# RISK-SENSITIVE DIVIDEND PROBLEMS 

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#### Abstract

We consider a discrete time version of the popular optimal dividend payout problem in risk theory. The novel aspect of our approach is that we allow for a risk averse insurer, i.e., instead of maximising the expected discounted dividends until ruin we maximise the expected utility of discounted dividends until ruin. This task has been proposed as an open problem in [13. The model in a continuous-time Brownian motion setting with the exponential utility function has been analysed in [14. Nevertheless, a complete solution has not been provided. In this work, instead we solve the problem in discrete time setup for the exponential and the power utility functions and give the structure of optimal history-dependent dividend policies. We make use of certain ideas studied earlier in [6], where Markov decision processes with general utility functions were treated. Our analysis, however, include new aspects, since the reward functions in this case are not bounded.


Key words : Markov decision process, Dividend payout, Risk aversion, Historydependent policy, Fixed point problem
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## 1. Introduction

The dividend payout problem in risk theory has been introduced by de Finetti 10 and has since then been investigated under various extensions during the decades up to now; see, for instance, [9, 14]. The task is to find in a given model for the free surplus process of an insurance company, a dividend payout strategy that maximises the expected discounted dividends until ruin. Typical models for the surplus process are compound Poisson processes, diffusion processes, general renewal processes or discrete time processes. The reader is referred to [2] and [4], where an excellent overview of recent results is provided.

In [13] the authors propose the problem of maximising the expected utility of discounted dividends until ruin instead of maximising the expected discounted dividends until ruin. This means that an insurance company is equipped with some utility function that helps it to measure the accumulated dividends paid to the shareholders. If this utility is increasing and concave, the company is risk averse (see Remark 2.2). To the best of our knowledge, there is only one work [14], in which this idea was taken up. More precisely, the authors in [14 consider a linear Brownian motion model for the free surplus process and apply the exponential utility function to evaluate the discounted dividends until ruin. It turns out that the mathematics involved in the analysis of this problem is quite different from the one used in the risk neutral case and only partial results could be obtained. In contrast to the same problem with a risk neutral insurance company, where the optimal dividend payout strategy is a barrier strategy (see e.g., [3]), the authors in [14] are not able to identify the structure of the optimal dividend policy rigorously. They show imposing some further assumptions that there is a time dependent optimal barrier.

We study the same problem but with a discrete time surplus process. The risk neutral problem within such a framework can be found in Section 9.2 in [5] or in Section 1.2 in [21]. By making use of the dynamic programming approach the authors in [5] and [21] prove that the optimal dividend payout policy is a stationary band-strategy. In [1], on the other hand,
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the authors consider a discrete time model that is formulated with the aid of a general Lévy surplus process but the dividend payouts are allowed only at random discrete time points. This version can again be solved by the dynamic programming arguments. However, the problem with a general utility function is more demanding. Like in the continuous time setting [14], it requires a sophisticated analysis. It is worth mentioning that Markov decision processes with general utility functions have been already studied in [6, 17. Moreover, there are also some papers, where the specific utility functions are considered. For example, Jaquette [15, 16] and Chung and Sobel [8] are among the first who examined discounted payoffs in Markov decision processes with the decision maker that is equipped with a constant risk aversion, i.e., grades her random payoffs with the help of the exponential utility function. The common feature of all the aforementioned papers is the fact that they deal with bounded rewards or costs. Therefore, their results cannot be directly applied to our case, where the payoffs are unbounded. We make use of the special structure of the underlying problem and show that the optimal dividend payout policy is a time dependent band-strategy. The value function itself can be characterised as a solution to a certain optimality equation. Furthermore, we also study the dividend payout model with the power utility function. As noted in [6], the original Markov decision process can then be viewed as a Markov decision process defined on the extended state space. We employ these techniques to solve our model, but only in the first step, where we use an approximation of the value function in the infinite time horizon by value functions in the finite time horizons. In contrast to the exponential utility case, we can only partly identify the structure of the optimal dividend payout policy. However, we are able to show that there is a barrier such that when the surplus is above the barrier, it is always optimal to pay down to a state below the barrier. The value function is again characterised as a solution to some optimality equation. Summing up, the optimal dividend payout problem with the exponential utility function can be solved completely in the discrete time case, in contrast to the continuous-time problem in [14], whilst for the case with the power utility function we are at least able to identify the important global structure of the optimal policy.

The paper is organised as follows. In the next section we introduce the model together with mild assumptions and general history-dependent policies. Section 3 is devoted to a study of the exponential utility case. We show first that the value function $J$ for discounted payoffs satisfies an optimality equation and give a lower and an upper bound for $J$. Then, we identify properties of the minimiser of the right-hand side of the optimality equation. This enables us to show that the minimiser indeed defines an optimal policy, which is a non-stationary band-policy. The nonstationarity is based only on the time-dependence. The power utility case is treated in Section 4. We pursue here a little different approach, but it also leads to an optimality equation. The policies obtained in this setting are really history-dependent. Nonetheless, we are still able to show that the optimal policy is of a barrier-type. In Section 5 we provide the policy improvement algorithm for the model with the exponential utility. Finally, Section 6 is devoted to concluding remarks and open issues.

## 2. The Model

We consider the financial situation of an insurance company at discrete times, say $n \in \mathbb{N}_{0}:=$ $0,1,2, \ldots$. Assume there is an initial surplus $x_{0}=x \in X:=\mathbb{Z}$ and $x_{0} \geq 0$. The surplus $x_{n+1}$ at time $n+1$ evolves according to the following equation

$$
\begin{equation*}
x_{n+1}=x_{n}-a_{n}+Z_{n+1}, \text { if } x_{n} \geq 0 \quad \text { and } \quad x_{n+1}=x_{n}, \text { if } x_{n}<0 . \tag{2.1}
\end{equation*}
$$

Here $a_{n} \in A\left(x_{n}\right):=\left\{0, \ldots, x_{n}\right\}$ denotes the dividends paid to the shareholders at time $n$, and $Z_{n+1}$ represents the income (possibly negative) of the company during the time interval from $n$ to $n+1$. More precisely, $Z_{n+1}$ is the difference between premium and claim sizes in the $(n+1)$-st time interval. Further, we assume that $Z_{1}, Z_{2}, \ldots$ are independent and identically distributed integer-valued random variables with distribution $\left(q_{k}\right)_{k \in \mathbb{Z}}$, i.e., $\mathbb{P}\left(Z_{n}=k\right)=q_{k}, k \in \mathbb{Z}$. A dividend payout problem in the risk theory can be viewed as a Markov decision process with the
state space $X$, the set of actions $A(x)$ available in state $x$ (for completeness, we put $A(x)=\{0\}$ for $x<0)$ and the transition probability $q(\cdot \mid x, a)$ of the next state, when $x$ is the current state and $a$ is the amount of dividend paid to the shareholders. Note that the dynamics of equation (2.1) implies that $q(y \mid x, a)=q_{y-x+a}$ for $x \geq 0$ and $q(x \mid x, a)=1$ if $x<0$. For the set of admissible pairs $D:=\{(x, a): x \in X, a \in A(x)\}$ we define the function $r: D \mapsto \mathbb{R}$ as $r(x, a)=a$ for $x \in X$.

The feasible history spaces are defined as follows $\Omega_{0}=X, \Omega_{k}=D^{k} \times X$ and $\Omega_{\infty}=D^{\infty}$. A policy $\pi=\left(\pi_{k}\right)_{k \in \mathbb{N}_{0}}$ is a sequence of mappings from $\Omega_{k}$ to $A$ such that $\pi_{k}\left(\omega_{k}\right) \in A\left(x_{k}\right)$, where $\omega_{k}=\left(x_{0}, a_{0}, \ldots, x_{k}\right) \in \Omega_{k}$. Let $\Gamma$ be the class of all functions $g: X \mapsto A$ such that $g(x) \in A(x)$. A Markov policy is $\pi=\left(g_{k}\right)_{k \in \mathbb{N}_{0}}$ where each $g_{k} \in \Gamma$. By $\Pi$ and $\Pi^{M}$ we denote the set of all history-dependent and Markov policies, respectively. By the Ionescu-Tulcea theorem [19], for each policy $\pi$ and each initial state $x_{0}=x$, a probability measure $\mathbb{P}_{x}^{\pi}$ and a stochastic process $\left(x_{k}, a_{k}\right)_{k \in \mathbb{N}_{0}}$ are defined on $\Omega_{\infty}$ in a canonical way, where $x_{k}$ and $a_{k}$ describe the state and the decision at stage $k$, respectively. By $\mathbb{E}_{x}^{\pi}$ we denote the expectation operator with respect to the probability measure $\mathbb{P}_{x}^{\pi}$.

Ruin occurs as soon as the surplus gets negative. The epoch $\tau$ of ruin is defined as the smallest integer $n$ such that $x_{n}<0$. The question arises as to how the risk-sensitive insurance company, equipped with some utility function will choose its dividend strategy. More precisely, we shall consider the following optimisation problem

$$
\sup _{\pi \in \Pi} \mathbb{E}_{x}^{\pi} U_{\gamma}\left(\sum_{k=0}^{\infty} \beta^{k} r\left(x_{k}, a_{k}\right)\right)=\sup _{\pi \in \Pi} \mathbb{E}_{x}^{\pi} U_{\gamma}\left(\sum_{k=0}^{\tau-1} \beta^{k} a_{k}\right), \quad x \geq 0
$$

where $\beta \in(0,1)$ is a discount factor and either
(1) $U_{\gamma}$ is the exponential utility function, i.e., $U_{\gamma}(x)=\frac{1}{\gamma} e^{\gamma x}$ with $\gamma<0$, or
(2) $U_{\gamma}$ is the power utility function, i.e., $U_{\gamma}(x)=x^{\gamma}$ with $\gamma \in(0,1)$.

Let $Z$ be a random variable with the same distribution as $Z_{1}$. Throughout the paper the following assumptions will be supposed to hold true.
(A1) $\mathbb{E} Z^{+}<+\infty$, where $Z^{+}=\max \{Z, 0\}$;
(A2) $\mathbb{P}(Z<0)>0$.
Assumption (A2) allows to avoid a trivial case, when the ruin will never occur under any policy $\pi \in \Pi$.

Remark 2.1. In our study, we assume that the random variables $\left\{Z_{n}\right\}$ only take integer values and the initial capital is also integer. From the proof of Lemma 1.9 in [21], it follows that in our problem we can restrict without loss of generality to the integer dividend payments.

Remark 2.2. If the function $U_{\gamma}$ is strictly concave and increasing as in our case, then the quantity $U_{\gamma}^{-1}\left(\mathbb{E}\left[U_{\gamma}(X)\right]\right)$ is called a certainty equivalent of the random variable $X$. From the optimisation's point of view it does not matter which value $U_{\gamma}^{-1}\left(\mathbb{E}\left[U_{\gamma}(X)\right]\right)$ or $\mathbb{E}\left[U_{\gamma}(X)\right]$ we study, because the inverse function $U_{\gamma}^{-1}$ is monotonic. However, the certainty equivalent has an important meaning. If we apply the Taylor expansion, then the certainty equivalent can be written as follows

$$
U_{\gamma}^{-1}\left(\mathbb{E}\left[U_{\gamma}(X)\right]\right) \approx \mathbb{E} X-\frac{1}{2} l(\mathbb{E} X) \operatorname{Var}[X],
$$

where

$$
l(y)=-\frac{U_{\gamma}^{\prime \prime}(y)}{U_{\gamma}^{\prime}(y)}
$$

is called the Arrow-Pratt function of absolute risk aversion. Hence, the second term accounts for the variability of $X$ (for a discussion see [7]). If $U_{\gamma}$ is concave like in our case, then $l(\cdot) \geq 0$ which means that the variance is subtracted. This fact implies that the decision maker is risk averse.

## 3. The Exponential Utility Function

In this section we assume that the insurer is risk averse and grades her random payoffs by taking the expectations of the exponential utility function of these random rewards. More precisely, we assume that the decision maker is equipped with the constant risk coefficient $\gamma<0$. The objective of the risk averse insurer is to maximise the expected discounted payoff function:

$$
\widetilde{J}_{\pi}(x)=\mathbb{E}_{x}^{\pi} U_{\gamma}\left(\sum_{k=0}^{\infty} \beta^{k} r\left(x_{k}, a_{k}\right)\right)
$$

and to find a policy $\pi^{*} \in \Pi$ (if exists) such that

$$
\begin{equation*}
\widetilde{J}(x):=\widetilde{J}_{\pi^{*}}(x)=\sup _{\pi \in \Pi} \widetilde{J}_{\pi}(x) \tag{3.1}
\end{equation*}
$$

for all $x \in X$. It is obvious that the optimal policy $\pi^{*}$ would depend on $\gamma$ and $\beta$. Clearly, $\widetilde{J}(x)=\frac{1}{\gamma}$ for every $x<0$.
3.1. Optimality equation and the properties of its largest minimiser. Our discounted model with the exponential utility function reveals some kind of non-stationarity that is implied by a discount factor. Therefore, one can extend the state space in the following way $\widetilde{X}:=X \times I$, with $I:=[\gamma, 0)$ (cf. also [11], [6]). If the process is in the state $(x, \theta)$ and the insurer selects an action $a \in A(x)$, then the probability of moving to a next state $\left(x^{\prime}, \theta^{\prime}\right)$ is $q\left(x^{\prime} \mid x, a\right)$, if $\theta^{\prime}=\theta \beta$ and is 0 , if $\theta^{\prime} \neq \theta \beta$. The second component of the state space keeps the track of the discount factor that changes over time in a deterministic way.

Furthermore, we can define an extended history-dependent policy $\sigma=\left(\sigma_{k}\right)_{k \in \mathbb{N}_{0}}$, where $\sigma_{k}$ is a mapping from the set of extended feasible histories up to the $k$ th day to the action set $A$ such that $\sigma_{k}\left(h_{k}\right) \in A\left(x_{k}\right)$ with $h_{k}=\left(x_{0}, \theta, a_{0}, \ldots, a_{k-1}, x_{k}, \theta \beta^{k}\right), k \geq 1$ and $h_{0}=\left(x_{0}, \theta\right)$. Recall that $\left(x_{m}, a_{m}\right) \in D$ for $m \in \mathbb{N}_{0}$. Let $\Xi$ be the set of all extended history-dependent policies. Note that for any $\sigma \in \Xi(\theta \in I$ is fixed $)$, there exists a policy $\pi \in \Pi$ that is equivalent to $\sigma$ in the following sense:

$$
\pi_{k}\left(\cdot \mid \omega_{k}\right):=\sigma_{k}\left(\cdot \mid h_{k}\right), \quad k \in \mathbb{N}_{0}
$$

Obviously, $\pi$ depends on $\theta \in I$. Therefore, for simplicity of notation we shall still use the original policies $\pi \in \Pi$, and the expectation operator $\mathbb{E}_{x}^{\pi}$, where $x$ is the first component of the initial state. The dependence on $\theta \in I$ will be denoted by adding the second variable to the value function.

For any initial state $(x, \theta) \in \widetilde{X}$ we define

$$
J(x, \theta):=\inf _{\pi \in \Pi} J_{\pi}(x, \theta), \quad \text { where } \quad J_{\pi}(x, \theta):=\mathbb{E}_{x}^{\pi}\left(\exp \left\{\theta \sum_{k=0}^{\infty} \beta^{k} r\left(x_{k}, a_{k}\right)\right\}\right)
$$

for $\pi \in \Pi$. Obviously, $0 \leq J_{\pi}(x, \theta) \leq 1$ for all $x \in X, \theta \in I$ and $\pi \in \Pi$. Observe next that our optimisation problem (3.1) is equivalent to the problem of minimising $J_{\pi}(x, \gamma)$ over $\pi \in \Pi$. By $C(\widetilde{X})$ we denote the space of bounded continuous real-valued functions on $\widetilde{X}$.

Theorem 3.1. For every $(x, \theta) \in \widetilde{X}$ the function $J$ is a solution to the following discounted optimality equation

$$
\begin{align*}
J(x, \theta) & =\min _{a \in A(x)}\left[e^{\theta r(x, a)} \sum_{x^{\prime} \in X} J\left(x^{\prime}, \theta \beta\right) q\left(x^{\prime} \mid x, a\right)\right]  \tag{3.2}\\
& =\min _{a \in\{0, \ldots, x\}}\left[e^{\theta a}\left(\sum_{k=a-x}^{\infty} J(x-a+k, \theta \beta) q_{k}+\sum_{k=-\infty}^{a-x-1} q_{k}\right)\right]
\end{align*}
$$

Proof. Clearly, $J(x, \theta)=1$ for $x<0$ and all $\theta \in I$. Consider the truncated payoff functions $r_{m}(x, a)=\min \{m, r(x, a)\}$ with $m \in \mathbb{N}$. From Proposition 3.1 in [11] there is a unique function $w_{m} \in C(\widetilde{X})$ such that

$$
\begin{equation*}
w_{m}(x, \theta)=\min _{a \in A(x)}\left[e^{\theta r_{m}(x, a)} \sum_{x^{\prime} \in X} w_{m}\left(x^{\prime}, \theta \beta\right) q\left(x^{\prime} \mid x, a\right)\right] \tag{3.3}
\end{equation*}
$$

and $w_{m}(x, \theta)=J_{m}(x, \theta)$ for all $(x, \theta) \in \widetilde{X}$. Here, $J_{m}(x, \theta)$ denotes the optimal discounted payoff function with $r$ replaced by $r_{m}$ in of $J(x, \theta)$. Clearly, the sequence $\left(w_{m}(x, \theta)\right)_{m \in N}$ is non-increasing for each $(x, \theta) \in \widetilde{X}$. Therefore, $\lim _{m \rightarrow \infty} w_{m}(x, \theta)=: w(x, \theta)$ exists. It is obvious that

$$
w_{m}(x, \theta)=J_{m}(x, \theta) \geq J(x, \theta), \quad \text { for }(x, \theta) \in \widetilde{X}
$$

Hence,

$$
\begin{equation*}
w(x, \theta) \geq J(x, \theta), \quad \text { for }(x, \theta) \in \widetilde{X} \tag{3.4}
\end{equation*}
$$

On the other hand, letting $m \rightarrow \infty$ in (3.3), making use of the dominated convergence theorem and the fact that $A(x)$ is finite for each $x \in X$, we infer that

$$
\begin{align*}
w(x, \theta) & =\lim _{m \rightarrow \infty} \min _{a \in A(x)}\left[e^{\theta r_{m}(x, a)} \sum_{x^{\prime} \in X} w_{m}\left(x^{\prime}, \theta \beta\right) q\left(x^{\prime} \mid x, a\right)\right] \\
& =\min _{a \in A(x)}\left[e^{\theta r(x, a)} \sum_{x^{\prime} \in X} w\left(x^{\prime}, \theta \beta\right) q\left(x^{\prime} \mid x, a\right)\right] \tag{3.5}
\end{align*}
$$

for $(x, \theta) \in \widetilde{X}$. Hence, for any $a \in A(x)$

$$
w(x, \theta) \leq e^{\theta r(x, a)} \sum_{x^{\prime} \in X} w\left(x^{\prime}, \theta \beta\right) q\left(x^{\prime} \mid x, a\right) .
$$

Iterating this inequality $(n-1)$ times we conclude that

$$
w(x, \theta) \leq \mathbb{E}_{x}^{\pi}\left(\exp \left\{\theta \sum_{k=0}^{n-1} \beta^{k} r\left(x_{k}, a_{k}\right)\right\} w\left(x_{n}, \theta \beta^{n}\right)\right) .
$$

Since $w \leq w_{n} \leq 1$, we have that

$$
w(x, \theta) \leq \mathbb{E}_{x}^{\pi}\left(\exp \left\{\theta \sum_{k=0}^{n-1} \beta^{k} r\left(x_{k}, a_{k}\right)\right\}\right)
$$

Letting $n \rightarrow \infty$ and applying the dominated convergence theorem we have that $w(x, \theta) \leq J_{\pi}(x, \theta)$ for $(x, \theta) \in \widetilde{X}$. Since the policy $\pi$ was chosen arbitrarily, we get that

$$
\begin{equation*}
w(x, \theta) \leq J(x, \theta), \quad \text { for }(x, \theta) \in \widetilde{X} \tag{3.6}
\end{equation*}
$$

Now the assertion follows from (3.4) and (3.6).
Remark 3.2. Among all functions $w$ which satisfy equation (3.2) and have the property that $w(x, \theta)=1$ for $x<0$ and $w(x, \theta) \in(0,1]$ for all $(x, \theta) \in \widetilde{X}$, the value function $J$ is the largest solution. This fact follows from the last part of the proof.
Remark 3.3. Theorem 3.1 was proved in the literature for the general state space, weakly continuous transition probabilities and bounded costs or rewards [6, 11]. However, we deal with unbounded payoffs, therefore we have to truncate them at the level $m$ and then let $m$ tend to infinity. Such a procedure may have a meaning from the numerical point of view.

Let us now consider the policy $\pi^{+}=(g, g \ldots)$, where $g(x)=x^{+}$for every $x \in X$. Hence, this policy asks the insurer to pay out everything at each time point until ruin occurs. Denote by $\vec{\pi}$ the " 1 -shifted" policy for $\pi=\left(\pi_{k}\right)_{k \in \mathbb{N}_{0}}$, that is, $\vec{\pi}=\left(\vec{\pi}_{k}\right)_{k \in \mathbb{N}_{0}}$, where

$$
\begin{equation*}
\vec{\pi}_{k}\left(\cdot \mid \omega_{k}\right)=\pi_{k+1}\left(\cdot \mid x_{0}, a_{0}, \omega_{k}\right) \quad \text { for } \quad \omega_{k} \in \Omega_{k} \quad \text { and } \quad k \in \mathbb{N}_{0} \tag{3.7}
\end{equation*}
$$

Lemma 3.4. For any $x \geq 0$ and $\theta \in I$ the following inequalities hold

$$
e^{\theta x} \underline{h}(\theta) \leq J(x, \theta) \leq e^{\theta x} \bar{h}(\theta),
$$

with

$$
\underline{h}(\theta):=\prod_{k=1}^{\infty} \mathbb{E}\left(\exp \left\{\theta \beta^{k} Z_{k}^{+}\right\}\right) \quad \text { and } \quad \bar{h}(\theta):=\sum_{m=-\infty}^{\infty} \mathbb{E}_{m}^{\pi^{+}}\left(\exp \left\{\theta \sum_{k=1}^{\tau-1} \beta^{k} x_{k}\right\}\right) q_{m},
$$

where $x_{1}=m$ and the empty sum is 0.
Proof. We start with the upper bound. Since $x \geq 0$, then $\tau \geq 1$. For the policy $\pi^{+}$we have that

$$
\begin{aligned}
J(x, \theta) & \leq J_{\pi^{+}}(x, \theta)=\mathbb{E}_{x}^{\pi^{+}}\left(\exp \left\{\theta \sum_{k=0}^{\tau-1} \beta^{k} x_{k}\right\}\right)=e^{\theta x} \mathbb{E}_{x}^{\pi^{+}}\left(\exp \left\{\theta \sum_{k=1}^{\tau-1} \beta^{k} x_{k}\right\}\right) \\
& =e^{\theta x} \mathbb{E}_{x}^{\pi^{+}}\left[\mathbb{E}_{x}^{\pi^{+}}\left(\exp \left\{\theta \sum_{k=1}^{\tau-1} \beta^{k} x_{k}\right\} \mid a_{0}, x_{1}\right)\right] \quad\left(\text { under } \pi^{+} \text {we have that } x_{1}=Z_{1}\right) \\
& =e^{\theta x} \mathbb{E}_{x}^{\pi^{+}}\left[\mathbb{E}_{Z_{1}}^{\pi^{+}}\left(\exp \left\{\theta \sum_{k=1}^{\tau-1} \beta^{k} x_{k}\right\}\right)\right] \\
& =e^{\theta x}\left(\sum_{m=-\infty}^{\infty} \mathbb{E}_{m}^{\pi^{+}}\left(\exp \left\{\theta \sum_{k=1}^{\tau-1} \beta^{k} x_{k}\right\}\right) q_{m}\right),
\end{aligned}
$$

where in the last equality we make use of the fact that $\mathbb{P}\left(Z_{1}=m\right)=q_{m}$ and $\vec{\pi}^{+}=\pi^{+}$.
On the other hand, the lower bound can be obtained as follows. First, we claim that for $\pi \in \Pi$ and $(x, \theta) \in \widetilde{X}$ with $x \geq 0$

$$
\begin{equation*}
e^{\theta x} \prod_{k=1}^{n-1} \mathbb{E}\left(\exp \left\{\theta \beta^{k} Z_{k}^{+}\right\}\right) \leq \mathbb{E}_{x}^{\pi}\left(\exp \left\{\theta \sum_{k=0}^{n-1} \beta^{k} r\left(x_{k}, a_{k}\right)\right\}\right) \tag{3.8}
\end{equation*}
$$

We proceed by induction. Clearly, $e^{\theta x} \leq \min _{a \in A(x)} e^{\theta r(x, a)}$. Assume now that 3.8 holds for some $n \geq 1$ and every $x \geq 0$ and $\theta \in I$. Let $\pi=\left(\pi_{n}\right)_{n \in \mathbb{N}_{0}}$ be any policy. Then, it follows that

$$
\begin{aligned}
& \mathbb{E}_{x}^{\pi}\left(\exp \left\{\theta \sum_{k=0}^{n} \beta^{k} r\left(x_{k}, a_{k}\right)\right\}\right) \geq e^{\theta r\left(x, \pi_{0}(x)\right)} \mathbb{E}_{x}^{\pi}\left[\mathbb{E}_{x}^{\pi}\left(\exp \left\{\theta \sum_{k=1}^{n} \beta^{k} r\left(x_{k}, a_{k}\right)\right\} \mid a_{0}, x_{1}\right)\right] \\
& \quad \geq \min _{a \in A(x)} e^{\theta r(x, a)} \mathbb{E}_{x}^{\pi}\left[1\left[x_{1} \geq 0\right] \mathbb{E}_{x_{1}}^{\vec{\pi}}\left(\exp \left\{\theta \sum_{k=1}^{n} \beta^{k} r\left(x_{k}, a_{k}\right)\right\}\right)+1\left[x_{1}<0\right]\right] \\
& \geq \min _{a \in A(x)} e^{\theta a} \mathbb{E}_{x}^{\pi}\left[1\left[x_{1} \geq 0\right] e^{\theta \beta x_{1}} \prod_{k=2}^{n} \mathbb{E}\left(\exp \left\{\theta \beta^{k} Z_{k}^{+}\right\}\right)+1\left[x_{1}<0\right]\right] \\
& \quad \geq \min _{a \in A(x)} e^{\theta a} \mathbb{E}_{x}^{\pi}\left[1\left[x_{1} \geq 0\right] e^{\theta \beta x_{1}}+1\left[x_{1}<0\right]\right] \prod_{k=2}^{n} \mathbb{E}\left(\exp \left\{\theta \beta^{k} Z_{k}^{+}\right\}\right)
\end{aligned}
$$

Furthermore, we have that

$$
\begin{aligned}
& \min _{a \in A(x)} e^{\theta a} \mathbb{E}_{x}^{\pi}\left[1\left[x_{1} \geq 0\right] e^{\theta \beta x_{1}}+1\left[x_{1}<0\right]\right] \\
& \quad \geq \min _{a \in A(x)} e^{\theta a}\left(\sum_{k=a-x}^{\infty} e^{\theta \beta(x-a+k)} q_{k}+\sum_{k=-\infty}^{a-x-1} q_{k}\right) \\
& \quad \geq \min _{a \in A(x)} e^{\theta a}\left(\sum_{k=1}^{\infty} e^{\theta \beta(x-a+k)} q_{k}+\sum_{k=-\infty}^{0} e^{\theta \beta(x-a)} q_{k}\right) \\
& \quad=\min _{a \in A(x)} e^{\theta a(1-\beta)}\left(\sum_{k=1}^{\infty} e^{\theta \beta(x+k)} q_{k}+\sum_{k=-\infty}^{0} e^{\theta \beta x} q_{k}\right) \\
& =e^{\theta x(1-\beta)} e^{\theta \beta x} \mathbb{E} e^{\theta \beta Z_{1}^{+}}=e^{\theta x} \mathbb{E} e^{\theta \beta Z_{1}^{+}}
\end{aligned}
$$

Hence, we conclude that

$$
\begin{aligned}
\mathbb{E}_{x}^{\pi}\left(\exp \left\{\theta \sum_{k=0}^{n} \beta^{k} r\left(x_{k}, a_{k}\right)\right\}\right) & \geq e^{\theta x} \mathbb{E} e^{\theta \beta Z_{1}^{+}} \prod_{k=2}^{n} \mathbb{E}\left(\exp \left\{\theta \beta^{k} Z_{k}^{+}\right\}\right) \\
& =e^{\theta x} \prod_{k=1}^{n} \mathbb{E}\left(\exp \left\{\theta \beta^{k} Z_{k}^{+}\right\}\right)
\end{aligned}
$$

Therefore, 3.8 holds for every $n \in \mathbb{N}, x \geq 0, \theta \in I$ and $\pi \in \Pi$. Now letting $n \rightarrow \infty$ in (3.8) and making use of the dominated convergence theorem we obtain the lower bound for $J(x, \theta)$.

Remark 3.5. Note that since $\beta<1$, we obtain by Jensen's inequality and assumption (A1) that

$$
\underline{h}(\theta) \geq \exp \left(\theta \sum_{k=1}^{\infty} \beta^{k} \mathbb{E} Z_{k}^{+}\right)>0, \quad \theta \in I
$$

This observation is essential in Theorem 3.8 , where we have to take the logarithm of $\underline{h}$. Also note that $\bar{h}(\theta) \leq 1$ since $\theta \in I$.

Let $x \geq 0$. For any $\theta \in I$ let us denote

$$
G(x, \theta):=\sum_{k=-x}^{\infty} J(x+k, \theta) q_{k}+\sum_{k=-\infty}^{-x-1} q_{k}
$$

then

$$
\begin{equation*}
J(x, \theta)=\min _{a \in A(x)}\left[e^{\theta a} G(x-a, \theta \beta)\right], \quad x \geq 0 \tag{3.9}
\end{equation*}
$$

Lemma 3.6. The function $J(\cdot, \theta)$ is decreasing for each $\theta \in I$ and

$$
J(x, \theta) \leq e^{\theta(x-v)} J(v, \theta)
$$

for $x \geq v \geq 0$.
Proof. Suppose $0 \leq v<x$. Then, it follows that

$$
\begin{aligned}
J(x, \theta) & =\min \left\{G(x, \theta \beta), \ldots, e^{\theta(x-v-1)} G(v+1, \theta \beta), \min \left\{e^{\theta(x-v)} G(v, \theta \beta), \ldots, e^{\theta(x-0)} G(0, \theta \beta)\right\}\right\} \\
& =\min \left\{G(x, \theta \beta), \ldots, e^{\theta(x-v-1)} G(v+1, \theta \beta), e^{\theta(x-v)} J(v, \theta)\right\} \\
& \leq e^{\theta(x-v)} J(v, \theta)
\end{aligned}
$$

Observe that for $v=x-1$ we obtain from the above inequality that

$$
J(x, \theta) \leq e^{\theta} J(x-1, \theta)<J(x-1, \theta)
$$

This fact finishes the proof.

Now let $f^{*}: \widetilde{X} \mapsto A$ be the largest minimiser of the right-hand side in 3.9 for $x \geq 0$ and let $f^{*}(x, \theta)=0$ for $x<0$ and all $\theta \in I$. Then, $f^{*}(\cdot, \theta) \in \Gamma$.

Lemma 3.7. For $x \geq 0$ it holds that $f^{*}\left(x-f^{*}(x, \theta), \theta\right)=0$ and

$$
J(x, \theta)=e^{\theta f^{*}(x, \theta)} J\left(x-f^{*}(x, \theta), \theta\right)
$$

Proof. By (3.9) we have that $J\left(x-f^{*}(x, \theta), \theta\right) \leq G\left(x-f^{*}(x, \theta), \theta \beta\right)$. By Lemma 3.6 (set $v:=$ $\left.x-f^{*}(x, \theta)\right)$ it follows that

$$
\begin{aligned}
J(x, \theta) & =e^{\theta f^{*}(x, \theta)} G\left(x-f^{*}(x, \theta), \theta \beta\right) \geq e^{\theta f^{*}(x, \theta)} J\left(x-f^{*}(x, \theta), \theta\right) \\
& \geq J(x, \theta) .
\end{aligned}
$$

Thus we have equality and, in particular, $J\left(x-f^{*}(x, \theta), \theta\right)=G\left(x-f^{*}(x, \theta), \theta \beta\right)$, which implies that $a=0$ minimises the expression $e^{\theta a} G\left(x-f^{*}(x, \theta)-a, \theta \beta\right)$. We claim that $a=0$ is the only minimiser of the above expression. Note that, if $f^{*}(x, \theta)=x$, then $J\left(x-f^{*}(x, \theta), \theta\right)=J(0, \theta)$ and the result holds true. If, on the other hand, $f^{*}(x, \theta)<x$, then

$$
J(x, \theta)=e^{\theta f^{*}(x, \theta)} G\left(x-f^{*}(x, \theta), \theta \beta\right)<e^{\theta\left(f^{*}(x, \theta)+a\right)} G\left(x-f^{*}(x, \theta)-a, \theta \beta\right)
$$

for $a=1, \ldots, x-f^{*}(x, \theta)$. This fact implies that $a=0$ is indeed the only minimiser and, consequently, $f^{*}\left(x-f^{*}(x, \theta), \theta\right)=0$.

Theorem 3.8. Let $\xi(\theta):=\sup \left\{x \in \mathbb{N}_{0}: f^{*}(x, \theta)=0\right\}$. Then $\xi^{*}:=\sup _{\theta \in I} \xi(\theta)<\infty$ and

$$
f^{*}(x, \theta)=x-\xi(\theta), \quad \text { for all } x>\xi(\theta) .
$$

Proof. Fix $\theta \in I$ and let $x \geq 0$ be such that $f^{*}(x, \theta)=0$. Note that such $x$ exists for each $\theta \in I$, because $f^{*}(0, \theta)=0$ for all $\theta \in I$. From (3.9) we obtain that

$$
\begin{equation*}
J(x, \theta)=\sum_{k=-x}^{\infty} J(x+k, \theta \beta) q_{k}+\sum_{k=-\infty}^{-x-1} q_{k} . \tag{3.10}
\end{equation*}
$$

Furthermore, by (3.10), Lemmas 3.4 and 3.6 we have that

$$
\begin{aligned}
J(x, \theta) & \geq \sum_{k=1}^{\infty} J(x+k, \theta \beta) q_{k}+\sum_{k=-\infty}^{0} J(x, \theta \beta) q_{k} \\
& \geq \sum_{k=1}^{\infty} e^{\theta \beta(x+k)} \underline{h}(\theta \beta) q_{k}+\sum_{k=-\infty}^{0} e^{\theta \beta x} \underline{h}(\theta \beta) q_{k} \\
& =e^{\theta \beta x} \underline{h}(\theta \beta)\left(\sum_{k=1}^{\infty} e^{\theta \beta k} q_{k}+\sum_{k=-\infty}^{0} q_{k}\right)=e^{\theta \beta x} \underline{h}(\theta)
\end{aligned}
$$

Hence,

$$
e^{\theta \beta x} \underline{h}(\theta) \leq J(x, \theta) \leq e^{\theta x} \bar{h}(\theta),
$$

which implies that

$$
x \leq \frac{\ln \bar{h}(\theta)-\ln \underline{h}(\theta)}{\theta(\beta-1)}=: s(\theta) .
$$

The function $s(\cdot)$ is continuous on $I$ and is finite for each $\theta \in I$. Additionally,

$$
\begin{aligned}
\lim _{\theta \rightarrow 0^{-}} s(\theta) & =\lim _{\theta \rightarrow 0^{-}}\left(\frac{\bar{h}^{\prime}(\theta)}{\bar{h}(\theta)}-\frac{\underline{h}^{\prime}(\theta)}{\underline{h}(\theta)}\right) /(\beta-1) \\
& =\left(\sum_{k=1}^{\infty} \beta^{k} \mathbb{E}\left(Z_{k}^{+}\right)-\sum_{m=0}^{\infty} \mathbb{E}_{m}^{\pi^{+}}\left(\sum_{k=1}^{\tau-1} \beta^{k} x_{k}\right) q_{m}\right) /(1-\beta)<+\infty
\end{aligned}
$$

which follows by assumption (A1). Thus, we have shown that $\sup _{\theta \in I} \xi(\theta)<+\infty$. Now let $x>\xi(\theta)$. We know from Lemma 3.7 that $f^{*}\left(x-f^{*}(x, \theta), \theta\right)=0$, which implies by definition of $\xi(\theta)$ that $f^{*}(x, \theta) \geq x-\xi(\theta)$. On the other hand, by (3.9) we obtain that

$$
\begin{aligned}
J(\xi(\theta), \theta) & \leq e^{\theta\left(f^{*}(x, \theta)-(x-\xi(\theta))\right)} G\left(\xi(\theta)-f^{*}(x, \theta)+(x-\xi(\theta)), \theta \beta\right) \\
& =J(x, \theta) e^{-\theta(x-\xi(\theta))} \leq J(\xi(\theta), \theta),
\end{aligned}
$$

where the last inequality follows from Lemma 3.6. Thus, because $f^{*}$ is the largest minimiser of (3.9), we obtain

$$
0=f^{*}(\xi(\theta), \theta) \geq f^{*}(x, \theta)-(x-\xi(\theta)) \geq 0
$$

which implies that $f^{*}(x, \theta)=x-\xi(\theta)$.
Lemma 3.9. Let $x_{0} \geq 0$. If $f^{*}\left(x_{0}, \theta\right)=a_{0}$ and $f^{*}\left(x_{0}+1, \theta\right)>0$, then $f^{*}\left(x_{0}+1, \theta\right)=a_{0}+1$.
Proof. By definition of $f^{*}$ we have that

$$
J\left(x_{0}, \theta\right)=e^{\theta a_{0}} G\left(x_{0}-a_{0}, \theta \beta\right)\left\{\begin{array}{l}
\leq e^{\theta a} G\left(x_{0}-a, \theta \beta\right), \text { for } a=0, \ldots, a_{0} \\
<e^{\theta a} G\left(x_{0}-a, \theta \beta\right), \text { for } a=a_{0}+1, \ldots, x_{0},
\end{array}\right.
$$

which further yields that

$$
e^{\theta\left(a_{0}+1\right)} G\left(x_{0}-a_{0}, \theta \beta\right)\left\{\begin{array}{l}
\leq e^{\theta(a+1)} G\left(x_{0}-a, \theta \beta\right), \text { for } a=0, \ldots, a_{0} \\
<e^{\theta(a+1)} G\left(x_{0}-a, \theta \beta\right), \text { for } a=a_{0}+1, \ldots, x_{0} .
\end{array}\right.
$$

Again from the definition of $f^{*}$ we obtain that

$$
\begin{aligned}
J\left(x_{0}+1, \theta\right) & =\min _{a \in A\left(x_{0}+1\right)} e^{\theta a} G\left(x_{0}+1-a, \theta \beta\right) \\
& =\min \left\{G\left(x_{0}+1, \theta \beta\right), \min _{a \in A\left(x_{0}\right)} e^{\theta(a+1)} G\left(x_{0}-a, \theta \beta\right)\right\} .
\end{aligned}
$$

Since $f^{*}\left(x_{0}+1, \theta\right)>0$, it holds that

$$
G\left(x_{0}+1, \theta \beta\right) \geq \min _{a \in A\left(x_{0}\right)} e^{\theta(a+1)} G\left(x_{0}-a, \theta \beta\right) .
$$

This fact, in turn, together with the previous observation yields by shifting the index that

$$
e^{\theta\left(a_{0}+1\right)} G\left(x_{0}-a_{0}, \theta \beta\right)\left\{\begin{array}{l}
\leq e^{\theta a} G\left(x_{0}+1-a, \theta \beta\right), \text { for } a=0, \ldots, a_{0}+1 \\
<e^{\theta a} G\left(x_{0}+1-a, \theta \beta\right), \text { for } a=a_{0}+2, \ldots, x_{0}+1 .
\end{array}\right.
$$

Thus, it follows that $f^{*}\left(x_{0}+1, \theta\right)=a_{0}+1$.
3.2. Optimal policy and its structure. Recall now that $\gamma$ is a constant risk averse coefficient of the insurer. Consider the following policy $\pi^{*}:=\left(\widetilde{g}_{0}, \widetilde{g}_{1}, \ldots\right)$, where $\widetilde{g}_{n}(\cdot):=f^{*}\left(\cdot, \gamma \beta^{n}\right)$. We note that $\pi^{*} \in \Pi^{M}$. Clearly, since $\gamma \in I$, then $\gamma \beta^{n} \in I$ for all $n \in \mathbb{N}_{0}$.
Corollary 3.10. Under policy $\pi^{*}$ the ruin occurs with probability 1, i.e., $\mathbb{P}_{x}^{\pi^{*}}(\tau<+\infty)=1$ for every $x \in X$.

Proof. Assume that the surplus process equals $x_{0} \in X$. If $x_{0} \geq 0$, then either $x_{0} \leq \xi(\gamma)$ or $x_{0}>\xi(\gamma)$. However, from Theorem 3.8 we know that in both cases the risk reserve (surplus) just after dividend payment is always less or equal to $\xi(\gamma) \leq \xi^{*}$. Therefore, the ruin will occur, if there appears a sequence of length $\xi^{*}+1$ of negative incomes. But the probability that such a sequence appears, equals

$$
\begin{equation*}
\mathbb{P}\left(Z_{1}<0, \ldots, Z_{\xi^{*}+1}<0\right)=\left(\sum_{m=-\infty}^{-1} q_{m}\right)^{\xi^{*}+1} \tag{3.11}
\end{equation*}
$$

which is positive by (A2). If the ruin has not occurred up to the $l$ th day, where $l:=\xi^{*}+1$, then again $x_{l} \leq \xi\left(\gamma \beta^{l}\right)$ or $x_{l}>\xi\left(\gamma \beta^{l}\right)$. But from Theorem 3.8 in both cases the risk reserve just after dividend payment is always less or equal to $\xi\left(\gamma \beta^{l}\right) \leq \xi^{*}$. The probability that there
exists a sequence of length $\xi^{*}+1$ of negative incomes is $\left(\sum_{m=-\infty}^{-1} q_{m}\right)^{\xi^{*}+1}$. Thus, considering the states $x_{k\left(\xi^{*}+1\right)}, k \in \mathbb{N}_{0}$, we may define the following events

$$
A_{k}=\left\{Z_{k\left(\xi^{*}+1\right)+1}<0, Z_{k\left(\xi^{*}+1\right)+2}<0 \ldots, Z_{(k+1)\left(\xi^{*}+1\right)}<0\right\}, \quad k \in \mathbb{N}_{0} .
$$

By the second Borel-Cantelli lemma $\mathbb{P}\left(A_{k}\right.$ i.o. $)=1$. Therefore, the ruin must occur.
Theorem 3.11. The Markov policy $\pi^{*}$ is optimal, i.e.,

$$
\widetilde{J}(x)=\widetilde{J}_{\pi^{*}}(x)=\frac{1}{\gamma} J_{\pi^{*}}(x, \gamma)=\frac{1}{\gamma} J(x, \gamma)
$$

for $x \in X$.
Proof. From Theorem 3.1 and the definition of $\pi^{*}$ we obtain for every $x \in X$ that

$$
\begin{aligned}
J(x, \gamma) & =\min _{a \in A(x)}\left[e^{\gamma r(x, a)} \sum_{x^{\prime} \in X} J\left(x^{\prime}, \gamma \beta\right) q\left(x^{\prime} \mid x, a\right)\right] \\
& =e^{\gamma \widetilde{g}_{0}(x)} \sum_{x^{\prime} \in X} J\left(x^{\prime}, \gamma \beta\right) q\left(x^{\prime} \mid x, \widetilde{g}_{0}(x)\right) .
\end{aligned}
$$

Assume that $x \geq 0$. Iterating the last equality $n$ times under the Markov policy $\pi^{*}$, we obtain that

$$
\begin{equation*}
J(x, \gamma)=\mathbb{E}_{x}^{\pi^{*}}\left(\exp \left\{\gamma \sum_{k=0}^{(\tau-1) \wedge(n-1)} \beta^{k} a_{k}\right\}\left(J\left(x_{n}, \gamma \beta^{n}\right) 1[\tau \geq n]+1[\tau<n]\right)\right) . \tag{3.12}
\end{equation*}
$$

Observe now that

$$
0 \leq \mathbb{E}_{x}^{\pi^{*}}\left(\exp \left\{\gamma \sum_{k=0}^{(\tau-1) \wedge(n-1)} \beta^{k} a_{k}\right\} J\left(x_{n}, \gamma \beta^{n}\right) 1[\tau \geq n]\right) \leq \mathbb{E}_{x}^{\pi^{*}} 1[\tau \geq n]=\mathbb{P}_{x}^{\pi^{*}}(\tau \geq n)
$$

But by Corollary 3.10, $\mathbb{P}_{x}^{\pi^{*}}(\tau \geq n) \rightarrow 0$ as $n \rightarrow \infty$. Hence, letting $n \rightarrow \infty$ in (3.12) and making use of the dominated convergence theorem we obtain that

$$
\inf _{\pi \in \Pi} J_{\pi}(x, \gamma)=J(x, \gamma)=\mathbb{E}_{x}^{\pi^{*}}\left(\exp \left\{\gamma \sum_{k=0}^{\tau-1} \beta^{k} a_{k}\right\}\right)=J_{\pi^{*}}(x, \gamma)
$$

for $x \in X$. The conclusion follows by multiplying the above display by the number $1 / \gamma$.
Definition 3.12. A function $g \in \Gamma$ is called a band-function, if there exists numbers $n \in \mathbb{N}_{0}$ and $c_{0}, \ldots, c_{n}, d_{1}, \ldots, d_{n} \in \mathbb{N}_{0}$ such that $d_{k}-c_{k-1} \geq 2$ for $k=1, \ldots, n, 0 \leq c_{0} \leq d_{1} \leq c_{1} \leq d_{2} \leq$ $\ldots \leq d_{n} \leq c_{n}$ and

$$
g(x)=\left\{\begin{array}{l}
0, \text { if } x \leq c_{0} \\
x-c_{k}, \text { if } c_{k}<x<d_{k+1} \\
0, \text { if } d_{k} \leq x \leq c_{k} \\
x-c_{n}, \text { if } x>c_{n}
\end{array}\right.
$$

A Markov policy $\pi=\left(g_{m}\right)_{m \in N_{0}}$ is called a band-policy, if $g_{m}$ is a band-function for every $n \in \mathbb{N}_{0}$.
Theorem 3.13. The optimal Markov policy $\pi^{*}$ is a band-policy.
Proof. Recall that $\widetilde{g}_{n}(\cdot)=f^{*}\left(\cdot, \gamma \beta^{n}\right)$ for $n \in \mathbb{N}_{0}$. By Theorem 3.8 we have $\widetilde{g}_{n}(x)=x-\xi\left(\gamma \beta^{n}\right)$ for all $x>\xi\left(\gamma \beta^{n}\right)$. For $x \leq \xi\left(\gamma \beta^{n}\right)$ we have to distinguish different cases. If $\widetilde{g}_{n}(x)=0$ for all $x=0, \ldots, \xi\left(\gamma \beta^{n}\right)$, then clearly $\widetilde{g}_{n}$ is a band-function. If there exists an $0<x_{0} \leq \xi\left(\gamma \beta^{n}\right)$ such that $\widetilde{g}_{n}(x)=0$ for $x=0, \ldots, x_{0}-1$ and $\widetilde{g}_{n}\left(x_{0}\right)>0$, then by Lemma 3.9 $\widetilde{g}_{n}\left(x_{0}\right)=1$. If further $\widetilde{g}_{n}\left(x_{0}+m\right)>0$ for $m=1, \ldots, \xi(\gamma)-x_{0}-1$ then by induction

$$
\widetilde{g}_{n}\left(x_{0}+m\right)=\widetilde{g}_{n}\left(x_{0}+m-1\right)+1=\ldots=\widetilde{g}_{n}\left(x_{0}\right)+m=m+1 .
$$

If $\widetilde{g}_{n}\left(x_{0}+1\right)=0$ we either have $\widetilde{g}_{n}(x)=0$ for $x=x_{0}+1, \ldots, \xi\left(\gamma \beta^{n}\right)$ or there exists an $x_{1}$ such that $x_{0}<x_{1} \leq \xi\left(\gamma \beta^{n}\right)$ and $\widetilde{g}_{n}\left(x_{0}+m\right)=0$ for $m=1, \ldots, x_{1}-x_{0}-1$ and $\widetilde{g}_{n}\left(x_{1}\right)>0$. Now we proceed in the same way as with $x_{0}$. After a finite number of steps we reach $\xi\left(\gamma \beta^{n}\right)$. In any case $\widetilde{g}_{n}$ is a band-function.

Remark 3.14. In the risk neutral dividend payout problem, the optimal policy is a stationary band-policy, i.e., it consists of the same band-function at all time points. The risk neutral problem can formally be obtained as a $\operatorname{limit} \lim _{\gamma \rightarrow 0} \frac{1}{\gamma}\left(e^{\gamma x}-1\right)$. Hence, the exponential utility function only implies some kind of non-stationarity of the optimal policy and thus does not really make it necessary to consider history-dependent policies.

## 4. The Power Utility Function

In this section we assume that the insurer is equipped with the power utility function $U_{\gamma}(x)=$ $x^{\gamma}$, where $\gamma$ is a fixed number from the interval $(0,1)$. The decision maker wishes to maximise the following expected discounted payoff

$$
\widehat{J}_{\pi}(x):=\mathbb{E}_{x}^{\pi} U_{\gamma}\left(\sum_{k=0}^{\infty} \beta^{k} r\left(x_{k}, a_{k}\right)\right)
$$

and to find a policy $\pi^{*} \in \Pi$ (if exists) such that

$$
\begin{equation*}
\widehat{J}(x):=\sup _{\pi \in \Pi} \widehat{J}_{\pi}(x)=\widehat{J}_{\pi^{*}}(x) \tag{4.1}
\end{equation*}
$$

for all $x \in X$. Clearly, $\widehat{J}(x)=0$ for $x<0$. In Lemma 4.4 we show that under our assumptions $\widehat{J}(x)<+\infty$ for each $x \geq 0$. Moreover, note that for $x \geq 0$

$$
\widehat{J}(x):=\sup _{\pi \in \Pi} \mathbb{E}_{x}^{\pi} U_{\gamma}\left(\sum_{k=0}^{\tau-1} \beta^{k} a_{k}\right) .
$$

4.1. Optimality equation and the properties of its largest maximiser. Contrary to the exponential utility function, the power utility function reveals certain non-separability, that is implied by the fact that the expectation operator is only linear. Therefore, we again extend the state space by defining the new state space $\widehat{X}:=X \times[0, \infty)$ (cf. [6]). In this case, the second component is responsible for the accumulated payoffs so far. If the process is in the state $(x, y)$ and the insurer selects an action $a \in A(x)$, then the probability of moving to a next state $\left(x^{\prime}, y^{\prime}\right)$ is $q\left(x^{\prime} \mid x, a\right)$, if $y^{\prime}=\frac{y+a}{\beta}$ and is 0 , if $y^{\prime} \neq \frac{y+a}{\beta}$. Hence, we can observe that the second component is again established in a deterministic way, but it differs from the previous case, since $y^{\prime}$ depends on the action chosen by the insurer.

Let us define the feasible extended histories of the process up to the $k$ th day as follows 11

$$
h_{0}=\left(x_{0}, y_{0}\right) \quad \text { and } \quad h_{k}=\left(x_{0}, y_{0}, a_{0}, x_{1}, y_{1}, \ldots, a_{k-1}, x_{k}, y_{k}\right), \quad k \geq 1,
$$

where $\left(x_{m}, a_{m}\right) \in D$ for each $m \in \mathbb{N}_{0}$ and with $y_{m+1}$ given by the recurrence equation

$$
y_{m+1}:=\frac{y_{m}+a_{m}}{\beta}, \quad m \in \mathbb{N}_{0} .
$$

Then, we can define, as usual, an extended history-dependent policy $\sigma=\left(\sigma_{k}\right)_{k \in \mathbb{N}_{0}}$, where $\sigma_{k}$ is a mapping from the set of feasible extended histories up to the $k$ th day to the action set $A$ such that $\sigma_{k}\left(h_{k}\right) \in A\left(x_{k}\right)$ with $h_{k}$ defined above. Let $\Xi$ be the set of all such policies. Note that for any $\sigma \in \Xi(y \geq 0$ is fixed $)$, there exists a policy $\pi \in \Pi$ that is equivalent to $\sigma$ in the following sense:

$$
\pi_{k}\left(\cdot \mid \omega_{k}\right):=\sigma_{k}\left(\cdot \mid h_{k}\right), \quad \omega_{k} \in \Omega_{k}, \quad k \in \mathbb{N}_{0}
$$

[^0]Obviously, $\pi$ must depend on $y$. Thus again, for simplicity of notation we shall still use the original set of policies $\Pi$, and the expectation operator $\mathbb{E}_{x}^{\pi}$, where $x$ is the first component of the initial state. The dependence on $y \geq 0$ of a policy will be indicated by writing the second variable to the value function.

In what follows, we put for $n \in \mathbb{N}, \pi \in \Pi, x \geq 0$ and $y \geq 0$

$$
J_{n, \pi}(x, y):=\mathbb{E}_{x}^{\pi}\left(\sum_{k=0}^{(n-1) \wedge(\tau-1)} \beta^{k} r\left(x_{k}, a_{k}\right)+y\right)^{\gamma}=\mathbb{E}_{x}^{\pi}\left(\sum_{k=0}^{(n-1) \wedge(\tau-1)} \beta^{k} a_{k}+y\right)^{\gamma}
$$

and

$$
J_{n}(x, y):=\sup _{\pi \in \Pi} J_{n, \pi}(x, y) .
$$

Moreover, for $\pi \in \Pi, x \geq 0$ and $y \geq 0$ we set

$$
J(x, y)=\sup _{\pi \in \Pi} J_{\pi}(x, y), \quad \text { where } \quad J_{\pi}(x, y)=\mathbb{E}_{x}^{\pi}\left(\sum_{k=0}^{\tau-1} \beta^{k} a_{k}+y\right)^{\gamma} .
$$

If $x<0$, then $J(x, y)=y^{\gamma}$ for $y \geq 0$. Obviously, $J(x, 0)=\widehat{J}(x)$.
Before we formulate our first result, we introduce a specific subset of policies $\widehat{\Pi} \subset \Pi$. Let $F$ be the set of functions $f: \widehat{X} \mapsto A$ such that $f(x, y) \in A(x)$ for all $y \geq 0$ and let $\left(f_{k}\right)_{k \in \mathbb{N}_{0}}$ be a sequence of functions with $f_{k} \in F$. Then, $\widehat{\Pi}$ is the set of all policies $\pi=\left(\pi_{k}\right)_{k \in \mathbb{N}_{0}}$ defined in the following way

$$
\pi_{k}\left(\omega_{k}\right):=f_{k}\left(x_{k}, y_{k}\right), \quad \omega_{k} \in \Omega_{k}, \quad k \in \mathbb{N}_{0},
$$

where $y_{k}:=\frac{y_{k-1}+a_{k-1}}{\beta}, k \in \mathbb{N}$, and $y_{0}:=y \geq 0$ is a fixed number. Furthermore, we shall identify a policy $\pi \in \widehat{\Pi}$ with the sequence $\left(f_{k}\right)_{k \in \mathbb{N}_{0}}$ by writing $\pi=\left(f_{k}\right)_{k \in \mathbb{N}_{0}}$.

Next for any function $h: \widehat{X} \mapsto \mathbb{R}$ we define an operator $T$ as follows

$$
\begin{equation*}
T h(x, y):=\beta^{\gamma} \max _{a \in A(x)}\left[\sum_{x^{\prime} \in X} h\left(x^{\prime}, \frac{a+y}{\beta}\right) q\left(x^{\prime} \mid x, a\right)\right] . \tag{4.2}
\end{equation*}
$$

Let $f \in F$ be the maximiser of the right-hand side in (4.2), i.e., $f(x)$ attains the maximum on the right-hand side of (4.2) for all $x \in X$. We also set

$$
T_{f} h(x, y):=\beta^{\gamma} \sum_{x^{\prime} \in X} h\left(x^{\prime}, \frac{f(x, y)+y}{\beta}\right) q\left(x^{\prime} \mid x, f(x, y)\right) .
$$

Note that $T_{f} h=T h$.
Theorem 4.1. For each $n \in \mathbb{N}_{0}$ the value function $J_{n}$ satisfies the equation

$$
\begin{equation*}
J_{n+1}=T J_{n} \tag{4.3}
\end{equation*}
$$

with $J_{n}(x, y)=y^{\gamma}$ for $x<0$ and $J_{0}(x, y)=y^{\gamma}$. Let $\bar{f}_{l} \in F$ be such that $J_{l+1}=T_{\bar{f}_{l}} J_{l}$ for $l=0, \ldots, n$. Then, $\bar{\pi}=\left(\bar{f}_{n}, \ldots, \bar{f}_{0}\right)$ is optimal for $J_{n+1}$, i.e., $J_{n+1}=J_{n+1, \bar{\pi}}$.
Proof. Let $n=1$. Then, by the definition of $J_{1}$ we have that

$$
J_{1}(x, y)=\sup _{\pi \in \Pi} \mathbb{E}_{x}^{\pi}\left(\beta^{0} a_{0}+y\right)^{\gamma}=\max _{a \in A(x)}(a+y)^{\gamma}=(x+y)^{\gamma} .
$$

On the other hand,

$$
T J_{0}(x, y)=\beta^{\gamma} \max _{a \in A(x)}\left(\frac{a+y}{\beta}\right)^{\gamma}=(x+y)^{\gamma}=T_{\bar{f}_{0}} J_{0}(x, y)
$$

with $\bar{f}_{0}(x, y)=x$.
Assume now that $J_{l}=T J_{l-1}=T_{\bar{f}_{l-1}} J_{l-1}$ with $\bar{f}_{l-1} \in F$ for all $l=1, \ldots, n$ and let $\left(\bar{f}_{n-1}, \ldots, \bar{f}_{0}\right)$ be an optimal policy for $J_{n}$. We show that $J_{n+1}=T J_{n}=T_{\bar{f}_{n}} J_{n}$ and $\left(\bar{f}_{n}, \ldots, \bar{f}_{0}\right)$ is optimal for $J_{n}$.

We have that

$$
J_{n+1}(x, y)=\sup _{\pi \in \Pi} \mathbb{E}_{x}^{\pi}\left(\sum_{k=1}^{n \wedge(\tau-1)} \beta^{k} a_{k}+a_{0}+y\right)^{\gamma}
$$

with the convention that the empty sum equals 0 (when $\tau=1$ ). Recall that $\vec{\pi}$ denotes a "1-shifted" policy, see (3.7). We further get the following

$$
\begin{align*}
J_{n+1}(x, y) & =\beta^{\gamma} \sup _{\pi \in \Pi} \mathbb{E}_{x}^{\pi}\left(\sum_{k=1}^{n \wedge(\tau-1)} \beta^{k-1} a_{k}+\frac{a_{0}+y}{\beta}\right)^{\gamma}  \tag{4.4}\\
& =\beta^{\gamma} \sup _{\pi \in \Pi} \mathbb{E}_{x}^{\pi}\left[\mathbb{E}_{x}^{\pi}\left(\left.\left(\sum_{k=1}^{n \wedge(\tau-1)} \beta^{k-1} a_{k}+\frac{a_{0}+y}{\beta}\right)^{\gamma} \right\rvert\, a_{0}, x_{1}\right)\right] \\
& \leq \beta^{\gamma} \sup _{a_{0} \in A(x)}\left[\sum_{m=-\infty}^{\infty} \sup _{\vec{\pi} \in \Pi} \mathbb{E}_{x-a_{0}+m}^{\vec{\pi}}\left(\sum_{k=1}^{n \wedge(\tau-1)} \beta^{k-1} a_{k}+\frac{a_{0}+y}{\beta}\right)^{\gamma} q_{m}\right] \\
& =\beta^{\gamma} \max _{a \in A(x)}\left[\sum_{m=-\infty}^{\infty} J_{n}\left(x-a+m, \frac{a+y}{\beta}\right) q_{m}\right]=T J_{n}(x, y) .
\end{align*}
$$

Let $\bar{f}_{n}$ be such that $T J_{n}(x, y)=T_{\bar{f}_{n}} J_{n}(x, y)$. Put $\bar{\pi}=\left(\bar{f}_{n}, \bar{f}_{n-1}, \ldots, \bar{f}_{0}\right)$. By induction assumption $\vec{\pi}=\left(\bar{f}_{n-1}, \ldots, \bar{f}_{0}\right)$ is optimal for $J_{n}$. Hence, it follows that

$$
\begin{align*}
T J_{n}(x, y) & =\beta^{\gamma} \max _{a \in A(x)}\left[\sum_{m=-\infty}^{\infty} J_{n \vec{\pi}}\left(x-a+m, \frac{a+y}{\beta}\right) q_{m}\right]  \tag{4.5}\\
& =\max _{a \in A(x)} \beta^{\gamma} \sum_{m=-\infty}^{\infty} \mathbb{E}_{x-a+m}^{\vec{\pi}}\left(\sum_{k=0}^{(n-1) \wedge(\tau-1)} \beta^{k} a_{k+1}+\frac{y+a}{\beta}\right)^{\gamma} q_{m} \\
& =\max _{a \in A(x)} \sum_{m=-\infty}^{\infty} \mathbb{E}_{x-a+m}^{\vec{\pi}}\left(\sum_{k=0}^{(n-1) \wedge(\tau-1)} \beta^{k+1} a_{k+1}+y+a\right)^{\gamma} q_{m} \\
& \leq \sup _{\pi \in \Pi} \mathbb{E}_{x}^{\pi}\left(\sum_{k=0}^{n \wedge(\tau-1)} \beta^{k} a_{k}+y\right)^{\gamma}=J_{n+1}(x, y) .
\end{align*}
$$

Thus, (4.4) and (4.5) yield that $J_{n+1}=T J_{n}$. The fact that $\bar{\pi}$ defined above is optimal for $J_{n+1}$ follows from repeating the calculations in (4.5) applied to $T_{\bar{f}_{n}} J_{n}$.

The next result can be concluded from Theorem 4.1.
Theorem 4.2. The function $J$ satisfies the following equation

$$
\begin{align*}
J(x, y) & =\beta^{\gamma} \max _{a \in A(x)}\left[\sum_{x^{\prime} \in X} J\left(x^{\prime}, \frac{a+y}{\beta}\right) q\left(x^{\prime} \mid x, a\right)\right]  \tag{4.6}\\
& =\beta^{\gamma} \max _{a \in A(x)}\left[\sum_{k=-\infty}^{\infty} J\left(x-a+k, \frac{a+y}{\beta}\right) q_{k}\right]
\end{align*}
$$

for $x \in \mathbb{N}_{0}$ and $y \geq 0$.
Proof. It is obvious that the sequence of functions $\left(J_{n}(x, y)\right)_{n \in \mathbb{N}_{0}}$ is increasing for each $(x, y) \in \widehat{X}$. Hence, $w(x, y):=\lim _{n \rightarrow \infty} J_{n}(x, y)$ exists for every $(x, y) \in \widehat{X}$. Since $J_{n} \leq J$, then it follows that $w \leq J$. On the other hand, for any policy $\pi \in \Pi$ we obtain that $J_{n, \pi} \leq J_{n}$. Letting $n \rightarrow \infty$,
making use of the monotone convergence theorem and taking the supremum over $\pi \in \Pi$, it is easily seen that

$$
J(x, y)=\sup _{\pi \in \Pi} \mathbb{E}_{x}^{\pi}\left(\sum_{k=0}^{\tau-1} \beta^{k} a_{k}+y\right)^{\gamma} \leq w(x, y) .
$$

Consequently, $w=J$. Equation (4.6) follows from (4.3) by letting $n \rightarrow \infty$ and replacing the maximum with the limit.
Remark 4.3. The counterpart of Theorem 4.2 is Theorem 4.1(a) in [6]. However, again as in the exponential case this theorem was proved for general state space, weakly continuous transitions and bounded costs.

The next lemma provides the following bounds for the function $J$.
Lemma 4.4. For any $x \in \mathbb{N}_{0}$ and $y \geq 0$ it follows that

$$
(x+y)^{\gamma} \leq J(x, y) \leq\left(x+\beta \frac{\mathbb{E} Z^{+}}{1-\beta}+y\right)^{\gamma} .
$$

Proof. Let $\pi$ be a policy such that $\pi_{0}\left(\cdot \mid \omega_{0}\right)=x$ and $\pi_{k}\left(\cdot \mid \omega_{k}\right)=0$ for $k \geq 1$. Then, $J(x, y) \geq$ $J_{\pi}(x, y)=(x+y)^{\gamma}$. The upper bound for the function $J$ is due to the Jensen inequality and Theorem 9.2.3(a) in [5] that gives the upper bound for the risk neutral setting.

For simplicity for any $x \in \mathbb{N}_{0}$ and $y \geq 0$ we define

$$
G(x, y):=\sum_{k=-\infty}^{\infty} J(x+k, y) q_{k}
$$

From (4.6) we obtain that

$$
\begin{equation*}
J(x, y)=\beta^{\gamma} \max _{a \in A(x)} G\left(x-a, \frac{a+y}{\beta}\right), \quad x \in \mathbb{N}_{0}, y \geq 0 . \tag{4.7}
\end{equation*}
$$

Lemma 4.5. For all $0 \leq v \leq x$ we have that $J(x, y) \geq J(x-v, y+v)$.
Proof. Suppose that $0<v \leq x$. For part (b) observe that

$$
\begin{aligned}
J(x, y)= & \max \left\{\left\{\beta^{\gamma} G\left(x, \frac{y}{\beta}\right), \ldots, \beta^{\gamma} G\left(x-v+1, \frac{y+v-1}{\beta}\right)\right\},\right. \\
& \left.\max \left\{\beta^{\gamma} G\left(x-v, \frac{y+v}{\beta}\right), \ldots, \beta^{\gamma} G\left(0, \frac{y+x}{\beta}\right)\right\}\right\} \\
= & \max \left\{\left\{\beta^{\gamma} G\left(x, \frac{y}{\beta}\right), \ldots, \beta^{\gamma} G\left(x-v+1, \frac{y+v-1}{\beta}\right)\right\},\right. \\
& \left.\beta^{\gamma} \max _{a \in A(x-v)} G\left(x-v-a, \frac{y+v+a}{\beta}\right)\right\} \geq J(x-v, y+v),
\end{aligned}
$$

where the last inequality is due to (4.7).
In what follows let $f^{*} \in F$ be the largest maximiser of the right-hand side in (4.7). For completeness, set $f^{*}(x, y)=0$ for $x<0$.
Lemma 4.6. For $x \in \mathbb{N}_{0}$ and $y \geq 0$ it follows that $f^{*}\left(x-f^{*}(x, y), y+f^{*}(x, y)\right)=0$.
Proof. By 4.7. we obtain $J(x, y) \geq \beta^{\gamma} G\left(x, \frac{y}{\beta}\right)$, which implies that

$$
J(x, y)=\beta^{\gamma} G\left(x-f^{*}(x, y), \frac{y+f^{*}(x, y)}{\beta}\right) \leq J\left(x-f^{*}(x, y), y+f^{*}(x, y)\right) \leq J(x, y)
$$

where the second inequality follows from Lemma 4.5. Hence,

$$
\beta^{\gamma} G\left(x-f^{*}(x, y), \frac{y+f^{*}(x, y)}{\beta}\right)=J\left(x-f^{*}(x, y), y+f^{*}(x, y)\right),
$$

which implies that $a=0$ maximises the expression $\beta^{\gamma} G\left(x-f^{*}(x, y)-a, \frac{y+f^{*}(x, y)+a}{\beta}\right)$. We claim that $a=0$ is the only maximiser of this expression. Obviously, if $f^{*}(x, y)=x$, then the result follows. If, on the other hand, $f^{*}(x, y)<x$, then

$$
J(x, y)=\beta^{\gamma} G\left(x-f^{*}(x, y), \frac{y+f^{*}(x, y)}{\beta}\right)>\beta^{\gamma} G\left(x-f^{*}(x, y)-a, \frac{y+f^{*}(x, y)+a}{\beta}\right)
$$

for $a=1, \ldots, x-f^{*}(x, y)$. This fact, in turn, implies that $a=0$ is the only maximiser, which concludes the proof.

Lemma 4.7. Let $\xi(y):=\sup \left\{x \in \mathbb{N}_{0}: f^{*}(x, y)=0\right\}$. Then $\xi^{*}:=\sup _{y \geq 0} \xi(y)<\infty$.
Proof. Fix $y \geq 0$. Let $x \in \mathbb{N}_{0}$ be such that $f^{*}(x, y)=0$. Clearly, such $x \in \mathbb{N}_{0}$ exists. From (3.9) we have that

$$
\begin{aligned}
J(x, y) & =\beta^{\gamma}\left(\sum_{k=-x}^{\infty} J\left(x+k, \frac{y}{\beta}\right) q_{k}+\left(\frac{y}{\beta}\right)^{\gamma} \sum_{k=-\infty}^{-x-1} q_{k}\right) \\
& \leq \beta^{\gamma}\left(\sum_{k=-x}^{\infty}\left(x+k+\frac{\beta}{1-\beta} \mathbb{E} Z^{+}+\frac{y}{\beta}\right)^{\gamma} q_{k}+\left(\frac{y}{\beta}\right)^{\gamma} \sum_{k=-\infty}^{-x-1} q_{k}\right) \quad \text { (by Lemma 4.4) } \\
& =\sum_{k=-x}^{\infty}\left(\beta x+\beta k+\frac{\beta^{2}}{1-\beta} \mathbb{E} Z^{+}+y\right)^{\gamma} q_{k}+y^{\gamma} \sum_{k=-\infty}^{-x-1} q_{k} \\
& \leq\left(\sum_{k=-x}^{\infty}\left(\beta x+\beta k+\frac{\beta^{2}}{1-\beta} \mathbb{E} Z^{+}+y\right) q_{k}+y \sum_{k=-\infty}^{-x-1} q_{k}\right)^{\gamma} \quad \text { (by the Jensen inequality) } \\
& \leq\left(\beta x+\beta \mathbb{E} Z^{+}+\frac{\beta^{2}}{1-\beta} \mathbb{E} Z^{+}+y\right)^{\gamma}=\left(\beta x+\frac{\beta}{1-\beta} \mathbb{E} Z^{+}+y\right)^{\gamma} .
\end{aligned}
$$

On the other hand, making use again of Lemma 4.4 we have that $J(x, y) \geq(x+y)^{\gamma}$ and, consequently,

$$
(x+y)^{\gamma} \leq\left(\beta x+\frac{\beta}{1-\beta} \mathbb{E} Z^{+}+y\right)^{\gamma}
$$

if and only if $x \leq \frac{\beta}{(1-\beta)^{2}} \mathbb{E} Z^{+}$, which is independent of $y$ and implies the result.
The next result is a counterpart of Lemma 3.9 and provides further properties of the function $f^{*} \in F$.
Lemma 4.8. Let $x_{0} \in \mathbb{N}_{0}$ and $y_{0} \geq 1$. If $f^{*}\left(x_{0}, y_{0}\right)=a_{0}$ and $f^{*}\left(x_{0}+1, y_{0}-1\right)>0$, then $f^{*}\left(x_{0}+1, y_{0}-1\right)=a_{0}+1$.
Proof. By definition of $f^{*}$ and 4.6) we have that

$$
J\left(x_{0}, y_{0}\right)=\beta^{\gamma} G\left(x_{0}-a_{0}, \frac{y_{0}+a_{0}}{\beta}\right)\left\{\begin{array}{l}
\geq \beta^{\gamma} G\left(x_{0}-a, \frac{y_{0}+a}{\beta}\right), \text { for } a=0, \ldots, a_{0} \\
>\beta^{\gamma} G\left(x_{0}-a, \frac{y_{0}+a}{\beta}\right), \text { for } a=a_{0}+1, \ldots, x_{0}
\end{array}\right.
$$

The above display implies that

$$
\beta^{\gamma} G\left(x_{0}-a_{0}, \frac{y_{0}+a_{0}}{\beta}\right)\left\{\begin{array}{l}
\geq \beta^{\gamma} G\left(x_{0}-(a-1), \frac{y_{0}+a-1}{\beta}\right), \text { for } a=1, \ldots, a_{0}+1  \tag{4.8}\\
>\beta^{\gamma} G\left(x_{0}-(a-1), \frac{y_{0}+a-1}{\beta}\right), \text { for } a=a_{0}+2, \ldots, x_{0}+1 .
\end{array}\right.
$$

On the other hand, we also obtain that

$$
\begin{aligned}
J\left(x_{0}+1, y_{0}-1\right) & =\max _{a \in A\left(x_{0}+1\right)} \beta^{\gamma} G\left(x_{0}+1-a, \frac{y_{0}-1+a}{\beta}\right) \\
& =\max \left\{\beta^{\gamma} G\left(x_{0}+1, \frac{y_{0}-1}{\beta}\right), \max _{a \in A\left(x_{0}\right)} \beta^{\gamma} G\left(x_{0}-a, \frac{y_{0}+a}{\beta}\right)\right\} .
\end{aligned}
$$

Since $f^{*}\left(x_{0}+1, y_{0}-1\right)>0$, we infer that

$$
\beta^{\gamma} G\left(x_{0}+1, \frac{y_{0}-1}{\beta}\right) \leq \max _{a \in A\left(x_{0}\right)} \beta^{\gamma} G\left(x_{0}-a, \frac{y_{0}+a}{\beta}\right) .
$$

This fact and (4.8) yield that

$$
\beta^{\gamma} G\left(x_{0}+1-\left(a_{0}+1\right), \frac{y_{0}-1+\left(a_{0}+1\right)}{\beta}\right) \geq \beta^{\gamma} G\left(x_{0}+1-a, \frac{y_{0}-1+a}{\beta}\right),
$$

for $a=0, \ldots, a_{0}+1$ and

$$
\beta^{\gamma} G\left(x_{0}+1-\left(a_{0}+1\right), \frac{y_{0}-1+\left(a_{0}+1\right)}{\beta}\right)>\beta^{\gamma} G\left(x_{0}+1-a, \frac{y_{0}-1+a}{\beta}\right),
$$

for $a=a_{0}+2, \ldots, x_{0}+1$. Thus, it follows that $f^{*}\left(x_{0}+1, y_{0}-1\right)=a_{0}+1$.
4.2. Optimal policy. Let $y_{0} \geq 0$ be fixed and let $x_{0} \in X$ be the initial state. Consider the following policy $\pi^{*}:=\left(\pi_{k}^{*}\right)_{k \in \mathbb{N}_{0}}$ generated by $f^{*}$ in the following way

$$
\begin{equation*}
\pi_{k}^{*}\left(\omega_{k}\right):=f^{*}\left(x_{k}, y_{k}\right), \quad \omega_{k} \in \Omega_{k} \quad k \in \mathbb{N}_{0} \tag{4.9}
\end{equation*}
$$

where

$$
\begin{equation*}
y_{k}:=\frac{y_{k-1}+f^{*}\left(x_{k-1}, y_{k-1}\right)}{\beta}=\frac{y_{0}+\sum_{m=0}^{k-1} \beta^{m} f^{*}\left(x_{m}, y_{m}\right)}{\beta^{k}}, \quad k \in \mathbb{N} . \tag{4.10}
\end{equation*}
$$

Obviously, $\pi \in \widehat{\Pi}$.
Corollary 4.9. Under policy $\pi^{*}$ ruin occurs with probability 1.
Proof. We proceed along similar lines as in the proof of Corollary 3.10. Let $\left(x_{0}, y_{0}\right) \in \widehat{X}$ with $x_{0} \geq 0$. Then, either $x_{0} \leq \xi^{*}$ or $x_{0}>\xi^{*}$. Observe that in both cases the risk reserve just after the dividend payment is less or equal to $\xi^{*}$. The first case is obvious. In the second case, we deduce from Lemma 4.6 that $f^{*}\left(x_{0}-f^{*}\left(x_{0}, y_{0}\right), y_{0}+f^{*}\left(x_{0}, y_{0}\right)\right)=0$, which means by Lemma 4.7 that $x_{0}-f^{*}\left(x_{0}, y_{0}\right) \leq \xi\left(y_{0}+f^{*}\left(x_{0}, y_{0}\right)\right) \leq \xi^{*}$. Hence, the ruin occurs, if there appears a sequence of length $\xi^{*}+1$ of negative incomes. The probability of such event equals $\left(\sum_{m=-\infty}^{-1} q_{m}\right)^{\xi^{*}+1}$, see also (3.11). If the ruin has not occurred up to the $l$ th day with $l=\xi^{*}+1$, then either $x_{l} \leq \xi^{*}$ or $x_{l}>\xi^{*}$. Now the remaining part follows from the proof of Corollary 3.10.
Theorem 4.10. For every $(x, y) \in \widehat{X}$ it holds that $J(x, y)=J_{\pi^{*}}(x, y)$.
Proof. From Theorem 4.2 and the definition of $\pi^{*}$ given in (4.9) we have that

$$
J(x, y)=\beta^{\gamma} \sum_{x_{1} \in X} J\left(x_{1}, y_{1}\right) q\left(x_{1} \mid x, f^{*}(x, y)\right)
$$

for $x \in \mathbb{N}_{0}$ and $y \geq 0$ with $y_{1}$ defined in 4.10) and $y_{0}:=y$. Hence,

$$
\begin{align*}
J(x, y) & =\beta^{\gamma} \sum_{x_{1} \geq 0} J\left(x_{1}, y_{1}\right) q\left(x_{1} \mid x, f^{*}(x, y)\right)+\beta^{\gamma} \sum_{x_{1}<0}\left(y_{1}\right)^{\gamma} q\left(x_{1} \mid x, f^{*}(x, y)\right)  \tag{4.11}\\
& =\beta^{\gamma} \sum_{x_{1} \geq 0} J\left(x_{1}, y_{1}\right) q\left(x_{1} \mid x, f^{*}(x, y)\right)+\sum_{x_{1}<0}\left(y+f^{*}(x, y)\right)^{\gamma} q\left(x_{1} \mid x, f^{*}(x, y)\right) .
\end{align*}
$$

Iterating (4.11) $(n-1)$ times and making use of the policy $\pi^{*}$ we arrive at the following equation

$$
\begin{align*}
& J(x, y)=\left(\beta^{\gamma}\right)^{n} \mathbb{E}_{x}^{\pi^{*}}\left(J\left(x_{n}, y_{n}\right) 1[\tau>n]\right)+  \tag{4.12}\\
& \quad \mathbb{E}_{x}^{\pi^{*}}\left[\left(y+\sum_{k=0}^{(n-1) \wedge(\tau-1)} \beta^{k} f^{*}\left(x_{k}, y_{k}\right)\right)^{\gamma} 1[\tau \leq n]\right] .
\end{align*}
$$

From Lemma 4.4 and the concavity of $x \mapsto x^{\gamma}$, we obtain the following bound for the first term in 4.12

$$
\begin{align*}
& \left(\beta^{\gamma}\right)^{n} \mathbb{E}_{x}^{\pi^{*}}\left(J\left(x_{n}, y_{n}\right) 1[\tau>n]\right) \leq\left(\beta^{\gamma}\right)^{n} \mathbb{E}_{x}^{\pi^{*}}\left(\left(x_{n}+y_{n}+\frac{\beta}{1-\beta} \mathbb{E} Z^{+}\right)^{\gamma} 1[\tau>n]\right) \\
& \quad \leq\left(\beta^{\gamma}\right)^{n} \mathbb{E}_{x}^{\pi^{*}}\left(x_{n}^{\gamma} 1[\tau>n]\right)+\left(\beta^{\gamma}\right)^{n}\left(\frac{\beta}{1-\beta} \mathbb{E} Z^{+}\right)^{\gamma}+\left(\beta^{\gamma}\right)^{n} \mathbb{E}_{x}^{\pi^{*}}\left(\left(y_{n}\right)^{\gamma} 1[\tau>n]\right) \tag{4.13}
\end{align*}
$$

By (4.10) the third term in 4.13 can be written as follows

$$
\begin{equation*}
\left(\beta^{\gamma}\right)^{n} \mathbb{E}_{x}^{\pi^{*}}\left(\left(y_{n}\right)^{\gamma} 1[\tau>n]\right)=\mathbb{E}_{x}^{\pi^{*}}\left[\left(y+\sum_{k=0}^{n-1} \beta^{k} f^{*}\left(x_{k}, y_{k}\right)\right)^{\gamma} 1[\tau>n]\right] \tag{4.14}
\end{equation*}
$$

Next by (2.1) we get for the first term in (4.13) the following

$$
\begin{align*}
0 & \leq\left(\beta^{\gamma}\right)^{n} \mathbb{E}_{x}^{\pi^{*}}\left(x_{n}^{\gamma} 1[\tau>n]\right) \leq\left(\beta^{n} \mathbb{E}_{x}^{\pi^{*}}\left(x_{n} 1[\tau>n]\right)\right)^{\gamma}  \tag{4.15}\\
& \leq\left(\beta^{n} \mathbb{E}_{x}^{\pi^{*}}\left[\left(x-\sum_{k=0}^{n-1} f^{*}\left(x_{k}, y_{k}\right)+\sum_{k=1}^{n} Z_{k}^{+}\right) 1[\tau>n]\right]\right)^{\gamma} \\
& \leq\left(\beta^{n} x+\beta^{n} n \mathbb{E} Z^{+}\right)^{\gamma}
\end{align*}
$$

By our assumption (A1) and 4.15 the first term in 4.13) converges to 0 as $n \rightarrow \infty$. Observe that the same remark also applies for the second term in 4.13. Summing up, from 4.12), 4.13 and 4.14 we obtain that

$$
J(x, y) \leq \lim _{n \rightarrow \infty} \mathbb{E}_{x}^{\pi^{*}}\left(y+\sum_{k=0}^{(n-1) \wedge(\tau-1)} \beta^{k} f^{*}\left(x_{k}, y_{k}\right)\right)^{\gamma}
$$

Now the monotone convergence theorem yields that $J(x, y) \leq J_{\pi^{*}}(x, y)$.
We close this section with a conclusion for our original model.
Corollary 4.11. Let $y_{0}:=0$. Then, $\pi^{*}$ is optimal for the original optimisation problem, i.e., $\widehat{J}(x)=\widehat{J}_{\pi^{*}}(x)$.

Remark 4.12. Note that in the case of a power utility, the optimal policy is history-dependent, but depends on the history only through the accumulated discounted dividends given by $\left(y_{k}\right)$ in (4.10).

Remark 4.13. It is well-known that the logarithmic utility function $U(x)=\log (x)$ can be obtained as a limit from the power utility since

$$
\lim _{\gamma \rightarrow 0} \frac{1}{\gamma}\left(x^{\gamma}-1\right)=\log (x)
$$

Indeed, the problem can then be treated for the logarithmic utility in a similar way. The optimality equation is given by

$$
J(x, y)=\log (\beta)+\max _{a \in A(x)}\left[\sum_{k=-\infty}^{\infty} J\left(x-a+k, \frac{a+y}{\beta}\right) q_{k}\right]
$$

and we can follow the same line of analysis. It is worth mentioning that the power and logarithmic utility functions are examples of the so-called HARA (hyperbolic absolute risk aversion) utilities, whereas the exponential utility function belongs to the CARA (constant absolute risk aversion) class of utilities. The reader is referred, for instance, to [5, 6, 6, 12] and references cited therein, for further properties of the aforementioned functions.

## 5. Howard's Policy Improvement

In this section we provide one numerical tool to solve these problems which is known under the name Howard policy improvement. We restrict the presentation here to the exponential utility function: Start with an arbitrary policy of the form $\pi=(f(\cdot, \gamma), f(\cdot, \gamma \beta), \ldots)$ induced by a decision rule $f$ where we assume that $f$ is such that $f(x, \theta) \geq x-s^{*}$ for all $x>s^{*}$ and all $\theta$. Note that we define $s^{*}:=\sup _{\theta \in[\gamma, 0)} s(\theta)$ (see Theorem 3.8). Take e.g. $f(x, \theta)=x^{+}$. We write $J_{f}:=J_{\pi}$. Next we compute the largest minimizer $h(x, \theta)$ of the expression

$$
a \mapsto e^{\theta a} G_{f}(x-a, \theta \beta), a \in\{0,1, \ldots, x\}
$$

where

$$
G_{f}(x, \theta):=\sum_{k=-x}^{\infty} J_{f}(x+k, \theta) q_{k}+\sum_{k=-\infty}^{-x-1} q_{k} .
$$

We claim now that
Lemma 5.1. The new decision rule $h$ and the corresponding value function $J_{h}$ have the following properties:
a) $h(x-h(x, \theta), \theta)=0$ for all $x$ and $\theta$.
b) $h(x, \theta) \geq x-s^{*}$ for all $x>s^{*}$ and all $\theta$.
c) $e^{\theta x} \underline{h}(\theta) \leq J_{h} \leq J_{f} \leq e^{\theta x} \bar{h}(\theta)$.

Proof. a) If $h(x, \theta)=0$ or $h(x, \theta)=x$ the statement is true. Now let $0<h(x, \theta)<x$ and suppose that $h(x-h(x, \theta), \theta)>0$, i.e. there is an $a^{*}>0$ s.t.

$$
e^{\theta a^{*}} G_{f}\left(x-h(x, \theta)-a^{*}, \theta \beta\right) \leq G_{f}(x-h(x, \theta), \theta \beta)
$$

On the other hand by the definition of $h$ we have for all $a>h(x, \theta)$ :

$$
e^{\theta h(x, \theta)} G_{f}(x-h(x, \theta), \theta \beta)<e^{\theta a} G_{f}(x-a, \theta \beta)
$$

Combining these inequalities leads to (note that $x-h(x, \theta)-a^{*} \geq 0$ )

$$
\begin{aligned}
e^{\theta h(x, \theta)} G_{f}(x-h(x, \theta), \theta \beta) & <e^{\theta\left(h(x, \theta)+a^{*}\right)} G_{f}\left(x-h(x, \theta)-a^{*}, \theta \beta\right) \\
& \leq e^{\theta h(x, \theta)} G_{f}(x-h(x, \theta), \theta \beta)
\end{aligned}
$$

which is a contradiction. Thus, the statement is shown.
b) We show first for $x>s^{*}$ and arbitrary $\theta$ that $h(x, \theta)>0$. In order to do this, consider the expression $e^{\theta a} G_{f}(x-a, \theta \beta)$ for $a=0$ and $a=f(x, \theta)$. By definition we obtain for $a=f(x, \theta)$ that

$$
e^{\theta f(x, \theta)} G_{f}(x-f(x, \theta), \theta \beta)=J_{f}(x, \theta) \leq e^{\theta x} \bar{h}(\theta)
$$

For $a=0$ we obtain that

$$
\begin{aligned}
G_{f}(x, \theta \beta) & =\sum_{k=-x}^{\infty} J_{f}(x+k, \theta \beta) q_{k}+\sum_{k=-\infty}^{-x-1} q_{k} \\
& \geq e^{\theta \beta x} \underline{h}(\theta \beta)\left(\sum_{k=-x}^{\infty} e^{\theta \beta k} q_{k}+\sum_{k=-\infty}^{-x-1} q_{k}\right) \\
& =e^{\theta \beta x} \underline{h}(\theta \beta) \mathbb{E} e^{\theta \beta Z^{+}}=e^{\theta \beta x} \underline{h}(\theta) .
\end{aligned}
$$

Furthermore, observe that

$$
e^{\theta \beta x} \underline{h}(\theta) \geq e^{\theta x} \bar{h}(\theta) \quad \Leftrightarrow \quad x>s(\theta)
$$

Thus, the inequality holds, in particular, if $x \geq s^{*}=\sup _{\theta \in[\gamma, 0)} s(\theta)$. This implies that 0 cannot be a minimiser, so $h(x, \theta)>0$ for all $x>s^{*}$ and all $\theta$. This fact and point (a) imply the conclusion.
c) From the definition of $h$ we obtain:

$$
J_{f}(x, \theta)=e^{\theta f(x, \theta)} G_{f}(x-f(x, \theta), \theta \beta) \geq e^{\theta h(x, \theta)} G_{f}(x-h(x, \theta), \theta \beta)
$$

Iterating this inequality yields

$$
J_{f}(x, \theta) \geq \mathbb{E}_{x}^{h}\left[\exp \left(\gamma \sum_{k=0}^{(\tau-1) \wedge(n-1)} \beta^{k} a_{k}\right)\left(J_{f}\left(x_{n}, \gamma \beta^{n}\right) 1[\tau \geq n]+1[\tau<n]\right)\right] .
$$

The property of $h$ shown in b) now implies that ruin occurs with probability 1 under $h$ and thus as in the proof of Theorem 3.11 we obtain with $n \rightarrow \infty$ that $J_{f} \geq J_{h}$.

From the proof it follows that in case $f \neq h$, the inequality $J_{h}(x, \theta) \leq J_{f}(x, \theta)$ is strict for at least one $(x, \theta)$. Now suppose no improvement is possible, i.e. $h=f$. Hence $J_{f}$ is another solution of (3.2). By Remark $3.2 J_{f} \leq J$. On the other hand by the definition of $J$ we have $J_{f} \geq J$ which implies $J=J_{f}$.

Finally if the iteration does not stop we obtain a non-increasing sequence $J_{f_{0}} \geq J_{f_{1}} \geq \ldots \geq J$. Denote $\underline{J}:=\lim _{n \rightarrow \infty} J_{f_{n}}$. Obviously $\underline{J} \geq J$. Next from the definition of an improvement:

$$
J_{f_{k+1}}(x, \theta) \leq \min _{a \in A(x)}\left[e^{\theta a} G_{f_{k}}(x-a, \theta \beta)\right] \leq J_{f_{k}}(x, \theta)
$$

Letting $k \rightarrow \infty$ we obtain (note that lim and min can be interchanged since $A(x)$ is always finite):

$$
\underline{J}(x, \theta) \leq \min _{a \in A(x)}\left[e^{\theta a}\left(\sum_{k=a-x}^{\infty} \underline{J}(x-a+k, \theta \beta) q_{k}+\sum_{k=-\infty}^{a-x-1} q_{k}\right)\right] \leq \underline{J}(x, \theta)
$$

hence $\underline{J}$ is another solution of (3.2) which implies that $\underline{J}=J$.

## 6. Concluding remarks

In this paper, we study the discrete time problem, suggested by Gerber and Shiu [13], of maximising the expected utility of discounted dividends until ruin. We restrict our attention to the integer-valued surplus process and to integer payments. To the best of our knowledge, the only paper that examines a similar issue (with the exponential utility) is [14], where the wealth of insurance company is driven by a Brownian motion with drift. However, the authors have not been able to solve the problem rigorously. Namely, assuming that a certain integral equation for the barrier function $b(t)$ has a desirable solution (see Standing Assumption in [14), they prove that $b(t)$ is indeed the barrier they search for (a barrier function is a band function with $n=0$ in Definition 3.1). Moreover, the numerical experiments provided in Section 1.4 in [14] are given without their convergence proofs. This fact and the lack of a complete solution in continuous time and any solution in discrete time to Gerber and Shiu's suggestion since 2004 indicate that the problem is not straightforward from the mathematical point of view. Firstly, similar as in [6, 14 ] we note that the optimal strategy is time dependent in a certain way. In order to get rid of non-stationarity we extend the state space to the two-dimensional space. Within such a new framework our problem becomes stationary. Secondly, since our dividend payments can be unbounded we cannot directly apply the results from [6, 11 to deduce that the value function satisfies the corresponding Bellman equation for the exponential and power utility functions. Nonetheless, we are able to show that in both cases the value iteration algorithm works (see Theorem 3.1 and Theorem 4.1) and in the exponential function case the Howard's policy improvement algorithm works (see Section 5). These facts, in turn, may have a significant meaning, when one thinks of numerical examples. Moreover, we are also able to describe the structure of optimal strategies for both utility cases and to prove for the exponential function case that the optimal policy is a band-policy.

Numerical experiments are difficult. Let us first recall that the maximisation of the expectation of discounted dividends in the model given by (2.1) with $\mathbb{P}\left(Z_{1}=1\right)=p=1-\mathbb{P}\left(Z_{1}=-N\right)$,
where $p \in(0,1)$ and $N \in \mathbb{N}$, was a challenging analytical problem. The reader may find the complete solution, for instance, in [18], where it was shown that the optimal policy is of barrier type. Our problem, as already mentioned, is non-stationary and non-separable. Therefore, the methods that solved analytically the risk neutral problem are useless here. Moreover, as was noted by Gerber and Shiu [13], in contrast to the risk neutral problem one can expect that in the model with exponential function the optimal policy is not of barrier type. This fact does not make easier potential calculations. Obviously, since obtaining an analytical solution is a challenge, one can think of numerical methods used in dynamic programming such as value iteration, policy improvement or others, see [20. However, our surplus process proceeds on the space $X=\mathbb{Z}$ and even the simple aforementioned case $\left(\mathbb{P}\left(Z_{1}=1\right)=p=1-\mathbb{P}\left(Z_{1}=-1\right)\right)$ requires some truncation of the state space to a finite one in order to obtain numerical results. In addition, in this model we meet one more obstacle that have not been treated so far, namely the exponential and power utility functions that imply non-stationarity and non-separability. Therefore, the problem of calculating numerically optimal strategies and value functions for models with these or other utility functions is left open.

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[^0]:    ${ }^{1}$ We use the same symbol $h_{k}$ as in the previous section to denote an extended feasible history of the process up to the $k$ th day. But there is no confusion, since $h_{k}$ in this subsection refers only to the power utility case. The same remark applies to the policy $\sigma$, the set $\Xi$ and the functions $J$ and $J_{\pi}$ defined below.

