Hedging Conditional Value at Risk with Options

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Abstract

We present a method of hedging Conditional Value at Risk of a position in stock using put options. The result leads to a linear programming problem that can be solved to optimise risk hedging.

Keywords: Conditional Value at Risk, Expected Shortfall, measures of risk, risk management

1. Introduction

One of the natural ideas to reduce risk of a position in stock is to buy put options. By doing so one can cut off the undesirable scenarios, while leaving oneself open to the positive outcomes. A choice of a high strike price of the put option does cut off more of the unfavourable states, but at the same time produces higher hedging costs. The question of how to balance the two trends so that the level of risk measured by Value at Risk (VaR) is minimised was investigated by Ahn, Boudoukh, Richardson and Whitelaw [2].

The Value at Risk, which is the worst case scenario of loss an investment might incur at a given confidence level, has established its position as one of the standard measures of risk, and is widely used throughout the field of finance and risk management. One of its shortcoming is that it neglects potential severity of unlikely events. Another, that it is not sub-additive, and is thus not a coherent risk measure [3]. Its most common modification to achieve these goals is the Conditional Value at Risk (CVaR) (also referred to as 'Expected Shortfall'), which takes into the account the average loss

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exceeding VaR. The CVaR is a coherent risk measure (the proof can be found in the work of Acerbi and Tasche [1]).

In this paper we show a mirror result to [2], using CVaR instead of VaR. It turns out that in such setting one can achieve closed form formulae for CVaR of stock hedged with puts. These can be used to optimise the position by solving a linear programming problem.

We restrict our attention to the Black–Scholes model and consider investments in stock and put options. The optimisation of CVaR can be carried out under more general assumptions, using also other securities (as an example see Rockafellar and Uryasev [6, 7]). One can also hedge CVaR dynamically (as in the work of Melnikov and Smirnov [5]), which provides slightly better results. Dynamic strategies though require constant rebalancing, which in practice can be costly. Advantages of our approach are as follows: its simplicity; closed form analytic formula for CVaR; protection against risk is very similar to the one attainable using dynamic strategies.

The paper is organised as follows. Section 2 recalls the results of Ahn, Boudoukh, Richardson and Whitelaw [2] for hedging of VaR with put options. This section serves also as preliminaries to the paper. In Section 3 we generalise the result to use CVaR instead of VaR. The main result of the paper is given in Theorem 4. The section ends with an example of its application. In Section 4 we compare our method to the results attainable using dynamic strategies. They turn out to be close. We finish the paper with a short conclusion in Section 5.

2. Hedging Value at Risk

In this section we set up our notations and recall the results of Ahn, Boudoukh, Richardson and Whitelaw [2].

Let X be a random variable, which represents a gain from an investment. For α in $(0, 1)$, we define the *Value at Risk* of X, at confidence level $1 - \alpha$, as $\mathrm{VaR}^\alpha(X) = -q^\alpha(X)$, where $q^\alpha(X)$ is the upper α-quantile of X.

We consider the Black–Scholes model, where the stock price evolves according to $dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$, with the money market account $dA(t) = rA(t)dt$. A European put option with strike price K and maturity T has payoff $P(T) = (K - S(T))^+$ and costs

$$P(0) = P(r, T, K, S(0), \sigma) = Ke^{-rT}N(-d_-) - S(0)N(-d_+), \qquad (1)$$

where

$$\begin{aligned} d_+ &= d_+(r,T,K,S(0),\sigma) = \frac{\ln\frac{S(0)}{K} + \left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \\ d_- &= d_-(r,T,K,S(0),\sigma) = d_+ - \sigma\sqrt{T}, \end{aligned} \tag{2}$$

and N is the standard normal cumulative distribution function.

Assume that we buy x shares of stock and z_i put options with strikes K_i, which cost $P_i(t)$ for $i = 1,\ldots,n$ and $t = 0, T$. Let $\mathbf{z}$, $\mathbf{1}$ and $\mathbf{P}(t)$ be vectors in $\mathbb{R}^n$ defined as

$$\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \qquad \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad \mathbf{P}(t) = \begin{bmatrix} P_1(t) \\ \vdots \\ P_n(t) \end{bmatrix}.$$

The value of our investment at time t is $V_{(x,\mathbf{z})}(t) = xS(t) + \mathbf{z}^T\mathbf{P}(t)$. The following theorem can be used to compute VaR for the discounted gain

$$X_{(x,\mathbf{z})} = e^{-rT}V_{(x,\mathbf{z})}(T) - V_{(x,\mathbf{z})}(0).$$

Theorem 1. *[2] If $z_i \geq 0$, for $i = 1,\ldots,n$, and $\mathbf{z}^T\mathbf{1} \leq x$, then*

$$\mathrm{VaR}^\alpha\left(X_{(x,\mathbf{z})}\right) = V_{(x,\mathbf{z})}(0) - e^{-rT}\left(xq^\alpha(S(T)) - \mathbf{z}^T\mathbf{q}^\alpha(-\mathbf{P}(T))\right), \tag{3}$$

where

$$\mathbf{q}^\alpha(-\mathbf{P}(T)) = -\begin{bmatrix} (K_1 - q^\alpha(S(T)))^+ \\ \vdots \\ (K_n - q^\alpha(S(T)))^+ \end{bmatrix}. \tag{4}$$

3. Hedging Conditional Value at Risk

One of the shortcomings of VaR is that it neglects the tail of the loss distribution. An improvement in this respect is the *Conditional Value at Risk*, defined as

$$\mathrm{CVaR}^\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}^\beta(X)d\beta = -\frac{1}{\alpha}\int_0^\alpha q^\beta(X)d\beta,$$

with a well known equivalent form

$$\mathrm{CVaR}^\alpha(X) = -\frac{1}{\alpha}\left[\mathbb{E}(X\mathbf{1}_{\{X\leq q^\alpha(X)\}}) + q^\alpha(X)(\alpha - \mathbb{P}(X \leq q^\alpha(X))\right]. \tag{5}$$

The CVaR also has the advantage of being a coherent risk measure [1, 3].

Our aim is to give a mirror result to Theorem 1, using CVaR as the risk measure. We start with a simple lemma.

Lemma 2. *For any* $q \in \mathbb{R}$,

$$\mathbb{E}\left(S(T)|W(T) \leq q\sqrt{T}\right) = \frac{1}{N(q)} S(0)e^{\mu T} N\left(q - \sigma\sqrt{T}\right).$$

Proof. Let $Z = W(T)/\sqrt{T}$. Since $\mathbb{P}(Z \leq q) = N(q) > 0$,

$$\begin{aligned}
\mathbb{E}\left(S(T)|Z \leq q\right) &= \frac{1}{P(Z \leq q)} \int_{-\infty}^{q} S(0)e^{\left(\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}x\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{N(q)} S(0)e^{\mu T} \int_{-\infty}^{q} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(x-\sigma\sqrt{T}\right)^2}{2}} dx \\
&= \frac{1}{N(q)} S(0)e^{\mu T} N\left(q - \sigma\sqrt{T}\right),
\end{aligned}$$

as required. ■

Let Z be a random variable with standard normal distribution $N(0, 1)$. To compute $\mathrm{CVaR}^{\alpha}(X_{(x,\mathbf{z})})$, we introduce notations

$$\begin{aligned}
d_{-}^{\mu} &= d_{-}(\mu, T, K, S(0), \sigma), \qquad & d_{+}^{\mu} &= d_{-}^{\mu} + \sigma\sqrt{T}, \\
d_{-}^{\mu,\alpha} &= \max\left(d_{-}^{\mu}, -q^{\alpha}(Z)\right), \qquad & d_{+}^{\mu,\alpha} &= d_{-}^{\mu,\alpha} + \sigma\sqrt{T},
\end{aligned}$$

$$P^{\alpha}(K) = Ke^{-\mu T} N(-d_{-}^{\mu,\alpha}) - S(0)N\left(-d_{+}^{\mu,\alpha}\right). \tag{6}$$

We first consider the case when we invest in puts with a single strike $K_1 = K$.

Proposition 3. *If* $\mathbf{z} = [z_1]$, *for* $z_1 = z \in [0, x]$, *then*

$$\mathrm{CVaR}^{\alpha}\left(X_{(x,\mathbf{z})}\right) = V_{(x,\mathbf{z})}(0) - \frac{1}{\alpha} e^{(\mu - r)T} \left[xS(0)N\left(q^{\alpha}(Z) - \sigma\sqrt{T}\right) + zP^{\alpha}(K)\right].$$

Proof. We first observe that

$$X_{(x,\mathbf{z})} = e^{-rT}\left(xS(T) + z\left(K - S(T)\right)^{+}\right) - V_{(x,\mathbf{z})}(0). \tag{7}$$

Since $z \leq x$, we see that

$$s \to e^{-rT}\left(xs + z\left(K - s\right)^{+}\right) - V_{(x,\mathbf{z})}(0) \tag{8}$$

is a non-decreasing function of s. Also $\xi \to S(0)\exp\left((\mu-\sigma^2/2)\,T+\sigma\sqrt{T}\xi\right)$ is increasing. Combining these two facts, taking $Z = W(T)/\sqrt{T}$,

$$\left\{X_{(x,\mathbf{z})} \le q^\alpha(X_{(x,\mathbf{z})})\right\} = \{S(T) \le q^\alpha(S(T))\} = \{Z \le q^\alpha(Z)\}\,. \tag{9}$$

We first prove the claim for $z < x$. Then (8) is strictly increasing, therefore $\mathbb{P}(X_{(x,\mathbf{z})} \le q^\alpha(X_{(x,\mathbf{z})})) = \mathbb{P}(S(T) \le q^\alpha(S(T))) = \alpha$, and

$$\begin{aligned}
\mathrm{CVaR}^\alpha(X_{(x,\mathbf{z})}) &= -\mathbb{E}\left(X_{(x,\mathbf{z})} | X_{(x,\mathbf{z})} \le q^\alpha(X_{(x,\mathbf{z})})\right) \\
&= -\mathbb{E}\left(X_{(x,\mathbf{z})} | Z \le q^\alpha(Z)\right) && \text{(see (9))} \\
&= V_{(x,z)}(0) - e^{-rT}x\mathbb{E}\left(S(T) | Z \le q^\alpha(Z)\right) && \text{(see (7))} \\
&\quad - e^{-rT}z\mathbb{E}\left((K-S(T))^+ | Z \le q^\alpha(Z)\right). && (10)
\end{aligned}$$

We now compute the last term in (10). Since $\{S(T) \le K\} = \{Z \le -d_-^\mu\}$,

$$\begin{aligned}
&\mathbb{E}\left((K-S(T))^+ | Z \le q^\alpha(Z)\right) \\
&= \frac{1}{\alpha}\int_{-\infty}^{\min(q^\alpha(Z), -d_-^\mu)} \left(K - S(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}\right)\frac{1}{\sqrt{2\pi}}e^{-x^2}dx \\
&= \frac{1}{\alpha}\int_{-\infty}^{-d_-^{\mu,\alpha}} K\frac{1}{\sqrt{2\pi}}e^{-x^2}dx - \frac{1}{\alpha}\int_{-\infty}^{-d_-^{\mu,\alpha}} S(0)e^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}x}\frac{1}{\sqrt{2\pi}}e^{-x^2}dx \\
&= \frac{1}{\alpha}KN(-d_-^{\mu,\alpha}) - \frac{1}{\alpha}\mathbb{P}(Z \le -d_-^{\mu,\alpha})\mathbb{E}\left(S(T) | Z \le -d_-^{\mu,\alpha}\right) \\
&= \frac{1}{\alpha}KN(-d_-^{\mu,\alpha}) - \frac{1}{\alpha}S(0)e^{\mu T}N\left(-d_-^{\mu,\alpha} - \sigma\sqrt{T}\right) \qquad \text{(by Lemma 2)} \\
&= \frac{1}{\alpha}e^{\mu T}\left(Ke^{-\mu T}N(-d_-^{\mu,\alpha}) - S(0)N\left(-d_+^{\mu,\alpha}\right)\right).
\end{aligned}$$

Substituting the above into (10) and applying Lemma 2 gives the claim.

We now need to consider the case when $z = x$. Since for any $\beta \in (0,1)$, $\lim_{z \nearrow x} q^\beta(X_{(x,\mathbf{z})}) = q^\beta(X_{(x,x)})$, we obtain

$$\begin{aligned}
\lim_{z \nearrow x} \mathrm{CVaR}^\alpha\left(X_{(x,\mathbf{z})}\right) &= \lim_{z \nearrow x} \frac{-1}{\alpha}\int_0^\alpha q^\beta(X_{(x,\mathbf{z})})d\beta \\
&= \frac{-1}{\alpha}\int_0^\alpha q^\beta(X_{(x,x)})d\beta = \mathrm{CVaR}^\alpha\left(X_{(x,x)}\right).
\end{aligned}$$

Hence the result follows from the fact that the formula for $\mathrm{CVaR}^\alpha(X_{(x,\mathbf{z})})$ in the claim is continuous with respect to z. ∎

We can now formulate our main result.

Theorem 4. *If $z_i \geq 0$ for $i = 1, \ldots, n$ and $z_1 + \ldots + z_n \leq x$, then*

$$\mathrm{CVaR}^{\alpha}(X_{(x,\mathbf{z})}) = V_{(x,\mathbf{z})}(0) - \frac{1}{\alpha} e^{(\mu - r)T} \left[x S(0) N\left(q^{\alpha}(Z) - \sigma\sqrt{T} \right) + \mathbf{z}^{\mathrm{T}} \mathbf{P}^{\alpha} \right], \tag{11}$$

where $\mathbf{P}^{\alpha} = (P^{\alpha}(K_1), \ldots, P^{\alpha}(K_n))$.

Proof. The proof follows from mirror arguments to the proof of Proposition 3. ■

We show how Theorem 4 can be applied. Assume that x is fixed. We investigate how to minimise $\mathrm{CVaR}^{\alpha}\left(X_{(x,\mathbf{z})}\right)$ by choosing $\mathbf{z}$. Assume that we invest V_0 and spend $c = V_0 - xS(0)$ on put options. By (11), minimising $\mathrm{CVaR}^{\alpha}\left(X_{(x,\mathbf{z})}\right)$ is equivalent to the problem:

$$\begin{array}{ll} \min -\mathbf{z}^{\mathrm{T}} \mathbf{P}^{\alpha} & \\ \text{subject to:} & \mathbf{z}^{\mathrm{T}} \mathbf{P}(0) = c, \\ & \mathbf{z}^{\mathrm{T}} \mathbf{1} \leq x, \\ & z_0, \ldots, z_n \geq 0. \end{array} \tag{12}$$

This is a linear programming problem, which can easily be solved numerically.

The result can be complemented by computing $\mathbb{E}\left(X_{(x,\mathbf{z})}\right)$ to give risk/return type analysis. A direct computation gives

$$\mathbb{E}\left(X_{(x,\mathbf{z})}\right) = e^{-rT}\left[xS(0)e^{\mu T} + \mathbf{z}^{\mathrm{T}} \mathbb{E}\left(\mathbf{P}(T)\right) \right] - V_{(x,z)}(0),$$

where

$$\mathbb{E}\left(\mathbf{P}(T)\right) = e^{\mu T} \begin{bmatrix} P(\mu, T, K_1, S(0), \sigma) \\ \vdots \\ P(\mu, T, K_n, S(0), \sigma) \end{bmatrix}.$$

Example 5. *Consider $S(0) = 100$, $\mu = 10\%$, $\sigma = 0.2$ and $r = 3\%$. Assume that we spend $V_0 = 1000$, investing in stock and put options with strike prices $K_1 = 80$, $K_2 = 90$, $K_3 = 100$, $K_4 = 110$, $K_5 = 120$ and expiry $T = 1$. We shall solve (12) for $\alpha = 0.05$, considering $c \in [0, 160]$.*

The choice of x depends on c, since $xS(0) + c = V_0$.

We compute the vectors:

$$\mathbf{P}(0) = \begin{bmatrix} 0.860 \\ 2.769 \\ 6.458 \\ 12.042 \\ 19.220 \end{bmatrix}, \qquad \mathbf{P}^{\alpha} = \begin{bmatrix} 0.366 \\ 0.819 \\ 1.271 \\ 1.724 \\ 2.176 \end{bmatrix}, \qquad \mathbb{E}(\mathbf{P}(T)) = \begin{bmatrix} 0.420 \\ 1.574 \\ 4.148 \\ 8.527 \\ 14.686 \end{bmatrix}.$$

The solutions to the problem (12) are:

c	x	z_1	z_2	z_3	z_4	z_5	$CVaR^\alpha$	$\mathbb{E}$
0	10	0	0	0	0	0	302.24	72.51
20	9.8	3.74	6.06	0	0	0	180.35	61.84
40	9.6	0	5.96	3.64	0	0	126.24	53.35
60	9.4	0	0.19	9.21	0	0	89.64	45.52
80	9.2	0	0	5.51	3.69	0	71.42	39.41
100	9	0	0	1.50	7.50	0	53.82	33.35
120	8.8	0	0	0	6.85	1.95	41.64	28.31
140	8.6	0	0	0	3.52	5.08	32.70	23.86
160	8.4	0	0	0	0.20	8.20	23.75	19.42

From the table we observe that for larger c we can afford to buy options with higher strike prices, which provide better protection, but are at the same time more expensive.

4. Comparison with dynamic hedging

An alternative to hedging with put options is to engage in a self financing strategy that will reduce the risk. In this section we explore the differences between this approach and our method.

Föllmer and Leukert [4] developed a method for dynamic optimisation of VaR. In [5], Melnikov and Smirnov (by combining techniques from [4] with [6, 7]) extend the method to the setting of dynamic optimisation of CVaR. They consider a contingent claim with a time T payoff H, and solve the following problem:

$$\begin{aligned} &\min_{\xi} \mathrm{CVaR}^\alpha(e^{-rT}(V_\xi(T) - H)), \\ &\text{subject to } V_\xi(0) \le V_0, \end{aligned} \tag{13}$$

where $V_\xi(t)$ is the time t value of a self financing strategy ξ, and $V_0 \le \mathbb{E}_*(H)$. (Here $\mathbb{E}_*$ stands for expectation with respect to the risk neutral measure.) Problem (13), in other words, is how to minimise the risk of a position in a contingent claim H, having available V_0 for hedging, which is smaller than the cost of the replicating strategy of the claim.

In our setting, we hedge a position in x shares of stock. We can take

$$H = e^{rT}V_0 - xS(T). \tag{14}$$

The interpretation of such choice of H is as follows. We borrow V_0 and buy x shares of stock. The remaining

$$c = V_0 - xS(0),$$

is spent on a self financing strategy ξ, which involves continuous time trading in stock and money market account. The combined position at time T is $-e^{rT}V_0 + xS(T) + V_\xi(T)$. After discounting, this is

$$-V_0 + e^{-rT}xS(T) + e^{-rT}V_\xi(T) = e^{-rT}(V_\xi(T) - H),$$

which fits the framework of problem (13).

The following theorem provides the solution to problem (13) for the payoff (14). It is a reformulation of Theorem 2.4 from [5] (adapted to our particular setting and notations).

Theorem 6. *[5] Let $K^* \in \mathbb{R}$ be a number satisfying*

$$c = x\mathbb{E}_* \left(e^{-rT}\left(K^* - S(T)\right)^+\right).$$

Let $b(K)$ be a function implicitly defined by

$$c = x\mathbb{E}_* \left(e^{-rT}\left(K - S(T)\right)^+ \mathbf{1}_{\{S_T > b(K)\}}\right), \tag{15}$$

and let

$$\mathfrak{c}(K) = \begin{cases} V_0 - xe^{-rT}K + \frac{xe^{-rT}}{\alpha}\mathbb{E}((K - S(T))^+ \mathbf{1}_{\{S_T \le b(K)\}}) & \text{for } K > K^* \\ V_0 - xe^{-rT}K & \text{for } K \le K^*. \end{cases} \tag{16}$$

Let H be defined by (14). Then the solution of problem (13) is

$$CVaR^\alpha(e^{-rT}(V_\xi(T) - H)) = \min_K \mathfrak{c}(K), \tag{17}$$

and the optimal strategy is the one replicating the contingent claim with the payoff

$$\left(K - S(T)\right)^+ \mathbf{1}_{\{S_T > b(K)\}}. \tag{18}$$

Since

$$\left(K - S(T)\right)^+ \mathbf{1}_{\{S_T \le b\}} = \left(b - S\left(T\right)\right)^+ + \left(K - b\right)\mathbf{1}_{\{S(T) \le b\}},$$

the term involving expectation in (16) is

$$
\begin{aligned}
&\mathbb{E}((K-S(T))^{+}\,\mathbf{1}_{\{S_T\le b\}}) = \\
&\quad = e^{\mu T}\mathbb{E}\left(e^{-\mu T}\left(b-S\left(T\right)\right)^{+}\right) + \left(K-b\right)\mathbb{E}\left(\mathbf{1}_{\{S(T)\le b\}}\right) \\
&\quad = e^{\mu T}P\left(\mu,T,b,S(0),\sigma\right) + \left(K-b\right)N\left(-d_{-}\left(\mu,T,b,S(0),\sigma\right)\right).
\end{aligned}
$$

Similarly, since

$$
(K-S(T))^{+}\,\mathbf{1}_{\{S_T>b\}} = (K-S(T))^{+} - (b-S(T))^{+} - (K-b)\,\mathbf{1}_{\{S(T)\le b\}},
$$

the constraint (15) is

$$
\begin{aligned}
c = &\, xP\left(r,T,K,S(0),\sigma\right) - xP\left(r,T,b,S(0),\sigma\right) \\
&- x\left(K-b\right)e^{-rT}N\left(-d_{-}\left(r,T,b,S(0),\sigma\right)\right).
\end{aligned}
$$

This means that we have analytic formulae for all the ingredients of Theorem 6, and thus problem (17) can be solved numerically with relative ease.

Example 7. *As in Example 5, consider* $S(0) = 100$*,* $\mu = 10\%$*,* $\sigma = 0.2$*,* $r = 3\%$ *and the hedging costs* $c = 20, 40, 60, \ldots, 160$*. The* K *solving (17) and the resulting optimal* $CVaR^{\alpha}\left(e^{-rT}(V_{\xi}(T) - H)\right)$ *are as follows:*

c	x	K	$CVaR^{\alpha}$
20	*9.8*	*87.06*	*172.06*
40	*9.6*	*94.43*	*120.23*
60	*9.4*	*99.84*	*89.25*
80	*9.2*	*104.41*	*67.85*
100	*9*	*108.53*	*52.10*
120	*8.8*	*112.40*	*40.12*
140	*8.6*	*116.12*	*30.84*
160	*8.4*	*119.78*	*23.59*

By comparing the values from tables in Examples 5 and 7, we see that optimal CVaR$^{\alpha}$ from dynamic hedging are close to CVaR$^{\alpha}$ for the static hedging with puts. Since the difference is small, an investor might prefer to buy a portfolio of puts and go for a static hedging position, rather than engage in a dynamic hedging strategy.

5. Conclusion

We have provided an analytic solution for CVaR of a position in stock hedged by put options. We have shown that the problem of minimising CVaR reduces to a linear programming problem that can easily be solved in practice. We have demonstrated that thus obtained results are close to the ones attainable using dynamic hedging.

References

[1] C. Acerbi, D. Tasche, *On the coherence of Expected Shortfall*, J. Banking & Finance, Vol. 26 (7), (2002), pp. 1487–1503

[2] D.-H. Ahn, J. Boudoukh, M. Richardson, R. F. Whitelaw, *Optimal Risk Management Using Options*, J. Finance, Vol. 54 (1), (1999), pp. 359–375.

[3] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath, *Coherent measures of risk*, Math. Fin. 9 (3), (1999), pp. 203–228.

[4] H. Föllmer, P. Leukert, *Efficient hedging: cost versus shortfall risk*, Finance and Stochastics 4 (2), (2000) pp 117–146.

[5] A. Melnikov, I. Smirnov *Dynamic hedging of conditional value-at-risk* Insurance: Mathematics and Economics 51 (2012), pp 182–190

[6] R.T. Rockafellar, S. Uryasev, *Optimization of conditional value-at-risk*, J. Risk 2 (3), (2000), 21–41.

[7] R.T. Rockafellar, S. Uryasev, *Conditional value-at-risk for general loss distributions*, J. Banking & Finance 26(7), (2002), pp 1443–1471