the strong economical power and influence of this subcoalition members. In such cases, if the selected subcoalition satisfies the conditions of being a management team, we may apply the corresponding M-web value. It is worth to remark that when management team M is a singleton, the M-web value provides an efficient solution of the digraph river game. The singleton management team happens, in particular, when the manager is located in the middle part of the river between possible islands, or if the river has no multiple sources or no delta and the only manager is located at the top or correspondingly at the bottom of the given digraph river structure. In these latter cases as solution we use the tree or sink value respectively. Otherwise, if no management team is selected, we may apply the AW-value. For the AW-value we can guarantee its efficiency only for the line-graph river structure. But if the river game appears to be convex and the river digraph is strongly cycle-free, i.e., there are no islands along the river bed, the AW-value is feasible. The distribution of water based on the application of the M-web value introduced via its properties formulated in terms of the axioms MWE, MWSE, and MWPE, which can be also equivalently characterized by MWE, MWFTE, and MEFSE, does not contradict both the Absolute Territorial Sovereignty (ATS) and the Absolute Territorial Integrity (ATI) legal principles. Due to efficiency properties MWE, MWSE, and MWPE, the M-web value of a superadditive river game provides individually rational payoffs to the players and therefore fully agrees with the ATS principle. At the same time the deletion link properties MWFTE and MWFSE to some extent reflects the ATI principle.

References

Ambec, S., & Sprumont, Y. (2002). Sharing a river. *Journal of Economic Theory, 107*, 453–462.
Béal, S., Rémila, E., & Solal, P. (2010). Rooted-tree solutions for tree games. *European Journal of Operational Research, 203*, 404–408.
Demange, G. (2004). On group stability in hierarchies and networks. *Journal of Political Economy, 112*, 754–778.
Faigle, U., & Kern, W. (1992). The Shapley value for cooperative games under precedence constraints. *International Journal of Game Theory, 21*, 249–266.
Gillies, D. B. (1953). *Some theorems on n-person games*. Ph.D. thesis, Princeton University.
Herings, P. J. J., van der Laan, G., & Talman, A. J. J. (2008). The average tree solution for cycle-free graph games. *Games and Economic Behavior, 62*, 77–92.
Khmelnitskaya, A. B. (2010). Values for rooted tree and sink-tree digraphs games and sharing a river. *Theory and Decision, 69*, 657–669.
Li, L., & Li, X. (2011). The covering values for acyclic digraph games. *International Journal of Game Theory, 40*, 697–718.
Myerson, R. B. (1977). Graphs and cooperation in games. *Mathematics of Operations Research, 2*, 225–229.
Shapley, L. S. (1971). Cores of convex games. *International Journal of Game Theory, 1*, 11–26.
van den Brink, R., van der Laan, G., & Vasil'ev, V. (2007). Component efficient solutions in line-graph games with applications. *Economic Theory, 33*, 349–364.