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# SOLVING THE COMBINED MODAL SPLIT AND TRAFFIC ASSIGNMENT PROBLEM WITH TWO TYPES OF TRANSIT IMPEDANCE FUNCTION

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## ABSTRACT

The gradient projection (GP) algorithm has been shown as a successful path-based algorithm for solving various traffic assignment problems. In this paper, the GP algorithm is adapted for solving the combined modal split and traffic assignment (CMSTA) problem, which can be viewed as an elastic demand traffic equilibrium problem (EDTEP) with two modes. Using the excess-demand formulation of EDTEP, the CMSTA problem is reformulated and solved by a modified GP algorithm. Numerical results based on a real bi-modal network in the city of Winnipeg, Canada are provided to demonstrate the efficiency and robustness of the modified path-based GP algorithm for solving the CMSTA problem. In addition, the CMSTA problem is investigated with two types of impedance function for the transit mode and with different degrees of dispersion for the modal split function. The computational results show the modified GP algorithm outperforms the classical Evan's algorithm for both types of transit impedance function, and it can be as efficient as the original GP algorithm for solving the traffic assignment problem with fixed demand.



**Keywords**: Combined modal split and traffic assignment problem; elastic demand; user equilibrium; gradient projection; bi-modal networks

# 1 INTRODUCTION

In the transportation literature, the gradient projection (GP) algorithm is a well-known path-based algorithm for solving various traffic assignment problems. Jayakrishnan et al. (1994) was the first to adapt the Goldstein-Levitin-Polyak (GLP) gradient projection method formulated by Bertsekas (1976) for general nonlinear multi-commodity problems to the user equilibrium (UE) traffic assignment problem (TAP) with fixed demand (Sheffi, 1985). Under an ingenious reformulation of the decision variables in terms non-shortest path flows, the travel demand between each origin-destination (O-D) pair is shifted from several non-shortest paths to the shortest path, in which the amount of flows to be shifted is determined by a quasi-Newton method and restricted by an efficient projection to the non-negative orthant to maintain feasibility (see Section 3 for a brief review of the path-based GP traffic assignment algorithm). In addition, GP allows for alternative flow update strategies (i.e., equilibrate path flows one O-D pair at a time) under an O-D decomposition scheme. These two features are keys to the computational efficiency of the path-based GP algorithm for solving the UE-TAP (Jayakrishnan et al., 1994; Chen et al., 2002). Chen et al. (2002) numerically demonstrated that GP indeed has computational advantage over the disaggregate simplicial decomposition (DSD) algorithm, another path-based traffic assignment algorithm developed by Larsson and Patriksson (1992). Improvements to the path-based GP algorithm were further explored by Sun et al., (1996), Chen and Jayakrishnan (1998), and Lee et al., (2002).

Given the successful application of GP to the UE-TAP, GP has been adopted to solve various network equilibrium problems: (a) the non-additive traffic equilibrium problem (Scott and Bernstein, 1997; Chen et al. 2012), (b) the capacitated traffic assignment problem (Nie et al., 2004; Prashker and Toledo, 2004), (c) the logit-based stochastic user equilibrium (SUE) model (Behkor and Toledo, 2005; Zhou et al., 2014), (d) the C-logit SUE problem (Xu et al. 2012; Zhou et al. 2012; Chen et al. 2013), (e) the simulation-based dynamic traffic assignment problem (Yang and Jayakrishnan, 2012), (f) the capacitated schedule-based transit assignment problem (Noh, 2013), (g) the multiclass percentile user equilibrium (PUE)-TAP (Wu and Nie, 2013), (h) the elastic-demand traffic equilibrium problem (EDTEP) (Ryu et al. 2014a), (i) the system optimal (SO)-TAP with continuously distributed value of time (Wu and Huang, 2014), and (j) the freight traffic assignment problem for road-rail intermodal networks (Uddin and Huynh, 2015). Most have reported promising results with a reasonable computational effort for each of the above applications.

In this paper, we adapt the path-based GP traffic assignment algorithm for solving the combined modal split and traffic assignment (CMSTA) problem, which can be considered as a special case of the EDTEP where users have both mode choice and route choice for determining their travel options (Sheffi, 1985). Using the excess-demand formulation of EDTAP, the CMSTA problem is formulated by using a random utility model (Ben-Akiva and Lerman, 1985) to determine the modal splits given the travel demand between each O-D pair in the network, and the user equilibrium (UE) principle (Wardrop, 1952) to assign the mode-specific O-D demand to the multi-modal transportation network (Abdulaal and LeBlanc, 1979). Our goal is to demonstrate that the GP algorithm can be easily modified via the excess-demand formulation of EDTEP to solve the CMSTA problem with two types of transit impedance function on a bi-modal network.

The remainder of this paper is organized as follows. Section 2 describes the CMSTA problem using the excess demand formulation of the EDTAP. Section 3 reviews the gradient projection (GP) concept and describes the modifications required for solving the CMSTA problem. Section 4 provides two sets of numerical experiments to examine the modified GP algorithm for solving the CMSTA problem with two types of transit impedance function on a real bi-modal network consisting of private cars and transit in the city of Winnipeg, Canada. Finally, some concluding remarks are provided in Section 5.

## 2 COMBINED MODAL SPLIT AND TRAFFIC ASSIGNMENT PROBLEM

In this section we review the excess-demand formulation of the EDTAP, show its application to the CMSTA problem, and provide a simple example as an illustration.

### 2.1 EXCESS DEMAND FORMULATION

The EDTEP accounts for both trip generation (i.e., travel choice) and traffic assignment (i.e., route choice) simultaneously by considering the equilibration between supply and demand (Sheffi, 1985). At equilibrium, the travel demand determined by the elastic demand function is consistent with the network level of service via the minimum O-D travel time for all O-D pairs. Gartner (1980a,b) summarized three approaches for modeling the generalized traffic equilibrium problem as an equivalent network in which the elastic demand functions are represented by appropriate generating links: (1) minimum-cost circulation, (2) zero-cost overflow, and (3) excess demand. In this paper, we adopt the excess demand formulation to model the CMSTA problem.

Minimize: $$\sum_{a\in \mathrm{A}} \int_0^{x_a=\sum_{rs\in \mathrm{RS}}\sum_{k\in \mathrm{K}_{rs}} f_k^{rs}\delta_{ka}^{rs}} t_a(w)dw + \sum_{rs\in \mathrm{RS}} \int_0^{e_{rs}} w_{rs}(v)dv \tag{1}$$

subject to: $$\sum_{k\in \mathrm{K}_{rs}} f_k^{rs} + e_{rs} = \overline{q}_{rs},\ \forall rs \in \mathrm{RS} \tag{2}$$

$$f_k^{rs} \geq 0,\ \ rs\in \mathrm{RS},\ \forall k \in \mathrm{K}_{rs} \tag{3}$$

$$e_{rs} \geq 0,\ \ \forall rs \in \mathrm{RS} \tag{4}$$

where A is the set of links; RS is the set of O-D pairs; $K_{rs}$ is the set of paths between O-D pair *rs*; $x_a$ is the flow on link *a*; $t_a(\cdot)$ is the travel time on link *a*; $f_k^{rs}$ is the flow on path *k* between O-D pair *rs*; $\delta_{ka}^{rs}$ is equal to 1 for link *a* on path *k* between O-D pair *rs* and 0 otherwise; $e_{rs}$ is the excess demand variable between O-D pair *rs*; $w_{rs}(\cdot)$ is the excess demand function between O-D pair *rs*; and $\bar{q}_{rs}$ is the upper bound demand between O-D pair *rs*.

The objective function in Eq. (1) consists of two terms: an user equilibrium (UE) term reflecting the *congestion* effect and an excess demand term reflecting the *elasticity* of O-D demands in terms of the network level of service (LOS). In essence, Eq. (1) is the objective function of the equivalent excess-demand reformulation in which the elastic demand problem is reformulated as a fixed demand problem through Eq. (2) by redefining the travel demand conservation of each O-D pair with the predefined upper bound demand and the excess demand variable. Eqs. (3) and (4) are the non-negativity constraints on the two sets of decision variables (i.e., excess demands and path flows).

## 2.2 APPLICATION TO THE CMSTA PROBLEM WITH TWO TYPES OF TRANSIT IMPEDANCE FUNCTION

The excess demand formulation above can be adapted to consider mode choice (instead of travel choice) and route choice as a combined modal split and traffic assignment (CMSTA) problem by defining the excess demand function as a modal split function as follows:

$$w_{rs}(q_{rs}^B) = \frac{1}{\theta}\ln\left(\frac{q_{rs}^B}{\bar{q}_{rs} - q_{rs}^B}\right) + \bar{c}_{rs}^B, \quad \forall rs \in \text{RS} \tag{5}$$

where $q_{rs}^B$ is the demand of mode *B* (or excess demand $e_{rs}$ ) between O-D pair *rs*; $\bar{c}_{rs}^B$ is the travel time of mode *B* between O-D pair *rs*; and $\theta$ is the logit parameter. At equilibrium, the travel costs between the two modes are equal and yield the following modal split for mode *B*:

$$\frac{1}{\theta}\ln\left(\frac{q_{rs}^B}{\bar{q}_{rs} - q_{rs}^B}\right) + \bar{c}_{rs}^B = c_{\bar{k}_{rs}}^{rs} \Rightarrow \frac{q_{rs}^B}{\bar{q}_{rs}} = \frac{1}{1+\exp\left(\theta\left(\bar{c}_{rs}^B - c_{\bar{k}_{rs}}^{rs}\right)\right)} = \frac{\exp\left(-\theta\bar{c}_{rs}^B\right)}{\exp\left(-\theta\bar{c}_{rs}^B\right)+\exp\left(-\theta c_{\bar{k}_{rs}}^{rs}\right)} \tag{6}$$

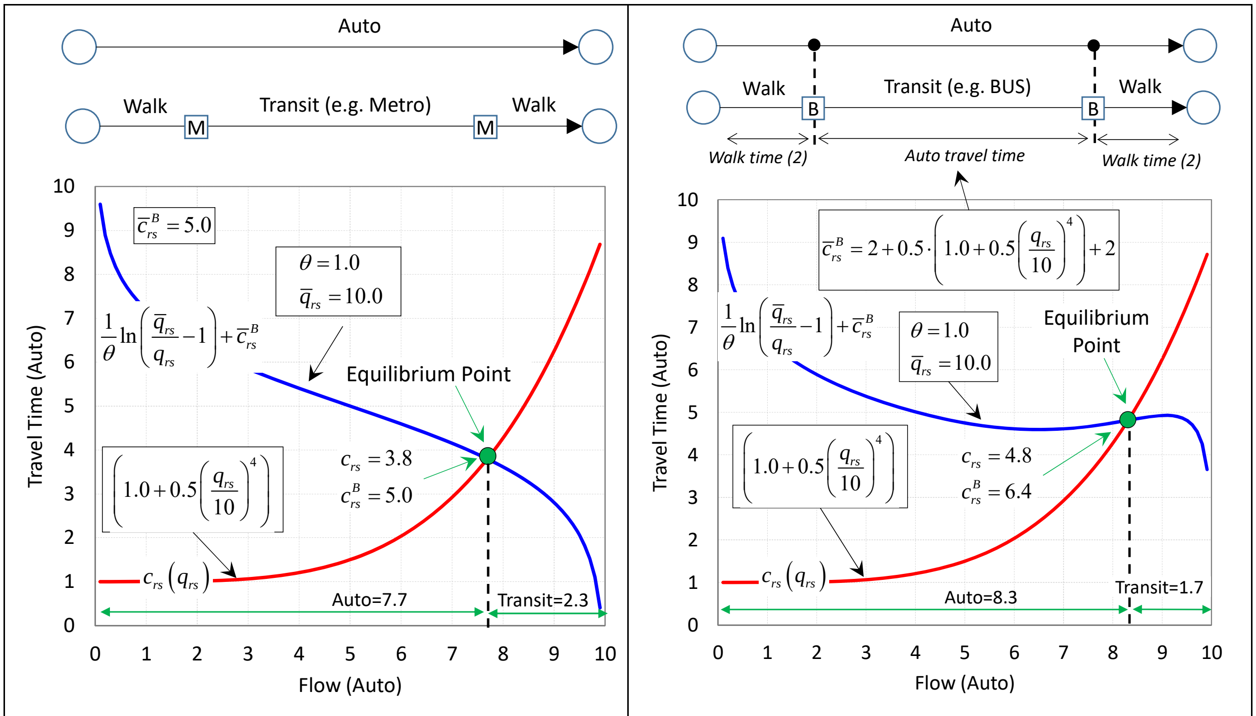
where $c_{\bar{k}_{rs}}^{rs} = \sum_{a\in A} t_a\left(x_a\right)\delta_{\bar{k}_{rs}a}^{rs}$ is the minimum auto travel time between O-D pair *rs*; $\bar{q}_{rs}$ becomes the total demand between O-D pair *rs*; the difference between $\bar{q}_{rs}$ and $q_{rs}^B$ is the auto demand between O-D pair *rs*, which is determined by the summation of path flows ( $\sum_{k\in K_{rs}} f_k^{rs}$ ) of O-D pair *rs*. Alternatively, Eq. (6) can be re-arranged to yield the modal split for the auto mode as follows:

$$\frac{1}{\theta}\ln\left(\frac{q_{rs}^{B}}{\bar{q}_{rs}-q_{rs}^{B}}\right)+\bar{c}_{rs}^{B}=c_{\bar{k}_{rs}}^{rs}\Rightarrow\frac{\bar{q}_{rs}-q_{rs}^{B}}{\bar{q}_{rs}}=\frac{1}{1+\exp\left(-\theta\left(c_{\bar{k}_{rs}}^{rs}-\bar{c}_{rs}^{B}\right)\right)}=\frac{\exp\left(-\theta c_{\bar{k}_{rs}}^{rs}\right)}{\exp\left(-\theta\bar{c}_{rs}^{B}\right)+\exp\left(-\theta c_{\bar{k}_{rs}}^{rs}\right)} \tag{7}$$

Note that $\bar{c}_{rs}^{B}$ in both Eq. (6) and Eq. (7) is assumed as a constant (i.e., fixed travel time), which may be suitable for metro or dedicated bus lane (e.g., bus rapid transit (BRT)). However, it may not be suitable for modeling buses sharing the highway network with passenger cars. In addition to the flow-dependent travel time for modeling congestion, other factors, such as approach to and from bus stations, waiting time, and bus fare, can be considered into the modified excess demand function as follows:

$$w_{rs}(q_{rs}^{B})=\frac{1}{\theta}\ln\left(\frac{q_{rs}^{B}}{\bar{q}_{rs}-q_{rs}^{B}}\right)+\psi_{rs}+\bar{c}_{rs}^{B}(\cdot),\quad \forall rs\in \mathrm{RS} \tag{8}$$

where $\psi_{rs}$ is a composite impedance parameter of all factors important to mode *B* between O-D pair *rs*; and $\bar{c}_{rs}^{B}(\cdot)$ is the flow-dependent travel time of mode *B* between O-D pair *rs*, which can be calculated the same way as the auto mode by summing up the link travel times of mode *B*. Figure 1 provides a graphical illustration of the equilibrated mode choice probability with these two types of transit impedance function (i.e., constant (or flow-independent) travel time and flow-dependent travel time).

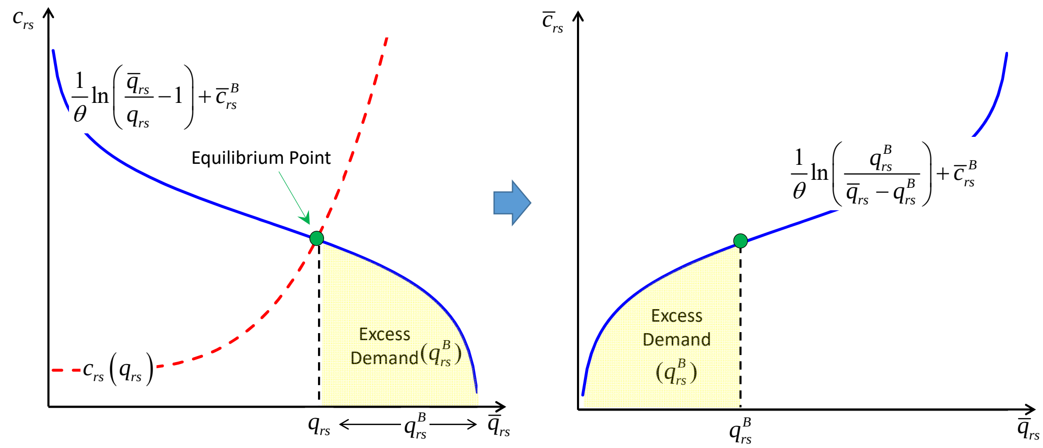



(a) Flow-independent cost for mode *B* (e.g., metro) (b) Flow-dependent cost for mode *B* (e.g., bus)

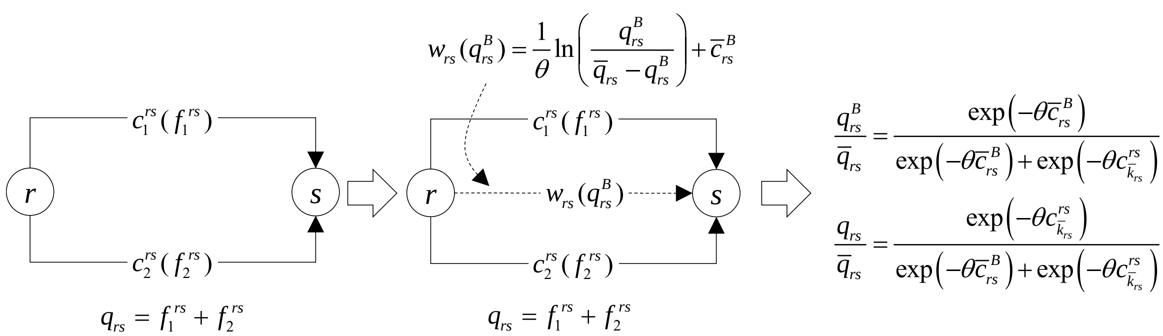
**Figure 1** Illustartion of mode choice equilibration with two types (flow-independent and flow-dependent) of travel times

### 2.3 AN ILLUSTRATION

This reformulation from excess demand to fixed total demand with modal split choice between O-D pair *rs* can be accomplished by using an appropriate modification of network representation as shown in Figure 2. Figure 2(a) redefines the decision variable from $q_{rs}$ to $q_{rs}^{B}$ (i.e., $\bar{q}_{rs} - q_{rs}$) and its corresponding excess demand function from $w_{rs}(q_{rs}) = \frac{1}{\theta}\ln\left(\frac{\bar{q}_{rs}}{q_{rs}} - 1\right) + \bar{c}_{rs}^{B}$ to $w_{rs}(q_{rs}^{B}) = \frac{1}{\theta}\ln\left(\frac{q_{rs}^{B}}{\bar{q}_{rs} - q_{rs}^{B}}\right) + \bar{c}_{rs}^{B}$, while Figure 2(b) provides an illustration of a bi-modal network along with its modified network representation as a fixed demand problem through appropriate cost functions and excess demand function. Similar idea has also been adopted for solving the logit-based SUE problem with elastic demand (Yu et al., 2014).




(a) Excess demand reformulation




(b) Bi-modal network representation

**Figure 2** Excess demand reformulation and its bi-modal network representation for O-D pair *rs*

## 3 PATH-BASED GRADIENT PROJECTION METHOD

In this section, we briefly review the basic flow update equations of the path-based gradient projection (GP) algorithm for solving a fixed demand traffic assignment problem, describe the modifications required for solving the CMSTA problem formulated as an excess demand traffic assignment problem, and provide a detailed step-by-step solution procedure for implementing the path-based GP algorithm for solving the CMSTA problem.

### 3.1 FLOW UPDATE EQUATIONS

The basic flow update equations of the path-based gradient projection (GP) algorithm for solving a fixed demand traffic assignment problem (Jayakrishnan et al., 1994) are as follows:

$$f_k^{rs}(n+1) = \max\left\{\left[f_k^{rs}(n) - \frac{\alpha}{s_k^{rs}(n)}\left(c_k^{rs}(n) - c_{\bar{k}_{rs}}^{rs}(n)\right)\right], 0\right\} \tag{9}$$

$$f_{\bar{k}_{rs}}^{rs}(n+1) = q_{rs} - \sum_{\substack{k\in \mathrm{K}_{rs} \\ k\neq \bar{k}_{rs}}} f_k^{rs}(n+1) \tag{10}$$

where $n$ is the iteration number; $\alpha$ is the step size; $f_k^{rs}(n)$ is flows on path $k$ between O-D pair $rs$ at iteration $n$, $s_k^{rs}(n)$ is a diagonal, positive-definite scaling factor on path $k$ between O-D pair $rs$ at iteration $n$; $c_k^{rs}(n)$ and $c_{\bar{k}_{rs}}^{rs}(n)$ are the travel times on path $k$ and shortest path $\bar{k}_{rs}$ between O-D pair $rs$ at iteration $n$; $\max\{\mathbf{f}, 0\}$ denotes the projection of the argument onto the non-negative orthant of the independent variables; and $f_k^{rs}(n+1)$ and $f_{\bar{k}_{rs}}^{rs}(n+1)$ are the updated flows on path $k$ and shortest path $\bar{k}_{rs}$ between O-D pair $rs$ at iteration $n$+1. The path travel time difference $\left(c_k^{rs}(n) - c_{\bar{k}_{rs}}^{rs}(n)\right)$ and the scaling factor ( $s_k^{rs}(n)$ ) are calculated to define the search direction as follows:

$$c_k^{rs} - c_{\bar{k}_{rs}}^{rs} = \sum_{a\in \mathrm{A}} t_a(x_a)\delta_{ka}^{rs} - \sum_{a\in \mathrm{A}} t_a(x_a)\delta_{\bar{k}_{rs}a}^{rs} \tag{11}$$

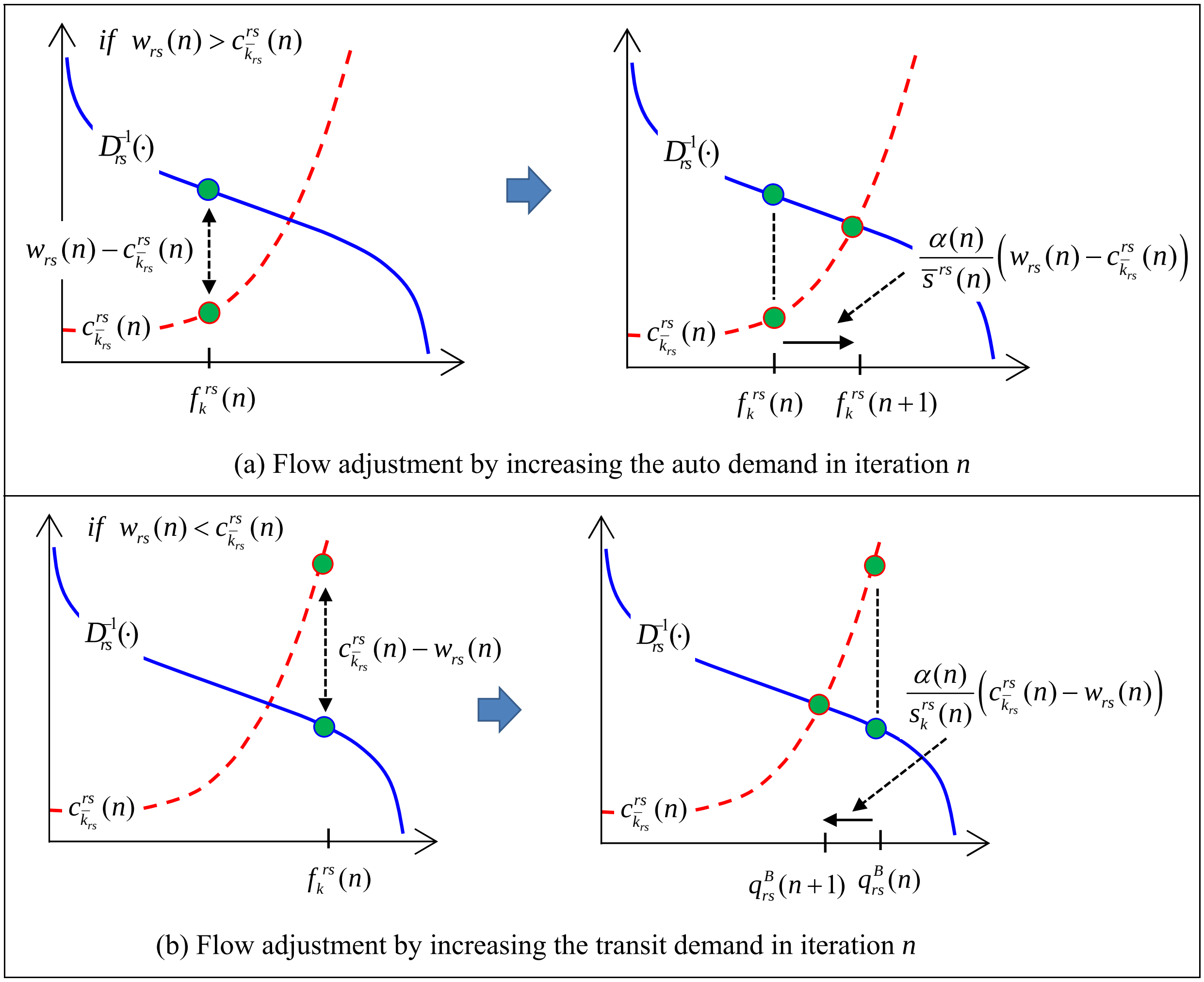
$$s_k^{rs} = \sum_{a\in \mathrm{A}} t'_a(x_a)\left(\delta_{ka}^{rs} - \delta_{\bar{k}_{rs}a}^{rs}\right)^2 \tag{12}$$

where $t_a(x_a)$ and $t'_a(x_a)$ are the travel time and first derivative travel time on link $a$; $\delta_{ka}^{rs}$ and $\delta_{\bar{k}_{rs}a}^{rs}$ are the path-link indicators on path $k$ and $\bar{k}_{rs}$ between O-D pair $rs$.

The above flow update equations make use of the special structure of the traffic equilibrium problem for fixed demand (i.e., decompose the problem by individual O-D pairs and redefine the decision variables in terms of non-shortest path flows by eliminating the demand conservation constraint). This enables GP to perform a simple projection onto the non-negative orthant without the need to solve a quadratic program to ensure feasibility, and it also allows for alternative flow update strategies (i.e., equilibrate path flows one O-D pair at a time). These two features are keys to the computational efficiency of the path-based GP algorithm for solving the UE-TAP (Jayakrishnan et al., 1994; Chen et al., 2002).

### 3.2 MODIFICATIONS FOR SOLVING THE CMSTA PROBLEM

In this section, we describe the modifications required in the path-based GP algorithm for solving the excess demand reformulation of CMSTA. In addition to the flow update equations for equilibrating path flows for the fixed demand described in Section 3.1, another set of flow update equations is needed for equilibrating the modal splits through the excess demand function. The modal split equilibration procedure determines an appropriate split for each O-D pair based on network congestion. The flow update equations for the modal split equilibration are graphically shown in Figure 3.




(a) Flow adjustment by increasing the auto demand in iteration *n*

(b) Flow adjustment by increasing the transit demand in iteration *n*

**Figure 3** Graphical illustration of the modal split adjustment

When the auto O-D cost is lower than the excess demand cost of alternative mode *B* (i.e., $c^{rs}_{\bar{k}_{rs}} < w_{rs}$), the modal split for auto should be increased (see Figure 3a), whereas the modal split for auto should be decreased (see Figure 3b) when the auto O-D cost is higher than the excess demand cost of the alternative mode *B* (i.e., $c^{rs}_{\bar{k}_{rs}} > w_{rs}$). When the two costs are equal (i.e., $c^{rs}_{\bar{k}_{rs}} = w_{rs}$), there is no need to readjust the modal split between the two modes. In essence, the modifications include the modal split equilibration procedure that determines an appropriate modal split for each O-D pair based on the network congestion level by equilibrating between excess demand (or modal split) function and O-D cost supply function. Meanwhile, the path flow equilibration procedure determines the flow allocations to the used paths to achieve a user equilibrium state.

### 3.3 SOLUTION PROCEDURE

The modified GP algorithm based on the excess-demand formulation for solving the CMSTA problem consists of initialization, column generation, equilibration (modal splits and path flows), and termination. The overall flowchart using the modified GP algorithm for solving the CMSTA problem is shown in Figure 4 and detailed algorithmic steps are provided as follows.

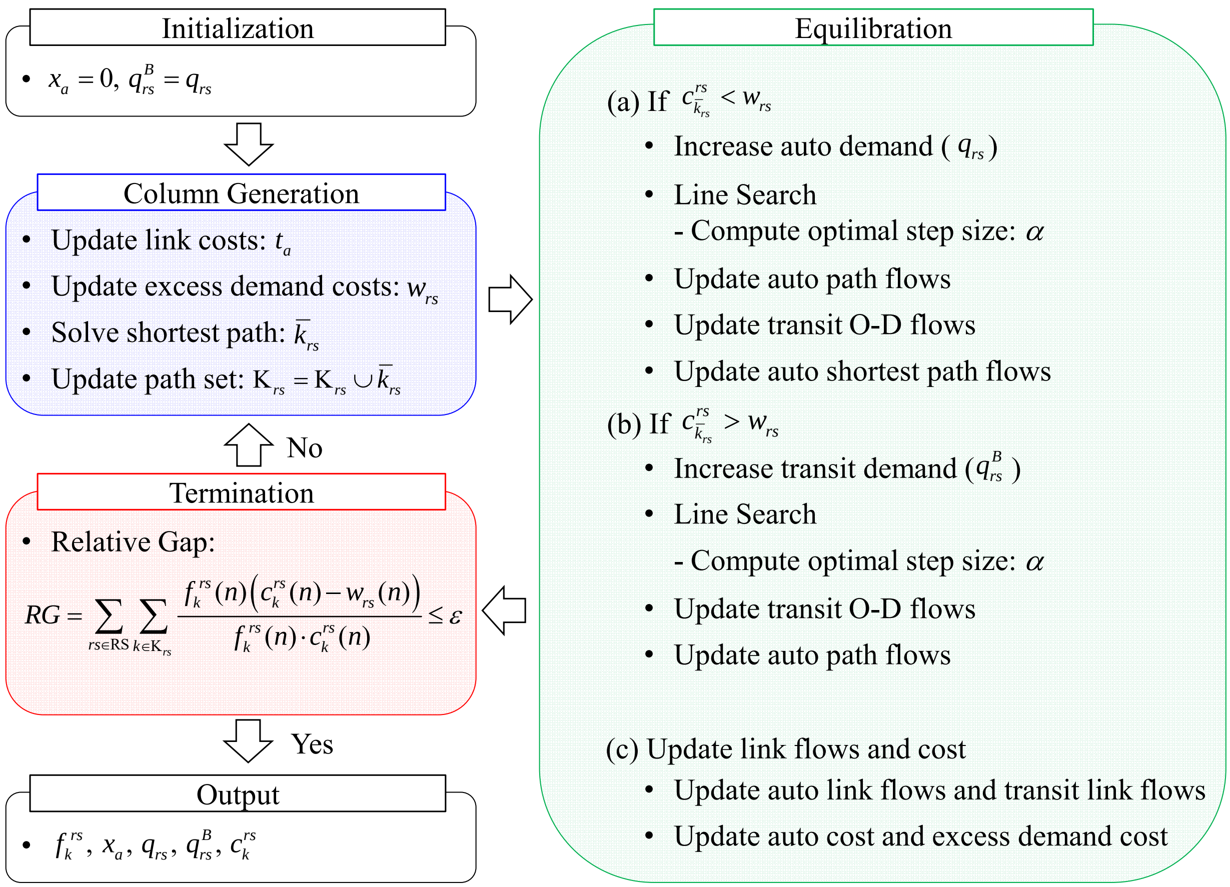



**Figure 4** Flowchart of the modified GP algorithm for solving the CMSTA problem

**Step 0: Initialization.** Initialize decision variables and initial path set

- $n = 0;\ x_a(n) = 0;\ q^B_{rs}(n) = q_{rs};\ f^{rs}_k = 0;\ K_{rs} = \varnothing$

**Step 1: Column Generation.** Generate shortest paths based on the current link travel times and augment the path set with new paths

- Update link travel times: $t_a(n) = t_a\left(x_a(n-1)\right)$
- Solve the shortest path problem: $\bar{k}_{rs}(n)$; $\mathrm{K}_a(n) = \mathrm{K}_a(n-1) \cup \bar{k}_{rs}(n)$

**Step 2: Equilibration.** Solve the CMSTA problem over the restricted set of paths generated thus far.

- Compute path cost and excess demand cost:

$$c_k^{rs}(n) = \sum_{a \in \mathrm{A}} t_a(n) \delta_{ka}^{rs},\ rs \in \mathrm{RS},\ \forall k \in \mathrm{K}_{rs},\ k \neq \bar{k}_{rs}(n);\ \ c_{\bar{k}}^{rs}(n) = \sum_{a \in \mathrm{A}} t_a(n) \delta_{\bar{k}_{rs}}^{rs},\ \forall rs \in \mathrm{RS}$$

$$w_{rs}(q_{rs}^B) = \frac{1}{\theta} \ln\left(\frac{q_{rs}^B(n)}{\bar{q}_{rs} - q_{rs}^B(n)}\right) + \bar{c}_{rs}^B(n), \qquad \forall rs \in \mathrm{RS}$$

- Compare the travel time between auto shortest path time and excess demand cost

| *if* $c_{\bar{k}}^{rs}(n) < w_{rs}(n)$ | *if* $c_{\bar{k}}^{rs}(n) > w_{rs}(n)$ |
|---|---|
| • Perform line search to determine step size $\alpha$<br>• Update auto path flows and transit flow<br>$f_k^{rs}(n+1) = \max\left\{\left[f_k^{rs}(n) - \frac{\alpha}{s_k^{rs}}\left(c_k^{rs}(n) - c_{\bar{k}_{rs}}^{rs}(n)\right)\right], 0\right\}$<br>$q_{rs}^B(n+1) = \max\left\{\left[q_{rs}^B(n) - \frac{\alpha}{s^{rs}}\left(w_{rs}(n) - c_{\bar{k}_{rs}}^{rs}(n)\right)\right], 0\right\}$<br>$f_{\bar{k}_{rs}}^{rs}(n+1) = \bar{q}_{rs} - \sum_{\substack{k \in \mathrm{K}_{rs}^A \\ k \neq \bar{k}_{rs}}} f_k^{rs}(n+1) - q_{rs}^B(n+1)$ | • Perform line search to determine step size $\alpha$<br>• Update auto path flows and transit flow<br>$f_k^{rs}(n+1) = \max\left\{\left[f_k^{rs}(n) - \frac{\alpha}{s_k^{rs}}\left(c_k^{rs}(n) - w_{rs}(n)\right)\right], 0\right\}$<br>$q_{rs}^B(n+1) = q_{rs} - \sum_{k \in \mathrm{K}_{rs}} f_k^{rs}(n+1)$ |

**Step 3: Termination.** Terminate the algorithm if it satisfies the stopping criterion.

- If $RG = \sum_{rs \in \mathrm{RS}} \sum_{k \in \mathrm{K}_{rs}} \frac{f_k^{rs}(n+1)\left(c_k^{rs}(n+1) - w_{rs}(n+1)\right)}{f_k^{rs}(n+1) \cdot c_k^{rs}(n+1)} \leq \varepsilon$, terminate; otherwise, go to Step 1.

**Remark 1**: Following the suggestions by Bertsekas et al. (1984), Jayakrishnan et al. (1994), Sun et al. (1996), and Chen et al. (2002), a unit stepsize is adopted for all iteration $n$, since the second derivative information for an automatic scaling (see Eq. (12)) and the one at-a-time flow update strategy are used in the equilibration procedure. This scheme has been found to be helpful in reducing the computational efforts. However, a line search step (e.g., self-adaptive strategies) can be used to determine a suitable stepsize to help better convergence, especially for highly accurate solutions are sought. Some specific methods include the self-regulated averaging (SRA) scheme (Liu et al., 2008), the self-adaptive scheme (He et al., 2002), and the self-adaptive Armijo scheme (Chen et al., 2013) have been developed for solving different traffic assignment problems (e.g., classical user equilibrium problem, stochastic user equilibrium problem with different discrete choice models, and nonadditive traffic equilibrium problem with different route cost structures). The requirements are different for different self-adaptive schemes. However, all schemes have been proven to be convergent. Hence, implementing an appropriate line search method with different self-adaptive schemes for the problem should consider the trade-off between the computational efforts (i.e., difficulty of evaluating or not requiring to evaluate the objective function) and rate of convergence. For example, the SRA scheme has been found to be effective compared to the method of successive averages (MSA) for problems without the need to evaluate complex objective functions, such as the C-logit SUE model with elastic demand (Xu and Chen, 2013), the paired combinatorial logit (PCL) SUE model with fixed and elastic demand (Chen et al., 2014; Ryu et al., 2014c), the weibit SUE model with fixed demand (Kitthamkesorn and Chen, 2014), the path-sized weibit SUE model with elastic demand (Kitthamkesorn et al., 2015), and the combined travel demand model (Yang et al., 2013a). For the self-adaptive scheme, He et al. (2002) embedded it in the modified Goldstein–Levitin–Polyak (GLP) projection method for solving asymmetric strongly monotone inequalities, while Chen et al. (2001) used it in the projection and contraction (PC) method for solving the nonadditive traffic equilibrium problem with route-specific cost. More recently, Chen et al. (2012) incorporated the self-adaptive scheme into the gradient projection (GP) algorithm for solving the nonadditive traffic equilibrium problem, while Xu et al. (2012) and Zhou et al. (2012) applied the self-adaptive GP for solving the C-logit SUE model with fixed demand. As for the self-adaptive Armijo scheme, Chen et al. (2013) demonstrated that it can be incorporated to different traffic assignment algorithms (i.e., the link-based Frank-Wolfe algorithm for solving the classical user equilibrium problem, the path-based disaggregate simplicial decomposition (DSD) algorithm for solving the multinomial logit SUE problem, and the path-based GP algorithm for solving the congestion-based C-logit SUE problem) to improve the computational efficiency.

## 4 NUMERICAL EXPERIMENTS

In this section, two sets of numerical experiments using a real network in the city of Winnipeg, Canada, are conducted to examine: (a) the convergence characteristics of the path-based GP algorithm for solving the CMSTA problem in a bi-modal network, and (b) the sensitivity of various parameters in the CMSTA problem. The bi-modal Winnipeg network, shown in Figure 5, consists of 154 zones, 1,067 nodes, 2,535 links, 4,345 O-D pair. The network structure, total O-D trip is 72,669 trips and link performance parameters are from the Emme software (INRO Consultants, 2013). Among the 2,535 links, 1,351 links have a transit line. To set up the excess demand function, we adopt equations (5) and (8) with two types of transit impedance function (i.e., flow-independent travel time and flow-dependent travel time):

Type 1: $\bar{c}_{rs}^{B}$ is obtained from a pre-assigned O-D travel time using the auto demand.

Type 2: $\bar{c}_{rs}^{B}$ is obtained from $\sum_{a \in A}\left(t_a(x_a)\delta_k^{rs}\eta_a^B + (\eta_a^B - 1)l_a / ws\right)$, where $\eta_a^B$ is equal to 1 for link a with a bus line and 0 otherwise; $l_a$ is the length on link a; and *ws* is the walking speed (i.e., 5 km/h or 4.56 ft./s).

The tolerance error of the relative gap is set at 1E-7, and the logit parameter ($\theta$) is set at 0.1. The path-based GP algorithm is coded in Intel Visual FORTRAN XE and run on a 3.60GHz processor and 16.00GB of RAM.

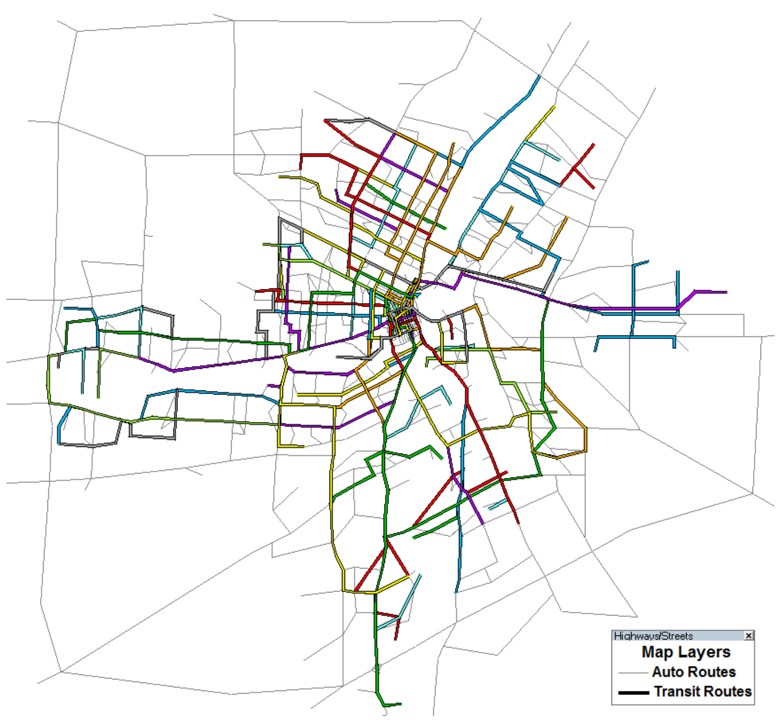



**Figure 5** Bi-modal Winnipeg network

### 4.1 CONVERGENCE CHARACTERISTICS

Evan's algorithm (Evans, 1976) is a classical algorithm for solving the combined distribution and assignment (CDA) problem as well as many variations of the elastic demand traffic equilibrium problem (e.g., Horowitz, 1989; Huang and Lam, 1992), including the combined modal split and traffic assignment problem (e.g., LeBlanc and Farhangian, 1981). Computational results revealed Evan's algorithm, also known as the partial linearization method, performed better than the complete linearization method of the Frank-Wolfe (FW) algorithm suggested by Florian et al. (1975) and Florian and Nguyen (1978). Although Evan's algorithm has better performance than FW, it still inherited the slow convergence required for highly accurate solution compared to the origin-based algorithm developed by Bar-Gera and Boyce (2003) and further enhanced by Xu et al. (2008) by streamlining the line search step.

Figure 6 provides the convergence comparisons between Evan's algorithm and modified GP algorithm for two types of transit impedance function (i.e., flow-independent and flow-dependent travel times used in Eqs. (5) and (8)). The results also revealed the slow convergence of the Evan's algorithm for both types of transit impedance function. Specifically, Evan's algorithm was not able to reach the highly accurate solution of a relative gap of 1E-7 for a maximum computational time of 100 seconds for both types of transit impedance functions. On the contrary, the modified GP algorithm can promise convergence for both types of transit impedance function albeit using different computational efforts. For the Type 1 travel time, it takes only 19 seconds to converge to a relative gap of 1E-7, whereas 63 seconds are needed for the Type 2 travel time to reach the same level of convergence. With the Type 1 travel time, the modal split is quickly adjusted because the excess demand function is only affected by the log term (i.e., the first term in Eq. (5)), whereas the Type 2 travel time requires both the log term and the flow-dependent travel time term in Eq. (8) to be adjusted at each iteration. Hence, the number of iterations and computational efforts required to reach the same convergence level increase nonlinearly. Note that the tolerance error used in the relative gap stopping criterion is much stricter (i.e., 1E-7 or three order of magnitude higher) than the typical one (i.e., 0.01% or 1E-4) required in practice (Boyce et al., 2004); this is to ensure that the traffic assignment results are sufficiently converged to achieve link-flow stability. If a relative gap of 1E-4 is used, the CPU times for both types of travel times reduce significantly (i.e., 4.5 seconds for Type 1 and 9.6 seconds for Type 2). The computational efforts required for solving the CMSTA problem in a medium-sized network with two modes are quite modest. Hence, the modified GP algorithm has the potential to solve large-scale networks with multiple modes.

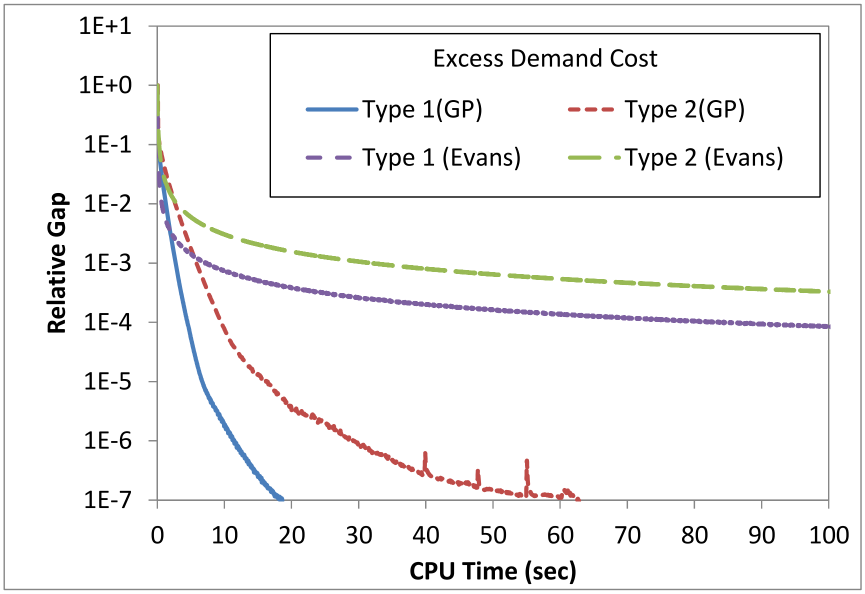



**Figure 6** Convergence characteristics of the CMSTA problem under two types of transit travel times

### 4.2 APPLICATION OF THE CMSTA PROBLEM

In this section, we examine the application of implementing the CMSTA model in the bi-modal Winnipeg network. Specifically, we examine the modal split equilibration between two modes under two types of travel times and the path flow equilibration for the auto mode. To demonstrate the equilibration procedure of the modified GP algorithm presented in the solution algorithm section, Figure 7 depicts the equilibration trajectories of O-D pair (38-2) for the bi-modal network. There are three paths for the auto mode and one transit route between O-D (38-2) in the main figure. The two subfigures ((a) and (b)) show the flow and cost equilibrations between the two modes as well as the path equilibration of the auto mode. Initially, the excess demand cost is higher than equilibrium path cost (i.e., costs of Paths 1, 2, and 3 are equal) in subfigure (b). The modal splits between the two modes and the auto path flows are iteratively adjusted (see subfigure (a)) using the flow update equations in Step 2 of the modified GP algorithm to reach an equilibrium (i.e., the costs between the two modes are equal as in Eq. (6), and the auto travel times on the used paths are equal according to the Wardrop equilibrium $c_1^{38-2} = c_2^{38-2} = c_3^{38-2}$) as shown in subfigure (b).

Figure 8 depicts the modal link flow differences on a color-coded GIS map for the two types of transit travel times. The red color indicates that the link flows from Type 2 flow-dependent travel time have higher flows than flows from Type 1 flow-independent travel time. The green color indicates the reverse (i.e., link flows from Type 1 flow-independent travel time have higher flows than flows from Type 2 flow-dependent travel time). Figures 8(a) and 8(b) display the link flow differences for auto and transit, respectively. For the transit mode, the differences tend to accumulate in the central area where the majority of the transit lines are located, while the differences for the auto model tend to be more dispersed to the outer ring roads. Overall, the results show that the two types of travel times used in the excess demand (or modal split) functions do have significant impacts on the link flow patterns for both modes.

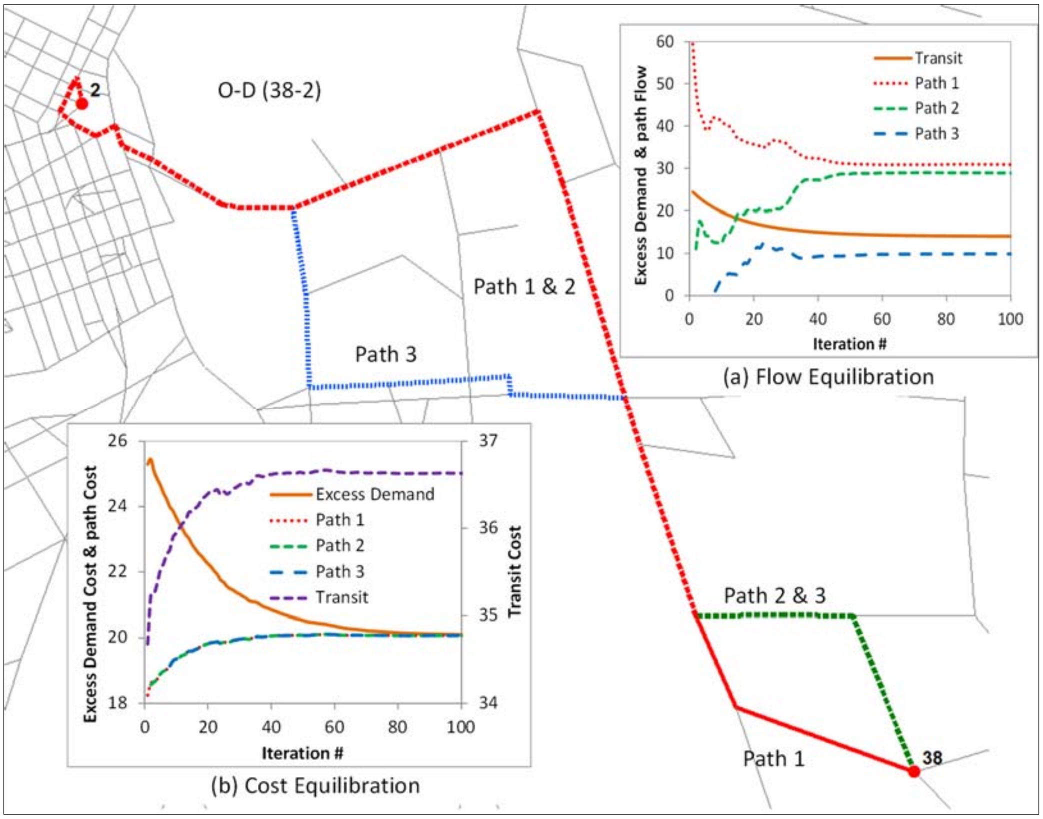
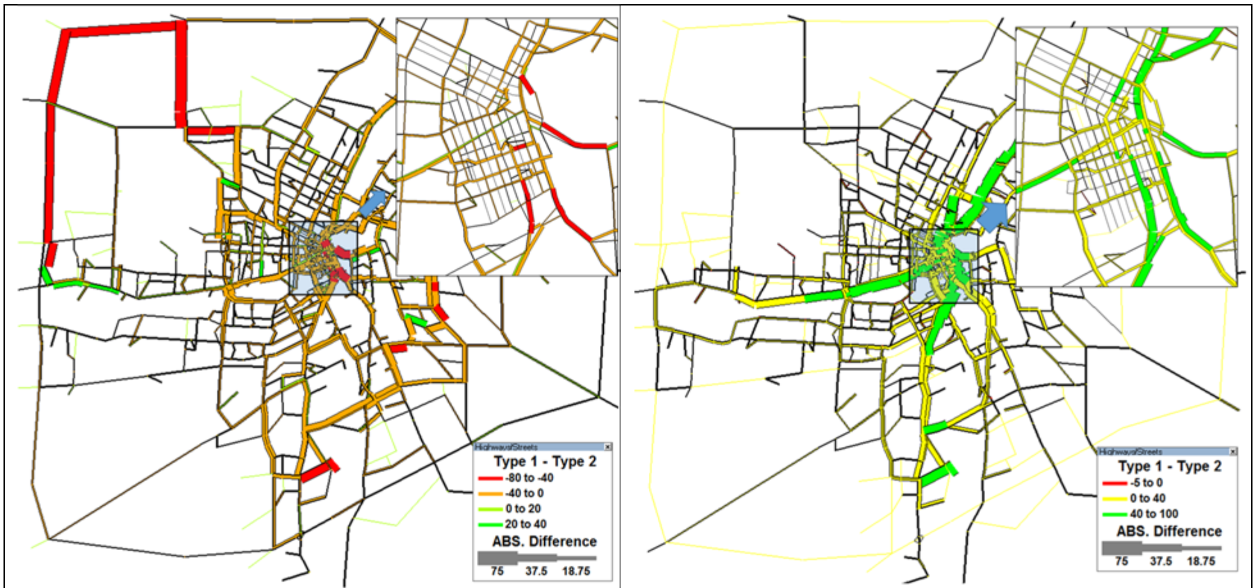



**Figure 7** Equilibration of modal splits and auto path flows for O-D pair (38-2)



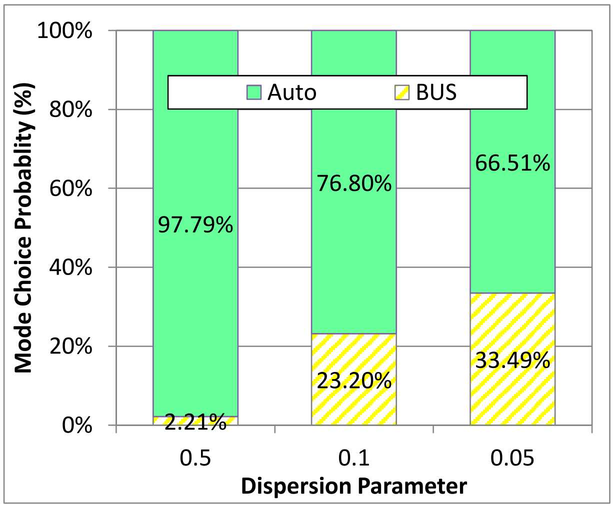

(a) Auto flow comparison

(b) Transit flow comparison

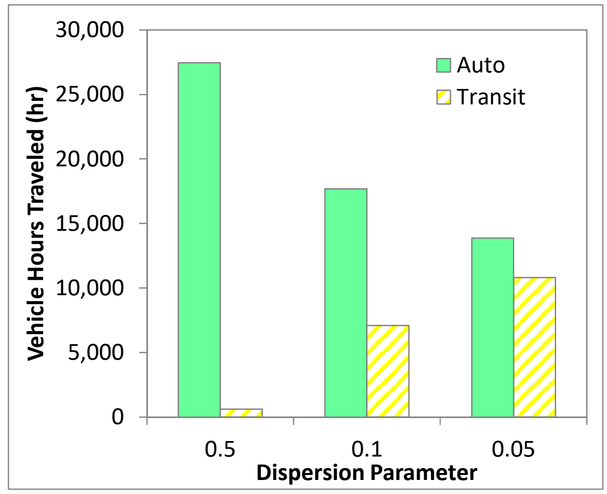
**Figure 8** Modal link flow patterns under two types of transit travel times

### 4.3 SENSITIVITY ANALYSIS

In this section, we test the sensitivity of the logit parameter ($\theta$) in the CMSTA problem using the flow-dependent transit travel time. Specifically, we examine the impact of three logit parameter values (i.e., 0.05, 0.1, and 0.5) on the modal split, vehicle hour traveled (VHT), and the flow allocation to the two modes. Figure 9 shows the modal splits and VHTs, while Figure 10 shows the link flow differences between two logit parameter values (i.e., 0.05 and 0.1, and 0.05 and 0.5) for both auto and transit modes. As can be seen, the modal splits and VHTs are highly influenced by the logit parameter value. When the logit parameter value is 0.5, almost all users (i.e., 97.8%) choose the auto mode. This suggests users are very sensitive to the cost difference between the two modes as indicated by Eqs. (6) and (7) using the Type 2 transit travel time. When the logit parameter value is 0.1 (i.e., one-fifth of 0.5), users are less sensitive to the cost difference, and more users (i.e., 23.2%) begin to choose transit. Similarly, when the logit parameter value is 0.05 (i.e., one-tenth of 0.5), 33.5% of the users choose transit. Hence, when more users choose transit due to the decreasing value of the logit parameter, the VHT values for the auto mode also decrease as indicated in Figure 9(b). However, the VHT decrease is not proportional to the split between the two modes (i.e., at a 66.5%/33.5% split between auto and transit, the VHT values for the two modes are nearly similar). This is because the transit mode has a higher travel time than that of the auto mode.



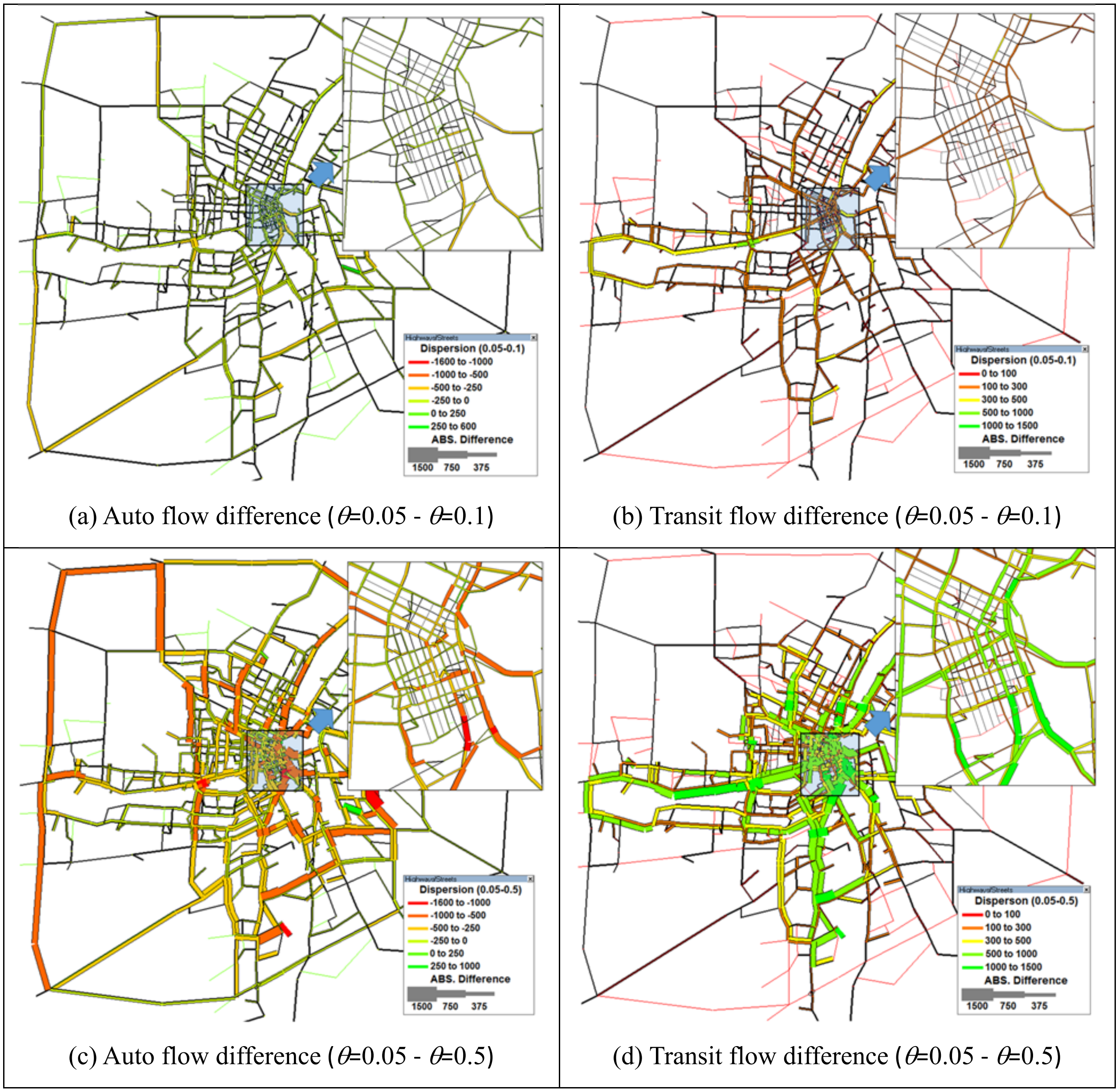

(a) Mode choice probability




(b) Vehicle hours traveled

**Figure 9** Effect of dispersion parameter on modal split and VHT

Figure 10 depicts the link flow differences between two logit parameter values for both auto and transit modes on a color-coded GIS map. Similar to the previous analysis, red color indicates that link flows using a smaller logit parameter value are larger than those of the larger logit parameter value, while green color indicates the reverse. Specifically, Figures 10(a) and 10(b) displays the color-coded link flow differences for the case of the link flow pattern with a logit parameter value of 0.05 minus the link flow pattern with a logit parameter value of 0.1 for auto and transit, respectively. Recall the modal splits for 0.05 as the logit parameter are 66.5% for auto and 33.5% for transit, and the modal splits for 0.1 as the logit parameter are 76.8% for auto and 23.2% for transit. Since the modal splits are quite different for these two logit parameter values, the link flow differences are also very different as shown in the color-coded GIS maps. Likewise, Figures 10(c) and 10(d) displays the color-coded link flow difference for the case of the link flow pattern with a logit parameter value of 0.05 minus the link flow pattern with a logit parameter value of 0.5 for auto and transit, respectively. Recall the modal splits for 0.5 as the logit parameter are 97.8% for auto and 2.2% for transit. The link flow difference in this case is even more dissimilar compared to the previous case as indicated by the thickness of the lines in the GIS map. Between the two modes, the link flow differences for the transit mode tend to accumulate in the central area where majority of the transit lines are located, while the link flow differences for the auto model tend to be more dispersed to the outer ring roads.




(a) Auto flow difference ($\theta$=0.05 - $\theta$=0.1)

(b) Transit flow difference ($\theta$=0.05 - $\theta$=0.1)

(c) Auto flow difference ($\theta$=0.05 - $\theta$=0.5)

(d) Transit flow difference ($\theta$=0.05 - $\theta$=0.5)

**Figure 10** Effect of logit parameter on flow allocation

## 5 CONCLUDING REMARKS

In this paper, we presented a modified path-based gradient projection (GP) algorithm for solving the combined modal split and traffic assignment (CMSTA) problem. The CMSTA problem considers both mode choice and route choice simultaneously when allocating the demand to the bi-modal transportation network. The mode choice step adopts the binary logit function to determine the modal split with two types of transit impedance function, and the route choice step allocates the auto O-D demand to the highway network according to the user equilibrium (UE) principle. The modification of the GP algorithm was achieved through a reformulation from excess demand to fixed total demand with the modal split choice between each O-D pair and accomplished by using an appropriate modification of network representation. In essence, the equilibration procedure was extended to include both modal split and path flow equilibrations. The modal split equilibration procedure determines an appropriate modal split for each O-D pair based on the network congestion level by equilibrating between the excess demand (or modal split) function and the O-D cost supply function, while the path flow equilibration procedure determines the flow allocations to the used paths to achieve a user equilibrium state.

We examined the efficiency and sensitivity of the modified GP algorithm using a real bi-modal network in the city of Winnipeg, Canada. Two types of transit impedance function were considered in the excess demand (or modal split) function. The results indicated that the type 2 (or flow-dependent) transit travel time requires more computational efforts due to the need to equilibrate both the log term and flow-dependent travel time term in the excess demand function. Overall, the results were encouraging and demonstrated that the modified GP algorithm could be as efficient as the original GP algorithm for solving the traffic assignment problem with fixed demand. The sensitivity test suggested that the logit parameter has a significant impact on the modal splits and link flow patterns as it indicates how sensitive is the users to the cost difference between the two modes (i.e., a large value means users are very sensitive to the cost difference, while a smaller value suggests users are less sensitive to the cost difference).

For future research, the modified GP algorithm should be tested on more transportation networks. It should also be extended to consider more than two modes using the multinomial logit function or the nested logit function to split the demand into multiple modes (e.g., auto, transit, and bicycle). In addition, mode (or vehicle) interactions should be considered if all modes share the same highway network, and route overlaps should be considered to correct for the overestimation of routes with significant couplings. It would also be interesting to see how the modified GP algorithm performs when other choice dimensions, such as the combined distribution and assignment problem (Xu et al., 2008; Ryu et al., 2014b,d; Yao et al., 2014) and the combined travel demand model (Yang and Chen, 2009; Zhou et al., 2009; Yang et al., 2013a,b), considered in the network equilibrium problem.

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