Stochastic Multiobjective Acceptability Analysis for the Choquet integral preference model and the scale construction problem

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Abstract

The Choquet integral is a preference model used in Multiple Criteria Decision Aiding (MCDA) to deal with interactions between criteria. The Stochastic Multiobjective Acceptability Analysis (SMAA) is an MCDA methodology used to take into account imprecision or lack of data in the problem at hand. For example, SMAA permits to compute the frequency that an alternative takes the k-th rank in the whole space of the admissible preference parameters, e.g. in case evaluations on the considered criteria are aggregated through the weighted sum model, in the space of weights compatible with the preference information supplied by the Decision Maker (DM). In this paper, we propose to integrate the SMAA methodology with the Choquet integral preference model in order to get robust recommendations taking into account the whole space of preference parameters compatible with the DM's preference information. In case the alternatives are evaluated by all the criteria on a common scale, the preference parameters are given by the capacity expressing the non-additive weights, representing the importance of criteria and their interaction. If the criteria are instead evaluated on different scales, besides the capacity, preference parameters include the common scale on which the evaluations of criteria have to be recoded to be compared. Our approach permits to explore the whole space of preference parameters being capacities and common scales compatible with the DM's preference information.

Keywords: MCDA, Choquet integral, SMAA, interacting criteria, evaluation scales.

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1 Introduction

In Multiple Criteria Decision Aiding (MCDA) (see [11] for a survey on MCDA), an alternative a_k, belonging to a finite set of l alternatives $A = \{a_1, a_2, \dots a_l\}$, is evaluated on the basis of a consistent family of n criteria $G = \{g_1, g_2, \dots g_n\}$ where, $g_i : A \to \mathbb{R}$ is an interval scale of measurement. In this paper, we shall consider also the case in which $g_i : A \to \mathcal{I}$, where $\mathcal{I} = \{[\alpha, \beta] \subseteq \mathbb{R}\}$ and $g_i(a_k) = [\alpha, \beta]$ meaning that the evaluation of alternative a_k on criterion g_i can be whichever value inside the interval $[\alpha, \beta]$. Obviously, if $\alpha = \beta$, then the evaluation of a_k on criterion g_i is given in a precise way. From here on, we will use the terms criterion g_i or criterion i interchangeably $(i = 1, 2, \dots, n)$ and, without loss of generality, we will assume that all the criteria have to be maximized.

In case the evaluation on criterion i are precise, i.e. $g_i(a_k) \in \mathbb{R}$ for all $k = 1, 2 \dots, l$, we define a marginal weak preference relation as follows:

$$a_k \text{ is at least as good as } a_h \text{ with respect to criterion } i \iff g_i(a_k) \geq g_i(a_h).$$

To give a recommendation for the decision making problem at hand, evaluations of alternatives on all criteria have to be aggregated. For this reason, three main approaches have been proposed:

- the Multi-Attribute Utility Theory (MAUT) [35],

- the outranking methods [46], among which the most well known are ELECTRE [47] and PROMETHEE [5, 6],

- the decision rule approach based on induction of logical "if..., then..." decision rules through Dominance-based Rough Set Approach (DRSA, see [22, 25, 50]).

In MAUT, an overall value function $U : \mathbb{R}^n \to \mathbb{R}$ with $U(g_1(a_k), \dots, g_n(a_k)) = U(a_k)$ is defined such that:

- a_k is indifferent to $a_h \iff U(a_k) = U(a_h)$,

- a_k is preferred to $a_h \iff U(a_k) > U(a_h)$,

and the principal aggregation model of value function is the multiple attribute additive utility:

$$U(a_k) = u_1(g_1(a_k)) + u_2(g_2(a_k)) + \dots + u_n(g_n(a_k)) \quad \text{with} \quad a_k \in A,$$

where u_i are non-decreasing marginal value functions for $i = 1, 2, \dots, n$.

Outranking methods are instead based on a binary relation S defined on A, where $a_k S a_h$ means

that "a_k is at least as good as a_h". A preference, an indifference and for some methods also an incomparability relation can be built on the basis of S.

A basic assumption of both MAUT and outranking methods is the absence of interaction (synergy or redundancy) of criteria, which is very often an unrealistic assumption or a too strong simplification. For example, in evaluating sport cars, a Decision Maker (DM) could consider criteria such as maximum speed, acceleration and price. From one side, maximum speed and acceleration are redundant criteria because, in general, speed cars also have a good acceleration. Therefore, even if these two criteria can be very important for a DM liking sport cars, their comprehensive importance is smaller than the sum of the importance of the two criteria considered separately. From the other side, the two criteria maximum speed and price lead to a synergy effect because a speed car having also a low price is very well appreciated. For such a reason, the comprehensive importance of these two criteria should be greater than the sum of the importance of the two criteria considered separately.

Within MCDA, the interaction of criteria has been dealt using non additive integrals the most well known of which are the Choquet integral [8] and the Sugeno integral [52] (see [15, 19, 20] for a comprehensive survey on the use of non-additive integrals in MCDA; see also [17, 18, 21, 27, 28] for some recently proposed extensions of non additive integrals useful in MCDA). Referring to the outranking methods, a notable contribution to deal with interaction of criteria has been given for ELECTRE methods in [12]. The decision rule approach instead can represent interaction of criteria in a natural way (see [24]).

One of the greatest drawbacks of the Choquet integral is the elicitation of the preference parameters. To deal with this issue, the DM can provide direct or indirect preference information [3, 39]. The direct preference information consists of providing all the values of parameters while the indirect preference information (see e.g. [30]) is based on giving some preference information related to preference among alternatives from which the parameters values are inferred.
Recently, an approach based on the determination of necessary and possible preference relations within the so-called Robust Ordinal Regression (ROR) [13, 26, 29] has been presented in [4].

In this paper, we have widely extended the work presented from us [1] in which we have applied the Stochastic Multiobjective Acceptability Analysis (SMAA) to explore the whole space of parameters compatible with some preference information provided by the DM related to the importance of criteria and to their interaction (for a survey on SMAA methods see [53]). In this paper, we have integrated the Choquet integral and the SMAA methods considering also preference information related to pairwise comparison of some reference alternatives. Moreover, we have also considered

alternatives whose evaluations on considered criteria may be given in terms of intervals of possible values.

Another drawback of the Choquet integral is that all evaluation criteria have to be defined on a common scale in order to compare alternatives on different criteria. This is a strong requirement for this methodology because, for example, in the car evaluation problem cited above, the DM should be able to compare the speed of a car with its acceleration estimating, for example, if the maximum speed of 200 km/h is as valuable as $35,000$ €.

This problem is quite well known in literature (see e.g. [40]). One solution has been proposed in [3] where a common scale was searched for through Monte Carlo simulation. In this paper, we shall show how SMAA permits to define a common scale on which one can compare evaluations on different criteria. Moreover, since this common scale, in general, is not unique, we shall describe how SMAA can explore the space of all compatible common scales and the corresponding preference orders that are obtained passing from one common scale to another one. The whole methodology permitting to define a common scale is another original contribution of this paper with respect to [1].

The paper is organized as follows. In Section 2, we present the basic concepts relative to interaction of criteria and to the Choquet integral. In Section 3, we briefly describe the SMAA methods. An extension of the SMAA method to the Choquet integral preference model is introduced in Section 4 and illustrated by several different examples in Section 5. Some conclusions and future directions of research are presented in Section 6.

2   The Choquet integral preference model

Let 2^G denote the power set of G (i.e. the set of all subsets of G). A set function $\mu : 2^G \to [0,1]$ is called a capacity (fuzzy measure) on 2^G if the following properties are satisfied:

1a) $\mu(\emptyset) = 0$ and $\mu(G) = 1$ (boundary conditions),

2a) $\forall \, S \subseteq T \subseteq G, \ \mu(S) \leq \mu(T)$ (monotonicity condition).

A capacity is called additive if $\mu(S \cup T) = \mu(S) + \mu(T)$, for any $S, T \subseteq G$ such that $S \cap T = \emptyset$.

If a capacity is additive, then it is uniquely determined by the value assigned to the singletons: $\mu(\{1\}), \mu(\{2\}) \ldots, \mu(\{n\})$. In such case $\forall \, T \subseteq G$, it results:

$$\mu(T) = \sum_{i \in T} \mu(\{i\}).$$

Whenever the capacity is non additive, one needs a value $\mu(T)$ for every subset $T \subseteq G$. More precisely, we have to define $2^{|G|} - 2$ coefficients, since the values $\mu(\emptyset) = 0$ and $\mu(G) = 1$ are already known.

The Möbius representation of a capacity μ (see [45]) is a function $m : 2^G \to \mathbb{R}$ ([48]) defined as follows:

$$\mu(S) = \sum_{T \subseteq S} m(T).$$

Note that if S is a singleton, i.e. $S = \{i\}$ with $i = 1, 2, \ldots, n$, then $\mu(\{i\}) = m(\{i\})$.

Moreover, if S is a couple (non-ordered pair) of criteria, i.e. $S = \{i, j\}$, then $\mu(\{i, j\}) = m(\{i\}) + m(\{j\}) + m(\{i, j\})$.

The Möbius representation $m(S)$ can be obtained from $\mu(S)$ as follows:

$$m(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \mu(T).$$

In terms of Möbius representation (see [7]), properties **1a)** and **2a)** are, respectively, restated as:

1b) $m(\emptyset) = 0, \ \sum_{T \subseteq G} m(T) = 1,$

2b) $\forall\, i \in G$ and $\forall R \subseteq G \setminus \{i\}, \ \sum_{T \subseteq R} m(T \cup \{i\}) \geq 0.$

In a multicriteria decision problem, the relevance of any criterion $g_i \in G$ can be analyzed considering g_i either as a singleton or in combination with all other criteria. As a result, the importance of a criterion $g_i \in G$ is evaluated not only by the capacity $\mu(\{i\})$, but also by considering $\mu(T \cup \{i\})$, $T \subseteq G \setminus \{i\}$, i.e. when the criterion interacts with all other criteria.

Given $a_k \in A$ and μ a capacity μ on 2^G, the Choquet integral [8] is defined as follows:

$$C_\mu(a_k) = \sum_{i=1}^{n} \left[\left(g_{(i)}(a_k) \right) - \left(g_{(i-1)}(a_k) \right) \right] \mu\left(N_i \right), \tag{1}$$

where $_{(\cdot)}$ stands for a permutation of the indices of criteria such that:

$g_{(1)}(a_k) \leq g_{(2)}(a_k) \leq \ldots \leq g_{(n)}(a_k)$, with $N_i = \{(i), \ldots, (n)\}$, $i = 1, 2, \ldots, n$, and $g_{(0)} = 0$.

In terms of Möbius representation [14], the Choquet integral may be reformulated as follows:

$$C_\mu(a_k) = \sum_{T \subseteq G} m(T) \min_{i \in T} g_i(a_k). \tag{2}$$

Within a Choquet integral preference model, the main difficult task is to infer $2^{|G|} - 2$ parameters, giving to them an adequate meaning from a decisional point of view. With the aim of reducing the number of parameters to be elicitated and of avoiding a too strict description of the interactions among criteria, in [16] the concept of fuzzy k-additive capacity has been introduced.

A capacity is called k-additive if $m(T) = 0$ for $T \subseteq G$ such that $|T| > k$ and there exists at least one $T \subseteq G$, with $|T| = k$, such that $m(T) > 0$. In particular, a 1-additive capacity is the standard additive capacity.

Within an MCDA context, it is easier and more straightforward to consider 2-additive capacities only, since in such case the DMs have to express a preference information on positive and negative interactions between two criteria, neglecting possible interactions among three, four and generally n criteria. Moreover, by considering 2-additive measures the computational issue of determining the parameters is weakened since only $n + \binom{n}{2}$ coefficients have to be assessed; specifically, in terms of Möbius representation, a value $m(\{i\})$ for every criterion i and a value $m(\{i,j\})$ for every couple of criteria $\{i, j\}$. For all these reasons, in the following we shall consider 2-additive capacities only. However, the methodology we are presenting can be applied to generally non-additive capacities.

The value that a 2-additive capacity μ assigns to a set $S \subseteq G$ can be expressed in terms of Möbius representation as follows:

$$\mu(S) = \sum_{i \in S} m(\{i\}) + \sum_{\{i,j\} \subseteq S} m(\{i,j\}), \ \forall S \subseteq G. \tag{3}$$

With regard to 2-additive capacities, properties **1b)** and **2b)** have, respectively, the following expressions:

1c) $m(\emptyset) = 0, \ \sum_{i \in G} m(\{i\}) + \sum_{\{i,j\} \subseteq G} m(\{i,j\}) = 1,$

2c)
$$\begin{cases} m(\{i\}) \geq 0, \ \forall i \in G, \\ m(\{i\}) + \sum_{j \in T} m(\{i,j\}) \geq 0, \ \forall i \in G \text{ and } \forall T \subseteq G \setminus \{i\}, \ T \neq \emptyset. \end{cases}$$

In this case, the Choquet integral of $a_k \in A$ is given by:

$$C_\mu(a_k) \ = \ \sum_{\{i\} \subseteq G} m(\{i\}) (g_i(a_k)) + \sum_{\{i,j\} \subseteq G} m(\{i,j\}) \min\{g_i(a_k), g_j(a_k)\}. \tag{4}$$

Finally, we recall the definitions of the importance and interaction indices for a couple of criteria.

The importance of criterion $i \in G$ expressed by the Shapley value [49] in case of a 2-additive capacity can be written as follows:

$$\varphi\left(\{i\}\right) = m\left(\{i\}\right) + \sum_{j \in G \setminus \{i\}} \frac{m\left(\{i,j\}\right)}{2}. \tag{5}$$

The interaction index [44] expressing the sign and the magnitude of the interaction in a couple of criteria $\{i,j\} \subseteq G$, in case of a 2-additive capacity μ, is given by:

$$\varphi\left(\{i,j\}\right) = m\left(\{i,j\}\right). \tag{6}$$

3   SMAA

Stochastic Multiobjective Acceptability Analysis (SMAA) methods [36, 38] are a family of MCDA methods aiming to get recommendations on the problem at hand taking into account uncertainty or imprecision on the considered data and preference parameters. Several SMAA methods have been developed to deal with different MCDA problems: SMAA-2 has been presented in [38] for ranking problems, SMAA-O [37] has been introduced for multicriteria problems with ordinal criteria and SMAA-TRI [54] for sorting problems. Other two recent contributions relating to SMAA and ROR have been presented in [33] and [34]. For a detailed survey on SMAA methods see [53].

Since in this paper we consider ranking problems, we only describe the SMAA-2 method.

In SMAA-2 the most commonly used utility function is the linear one:

$$u(a_k, w) = \sum_{i=1}^{n} w_i g_i(a_k).$$

In order to take into account imprecision or uncertainty, SMAA-2 considers two probability distributions $f_W(w)$ and $f_\chi(\xi)$, respectively, on W and χ, where $W = \{(w_1, \ldots, w_n) \in \mathbb{R}^n : w_i \geq 0 \text{ and } \sum_{i=1}^{n} w_i = 1\}$ and χ is the evaluation space.

First of all, SMAA-2 introduces a ranking function relative to the alternative a_k:

$$rank(k, \xi, w) = 1 + \sum_{h \neq k} \rho\left(u(\xi_h, w) > u(\xi_k, w)\right),$$

where $\rho(false) = 0$ and $\rho(true) = 1$.

Then, for each alternative a_k, for each evaluations of alternatives $\xi \in \chi$ and for each rank $r = 1, \ldots, l$, SMAA-2 computes the set of weights of criteria for which alternative a_k assumes rank r:

$$W_k^r(\xi) = \{w \in W : rank(k, \xi, w) = r\}.$$

SMAA-2 is based on the computation of three indices:

- The rank acceptability index measures for each alternative a_k and for each rank r the variety of different parameters compatible with the DM's preference information giving to a_k the rank r and is given by:

$$b_k^r = \int_{\xi \in \chi} f_\chi(\xi) \int_{w \in W_k^r(\xi)} f_W(w) \, dw \, d\xi.$$

b_k^r gives the probability that alternative a_k has rank k and it is within the range $[0, 1]$.

- The central weight vector describes the preferences of a typical DM giving to a_k the best position and is defined as follows:

$$w_k^c = \frac{1}{b_k^1} \int_{\xi \in \chi} f_\chi(\xi) \int_{w \in W^1(\xi)} f_W(w) w \, dw \, d\xi;$$

- The confidence factor is defined as the frequency of an alternative to be the preferred one with the preferences expressed by its central weight vector and is given by:

$$p_k^c = \int_{\substack{\xi \in \chi : u(\xi_k, w_k^c) \geq u(\xi_h, w_k^c) \\ \forall h=1,\dots,l}} f_\chi(\xi) \, d\xi.$$

In the following, we shall consider also the frequency that an alternative a_h is preferred to an alternative a_k in the space of the preference parameters (weight vectors in case of SMAA-2), i.e.

$$p(a_h, a_k) = \int_{w \in W} f_W(w) \int_{\xi \in \chi : u(\xi_h, w) \geq u(\xi_k, w)} f_\chi(\xi) d\xi \, dw.$$

From a computational point of view, the considered indices are evaluated by the multidimensional integrals approximated by using the Monte Carlo method.

After performing Monte Carlo simulations, the aforementioned indices are evaluated simultaneously in order to help the DM to choose the best alternative of the decision problem under consideration.

4 An extension of the SMAA method to the Choquet integral preference model

In this section, we shall describe how integrate the SMAA methodology within the Choquet integral preference model in three different cases:

case 1) the evaluations on the criteria are on a common scale and are given in a precise way, that is $g_i(a_k) \in \mathbb{R}$ for all i and for all k,

case 2) the evaluations on criteria are on a common scale, but they can be given in an imprecise way, that is $g_i(a_k) \in [\alpha_i^k, \beta_i^k]$ and $\alpha_i^k \leq \beta_i^k$, for some i and for some k,

case 3) the evaluations on the criteria are on different scales and therefore a common scale has to be constructed (for the sake of simplicity in this case we have supposed that evaluations of alternatives on the considered criteria are given in a precise way).

In Section 2, we have observed that the use of the Choquet integral in terms of Möbius representation with a 2-additive capacity involves the knowledge of $n + \binom{n}{2}$ parameters. In order to get these parameters, the DM is therefore asked to provide some preference information in a direct or an indirect way. In the context of the Choquet integral preference model, the direct preference information consists of providing the capacity involved in the considered method while the indirect preference information consists in asking the DM some preference information regarding comparisons of alternatives or importance and interaction of criteria from which eliciting the capacity. Generally, the indirect preference information requires less cognitive effort from the DM, and for this reason it is widely used in MCDA (see for example [30, 31] if the preference model is a value function, [42, 43] if the preference model is an outranking relation and [4] if the preference model is the Choquet integral). Notice that the use of the indirect preference information is intrinsic in the decision rule approach [22].

If the preference model is the Choquet integral, the DM can provide the preference information presented in the following, together with its formulation in terms of linear constraints:

- Comparisons related to importance and interaction of criteria:

    - criterion i is at least as important as j ($i \succsim j$): $\varphi(\{g_i\}) \geq \varphi(\{g_j\})$;

    - criterion i is more important than criterion j ($i \succ j$): $\varphi(\{g_i\}) > \varphi(\{g_j\})$;

    - criteria i and j have the same importance ($i \sim j$): $\varphi(\{g_i\}) = \varphi(\{g_j\})$;

    - criteria i and j are synergic: $\varphi(\{g_i, g_j\}) > 0$;

    - criteria i and j are redundant: $\varphi(\{g_i, g_j\}) < 0$.

- Comparisons between couples or quadruples of alternatives:

– alternative a_k is at least as good as alternative a_h $(a_k \gtrsim a_h)$: $C_\mu(a_k) \geq C_\mu(a_h)$;

– alternative a_k is preferred to alternative a_h $(a_k \succ a_h)$: $C_\mu(a_k) > C_\mu(a_h)$;

– alternative a_k and a_h are indifferent $(a_k \sim a_h)$: $C_\mu(a_k) = C_\mu(a_h)$;

– alternative a_k is preferred to alternative a_h more than alternative a_s is preferred to alternative a_t $((a_k, a_h) \succ^* (a_s, a_t))$: $C_\mu(a_k) - C_\mu(a_h) > C_\mu(a_s) - C_\mu(a_t)$;

– the difference of preference between a_k and a_h is the same of the difference of preference between a_s and a_t $((a_k, a_h) \sim^* (a_s, a_t))$: $C_\mu(a_k) - C_\mu(a_h) = C_\mu(a_s) - C_\mu(a_t)$.

For descriptive purposes, we distinguish three sets of constraints:

- Monotonicity and boundary constraints

$$
\left.
\begin{aligned}
& m(\{\emptyset\}) = 0, \ \sum_{g_i \in G} m(\{g_i\}) + \sum_{\{g_i, g_j\} \subseteq G} m(\{g_i, g_j\}) = 1, \\
& m(\{g_i\}) \geq 0, \ \forall g_i \in G, \\
& m(\{g_i\}) + \sum_{g_j \in T} m(\{g_i, g_j\}) \geq 0, \forall g_i \in G \text{ and } \forall T \subseteq G \setminus \{g_i\}, T \neq \emptyset
\end{aligned}
\right\} \quad (E^{MB})
$$

- Constraints related to importance and interaction of criteria:

$$
\left.
\begin{aligned}
& \varphi(\{g_i\}) \geq \varphi(\{g_j\}), \quad \text{if } i \gtrsim j, \\
& \varphi(\{g_i\}) \geq \varphi(\{g_j\}) + \varepsilon, \quad \text{if } i \succ j, \\
& \varphi(\{g_i\}) = \varphi(\{g_j\}), \quad \text{if } i \sim j, \\
& \varphi(\{g_i, g_j\}) \geq \varepsilon, \quad \text{if criteria } i \text{ and } j \text{ are synergic with } i, j \in G, \\
& \varphi(\{g_i, g_j\}) \leq -\varepsilon, \quad \text{if criteria } i \text{ and } j \text{ are redundant with } i, j \in G,
\end{aligned}
\right\} \quad (E^C)
$$

- Constraints related to comparisons between alternatives:

$$
\left.
\begin{aligned}
& C_\mu(a_k) \geq C_\mu(a_h), \quad \text{if } a_k \gtrsim a_h, \\
& C_\mu(a_k) \geq C_\mu(a_h) + \varepsilon, \quad \text{if } a_k \succ a_h, \\
& C_\mu(a_k) = C_\mu(a_h) \quad \text{if } a_k \sim a_h, \\
& C_\mu(a_k) - C_\mu(a_h) \geq C_\mu(a_s) - C_\mu(a_t) + \varepsilon, \quad \text{if } (a_k, a_h) \succ^* (a_s, a_t), \\
& C_\mu(a_k) - C_\mu(a_h) = C_\mu(a_s) - C_\mu(a_t), \quad \text{if } (a_k, a_h) \sim^* (a_s, a_t),
\end{aligned}
\right\} \quad (E^A)
$$

where the strict inequalities have been transformed in weak inequalities in E^C and E^A by adding an auxiliary variable ε.

We shall call *compatible model*, a capacity whose Möbius representation satisfies the set of constraints $E^{DM} = E^{MB} \cup E^C \cup E^A$ where E^C or E^A could be eventually empty if the DM does not provide any information on importance and interaction of criteria, or comparison of alternatives, respectively.

In order to check if there exists at least one compatible model, one has to solve the following linear programming problem:

$$
\max \varepsilon = \varepsilon^* \quad s.t.
$$
$$
E^{DM}.
$$
(7)

If E^{DM} is feasible and $\varepsilon^* > 0$, then there exists at least one model compatible with the preference information provided by the DM. If E^{DM} is infeasible or $\varepsilon^* \leq 0$, then one can check which is the minimum set of constraints causing the infeasibility using one of the techniques described in [41].

Each compatible model restores all the information provided by the DM but, generally, gives different recommendations regarding the alternatives that are not provided as example by the DM. For this reason, the choice of only one compatible model could be considered arbitrary and meaningless. The SMAA methodology can then be applied in order to take into account the whole set of models compatible with the preference information provided by the DM computing the above recalled indices i.e. the rank acceptability indices, central weight vectors and frequency of the preference. Notice that within the Choquet integral preference model we do not work with weights but with capacities expressed in terms of Möbius representations; for this reason, the Möbius representation of the central capacity will be the equivalent of the central weight vector in SMAA. Also ROR [13, 29], and Non-Additive ROR (NAROR) in case the evaluations on criteria g_i from G are aggregated through the Choquet integral [4], take into account the whole set of compatible models. More precisely within NAROR, and, more generally, within ROR, two preference relations are taken in consideration: for all $a_h, a_k \in A$

- *necessary preference*, which holds whenever alternative a_h is preferred to alternative a_k for all compatible models,

- *possible preference*, which holds whenever alternative a_h is preferred to alternative a_k for at least one compatible model.

Observe that because the computation of the necessary and possible preference relations involve to solve two linear programming problems for each pair of alternatives in A, we could use the SMAA methodology to approximate the two preference relations. In fact:

- if a_h is necessarily preferred to a_k, then the sum of the frequencies of the preference of a_h over a_k and of the indifference between a_h and a_k is 100%,

- if the frequency of the preference of a_h over a_k is greater than zero, then a_h is possibly preferred to a_k.

The vice versa of these statements are not true because, from one side, even if the frequency of the preference of a_h over a_k is 100%, then there could exist one non-sampled compatible model for which a_k is preferred to a_h; from the other side, even if a_h is possibly preferred to a_k, it is possible that for all sampled compatible models a_k is at least as good as a_h and therefore the frequency of the preference of a_h over a_k is 0%. Observe that the larger is the sample of compatible models, the better the approximation of the necessary and the possible preference relations obtained with SMAA, such that, in case of an enough large sample of compatible models, one can reasonably accept the approximations of SMAA as results of the ROR approach. Another result of ROR that can be approximated through SMAA is the interval of ranking positions of an alternative given by the extreme ranking analysis [32]. Also in this case, the larger the set of compatible models, the more accurate is the approximation.

Now, we shall describe how the SMAA methodology can be integrated with the Choquet integral in each of the three cases above considered.

case 1) Since the evaluations on considered criteria are on a common scale, they are given in a precise way, and the preference information provided by the DM are related only to the importance and interaction of criteria defined by the linear constraints in E^{DM}, we apply the Hit-and-Run method [51, 55] in order to sample some compatible models.

The Hit-And-Run sampling starts from the choice of one point (the Möbius representation of one capacity) inside the polytope E^{DM} delimited by the constraints translating the DM's preference information. At each iteration, a random direction is sampled from the unit hypersphere that, with the considered position, generates a line. Finally, one point inside the segment whose extremes are the intersection of the line with its boundaries and the current point is sampled.

At each iteration of the Hit-and-Run method, one can store the sampled compatible model. Then the Choquet integral of every considered alternative is evaluated with respect to the stored

compatible model, such that at each iteration a complete ranking of alternatives is obtained.

In this way, at the end of all the iterations, on the basis of the obtained rankings of the considered alternatives, one can compute the frequency of the preference and of the indifference between two alternatives, the Möbius representation of the central capacity for each alternative that arrived first at least once and also the rank acceptability index of each alternative with respect to all the possible ranking positions.

case 2) In this case, the criteria evaluations may be given as intervals of possible values, that is, $g_i(a_k)$ can be whichever value inside the interval $\left[\alpha_i^k, \beta_i^k\right]$ for some i and some k.

At each iteration, one samples an evaluation matrix M whose element M_{ki}, $k = 1, \ldots, l$ and $i = 1, \ldots, n$ is taken in a random way inside the interval $\left[\alpha_i^k, \beta_i^k\right]$. Then, one samples a model compatible with the preference information provided by the DM. Observe that the constraints related to preference information in terms of importance and interaction of criteria, i.e. the constraints E^C, are not dependent from the sampled evaluations. Instead, the constraints related to preference information in terms of pairwise comparisons between alternatives, i.e. the constraints in E^A, are dependent from the sampled evaluations. Thus, we distinguish two hypothesis: $E^A = \emptyset$ or $E^A \neq \emptyset$.

In the first hypothesis ($E^A = \emptyset$), the DM is not able or does not want to provide any preference information on comparisons of alternatives. Therefore, the compatible model sampled at each iteration is independent from the sampled evaluation matrix. For this reason, in order to sample models compatible with the preference information provided by the DM, one can use the Hit-and-Run method as done in case 1). The only difference with respect to the first case, is that the Choquet integral of each alternative does not depend only from the compatible model sampled in that iteration, but it depends also from the evaluation matrix that has been sampled at the beginning of that iteration.

In the second hypothesis ($E^A \neq \emptyset$), the DM is able to provide some preference information on comparisons of alternatives. For this reason, one cannot use the Hit-and-Run method because the set E^A depends from the sampled evaluation matrix and therefore at each iteration one has to sample a compatible model from a different set of constraints E^{DM}. Due to the change of the constraints in E^{DM} at each iteration, it is possible that for certain sampled evaluation matrices there is no model compatible with the DM's preference information. For this reason, at each iteration, before sampling a compatible model, one needs to check if the set of constraints

E^{DM} is feasible. In this case, one can sample a compatible model and consequently obtain the complete ranking of the considered alternatives by applying the Choquet integral with the sampled evaluation matrix and the sampled compatible model.

In both these hypotheses, after all the iterations, one can compute the frequency of the preference or indifference in each couple of alternatives, the Möbius representation of the central capacity and the rank acceptability index for each alternative. The only difference between the two hypotheses is that, in the second one, to compute the frequency of the preference for each couple of alternatives, one has to consider only those iterations in which, the set of constraints E^{DM} corresponding to the sampled evaluation matrix is feasible.

case 3) In the third case, we suppose that the evaluations with respect to considered criteria are on heterogeneous scales. For example, in evaluating a sport car, one can consider criteria such as maximum speed, acceleration, price, comfort etc. and each of them has a different scale. In this case, one can not apply directly the Choquet integral to aggregate the preferences of the DM because, a requisite of the method is that all considered criteria have a common scale.

In order to deal with this drawback, we propose to construct at each iteration a common scale with a procedure having the following steps for each criterion g_i:

- sampling uniformly from the interval $[0, 1]$, l' different real numbers $x_1, \ldots, x_{l'}$ supposing that the different evaluations on g_i are l', with $l' \leq l$,

- ordering the l' numbers in an increasing way, $x_{i(1)} < \ldots < x_{i(l')}$,

- assigning $x_{i(h)}$ to the alternatives having the h-th evaluation, in an increasing order, on g_i.

Supposing to deal with the aforementioned car evaluation problem, and looking at the evaluations of the considered cars on criterion acceleration shown in Table 1, we proceed as follows:

- Because the evaluations of the 10 alternatives on criterion acceleration are all different, we sample 10 different real numbers from the interval $[0, 1]$. For example, $x_1 = 0.81$, $x_2 = 0.90$, $x_3 = 0.12$, $x_4 = 0.91$, $x_5 = 0.63$, $x_6 = 0.09$, $x_7 = 0.27$, $x_8 = 0.54$, $x_9 = 0.95$, $x_{10} = 0.96$.

- We order the 10 numbers in an increasing way: $x_{(1)} = 0.09 < x_{(2)} = 0.12 < x_{(3)} = 0.27 < x_{(4)} = 0.54 < x_{(5)} = 0.63 < x_{(6)} = 0.81 < x_{(7)} = 0.90 < x_{(8)} = 0.91 < x_{(9)} = 0.95 < x_{(10)} = 0.96$.

- We assign value $x_{(1)} = 0.09$ to PEUGEOT 208 1.6 8V, value $x_{(2)} = 0.12$ to FIAT 500 0.9 and so on (see the third column of Table 1).

Table 1: Car evaluation with respect to the criterion acceleration and the corresponding scale

Cars	Acceleration	Scale value
PEUGEOT 208 1.6 8V	10.9	0.09
Citroen C3	13.5	0.90
FIAT 500 0.9	11	0.12
SKODA Fabia 1.2	14.2	0.95
LANCIA Ypsilon 5p	11.4	0.54
RENAULT Clio 1.5 dCi 90	11.3	0.27
SEAT Ibiza ST 1.2	14.6	0.96
ALFA ROMEO MiTo 1.3	12.9	0.81
TOYOTA Yaris 1.5	11.8	0.63
VOLKSWAGEN Polo 1.2	13.9	0.91

The values $x_{i(r)}, i = 1, \ldots, n$ and $r = 1, \ldots, l'$, become the evaluations of the considered alternatives on the different criteria. In this way, evaluations on all criteria are expressed on the same common scale and therefore, having the Möbius representation of a capacity compatible with the DM's preferences, one can compute the Choquet integral of all alternatives.

As described in case 2), the sampling of the scale will influence the sampling of compatible models only if the DM will provide some preference information on the comparison of alternatives. For this reason, one can distinguish, again, between $E^A = \emptyset$ and $E^A \neq \emptyset$.

If the DM does not provide any preference information in terms of comparisons between alternatives ($E^A = \emptyset$), then one can proceed exactly as done in case of imprecise evaluations, that is using the Hit-and-Run for sampling compatible models and obtaining at each iteration a complete ranking of the considered alternatives by applying the Choquet integral with the sampled common scale and the sampled compatible model.

If the DM provides some preference information in terms of pairwise comparisons of alternatives ($E^A \neq \emptyset$), then the sampling of a model compatible with the DM's preference information depends from the sampled common scale. For this reason, at each iteration one can sample a common scale and a model compatible with it. In this case, different options are possible:

- After all iterations, compute the rank acceptability indices, the Möbius representation of the central capacity for each alternative and the preference matrix. This is analogous to the case of imprecise evaluations presented previously because the sampling of a common scale at the beginning of each iteration is conceptually analogous to the sampling of an evaluation matrix.

- Sampling a certain number of possible common scales $S_1, \ldots, S_{iter}$, considering the corresponding feasible sets of constraints $E_1^{DM}, \ldots, E_{iter}^{DM}$ and denoted by $\varepsilon_1, \ldots, \varepsilon_{iter}$, the solutions of the linear programming problems

$$
\left.\begin{array}{ll} \max \varepsilon & s.t. \\[1em] E_1^{DM} & \end{array}\right\}, \ldots \ldots \ldots \ldots, \left.\begin{array}{ll} \max \varepsilon & s.t. \\[1em] E_{iter}^{DM} & \end{array}\right\} \tag{8}
$$

we can present to the DM the scale S_k such that $\varepsilon_k = \max\{\varepsilon_1, \ldots, \varepsilon_{iter}\}$, that is the most discriminant one admitting a representation of DM's preference information in terms of the Choquet integral preference model.

After obtaining the most discriminant common scale, the decision aiding process can continue in one of the following ways:

- asking the capacities directly to the DM,

- inducing one capacity compatible with the preference information given by the DM [39],

- considering the whole set of capacities compatible with the preference information provided by the DM taking into account necessary and possible preference relations using NAROR [4],

- applying the SMAA methodology as in case the evaluations of alternatives on considered criteria are given on a common scale from the beginning.

5   Some examples

The whole methodology presented in the previous section will be illustrated by several didactic examples. In the following, where not differently stated, we shall consider uniform probability distributions f_W and f_χ, respectively, on W and χ.

5.1   An example with all criteria expressed on the same common scale

Let us consider a set of 18 alternatives evaluated on the basis of 4 criteria: $G = \{g_1, g_2, g_3, g_4\}$. The criteria are on a $[0, 20]$ scale and the scores of every alternative are reported in Table 2.

Firstly, the DM expresses his/her preference information on the importance and interaction of criteria that can be synthesized as follows:

Table 2: Evaluation matrix

Criteria	Alternatives								
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
g_1	15	7	18	9	12	8	14	8	3
g_2	12	8	8	16	5	3	20	13	17
g_3	10	14	4	4	14	7	5	15	2
g_4	7	16	12	16	14	20	10	6	14
Criteria	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
g_1	4	16	8	17	8	20	12	14	9
g_2	20	7	11	12	6	7	4	11	13
g_3	8	14	5	6	7	4	15	12	12
g_4	9	10	19	8	19	12	13	9	6

- g_1 is more important than g_2,

- g_3 is more important than g_4,

- there is a positive interaction between criteria g_1 and g_2,

- there is a positive interaction between criteria g_2 and g_3,

- there is a negative interaction between criteria g_2 and g_4.

As explained in Section 4, we use the Hit-and-Run procedure considering a number of iterations equal to $2,000,000$.

For each iteration, we sample a model compatible with the preference information provided by the DM, and therefore we compute the Choquet integral of each alternative obtaining a complete ranking. At the end of all iterations, we compute the rank acceptability index b_k^r for each $k, r = 1, \ldots, l$ and the Möbius representation of the central capacity for each alternative a_k that can get the first rank at least once, as shown, respectively, in Tables 3 and 4. In particular, looking at Table 3, we observe that alternatives a_3, a_4, a_9, a_{10} and a_{18} can never be ranked first, a_{11} has reached the first position more than all other alternatives ($b_{11}^1 = 26.22$) and a_9 is instead the alternative that reached almost always the last position in the obtained rankings ($b_9^{18} = 93.18$).

For each pair of alternatives (a_h, a_k) we compute also the frequency of the preference of a_h over a_k as shown in Table 6. Looking at the results, alternative a_{11} can be still considered the best because

Table 3: Rank acceptability indices considering precise evaluations on the considered criteria and preference information only on importance and interaction of criteria

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}	b_k^{11}	b_k^{12}	b_k^{13}	b_k^{14}	b_k^{15}	b_k^{16}	b_k^{17}	b_k^{18}
a_1	1.76	9.46	16.16	10.66	13.72	12.75	10.64	7.81	4.98	3.85	2.89	2.35	1.37	1.15	0.45	0.02	0.00	0.00
a_2	10.88	6.03	6.19	8.15	9.32	9.00	7.11	9.02	7.34	8.75	5.75	5.86	2.87	1.94	1.38	0.41	0.00	0.00
a_3	0.00	4.37	3.85	3.10	3.10	5.24	6.45	5.01	6.10	6.19	6.97	9.31	6.23	8.23	9.73	10.47	5.51	0.14
a_4	0.00	0.03	0.07	0.17	0.45	0.72	1.02	2.10	2.69	4.70	5.26	8.49	16.93	13.06	16.65	18.44	9.21	0.00
a_5	7.60	11.15	12.89	9.43	8.21	7.68	6.83	9.69	6.97	5.97	5.69	3.02	2.62	1.79	0.44	0.03	0.00	0.00
a_6	1.60	1.30	1.29	1.42	1.93	2.31	2.94	3.42	3.12	5.56	4.62	5.40	6.13	8.79	9.05	20.47	19.69	0.96
a_7	13.77	4.54	5.00	5.72	5.00	6.24	7.81	6.31	9.52	7.66	6.99	7.30	6.34	3.49	3.25	1.01	0.05	0.00
a_8	3.31	3.11	3.94	4.78	7.53	10.01	10.04	10.69	9.42	9.88	11.04	5.03	3.48	3.21	2.83	1.65	0.03	0.00
a_9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.03	0.08	0.15	0.26	1.05	5.26	93.18
a_{10}	0.00	0.00	0.00	0.00	0.00	0.02	0.07	0.19	0.78	1.32	2.66	2.93	6.32	7.41	6.84	12.78	53.79	4.88
a_{11}	26.22	17.56	15.43	15.32	9.66	6.37	5.69	2.29	0.98	0.33	0.12	0.03	0.00	0.00	0.00	0.00	0.00	0.00
a_{12}	1.14	1.01	1.67	1.67	1.66	2.01	2.91	4.11	5.34	5.38	9.17	10.31	11.59	21.09	15.51	5.13	0.31	0.00
a_{13}	0.59	4.56	6.53	8.77	7.27	5.96	7.18	7.05	8.10	9.06	6.20	5.31	5.05	5.78	4.36	6.20	1.53	0.51
a_{14}	0.67	2.43	2.24	1.82	2.63	4.50	5.45	4.66	8.05	6.73	8.51	7.40	16.40	8.98	12.33	7.08	0.12	0.00
a_{15}	12.84	5.54	3.16	3.70	5.05	5.97	4.50	5.11	5.33	5.46	7.06	5.45	5.60	6.16	7.46	7.10	4.17	0.33
a_{16}	3.02	8.29	10.75	10.93	8.96	6.73	6.79	7.22	8.80	6.58	5.17	5.26	3.16	3.38	3.40	1.47	0.08	0.00
a_{17}	16.60	20.65	10.81	13.48	12.62	9.63	6.48	3.74	2.85	1.69	0.84	0.35	0.22	0.03	0.00	0.00	0.00	0.00
a_{18}	0.00	0.00	0.02	0.87	2.88	4.85	8.07	11.56	9.64	10.90	11.08	16.16	5.61	5.37	6.05	6.69	0.24	0.00

it is preferred to all other alternatives with a frequency of at least the 63.31%, while a_9 can be considered surely the worst alternative because all alternatives are preferred to it with a frequency at least equal to the 94.77%.

Computing the Möbius representation of the barycenter of compatible capacities shown in Table 5, and applying the Choquet integral, we find the following complete ranking of the considered alternatives:

$$a_{11} \succ a_{17} \succ a_1 \succ a_5 \succ a_{16} \succ a_2 \succ a_7 \succ a_8 \succ a_{13} \succ a_{15} \succ a_{18} \succ a_3 \succ a_{14} \succ a_{12} \succ a_6 \succ a_4 \succ a_{10} \succ a_9.$$

Let us suppose the DM gives the following further preference information in terms of comparisons of alternatives: $a_{16} \succ a_2$, $a_3 \succ a_{14}$ and $a_{13} \succ a_8$.

Using the procedure described in Section 4, we obtain the rank acceptability indices, the Möbius representations of the central capacities and the frequency of the preference between pairs of alternatives shown, respectively, in Tables 17, 18 and 20 in the Appendix.

Considering this preference information, we get some new results. Looking at Table 17, alternative a_9 is almost always the worst because it has reached the last position in the rank with a frequency of the 96.89%, but, differently from the previous case, a_{11} is no more the alternative obtaining the first

Table 4: Möbius representation of the central capacity for alternatives being the first at least once in the final ranking obtained by the Choquet integral, considering precise evaluations on considered criteria and preference information only on importance and interaction of criteria

Alt/Möbius	m({1})	m({2})	m({3})	m({4})	m({1,2})	m({1,3})	m({1,4})	m({2,3})	m({2,4})	m({3,4})
a_1	0.36	0.09	0.18	0.18	0.21	-0.06	-0.09	0.17	-0.04	0.01
a_2	0.25	0.11	0.37	0.31	0.08	-0.09	-0.07	0.08	-0.07	0.04
a_5	0.12	0.12	0.19	0.17	0.05	0.10	0.09	0.06	-0.08	0.17
a_6	0.19	0.11	0.48	0.57	0.04	0.07	-0.07	0.04	-0.09	-0.33
a_7	0.28	0.20	0.22	0.19	0.18	-0.06	0.02	0.06	-0.09	-0.01
a_8	0.25	0.10	0.50	0.21	0.08	-0.16	0.04	0.15	-0.05	-0.11
a_{11}	0.25	0.10	0.24	0.22	0.06	0.15	-0.02	0.07	-0.05	-0.02
a_{12}	0.20	0.11	0.45	0.49	0.12	-0.01	-0.04	0.06	-0.05	-0.34
a_{13}	0.43	0.07	0.23	0.13	0.24	-0.17	-0.06	0.07	-0.03	0.08
a_{14}	0.16	0.11	0.32	0.48	0.07	0.11	-0.07	0.14	-0.08	-0.24
a_{15}	0.49	0.11	0.36	0.22	0.07	-0.23	-0.01	0.09	-0.06	-0.04
a_{16}	0.14	0.09	0.47	0.22	0.04	-0.02	0.16	0.04	-0.06	-0.07
a_{17}	0.18	0.10	0.15	0.16	0.15	0.10	0.02	0.17	-0.04	0.01

Table 5: Möbius representation of the barycenter of compatible capacities considering precise evaluations on considered criteria and preference information only on importance and interaction of criteria

m({1})	m({2})	m({3})	m({4})	m({1,2})	m({1,3})	m({1,4})	m({2,3})	m({2,4})	m({3,4})
0.26	0.12	0.27	0.22	0.098	0.008	-0.001	0.09	-0.06	-0.07

rank more than all other alternatives. In fact, a_{15} has been the first with a frequency of the 35.98%, while a_{11} has been the first with a frequency of the 30.33%. Looking at Table 20, we can see that in the space of compatible preference models, alternative a_{11} is preferred to a_{15} with a frequency of the 53.63%. From Table 18 we still get that, in average, the main differences in the Möbius representation of central capacities of a_{11} and a_{15} are the following:

- $m(\{1\}) = 0.35$ for a_{11} being the first and $m(\{1\}) = 0.52$ for a_{15} being the first,

- $m(\{3\}) = 0.20$ for a_{11} being the first and $m(\{3\}) = 0.31$ for a_{15} being the first,

- $m(\{1,3\}) = 0.08$ for a_{11} being the first and $m(\{1,3\}) = -0.2$ for a_{15} being the first.

Also in this case, we have computed the Möbius representations of the barycenter of the compatible capacities shown in Table 19 of the Appendix, and again applying the Choquet integral considering the precise evaluations on the considered criteria, we get the following complete ranking of the considered alternatives:

Table 6: Frequency of the preference between pairs of alternatives considering precise evaluations on considered criteria and preference information only on importance and interaction of criteria

Alt/Alt	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
a_1	0.00	56.46	82.31	94.54	50.14	85.66	67.36	70.21	99.94	99.94	24.69	86.75	82.95	83.25	71.03	54.40	19.54	92.90
a_2	43.54	0.00	68.55	90.84	42.85	91.78	54.37	59.23	99.97	98.56	24.05	86.02	60.76	86.17	61.69	50.39	30.49	75.18
a_3	17.69	31.45	0.00	70.87	30.27	63.25	26.46	36.13	99.33	84.09	13.18	58.31	28.54	54.25	13.88	33.60	15.02	47.75
a_4	5.46	9.16	29.13	0.00	9.26	46.67	8.76	11.70	100.00	78.89	1.17	26.52	16.96	28.28	25.23	12.64	1.93	21.43
a_5	49.86	57.15	69.73	90.74	0.00	91.72	59.45	66.32	99.99	99.32	18.10	85.11	63.90	86.92	62.57	76.10	33.99	82.88
a_6	14.34	8.22	36.75	53.33	8.28	0.00	24.38	20.54	99.03	73.12	4.41	35.32	28.96	10.39	33.08	10.31	8.08	28.92
a_7	32.64	45.63	73.54	91.24	40.55	75.62	0.00	55.11	99.80	97.72	25.14	77.58	64.36	70.14	61.07	44.18	24.53	69.11
a_8	29.79	40.77	63.87	88.30	33.68	79.46	44.89	0.00	100.00	98.97	12.26	76.40	52.87	73.15	55.81	37.86	14.60	78.19
a_9	0.06	0.03	0.67	0.00	0.01	0.97	0.20	0.00	0.00	5.23	0.00	0.00	1.19	0.13	0.70	0.03	0.00	0.08
a_{10}	0.06	1.44	15.91	21.11	0.68	26.88	2.28	1.03	94.77	0.00	0.01	16.18	6.90	13.10	14.26	1.54	0.01	1.74
a_{11}	75.31	75.95	86.82	98.83	81.90	95.59	74.86	87.74	100.00	99.99	0.00	94.78	81.70	94.66	77.34	85.12	63.31	98.66
a_{12}	13.25	13.98	41.69	73.48	14.89	64.68	22.42	23.60	100.00	83.82	5.22	0.00	28.98	36.20	37.16	19.12	7.47	33.20
a_{13}	17.05	39.24	71.46	83.04	36.10	71.04	35.64	47.13	98.81	93.10	18.30	71.02	0.00	64.55	52.97	39.93	17.03	58.90
a_{14}	16.75	13.83	45.75	71.72	13.08	89.61	29.86	26.85	99.87	86.90	5.34	63.80	35.45	0.00	41.22	17.65	8.79	37.08
a_{15}	28.97	38.31	86.12	74.77	37.43	66.92	38.93	44.19	99.30	85.74	22.66	62.84	47.03	58.78	0.00	40.69	24.17	54.33
a_{16}	45.60	49.61	66.40	87.36	23.90	89.69	55.82	62.14	99.97	98.46	14.88	80.88	60.07	82.35	59.31	0.00	30.80	77.23
a_{17}	80.46	69.51	84.98	98.07	66.01	91.92	75.47	85.40	100.00	99.99	36.69	92.53	82.97	91.21	75.83	69.20	0.00	100.00
a_{18}	7.10	24.82	52.25	78.57	17.12	71.08	30.89	21.81	99.92	98.26	1.34	66.80	41.10	62.92	45.67	22.77	0.00	0.00

$$a_{11} \succ a_{15} \succ a_{17} \succ a_1 \succ a_{13} \succ a_7 \succ a_3 \succ a_5 \succ a_{16} \succ a_2 \succ a_8 \succ a_{18} \succ a_4 \succ a_{12} \succ a_{14} \succ a_6 \succ a_{10} \succ a_9.$$

We observe that, alternative a_{15} is in the second position, while it was in the middle of the ranking obtained considering the barycenter in case there were not preference information on comparisons between alternatives.

5.2 Considering imprecision in the evaluations on considered criteria

At this stage, we slightly modify the evaluation matrix shown in Table 2, by introducing imprecision on the evaluation criteria (see Table 7). In this example, we suppose that the evaluations of considered alternatives on each criterion are integer numbers within an interval (for example, the evaluation of a_1 on criterion g_1 can be 14, 15 or 16), but this is not a specific requirement for our model i.e., in general, we can sample values from the whole interval. We can consider this as a specific probability distribution $f_\chi(\xi)$ concentrating uniformly the mass only on the integers in the interval of evaluations on considered criteria.

Firstly, we shall take into account only the preference information in terms of importance and interaction of criteria as presented in Section 5.1. As explained in Section 4, the set of constraints

Table 7: Imprecise evaluations of alternatives on considered criteria

Criteria	Alternatives								
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
g_1	[14,16]	[6,8]	[17,19]	[8,10]	[11,13]	[7,9]	[13,15]	[7,9]	3
g_2	[11,13]	[7,9]	[7,9]	[15,17]	5	3	[18,20]	[12,14]	[16,18]
g_3	[9,11]	[13,15]	4	[3,5]	[13,15]	[6,9]	5	[14,15]	2
g_4	[6,9]	[15,17]	[11,13]	[15,17]	[13,15]	[18,20]	[9,11]	6	[13,15]

Criteria	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
g_1	4	[15,17]	[7,9]	[16,18]	[7,9]	[18,20]	[11,13]	[13,15]	[8,10]
g_2	[18,20]	7	[10,12]	[11,13]	[6,8]	[6,9]	4	[10,12]	[12,14]
g_3	[7,9]	[13,15]	5	[5,7]	[6,9]	[3,5]	[14,16]	[11,13]	[11,13]
g_4	[8,10]	[9,11]	[18,20]	8	[18,20]	[11,13]	[12,15]	[8,10]	[5,7]

E^{DM} is the same during all the iterations and therefore we can apply the Hit-and-Run method to sample $2,000,000$ compatible models. At the end of all iterations we obtain the rank acceptability index of each alternative for each possible rank, the Möbius representations of the central capacities and the preference matrix shown, respectively, in Tables 21, 22 and 24 in the Appendix. Comparing results in Tables 3 and 21, we observe that the imprecision on the evaluations of alternatives with respect to considered criteria has increased the uncertainty for the ranking position of the alternatives. In fact, in case of exact evaluations, five alternatives cannot reach the first position, while considering imprecise interval evaluations, only two alternatives (a_9 and a_{10}) cannot arrive first in the final ranking. Nevertheless, the best alternative should be still chosen from a_{11}, a_{17}, and a_7 because they have the greatest rank acceptabilities for the first position ($b_{11}^1 = 21.06\%$, $b_{17}^1 = 15.98\%$ and $b_7^1 = 11.43\%$) while the worst one is almost surely a_9 because it has the greatest rank acceptability for the last position ($b_9^{18} = 90.05\%$). Table 24 supports the idea that a_{11} is the best alternative because it is preferred to all other alternatives with a frequency at least equal to the 57.60%. Table 22 gives the Möbius representation of the central capacities ranking the considered alternatives in the first position.

As already done previously, we find the barycenter of the Möbius representations of the central capacities shown in Table 23 in the Appendix and representing the average preferences of the DM. Because, we sampled uniformly at each iteration an evaluation matrix, the average evaluation matrix is that one in which each element is the middle point of each interval. Applying the Choquet integral considering the average evaluation matrix and the barycenter shown in Table 23, we get the following ranking of alternatives:

$a_{11} \succ a_{17} \succ a_1 \succ a_5 \succ a_2 \succ a_{16} \succ a_7 \succ a_8 \succ a_{13} \succ a_{15} \succ a_3 \succ a_{18} \succ a_{14} \succ a_{12} \succ a_4 \succ a_6 \succ a_{10} \succ a_9.$

Introducing the preference information in terms of comparison between alternatives presented in Section 5.1, we find the rank acceptability indices, the Möbius representations of the central capacities and the preference matrix shown in Tables 25, 26 and 28 reported in the Appendix. Moreover, considering the barycenter of the Möbius representations of the central capacities and applying the Choquet integral we get the following ranking of considered alternatives:

$a_{17} \succ a_7 \succ a_{11} \succ a_1 \succ a_{13} \succ a_{15} \succ a_5 \succ a_3 \succ a_{16} \succ a_8 \succ a_{18} \succ a_2 \succ a_4 \succ a_{12} \succ a_{14} \succ a_6 \succ a_{10} \succ a_9.$

5.3 An example with the criteria expressed on different scales

In this section, we deal with a decision making problem in which the evaluation of alternatives on considered criteria are expressed on heterogeneous scales.

From the city-car segment market, we select ten cars evaluated on the basis of the following criteria: price (in Euro), acceleration (0/100 km/h in seconds), maximum speed (in km/h) and consumption (in l/100km) (see Table 8).

Table 8: Evaluation matrix

Cars	Price Euro	Acceleration 0/100 km/h	Max speed km/h	Consumption l/100km
PEUGEOT 208 1.6 8V e-HDi 92 CV Stop&Start 3p. Allure	17,800	10.9	185	3.8
Citroen C3 1.4 HDi 70 Seduction	15,750	13.5	163	3.8
FIAT 500 0.9 TwinAir Turbo Street	15,050	11	173	4
SKODA Fabia 1.2 TDI CR 75 CV 5p. GreenLine	15,260	14.2	172	3.4
LANCIA Ypsilon 5p 1.3 MJT 95 CV 5p. S&S Gold	16,300	11.4	183	3.8
RENAULT Clio 1.5 dCi 90 CV 3p. Dynamique	16,050	11.3	176	4
SEAT Ibiza ST 1.2 TDI CR Ecomotive	15,700	14.6	173	3.4
ALFA ROMEO MiTo 1.3 JTDm 85 CV S&S Progression	17,500	12.9	174	3.5
TOYOTA Yaris 1.5 Hybrid 5p. Lounge	17,800	11.8	165	3.2
VOLKSWAGEN Polo 1.2 TDI 5p. BlueMotion 89g	17,060	13.9	173	3.4

The DM supplies the following preference information in terms of importance and interaction of criteria as well as in terms of comparisons of alternatives:

- Preference between alternatives: $a_5 \succ a_1$, $a_7 \succ a_6$, $a_2 \succ a_3$,

- Comparisons of importance of criteria: $g_1 \succ g_2$, $g_4 \succ g_3$,

- Positive interaction between criteria g_3 and g_4 and negative interaction between criteria g_2 and g_3.

As explained in Section 4, at each iteration we sample a common scale, and, if the set of constraints E^{DM}, that in this case will depend from the sampled scale, is feasible, then we sample a capacity compatible with these constraints.

At the end of all the iterations, we shall get the rank acceptability indices, the Möbius representations of the central capacities for each alternative and the preference matrix shown respectively in Tables 9, 10 and 11.

Looking at Table 9, we observe that car a_7 is the most preferred by the DM ($b_7^1 = 71.82\%$) followed by a_4, while a_3 is most frequently the least preferred car ($b_3^{10} = 40.68\%$) and a_1, a_2, a_3, a_6 and a_{10} can never arrive first. Table 10 gives the Möbius representations of the central capacities ranking considered alternatives in the first position at least once, while from Table 11, giving the frequency of the preference between pairs of alternatives, we observe that a_7 is preferred to all other alternatives with a frequency at least equal to 71.91%.

Table 9: Rank acceptability indices sampling simultaneously compatible capacities and scales

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}
a_1	0	0	0	0.017	0.14	1.68	17.45	25.28	30.11	25.30
a_2	0	0	5.32	10.41	21.22	52.43	9.74	0.80	0.04	0
a_3	0	0	0	0.18	0.54	1.41	5.25	22.96	28.94	40.68
a_4	28.06	71.14	0.71	0.07	0.004	0	0	0	0	0
a_5	0.04	0.189	17.15	20.88	39.46	14.50	5.28	1.73	0.74	0
a_6	0	0	0.1	0.63	1.67	6.27	36.4	28.26	19.9	6.74
a_7	71.82	28.09	0.07	0.002	0	0	0	0	0	0
a_8	0.058	0.252	15.62	43.49	21.7	13.12	3.899	1.57	0.259	0.013
a_9	0.006	0.004	0.41	1.21	4.27	7.036	20.739	19.101	19.95	27.25
a_{10}	0	0.31	60.59	23.08	10.96	3.52	1.22	0.26	0.02	0

Table 10: Möbius representations of central capacities sampling simultaneously compatible capacities and scales

Alt/Möbius	m({1})	m({2})	m({3})	m({4})	m({1,2})	m({1,3})	m({1,4})	m({2,3})	m({2,4})	m({3,4})
a_4	0.14	0.21	0.19	0.14	0.077	0.001	0.22	-0.09	-0.01	0.096
a_5	0.17	0.22	0.21	0.17	0.002	0.001	0.19	-0.109	0.001	0.109
a_7	0.14	0.21	0.19	0.14	0.06	0.0006	0.23	-0.1	-0.007	0.099
a_8	0.17	0.22	0.21	0.18	0.015	0.002	0.18	-0.108	-0.001	0.108
a_9	0.18	0.15	0.13	0.51	0.012	0.008	-0.0004	-0.067	-0.009	0.067

Since there are different common compatible scales, we propose the DM the most discriminant common scale presented in Table 12. After the DM accepts the common scale, we apply SMAA sampling capacities compatible with the preference information provided by the DM, computing the

Table 11: Frequency of the preference between pairs of alternatives considering a simulation sampling of random capacities and common scales

Alt/Alt	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
a_1	0	1.64	58.82	0	0	31.02	0	0.619	48.18	0.31
a_2	98.35	0	100	0	25.72	95.56	0	24.59	91.2	11.08
a_3	41.17	0	0	0	2.99	16.61	0	2.38	36.3	0.67
a_4	99.99	100	100	0	99.77	100	28.08	99.67	99.98	99.65
a_5	100	74.27	97	0.22	0	97.7	0.067	39.38	90.88	23.12
a_6	68.97	4.43	83.38	0	2.29	0	0	4.97	57.32	2.21
a_7	99.99	100	100	71.91	99.93	100	0	99.91	99.99	100
a_8	99.38	75.4	97.61	0.32	60.61	95.02	0.082	0	96.81	23.74
a_9	51.81	8.79	63.69	0.01	9.11	42.67	0.007	3.18	0	0.85
a_{10}	99.68	88.91	99.32	0.34	76.87	97.78	0	76.25	99.14	0

rank acceptability indices, the Möbius representations of the central capacities and the preference matrix displayed, respectively, in Tables 13, 14 and 16. Applying the Choquet integral with respect to the Möbius representation of the barycenter of the compatible capacities shown in Table 15, and considering the most discriminating common scale we get the following ranking of the considered alternatives:

$$a_7 \succ a_4 \succ a_5 \succ a_8 \succ a_2 \succ a_{10} \succ a_3 \succ a_1 \succ a_9 \succ a_6.$$

Table 12: Evaluations of alternatives on considered criteria expressed on the most discriminating common scale

Alt	Price	Acceleration	Max speed	Consumption
a_1	0.0193	0.1135	0.6644	0.5638
a_2	0.7066	0.2816	0.0717	0.5638
a_3	0.9955	0.1187	0.3358	0.0878
a_4	0.7377	0.4395	0.09	0.6603
a_5	0.4261	0.2167	0.6397	0.5638
a_6	0.4802	0.2017	0.6284	0.0878
a_7	0.7105	0.8193	0.3358	0.6603
a_8	0.2811	0.2606	0.6164	0.6138
a_9	0.0199	0.248	0.0868	0.9567
a_{10}	0.3982	0.2835	0.3358	0.6603

Table 13: Rank acceptability indices taking into account evaluations of alternatives on considered criteria expressed on the most discriminating common scale shown in Table 12

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}
a_1	0	0	0.21	4.09	4.61	6.82	16.89	24.35	28.87	14.11
a_2	0	0	27.23	13.39	13.17	21.76	15.13	8.53	0.75	0
a_3	0	0	0	7.55	5.34	6.41	20.99	19.99	25.36	14.32
a_4	0	65.12	10.82	10.44	6.72	5.17	1.69	0.001	0	0
a_5	0.52	21.71	34.57	23.30	12.25	4.92	2.17	0.53	0	0
a_6	0	0	0	0	0.001	0.502	7.71	28.22	27.76	35.79
a_7	97.11	1.809	0.85	0.21	0.003	0	0	0	0	0
a_8	1.12	7.79	19.78	21.83	17.31	20.66	10.45	1.02	0	0
a_9	1.22	3.43	3.17	1.98	2.7	5.57	13.58	15.42	17.12	35.76
a_{10}	0	0.11	3.32	17.16	37.86	28.16	11.35	1.89	0.109	0

Table 14: Möbius representation of the central capacities, taking into account evaluations of alternatives on considered criteria expressed on the most discriminating common scale, shown in Table 12

Alt/Möbius	$m(\{1\})$	$m(\{2\})$	$m(\{3\})$	$m(\{4\})$	$m(\{1,2\})$	$m(\{1,3\})$	$m(\{1,4\})$	$m(\{2,3\})$	$m(\{2,4\})$	$m(\{3,4\})$
a_5	0.089	0.106	0.155	0.2	-0.016	0.087	0.09	-0.03	-0.015	0.336
a_7	0.222	0.201	0.163	0.244	0.038	0.002	0.102	-0.06	-0.007	0.09
a_8	0.145	0.098	0.167	0.28	0.017	-0.017	-0.027	-0.033	-0.012	0.38
a_9	0.269	0.152	0.099	0.685	-0.005	0.016	-0.187	-0.03	-0.05	0.055

Table 15: Möbius representation of the barycenter of capacities taking into account evaluations of alternatives on the considered criteria expressed on the most discriminant common scale shown in Table 12

m({1})	m({2})	m({3})	m({4})	m({1,2})	m({1,3})	m({1,4})	m({2,3})	m({2,4})	m({3,4})
0.22	0.19	0.16	0.25	0.037	0.003	0.097	-0.05	-0.008	0.09

Table 16: Frequency of the preference between pairs of alternatives taking into account evaluations of alternatives on the most discriminant common scale shown in Table 12

Alt/Alt	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
a_1	0	18.33	48.3	6.52	0	64.7	0.21	0	56.69	9.97
a_2	81.66	0	100	0	30.88	98.49	0	43.23	80.83	52.06
a_3	51.69	0	0	0	8.66	74.71	0	15.11	58.21	17.68
a_4	93.47	100	100	0	69.57	99.99	0	75.77	92.34	87.75
a_5	100	69.11	91.33	30.42	0	100	1.27	77.77	90.54	88.35
a_6	35.29	1.5	25.28	0.004	0	0	0	0.18	47.07	0.022
a_7	99.7843	100	100	100	98.72	100	0	98.55	98.75	100
a_8	100	56.76	84.88	24.22	22.22	99.81	1.44	0	90.09	65.22
a_9	43.308	19.16	41.78	7.65	9.45	52.92	1.24	9.904	0	11.73
a_{10}	90.02	47.93	82.31	12.24	11.64	99.97	0	34.77	88.26	0

6 Conclusions

In this paper, we have integrated the Stochastic Multiobjective Acceptability Analysis (SMAA) to the Choquet integral preference model extending a work already published by the authors [1]. We have proposed to explore the space of the parameters compatible with some preference information provided by the DM using SMAA. In particular, we have considered the DM's preference information not only in terms of relative importance of criteria and interaction between them , but differently from [1], also in terms of pairwise comparison between alternatives and comparisons of intensity of preferences between pairs of alternatives. Moreover, again differently from [1], we have considered also imprecise evaluations of alternatives on the considered criteria expressed in terms of intervals of possible values.

Finally, we have proposed a methodology to construct the common scale required by the Choquet integral; this is very useful in case the criteria for the decision problem at hand are defined on different

scales. Such aspect of the methodology we are proposing, constitutes another original contribution with respect to [1]. We have provided several different examples in which the proposed methodology has been applied. We envisage the following future developments:

- application of SMAA methodology to some extensions of the classical Choquet integral, e.g. the bipolar Choquet integral [17, 18] (see also [23]), the level dependent Choquet integral [21], the robust Choquet integral [28];

- implementation of the SMAA methodology to the Choquet integral in presence of hierarchy of criteria [2] within the so called multiple criteria hierarchy process [9, 10].

We believe that the methodology we are proposing can greatly contribute to extend and to improve the use of the Choquet integral preference model in Multiple Criteria Decision Aiding.

References

[1] S. Angilella, S. Corrente, and S. Greco. SMAA-Choquet: Stochastic multicriteria acceptability analysis for the Choquet integral. In S. Greco et al., editor, *Advances in Computational Intelligence*, volume 300 of *Communications in Computer and Information Science*, pages 248–257. Springer Berlin Heidelberg, 2012.

[2] S. Angilella, S. Corrente, S. Greco, and R. Słowiński. Multiple criteria hierarchy process for the Choquet integral. In R.C. Purshouse et al., editor, *EMO 2013*, volume 7811 of *LNCS*, pages 475–489. Springer Berlin Heidelberg, 2013.

[3] S. Angilella, S. Greco, F. Lamantia, and B. Matarazzo. Assessing non-additive utility for multicriteria decision aid. *European Journal of Operational Research*, 158(3):734–744, 2004.

[4] S. Angilella, S. Greco, and B. Matarazzo. Non-additive robust ordinal regression: A multiple criteria decision model based on the Choquet integral. *European Journal of Operational Research*, 201(1):277–288, 2010.

[5] J.P. Brans and B. Mareschal. PROMETHEE Methods. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 163–196. Springer, Berlin, 2005.

[6] J.P. Brans, B. Mareschal, and Ph. Vincke. PROMETHEE: a new family of outranking methods in multicriteria analysis. In J.P. Brans, editor, *Operational Research, IFORS 84*, pages 477–490. North Holland, Amsterdam, 1984.

[7] A. Chateauneuf and J.Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of möbius inversion. *Mathematical Social Sciences*, 17:263–283, 1989.

[8] G. Choquet. Theory of capacities. *Ann. Inst. Fourier*, 5(131-295):54, 1953.

[9] S. Corrente, S. Greco, and R. Słowiński. Multiple Criteria Hierarchy Process in Robust Ordinal Regression. *Decision Support Systems*, 53(3):660–674, 2012.

[10] S. Corrente, S. Greco, and R. Słowiński. Multiple Criteria Hierarchy Process with ELECTRE and PROMETHEE. *Omega*, 41:820–846, 2013.

[11] J.R. Figueira, S. Greco, and M. Ehrgott, editors. *Multiple Criteria Decision Analysis: State of the Art Surveys*. Springer, Berlin, 2010.

[12] J.R. Figueira, S. Greco, and B. Roy. ELECTRE methods with interaction between criteria: An extension of the concordance index. *European Journal of Operational Research*, 199(2):478 – 495, 2009.

[13] J.R. Figueira, S. Greco, and R. Słowiński. Building a set of additive value functions representing a reference preorder and intensities of preference: GRIP method. *European Journal of Operational Research*, 195(2):460–486, 2009.

[14] I. Gilboa and D. Schmeidler. Additive representations of non-additive measures and the Choquet integral. *Ann. Operational Research*, 52:43–65, 1994.

[15] M. Grabisch. The application of fuzzy integrals in multicriteria decision making. *European Journal of Operational Research*, 89:445–456, 1996.

[16] M. Grabisch. *k*-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[17] M. Grabisch and C. Labreuche. Bi-capacities-I: definition, Möbius transform and interaction. *Fuzzy Sets and Systems*, 151(2):211–236, 2005.

[18] M. Grabisch and C. Labreuche. Bi-capacities-II: the Choquet integral. *Fuzzy Sets and Systems*, 151(2):237–259, 2005.

[19] M. Grabisch and C. Labreuche. Fuzzy measures and integrals in MCDA. In J.R. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 563–604. Springer, Berlin, 2005.

[20] M. Grabisch and C. Labreuche. A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid. *4OR*, 6(1):1–44, 2008.

[21] S. Greco, B. Matarazzo, and S. Giove. The Choquet integral with respect to a level dependent capacity. *Fuzzy Sets and Systems*, 175:1–35, 2011.

[22] S. Greco, B. Matarazzo, and R. Słowiński. Rough sets theory for multicriteria decision analysis. *European Journal of Operational Research*, 129(1):1–47, 2001.

[23] S. Greco, B. Matarazzo, and R. Słowiński. Bipolar Sugeno and Choquet integrals. In *Proceedings of the EUROFUSE 02 Workshop on Information Systems*, pages 191–196, 2002.

[24] S. Greco, B. Matarazzo, and R. Słowiński. Axiomatic characterization of a general utility function and its particular cases in terms of conjoint measurement and rough-set decision rules. *European Journal of Operational Research*, 158(1):271–292, 2004.

[25] S. Greco, B. Matarazzo, and R. Słowiński. Decision Rule Approach. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 507–562. Springer, Berlin, 2005.

[26] S. Greco, V. Mousseau, and R. Słowiński. Ordinal regression revisited: multiple criteria ranking using a set of additive value functions. *European Journal of Operational Research*, 191(2):416–436, 2008.

[27] S. Greco and F. Rindone. Bipolar fuzzy integrals. *Fuzzy Sets and Systems*, 2013, to appear. `http://dx.doi.org/10.1016/j.fss.2012.11.021`.

[28] S. Greco and F. Rindone. Robust integrals. *Fuzzy Sets and Systems*, 2013, to appear. `http://dx.doi.org/10.1016/j.fss.2013.01.008`.

[29] S. Greco, R. Słowiński, J.R. Figueira, and V. Mousseau. Robust Ordinal Regression. In M. Ehrgott, J.R. Figueira, and S. Greco, editors, *Trends in Multiple Criteria Decision Analysis*, pages 241–283. Springer, New York, 2010.

[30] E. Jacquet-Lagreze and J. Siskos. Assessing a set of additive utility functions for multicriteria decision-making, the UTA method. *European Journal of Operational Research*, 10(2):151–164, 1982.

[31] E. Jacquet-Lagrèze and Y. Siskos. Preference disaggregation: 20 years of MCDA experience. *European Journal of Operational Research*, 130(2):233–245, 2001.

[32] M. Kadziński, S. Greco, and R. Słowiński. Extreme ranking analysis in robust ordinal regression. *Omega*, 40(4):488–501, 2012.

[33] M. Kadziński and T. Tervonen. Robust multi-criteria ranking with additive value models and holistic pair-wise preference statements. *European Journal of Operational Research*, 2013, to appear.

[34] M. Kadziński and T. Tervonen. Stochastic ordinal regression for multiple criteria sorting problems. *Decision Support Systems*, 2013, to appear.

[35] R.L. Keeney and H. Raiffa. *Decisions with multiple objectives: Preferences and value tradeoffs.* J. Wiley, New York, 1976.

[36] R. Lahdelma, J. Hokkanen, and P. Salminen. SMAA - stochastic multiobjective acceptability analysis. *European Journal of Operational Research*, 106(1):137–143, 1998.

[37] R. Lahdelma, K. Miettinen, and P. Salminen. Ordinal criteria in stochastic multicriteria acceptability analysis (SMAA). *European Journal of Operational Research*, 147(1):117–127, 2003.

[38] R. Lahdelma and P. Salminen. SMAA-2: Stochastic multicriteria acceptability analysis for group decision making. *Operations Research*, 49(3):444–454, 2001.

[39] J.L. Marichal and M. Roubens. Determination of weights of interacting criteria from a reference set. *European Journal of Operational Research*, 124(3):641–650, 2000.

[40] F. Modave and M. Grabisch. Preference representation by the Choquet integral: The commensurability hypothesis. *Proceedings of the Seventh International Conference IPMU Paris, July 610, 1998*, pages 164–171, 1998.

[41] V. Mousseau, J. R. Figueira, L. Dias, C. Gomes da Silva, and J. Climaco. Resolving inconsistencies among constraints on the parameters of an MCDA model. *European Journal of Operational Research*, 147(1):72–93, 2003.

[42] V. Mousseau and R. Słowiński. Inferring an ELECTRE TRI model from assignment examples. *Journal of Global Optimization*, 12(2):157–174, 1998.

[43] V. Mousseau, R. Słowiński, and P. Zielniewicz. A user-oriented implementation of the ELECTRE-TRI method integrating preference elicitation support. *Computers and Operations Research*, 27:757–777, 2000.

[44] S. Murofushi, T. Soneda. Techniques for reading fuzzy measures (iii): interaction index. *9th Fuzzy Systems Symposium, Sapporo, Japan*, pages 693–696, 1993.

[45] G.C. Rota. On the foundations of combinatorial theory. I. Theory of Möbius functions. *Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 2:340–368, 1964.

[46] B. Roy. *Multicriteria Methodology for Decision Aiding*. Kluwer Academic, Dordrecht, 1996.

[47] B. Roy. ELECTRE Methods. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, pages 133–162. Springer, Berlin, 2005.

[48] G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

[49] L.S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Contributions to the Theory of Games II*, pages 307–317. Princeton University Press, Princeton, 1953.

[50] R. Słowiński, S. Greco, and B. Matarazzo. Rough sets in decision making. In R.A. Meyers, editor, *Encyclopedia of Complexity and Systems Science*, pages 7753–7786, New York, 2009. Springer.

[51] R.L. Smith. Efficient Monte Carlo procedures for generating points uniformly distributed over bounded regions. *Operations Research*, 32:1296–1308, 1984.

[52] M. Sugeno. *Theory of fuzzy integrals and its applications*. Tokyo institute of Technology, 1974.

[53] T. Tervonen and J.R. Figueira. A survey on stochastic multicriteria acceptability analysis methods. *Journal of Multi-Criteria Decision Analysis*, 15(1-2):1–14, 2008.

[54] T. Tervonen, J.R. Figueira, R. Lahdelma, D.A. Juscelino, and P. Salminen. A stochastic method for robustness analysis in sorting problems. *European Journal of Operational Research*, 192(1):236–242, 2009.

[55] T. Tervonen, G. Van Valkenhoef, N. Bastürk, and D. Postmus. Hit-and-run enables efficient weight generation for simulation-based multiple criteria decision analysis. *European Journal of Operational Research*, 224:552–559, 2013.

Appendix

Table 17: Rank acceptability indices considering preference information both in terms of comparison of alternatives and importance and interaction of criteria

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}	b_k^{11}	b_k^{12}	b_k^{13}	b_k^{14}	b_k^{15}	b_k^{16}	b_k^{17}	b_k^{18}
a_1	1.25	7.05	20.65	15.61	22.70	14.64	6.70	4.49	4.33	2.02	0.49	0.04	0.02	0.01	0.00	0.00	0.00	0.00
a_2	0.00	0.02	0.89	1.72	2.86	3.63	3.53	4.40	5.90	21.34	15.26	17.32	8.53	7.24	5.72	1.61	0.03	0.00
a_3	0.00	14.80	11.54	7.51	5.84	9.47	14.00	8.32	9.18	7.55	5.30	4.85	1.00	0.50	0.14	0.00	0.00	0.00
a_4	0.00	0.00	0.00	0.03	0.10	0.28	0.42	2.95	2.10	3.77	4.97	9.02	28.28	11.80	14.35	19.25	2.67	0.00
a_5	4.06	5.81	7.42	8.10	7.92	7.97	8.70	25.07	11.06	8.34	3.53	1.52	0.46	0.03	0.00	0.00	0.00	0.00
a_6	0.00	0.00	0.01	0.03	0.02	0.04	0.12	0.24	0.42	1.42	2.63	4.17	4.58	11.66	8.52	40.58	23.45	2.12
a_7	17.21	7.07	6.83	9.79	8.15	9.44	13.66	6.32	9.19	6.23	3.85	1.32	0.62	0.28	0.03	0.00	0.00	0.00
a_8	0.00	0.00	0.01	0.07	0.54	2.18	3.95	11.22	10.55	20.45	26.58	8.63	4.71	4.69	4.12	2.31	0.01	0.00
a_9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.04	0.27	2.78	96.89
a_{10}	0.00	0.00	0.00	0.00	0.00	0.02	0.06	0.13	0.37	0.50	1.09	1.00	3.18	5.37	5.19	11.73	70.38	0.99
a_{11}	30.33	18.57	14.55	14.01	9.74	6.68	5.74	0.34	0.04	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a_{12}	0.00	0.00	0.02	0.05	0.10	0.18	0.27	0.66	1.08	2.05	3.82	6.98	12.93	37.34	28.29	5.88	0.36	0.00
a_{13}	0.98	8.39	15.35	17.00	16.57	11.90	12.89	10.13	4.71	1.24	0.39	0.30	0.12	0.02	0.00	0.00	0.00	0.00
a_{14}	0.00	0.00	0.00	0.00	0.04	0.08	0.11	0.37	0.91	2.85	6.24	8.11	26.64	14.16	27.83	12.35	0.31	0.00
a_{15}	35.98	12.28	4.75	6.14	7.22	10.41	5.66	5.24	4.32	2.77	2.75	1.16	0.58	0.34	0.33	0.06	0.00	0.00
a_{16}	0.45	3.89	6.58	7.90	8.06	6.90	7.72	12.19	25.64	6.21	7.45	3.33	2.49	1.13	0.06	0.00	0.00	0.00
a_{17}	9.73	22.11	11.40	12.01	9.81	14.51	13.98	3.13	1.97	0.87	0.45	0.02	0.01	0.00	0.00	0.00	0.00	0.00
a_{18}	0.00	0.00	0.00	0.04	0.33	1.69	2.50	4.81	8.22	12.38	15.18	32.23	5.83	5.42	5.39	5.98	0.01	0.00

Table 18: Möbius representations of central capacities for alternatives taking into account precise evaluations on considered criteria and preference information both in terms of comparison of alternatives and importance and interaction of criteria

Alt/Möbius	m({1})	m({2})	m({3})	m({4})	m({1,2})	m({1,3})	m({1,4})	m({2,3})	m({2,4})	m({3,4})
a_1	0.38	0.10	0.14	0.11	0.18	0.02	-0.02	0.13	-0.04	0.01
a_5	0.26	0.10	0.20	0.11	0.06	-0.05	0.11	0.04	-0.06	0.23
a_7	0.32	0.22	0.18	0.15	0.12	-0.02	0.06	0.05	-0.08	0.00
a_{11}	0.35	0.10	0.20	0.16	0.07	0.08	0.00	0.06	-0.05	0.03
a_{13}	0.44	0.07	0.16	0.08	0.23	-0.08	-0.01	0.05	-0.03	0.08
a_{15}	0.52	0.10	0.31	0.17	0.06	-0.20	0.03	0.07	-0.05	-0.01
a_{16}	0.31	0.07	0.39	0.12	0.04	-0.20	0.22	0.02	-0.04	0.07
a_{17}	0.24	0.10	0.11	0.10	0.16	0.10	0.06	0.13	-0.04	0.04

Table 19: Möbius representation of the barycenter of compatible capacities taking into account precise evaluations on considered criteria and preference information both in terms of comparison of alternatives and importance and interaction of criteria

m({1})	m({2})	m({3})	m({4})	m({1,2})	m({1,3})	m({1,4})	m({2,3})	m({2,4})	m({3,4})
0.39	0.12	0.23	0.15	0.086	-0.045	0.033	0.067	-0.053	0.02

Table 20: Frequency of the preference between pairs of alternatives taking into account precise evaluations on considered criteria and preference information both in terms of comparison of alternatives and importance and interaction of criteria

Alt/Alt	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
a_1	0.00	90.34	60.44	99.91	71.66	99.66	57.00	98.28	100.00	100.00	24.60	99.68	56.53	99.71	41.99	74.29	36.57	100.00
a_2	9.66	0.00	19.32	79.09	2.42	94.84	15.90	46.24	99.94	96.38	1.35	83.35	12.29	88.01	12.78	0.00	6.29	60.07
a_3	39.56	80.68	0.00	97.31	58.69	99.86	45.88	84.21	100.00	98.58	28.08	98.54	36.69	100.00	2.56	62.93	37.81	87.02
a_4	0.09	20.91	2.69	0.00	6.07	70.65	0.05	14.80	100.00	93.23	0.09	57.33	0.42	53.37	2.23	8.81	0.03	19.30
a_5	28.34	97.58	41.31	93.93	0.00	99.73	37.63	81.39	99.99	98.96	6.25	97.35	35.18	99.26	30.14	77.75	20.32	89.19
a_6	0.34	5.16	0.14	29.35	0.27	0.00	1.77	10.55	97.88	73.47	0.00	22.66	0.85	6.08	0.00	0.39	0.17	14.21
a_7	43.00	84.10	54.12	99.95	62.37	98.23	0.00	91.99	100.00	100.00	32.00	99.10	46.98	97.86	38.11	65.33	39.23	97.40
a_8	1.72	53.76	15.79	85.20	18.61	89.45	8.01	0.00	100.00	99.06	0.41	85.96	0.00	84.80	9.68	22.92	1.66	68.45
a_9	0.00	0.06	0.00	0.00	0.01	2.12	0.00	0.00	0.00	0.99	0.00	0.00	0.00	0.32	0.00	0.02	0.00	0.00
a_{10}	0.00	3.62	1.42	6.77	1.04	26.53	0.00	0.94	99.01	0.00	0.00	11.80	0.02	14.84	1.04	1.89	0.00	1.00
a_{11}	75.40	98.65	71.92	99.91	93.75	100.00	68.00	99.59	100.00	100.00	0.00	100.00	71.33	100.00	53.63	95.71	72.88	100.00
a_{12}	0.32	16.65	1.46	42.67	2.65	77.34	0.90	14.04	100.00	88.20	0.00	0.00	0.97	50.18	1.09	4.92	0.10	18.24
a_{13}	43.47	87.71	63.31	99.58	64.82	99.15	53.02	100.00	100.00	99.98	28.67	99.03	0.00	98.96	33.03	68.69	39.70	98.89
a_{14}	0.29	11.99	0.00	46.63	0.74	93.92	2.14	15.20	99.68	85.16	0.00	49.82	1.04	0.00	0.40	1.66	0.07	19.28
a_{15}	58.01	87.22	97.44	97.77	69.86	100.00	61.89	90.32	100.00	98.96	46.37	98.91	66.97	99.60	0.00	73.79	53.59	90.86
a_{16}	25.71	100.00	37.07	91.19	22.25	99.61	34.67	77.08	99.98	98.11	4.29	95.08	31.31	98.34	26.21	0.00	18.58	83.79
a_{17}	63.43	93.71	62.19	99.97	79.68	99.83	60.77	98.34	100.00	100.00	27.12	99.90	60.30	99.93	46.41	81.42	0.00	100.00
a_{18}	0.00	39.93	12.98	80.70	10.81	85.79	2.60	31.55	100.00	99.00	0.00	81.76	1.11	80.72	9.14	16.21	0.00	0.00

Table 21: Rank acceptability indices taking into account imprecise evaluations and preference information both in terms of comparison of alternatives and importance and interaction of criteria

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}	b_k^{11}	b_k^{12}	b_k^{13}	b_k^{14}	b_k^{15}	b_k^{16}	b_k^{17}	b_k^{18}
a_1	7.80	11.66	12.23	10.72	11.48	10.95	9.26	7.45	5.23	3.89	3.05	2.22	1.63	1.25	0.83	0.33	0.03	0.00
a_2	9.42	7.08	7.25	7.84	8.24	8.19	7.56	7.69	6.93	7.52	6.19	5.50	3.96	2.98	2.36	1.15	0.11	0.01
a_3	2.16	4.16	4.12	4.34	4.94	5.64	6.01	5.69	6.05	6.32	7.07	7.76	7.43	7.82	8.40	7.87	3.97	0.26
a_4	0.10	0.29	0.51	0.82	1.20	1.55	2.07	3.25	3.99	5.40	6.84	9.38	12.79	14.08	15.01	14.78	7.96	0.00
a_5	6.34	9.60	10.55	9.78	8.52	7.84	7.50	8.30	7.63	6.38	5.68	4.53	3.32	2.58	1.12	0.30	0.02	0.00
a_6	0.50	0.92	1.25	1.36	1.69	2.23	2.81	3.25	3.92	4.77	4.90	5.47	6.82	8.83	11.15	20.99	17.64	1.49
a_7	11.43	6.15	5.87	5.83	5.74	6.12	6.89	6.70	7.95	8.05	7.38	6.97	5.79	4.14	3.07	1.71	0.20	0.01
a_8	1.87	2.58	3.53	4.74	6.48	8.21	9.03	9.63	9.40	9.53	9.50	6.99	5.38	4.53	4.30	3.26	0.65	0.38
a_9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.04	0.07	0.12	0.20	0.34	0.74	1.23	7.17	90.05
a_{10}	0.00	0.00	0.00	0.00	0.00	0.02	0.06	0.15	0.49	1.04	1.84	2.65	4.49	6.51	8.41	16.19	51.76	6.37
a_{11}	21.06	16.32	15.09	13.48	10.60	7.99	6.35	3.62	2.28	1.38	0.82	0.50	0.28	0.15	0.05	0.01	0.00	0.00
a_{12}	1.64	1.92	1.84	1.84	2.09	2.57	3.27	4.31	5.31	6.53	8.42	9.65	11.80	14.47	14.38	8.17	1.80	0.00
a_{13}	3.17	5.75	6.78	7.65	6.95	6.69	6.79	6.76	7.56	7.42	6.44	5.67	5.19	5.28	4.89	4.70	1.77	0.55
a_{14}	3.34	3.08	2.99	3.28	4.01	5.03	5.79	6.23	7.87	7.69	8.33	8.50	11.74	9.55	7.79	4.14	0.62	0.00
a_{15}	7.53	5.50	4.79	4.78	5.26	5.55	5.50	5.36	5.64	5.95	6.38	6.61	6.30	6.41	6.68	6.29	4.96	0.53
a_{16}	7.60	9.15	9.20	8.66	7.78	7.08	6.84	7.37	7.25	6.09	5.51	4.89	4.01	3.61	3.49	1.29	0.16	0.03
a_{17}	15.98	15.49	13.02	12.88	11.65	9.29	7.02	4.66	3.43	2.49	1.65	1.05	0.71	0.44	0.20	0.05	0.01	0.00
a_{18}	0.06	0.36	0.98	2.03	3.37	5.06	7.26	9.56	9.06	9.52	9.91	11.53	8.15	7.02	7.13	7.54	1.17	0.31

Table 22: Möbius representation of central capacities taking into account imprecise evaluations of alternatives on considered criteria and including preference information in terms of importance and interaction of criteria

Alt/Möbius	m({1})	m({2})	m({3})	m({4})	m({1,2})	m({1,3})	m({1,4})	m({2,3})	m({2,4})	m({3,4})
a_1	0.29	0.11	0.18	0.18	0.16	0.00	-0.03	0.14	-0.05	0.01
a_2	0.26	0.11	0.37	0.30	0.08	-0.09	-0.07	0.08	-0.07	0.03
a_3	0.49	0.12	0.37	0.24	0.07	-0.25	0.01	0.07	-0.06	-0.06
a_4	0.24	0.16	0.32	0.41	0.11	-0.03	-0.02	0.04	-0.03	-0.20
a_5	0.16	0.12	0.26	0.21	0.06	0.07	0.07	0.06	-0.08	0.07
a_6	0.21	0.13	0.44	0.58	0.04	0.05	-0.09	0.04	-0.11	-0.28
a_7	0.30	0.20	0.22	0.21	0.18	-0.08	0.02	0.06	-0.09	-0.01
a_8	0.25	0.10	0.45	0.20	0.08	-0.14	0.04	0.17	-0.04	-0.10
a_{11}	0.26	0.10	0.26	0.22	0.07	0.10	-0.02	0.08	-0.05	-0.02
a_{12}	0.24	0.13	0.37	0.51	0.10	-0.01	-0.09	0.06	-0.07	-0.25
a_{13}	0.45	0.13	0.26	0.18	0.15	-0.16	-0.07	0.08	-0.05	0.03
a_{14}	0.21	0.12	0.35	0.49	0.08	0.05	-0.09	0.08	-0.09	-0.21
a_{15}	0.50	0.12	0.36	0.24	0.07	-0.23	-0.02	0.08	-0.06	-0.04
a_{16}	0.18	0.11	0.35	0.22	0.05	0.01	0.06	0.05	-0.07	0.03
a_{17}	0.20	0.10	0.18	0.17	0.13	0.08	0.02	0.15	-0.04	0.00
a_{18}	0.16	0.10	0.24	0.17	0.11	0.06	0.05	0.23	-0.06	-0.06

Table 23: Möbius representation of the barycenter of the compatible capacities taking into account interval evaluations and preference information of alternatives on considered criteria in terms of importance and interaction of criteria

m({1})	m({2})	m({3})	m({4})	m({1,2})	m({1,3})	m({1,4})	m({2,3})	m({2,4})	m({3,4})
0.27	0.12	0.27	0.23	0.10	-0.007	-0.005	0.09	-0.06	-0.013

Table 24: Frequency of the preference between pairs of alternatives taking into account imprecise evaluations of alternatives on considered criteria and preference information expressed in terms of importance and interaction of criteria only

Alt/Alt	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}
a_1	0.00	58.70	78.99	91.93	54.65	86.95	67.84	74.97	99.56	99.69	33.11	84.96	77.10	77.84	72.88	56.05	34.62	88.88
a_2	41.30	0.00	64.82	84.80	44.84	91.95	53.41	63.46	99.91	98.53	26.97	80.98	57.54	74.49	61.01	47.78	32.65	73.23
a_3	21.01	35.18	0.00	70.30	34.29	67.41	30.46	43.17	99.11	86.97	17.40	59.37	33.71	50.44	38.55	36.42	19.39	51.98
a_4	8.07	15.20	29.70	0.00	14.00	53.61	12.01	21.74	100.00	82.30	4.64	34.27	19.97	25.77	27.52	16.88	5.18	29.96
a_5	45.35	55.16	65.71	86.00	0.00	92.65	56.97	68.91	99.88	99.28	22.34	81.66	60.05	77.52	61.78	56.91	34.60	79.78
a_6	13.05	8.05	32.59	46.39	7.35	0.00	23.11	21.81	98.43	74.52	4.75	30.60	26.35	13.00	30.55	8.61	8.25	28.71
a_7	32.16	46.59	69.54	87.99	43.03	76.89	0.00	58.11	99.50	97.24	27.80	75.82	59.52	63.88	62.11	44.89	27.60	68.24
a_8	25.03	36.54	56.83	78.26	31.09	78.19	41.89	0.00	99.10	97.76	12.07	70.11	47.26	60.70	52.63	33.71	14.22	69.80
a_9	0.44	0.09	0.89	0.00	0.12	1.57	0.50	0.90	0.00	7.34	0.08	0.01	1.49	0.10	1.04	0.26	0.16	1.15
a_{10}	0.31	1.47	13.03	17.70	0.72	25.48	2.76	2.24	92.66	0.00	0.06	13.69	6.20	8.05	12.53	1.46	0.12	3.40
a_{11}	66.89	73.03	82.60	95.36	77.66	95.25	72.20	87.93	99.92	99.94	0.00	91.39	77.31	88.28	77.35	77.99	57.60	95.87
a_{12}	15.04	19.02	40.63	65.73	18.34	69.40	24.18	29.89	99.99	86.31	8.61	0.00	29.68	29.69	37.78	21.38	10.72	37.54
a_{13}	22.90	42.46	66.29	80.03	39.95	73.65	40.48	52.74	98.51	93.80	22.69	70.32	0.00	59.91	56.87	42.00	22.21	62.07
a_{14}	22.16	25.51	49.56	74.23	22.48	87.00	36.12	39.30	99.90	91.95	11.72	70.31	40.09	0.00	46.50	26.17	15.16	48.14
a_{15}	27.12	38.99	60.83	72.48	38.22	69.45	37.89	47.37	98.96	87.47	22.65	62.22	43.13	53.50	0.00	40.20	24.45	55.56
a_{16}	43.95	52.22	63.58	83.12	43.09	91.39	55.11	66.29	99.74	98.54	22.01	78.62	58.00	73.83	59.80	0.00	34.14	75.92
a_{17}	65.29	67.35	80.61	94.82	65.40	91.75	72.40	85.78	99.84	99.88	42.40	89.28	77.79	84.84	75.55	65.86	0.00	97.38
a_{18}	11.12	26.77	48.02	70.04	20.22	71.29	31.76	30.20	98.85	96.60	4.13	62.46	37.93	51.86	44.44	24.08	2.62	0.00

Table 25: Rank acceptability indices taking into account imprecise evaluations of alternatives on considered criteria, preference information in terms of importance and interaction of criteria and comparisons between alternatives

Alt	b_k^1	b_k^2	b_k^3	b_k^4	b_k^5	b_k^6	b_k^7	b_k^8	b_k^9	b_k^{10}	b_k^{11}	b_k^{12}	b_k^{13}	b_k^{14}	b_k^{15}	b_k^{16}	b_k^{17}	b_k^{18}
a_1	19.03	22.62	23.26	15.84	9.28	4.99	2.73	1.29	0.62	0.22	0.09	0.03	0.01	0.00	0.00	0.00	0.00	0.00
a_2	0.00	0.01	0.04	0.13	0.43	1.60	3.84	7.12	10.46	14.90	17.97	19.02	12.43	7.38	4.10	0.55	0.02	0.00
a_3	0.63	1.55	2.56	3.93	6.36	9.95	13.55	13.66	12.51	11.35	11.17	9.05	2.88	0.80	0.05	0.00	0.00	0.00
a_4	0.00	0.01	0.04	0.10	0.26	0.64	1.28	2.63	3.55	4.89	6.33	8.90	14.84	20.53	23.38	10.84	1.78	0.00
a_5	0.47	1.62	3.70	7.17	10.58	12.19	12.14	12.65	12.70	10.04	7.57	5.02	2.57	1.13	0.41	0.03	0.00	0.00
a_6	0.00	0.00	0.00	0.00	0.00	0.01	0.02	0.04	0.08	0.19	0.43	0.84	1.96	4.75	10.73	44.11	36.42	0.41
a_7	22.46	15.47	15.55	18.16	11.48	6.59	4.04	2.61	1.75	0.96	0.51	0.25	0.10	0.04	0.01	0.00	0.00	0.00
a_8	0.00	0.01	0.09	0.52	2.14	7.74	11.46	13.98	12.38	12.80	11.37	9.61	7.85	5.49	3.53	1.02	0.01	0.00
a_9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.59	99.40
a_{10}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.03	0.07	0.17	0.60	1.91	5.88	30.78	60.35	0.19
a_{11}	25.30	22.25	18.37	14.01	9.37	5.43	3.37	1.37	0.40	0.10	0.03	0.01	0.00	0.00	0.00	0.00	0.00	0.00
a_{12}	0.00	0.00	0.00	0.02	0.05	0.15	0.40	1.04	2.07	3.63	5.81	9.13	16.47	26.31	27.59	6.90	0.44	0.00
a_{13}	2.29	6.10	10.01	15.94	23.21	17.91	11.67	6.41	3.78	1.70	0.66	0.22	0.07	0.02	0.00	0.00	0.00	0.00
a_{14}	0.00	0.00	0.00	0.00	0.02	0.09	0.39	1.51	3.35	5.66	8.71	13.71	23.24	20.85	17.91	4.23	0.32	0.00
a_{15}	2.91	4.15	5.40	7.43	10.53	13.45	12.39	9.96	8.22	7.05	6.43	5.05	3.38	2.07	1.24	0.31	0.05	0.00
a_{16}	0.47	1.03	1.69	2.93	5.28	8.65	12.29	15.63	17.18	14.82	10.75	5.60	2.60	0.99	0.07	0.00	0.00	0.00
a_{17}	26.42	25.15	19.06	12.73	7.65	4.48	2.69	1.20	0.46	0.12	0.03	0.01	0.00	0.00	0.00	0.00	0.00	0.00
a_{18}	0.00	0.03	0.23	1.10	3.36	6.12	7.74	8.91	10.48	11.55	12.07	13.37	10.98	7.74	5.08	1.21	0.02	0.00

Table 26: Möbius representation of central capacities for alternatives taking into account imprecise evaluations of alternatives on considered criteria, preferences on importance and interaction of criteria and comparisons between alternatives

Alt/Möbius	m({1})	m({2})	m({3})	m({4})	m({1,2})	m({1,3})	m({1,4})	m({2,3})	m({2,4})	m({3,4})
a_1	0.31	0.18	0.19	0.19	0.10	0.03	0.00	0.10	-0.09	-0.01
a_3	0.40	0.15	0.26	0.21	0.09	-0.11	0.03	0.09	-0.06	-0.05
a_4	0.27	0.14	0.27	0.24	0.08	-0.01	0.01	0.07	0.06	-0.15
a_5	0.28	0.16	0.20	0.21	0.06	0.05	0.04	0.06	-0.11	0.03
a_7	0.32	0.20	0.21	0.20	0.11	-0.04	0.01	0.09	-0.09	-0.01
a_{11}	0.30	0.16	0.17	0.19	0.09	0.09	0.01	0.08	-0.08	-0.02
a_{12}	0.30	0.10	0.30	0.30	0.07	-0.03	-0.02	0.09	0.04	-0.15
a_{13}	0.34	0.19	0.21	0.19	0.11	-0.04	0.00	0.09	-0.09	-0.01
a_{15}	0.39	0.15	0.25	0.21	0.09	-0.09	0.02	0.09	-0.07	-0.04
a_{16}	0.31	0.16	0.25	0.22	0.06	-0.03	0.04	0.06	-0.11	0.04
a_{17}	0.29	0.18	0.18	0.19	0.10	0.07	0.00	0.10	-0.08	-0.01
a_{18}	0.28	0.20	0.21	0.18	0.10	0.03	-0.02	0.12	-0.11	0.00

Table 27: Möbius representation of the barycenter of the compatible capacities taking into account interval evaluations of alternatives on considered criteria, preference information on importance and interaction of criteria and comparisons between alternatives

m({1})	m({2})	m({3})	m({4})	m({1,2})	m({1,3})	m({1,4})	m({2,3})	m({2,4})	m({3,4})
0.31	0.18	0.19	0.19	0.097	0.034	0.008	0.091	-0.08	-0.014

Table 28: Frequency of the preference between pairs of alternatives taking into account imprecise evaluations of alternatives on considered criteria, preferences on importance and interaction of criteria and comparisons between alternatives

Alt/Alt	a₁	a₂	a₃	a₄	a₅	a₆	a₇	a₈	a₉	a₁₀	a₁₁	a₁₂	a₁₃	a₁₄	a₁₅	a₁₆	a₁₇	a₁₈
a₁	0.00	99.09	92.23	99.70	91.39	99.98	55.80	98.53	100.00	100.00	47.59	99.88	80.36	99.85	85.65	94.32	44.09	99.16
a₂	0.91	0.00	29.52	78.07	14.31	99.20	2.96	37.69	99.99	99.33	0.25	83.54	6.58	77.66	23.08	0.00	0.39	45.69
a₃	7.77	70.48	0.00	93.07	43.69	99.57	8.24	61.99	100.00	99.87	8.77	95.18	18.31	100.00	35.82	53.00	8.02	66.50
a₄	0.30	21.93	6.93	0.00	9.17	86.72	0.23	16.44	100.00	95.37	0.45	51.96	0.97	42.25	6.56	11.77	0.23	21.00
a₅	8.61	85.69	56.31	90.83	0.00	99.99	15.78	65.84	100.00	99.85	2.72	94.92	27.39	94.32	46.07	62.74	5.17	73.95
a₆	0.02	0.80	0.43	13.28	0.01	0.00	0.14	2.68	99.59	62.13	0.00	9.16	0.13	4.64	0.78	0.04	0.00	2.85
a₇	44.20	97.04	91.76	99.77	84.22	99.86	0.00	96.75	100.00	100.00	43.89	99.73	76.56	99.37	84.76	90.05	40.20	96.87
a₈	1.47	62.31	38.01	83.56	34.16	97.32	3.25	0.00	100.00	99.93	2.64	86.11	0.00	79.90	29.48	42.28	1.66	58.17
a₉	0.00	0.01	0.00	0.00	0.00	0.41	0.00	0.00	0.00	0.20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
a₁₀	0.00	0.67	0.13	4.63	0.15	37.87	0.00	0.07	99.80	0.00	0.00	4.19	0.00	3.89	0.28	0.19	0.00	0.19
a₁₁	52.41	99.75	91.23	99.55	97.28	100.00	56.11	97.36	100.00	100.00	0.00	99.94	77.93	99.96	84.83	97.71	47.04	98.97
a₁₂	0.12	16.46	4.82	48.04	5.08	90.84	0.27	13.89	100.00	95.81	0.06	0.00	0.60	39.11	5.38	7.14	0.04	17.50
a₁₃	19.64	93.42	81.69	99.03	72.61	99.87	23.44	100.00	100.00	100.00	22.07	99.40	0.00	99.07	69.73	82.77	18.51	91.79
a₁₄	0.15	22.34	0.00	57.75	5.68	95.36	0.63	20.10	100.00	96.11	0.04	60.89	0.93	0.00	7.73	9.83	0.05	24.04
a₁₅	14.35	76.92	63.57	93.44	53.93	99.22	15.24	70.52	100.00	99.72	15.17	94.62	30.27	92.27	0.00	63.15	14.14	72.99
a₁₆	5.68	100.00	47.00	88.23	37.26	99.96	9.95	57.72	100.00	99.81	2.29	92.86	17.23	90.17	36.85	0.00	4.06	65.09
a₁₇	55.82	99.61	91.98	99.77	94.83	100.00	59.80	98.34	100.00	100.00	52.96	99.96	81.49	99.95	85.86	95.94	0.00	99.46
a₁₈	0.84	54.31	33.50	79.00	26.05	97.15	3.13	41.83	100.00	99.81	1.03	82.50	8.21	75.96	27.01	34.91	0.54	0.00