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Abstract

We consider a production-inventory system facing stochastic product returns that can either be disposed upon arrival or placed in a serviceable inventory, where serviceable products can be disposed at any time. In an M/M/1 make-to-stock queue setting, we establish that the optimal control policy is a threshold policy with three policy parameters and we derive closed-form results for the optimal thresholds and costs. For several situations, we establish that either the disposal upon arrival (DUA) option or the disposal of serviceable products (SD) option is sufficient to achieve optimality. We also present numerical examples for which it is useful to have both disposal options. Moreover, we explore four extensions for which the two options are complementary (limited secondary market, manufacturing start-up cost, Markov modulated demand and positive remanufacturing lead time).

Keywords: Inventory; Reverse logistics; Queuing; Markov decision process

1. Introduction

In this paper, we consider a hybrid system including two disposal options (see Figure 1). These disposal options have the objective to avoid excess inventories.

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We assume that all returns can be remanufactured to an as good as new product, however it would be straightforward to relax this assumption by introducing a probability for a return to be remanufacturable.

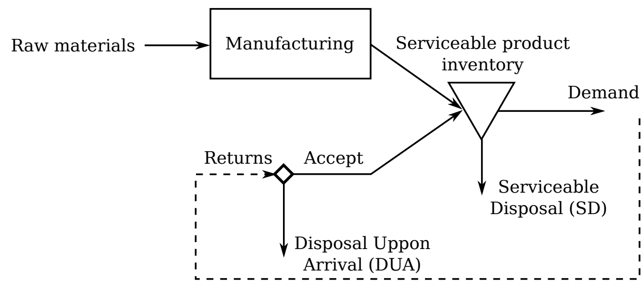


Figure 1: A make-to-stock system with two disposal options.

Excess inventories may appear for a variety of reasons. Return take-back obligations may lead to overstock when the flow of returns is large compared to the demand. Surplus of stock also may appear in case of volatility and random shifts in demand, bullwhip effect or inadequate information and forecasting systems (Angelus, 2011). Clearing mechanisms are also necessary when it is too costly to switch off or reduce the production rate (Germs et al., 2016). Disposal implies costs but may also generate revenues, for instance when the excess inventory is sold to a secondary market (Angelus, 2011).

The first disposal option considered in this paper is dedicated to product returns and will be referred to as the Disposal Upon Arrival (DUA) option. When a return arrives, it can either be disposed upon arrival or remanufactured and placed in the serviceable product inventory with newly manufactured products. The second disposal option can be used for any serviceable product and will be referred to as the Serviceable Disposal (SD) option. This second disposal option can among others be found in the literature of clearing mechanisms and secondary markets (see e.g. Germs et al. (2016); Angelus (2011)).

Our contributions. To the best of our knowledge, this paper presents the first model that makes a distinction between the two disposal options presented above (DUA and SD), when disposal is not related to the quality of products. The existing models with multiple disposal options are related to quality inspections and not to inventory considerations. This paper also contributes to the literature of make-to-stock queues by characterizing the optimal policy for an M/M/1 make-to-stock queue with returns that includes a disposal option (DUA or SD).

For the queuing model under consideration, we establish that the optimal policy can be described by three policy parameters and we derive closed-form results for the optimal thresholds and costs. We show that a single disposal option (either DUA or SD) is sufficient to achieve optimality, except in a situation which is not realistic in practice. Finally, we explore four extensions where the two options are complementary (limited secondary market, manufacturing start-up cost, Markov modulated demand and positive remanufacturing lead time).

2. Literature review

Our work is related to two streams of literature, one dealing with disposal options in hybrid systems combining manufacturing and remanufacturing, the other dealing with make-to-stock queues.

Disposal options in hybrid systems. There is a lot of literature available on combined planning and control of manufacturing and remanufacturing. For some recent extensive reviews, we refer the reader to Ilgin and Gupta (2010); Akçalı and Çetinkaya (2011). Hereafter we focus on models that include disposal options in hybrid systems.

In a periodic review setting, Simpson (1978) considers the coordination of manufacturing, remanufacturing and disposal decisions. All returned items are accepted and placed in a remanufacturable inventory, before being remanufactured or disposed. For zero manufacturing and remanufacturing lead times, Simpson proves that the optimal policy is a simple policy with a threshold level for each decision. When lead times are positive, the optimal policy has not yet been characterized but some special cases have been investigated. When the lead times are equal, Inderfurth (1997) partially characterizes the optimal policy. He also provides some results when the manufacturing lead time exceeds the remanufacturing lead time by one period, and the accepted returns are remanufactured directly without delay. DeCroix (2006) extends the results of Simpson (1978) and Inderfurth (1997) for a multi-stage serial system where returned items can enter at different stages. Li et al. (2010) generalize Simpson (1978) by including start-up costs for manufacturing and disposal.

Several papers investigate heuristic policies to control the flow of returned products. Aras et al. (2006) and van der Laan and Salomon (1997) consider a policy that rejects returned products if the remanufacturable inventory exceeds some threshold. In another paper, van der Laan and Salomon (1997) introduce a threshold policy based on the inventory position (net serviceable inventory plus products that are currently manufactured and remanufactured). For an M/M/C/K repair shop, van der Laan et al. (1996) study a policy which rejects returns if the number of items waiting for inspection or repair exceed some threshold.

Despite the abundance of papers on reverse logistics, only very few papers consider multiple disposal options. The four papers that we found all include at least one disposal option related to the quality of the returns, not allowing their reuse or remanufacturing. Inderfurth (2005) considers two disposal options: one at arrival of a return and the other for accepted returns waiting to be remanufactured, where the former is based on a check whether or not a return can be remanufactured at all. Zikopoulos and Tagaras (2008) consider a one period problem. Apart from being able to collect whatever number of returns desired, there are three disposal options where two of them are related to the quality of the returns. The quality of all returns is checked, based on which returns that are not good enough are disposed. However, this test is not perfect. Of all accepted returns the actual condition becomes clear after disassembly, where the returns that have been incorrectly accepted via the entrance test are disposed. Excessively remanufactured returns are disposed at the end of the single period considered. Zikopoulos and Tagaras (2007) consider a one period problem with three disposal options, where it is possible to buy as many returns as desired from each of two different sources. Unlike in the above paper of the authors, there is a (perfect) test on the quality of the collected returns, after which a certain number of returns is remanufactured. If enough good returns have been processed the remaining not yet remanufactured returns are disposed. Hereafter actual demand becomes known. In case of excess remanufactured retruns, these are disposed. So essentially all three types of disposal are completely predetermined once it has been decided how many returns to collect from each of the two sources. Ketzenberg (2009) allows two disposal options: one disposal option concerns disposal of returns at arrival and one disposal option for accepted returns that are not remanufactured during a period because this is not deemed to be useful. Note that the author decides on the two disposal quantities based on actual demand where the production decision is made before actual demand and arrival of returns is known.

Make-to-stock queues. In make-to-stock systems, goods can be produced and stored in anticipation of demand. In real manufacturing systems, the production facility can produce one item at a time, or at most a finite number of items at a time. In such systems, the order lead times are load-dependent. Make-to-stock systems with production modeled by servers will be referred to as *make-to-stock queues*. An M/M/1 make-to-stock queue is a make-to-stock system with one server, an exponentially distributed manufacture time and demand arriving according to a Poisson process.

The expression make-to-stock queue was first introduced by Wein (1992) who analyses an M/M/1 make-to-stock queue with backorders. However, the use of queuing theory to study production/inventory systems is much older and seems to have been suggested by (Morse, 1958) who investigates in his seminal book the relationships between inventory control and queuing theory. Since then, a vast literature has been dedicated to make-to-stock queues (see e.g. de Véricourt et al. (2002); Gayon et al. (2009); Benjaafar et al. (2011)). The books of Buzacott and Shanthikumar (1993) and Zipkin (2000) provide good introductions to make-to-stock queues.

More recently, several papers have included product returns in make-to-stock queues. Vercraene and Gayon (2013) consider an M/M/1 make-to-stock queue with product returns that can be directly be placed in the serviceable inventory. They show that the optimal policy is a simple production threshold policy with a closed form formula for the optimal base-stock level. Kim et al. (2013) and Vercraene et al. (2014) includes a remanufacturing lead time modelled by exponentially distributed single server. They both show that the optimal policy is characterized by two state-dependent base-stock thresholds for manufacturing and remanufacturing and one state-dependent return acceptance threshold. Vercraene et al. (2014) introduce several heuristic control rules and compare them with the optimal policy. Fathi et al. (2015) includes a DUA option based on a quality inspection of product returns. Returns are accepted if the remanufacturing lead time is smaller than some threshold.

3. Make-to-stock queuing model with returns

In this section we introduce the make-to-stock queuing model with returns and formulate the optimal control problem as a continuous-time Markov decision problem.

Notations and assumptions. The system under consideration is illustrated in Figure 2. We consider a single item production-inventory system where cus-

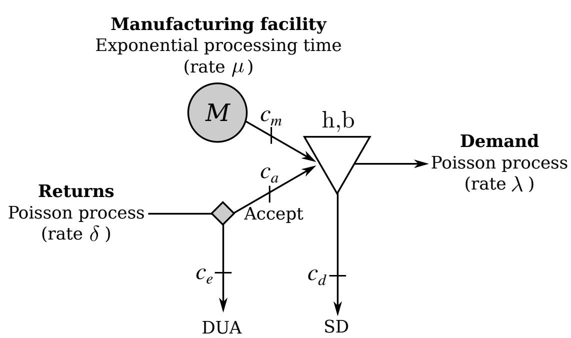


Figure 2: Make-to-stock queuing model with returns

tomer demand and returned products arrive according to independent Poisson processes with rates λ and δ. A returned product can either be accepted upon arrival with cost c_a or rejected (= disposed upon arrival) with cost c_e. The net rejection cost $c_{dua} = c_e - c_a$ can be positive or negative. After acceptance, returns can immediately be put in the serviceable stock (or after a negligible delay). The serviceable inventory can also be replenished via a single manufacturing facility with an exponentially distributed lead time μ and a per unit manufacturing cost $c_m \geq 0$. Serviceable products can be used to serve customer demand but can also be disposed with a per unit disposal cost c_d (positive or negative). We assume that any number of serviceable products can be disposed at any time. Unsatisfied demand is backlogged.

Let $\rho = \lambda/(\mu + \delta)$ and $\rho_m = \lambda/\mu$. To ensure the stability of the system, we assume that $\rho < 1$, i.e. that the demand rate is smaller than the sum of the manufacturing and return rates.

The state of the system can be summarized by $X(t)$, the net inventory level at time t. The system incurs in state x a cost rate per unit of time $C(x) = h[x]^+ + b[x]^-$, where h is the unit inventory holding cost per unit of time, b is the unit backlog cost per unit of time, $[x]^+ = \max\{0, x\}$, and $[x]^- = -\min\{0, x\}$.

Control policy π specifies at any time when to manufacture new products, when to accept returns and when to dispose serviceable products. The expected discounted cost over an infinite horizon for a policy π, with x the state of the system at $t = 0$ and $\alpha > 0$ the discount rate, is

$$\begin{aligned} v^\pi(x) = E_x^\pi & \left[\int_0^{+\infty} e^{-\alpha t} C(X(t)) \mathrm{d}t \right] \\ & + E_x^\pi \left[\sum_{i=1}^{\infty} \left(\begin{array}{c} e^{-\alpha\phi_a(i)} c_a + e^{-\alpha\phi_e(i)} c_e \\ + e^{-\alpha\phi_m(i)} c_m + e^{-\alpha\phi_d(i)} c_d \end{array} \right) \right]. \end{aligned}$$

where $\phi_a(i)$, $\phi_e(i)$, $\phi_m(i)$ and $\phi_d(i)$ respectively represent the i^{th} event time when either a return is accepted, a return is disposed upon arrival, a product is manufactured or a serviceable product is disposed. Function v^π will be referred to as the value function.

The objective is to find an optimal policy π^* that minimizes the expected discounted cost over an infinite horizon. Let $v^\star$ be the optimal value function defined by

$$v^\star(x) = \min_\pi \{v^\pi(x)\}.$$

MDP formulation. The optimization problem can be formulated as a continuous time Markov decision process (MDP); see Puterman (1994) for an introduction. Let $v^\star(x)$ be the optimal expected discounted cost over the time interval $[0,\infty)$ when the level of serviceable stock at time 0 is x. Let $w^\star(x)$ be the optimal expected discounted cost over the time interval $(0,\infty)$, with no disposal decision at time $t = 0$.

We uniformize the continuous-time MDP with rate $\eta = \lambda + \mu + \delta$ and transform it into a discrete time MDP (see Puterman (1994), chapter 9). Let $\tau = \eta + \alpha$. Let $v_k(x)$ and $w_k(x)$ be value functions defined as follows, for all integer x and $k \geq 0$:

$$v_0(x) = w_0(x) = 0, \tag{1}$$

$$w_k(x) = \mathcal{T} v_{k-1}(x), \tag{2}$$

$$v_k(x) = T_d w_k(x) \tag{3}$$

where operators $\mathcal{T}$ and T_d are defined by:

$$\mathcal{T} f(x) = \frac{1}{\tau}\begin{bmatrix} C(x) + \lambda T_a f(x) \\ +\mu T_m f(x) + \delta T_r f(x) \end{bmatrix} \tag{4}$$

$$T_d f(x) = \min_{n \in \{0,\ldots,[x]^+\}} \{f(x-n) + nc_d\}, \qquad (SD\,option) \tag{5}$$

$$T_a f(x) = f(x-1), \qquad (demand\,arrival) \tag{6}$$

$$T_r f(x) = \min\{f(x) + c_e, f(x+1) + c_a\}, \qquad (DUA\,option) \tag{7}$$

$$T_m f(x) = \min\{f(x), f(x+1) + c_m\}. \qquad (manufacturing) \tag{8}$$

The optimal value functions $v^\star$ and $w^\star$ can be shown to be the limits of the sequences of value functions (v_k) and (w_k) when k goes to infinity (see Puterman (1994), chapter 6).

We also define a relaxed problem which has exactly the same formulation except that T_d is replaced by

$$\tilde{T}_d f(x) = \min_{n \in \mathbb{N}} \{ f(x-n) + nc_d \}. \tag{9}$$

In the relaxed problem, we are allowed to dispose a serviceable product that we don't have by increasing the backlog. We will use the tilde symbol for the relaxed problem: $\tilde{\pi}^\star$, $\tilde{v}_k$, $\tilde{w}_k$, etc. The non-relaxed problem will be referred to as the original problem.

Let Δ be the operator such that $\Delta f(x) = f(x+1) - f(x)$. A value function f is convex if $\Delta f(x)$ is non-decreasing in x.

4. Characterization of the optimal policy

In this section, we first establish the structure of the optimal policy for the relaxed problem. Then we show that the optimal policy for the relaxed problem is a feasible and optimal solution for the original problem under a weak assumption. Among others, we will prove that there exists an optimal policy of the following form.

Definition 1 (Threshold policy). *An (S_a, S_m, S_d) policy states*

- *to accept product returns if and only if $x < S_a$,*
- *to manufacture products if and only if $x < S_m$, and*
- *to dispose $x - S_d$ serviceable products when $x > S_d$.*

Relaxed problem. The following lemma establishes that operators $\tilde{T}_d$ and $\mathcal{T}$ preserve convexity. The proof is given in Appendix A.

Lemma 1. *If f is convex, then $\tilde{T}_d f$ and $\mathcal{T} f$ are convex.*

By Lemma 1, if $\tilde{v}_{k-1}$ is convex, then $\tilde{w}_k = \mathcal{T}\tilde{v}_{k-1}$ is convex and $\tilde{v}_k = \tilde{T}_d \tilde{w}_k$ is convex. As $\tilde{v}_0 = 0$ is convex, we can conclude by induction that $\tilde{v}^\star = \lim_{k\to\infty} \tilde{v}_k$ and

$\tilde{w}^\star = \lim_{k\to\infty} \tilde{w}_k$ are convex. It follows that threshold policy $\tilde{\pi}^\star = (\tilde{S}_a^\star, \tilde{S}_m^\star, \tilde{S}_d^\star)$ is optimal, with

$$\tilde{S}_a^\star = \min[x : \Delta\tilde{v}^\star(x) \geq c_{dua}] \qquad \text{(returns acceptance)} \qquad (10)$$

$$\tilde{S}_m^\star = \min[x : \Delta\tilde{v}^\star(x) \geq -c_m] \qquad \text{(manufacturing)} \qquad (11)$$

$$\tilde{S}_d^\star = \min[x : \Delta\tilde{v}^\star(x) \geq c_d] \qquad \text{(disposal)} \qquad (12)$$

Lemma 2 (Discounted cost optimal policy for the relaxed problem). *The optimal value function $\tilde{v}^\star$ is convex and the optimal policy is (S_a, S_m, S_d).*

The recurrent region of the induced Markov chain is the set of states $\{x \in \mathbb{Z} : x \leq \tilde{S}^\star\}$ where $\tilde{S}^\star = \min(\tilde{S}_d^\star, \max(\tilde{S}_a^\star, \tilde{S}_m^\star))$. In what follows, we will show by contradiction that $\tilde{S}_d^\star \geq 0$ and $\max(\tilde{S}_a^\star, \tilde{S}_m^\star) \geq 0$, under some mild assumptions on the cost parameters.

Lemma 3. *If $b/\alpha > -c_d$, then $\tilde{S}_d^\star \geq 0$.*

Lemma 3 shows that we should not dispose products when there are backorders if the disposal revenue per product $-c_d$ is smaller than the cost of backlogging a product forever $\left(\frac{b}{\alpha} = b\int_0^{+\infty} e^{-\alpha t}\right)$. This result is proven by contradiction in Appendix B. It implies that the optimal policy for the relaxed problem is feasible and optimal for the original problem, under the assumption that $b/\alpha > -c_d$.

Lemma 4. *If $b/\alpha > \min(c_m, -c_{dua})$ and $\tilde{S}_d^\star \geq 0$, then $\max(\tilde{S}_a^\star, \tilde{S}_m^\star) \geq 0$.*

Lemma 4 shows that we should either manufacture or accept returns when there are backorders if the cost for backlogging a unit forever is higher than the manufacturing cost or the marginal acceptance cost. This result is also proven by contradiction, see Appendix C.

Original problem. We can now state our main result which results from the previous lemmas. When $b/\alpha > -c_d$, Lemma 3 implies that the optimal policy for the relaxed problem $\tilde{\pi}^\star = (\tilde{S}_a^\star, \tilde{S}_m^\star, \tilde{S}_d^\star)$, is feasible and optimal for the original problem. It will be denoted by $\pi^\star = (S_a^\star, S_m^\star, S_d^\star)$ in the rest of the paper. Moreover $v^\star = \tilde{v}^\star$ and thus $v^\star$ is convex.

Theorem 1 (Discounted cost optimal policy). *If $b/\alpha > -c_d$, then $v^\star$ is convex and the optimal policy is (S_a, S_m, S_d), with $S_d \geq 0$. If we additionally assume that $b/\alpha > \min(-c_{dua}, c_m)$, then $\max(S_a, S_m) \geq 0$.*

Threshold definitions (10)-(12) and convexity of the optimal value function immediately imply the following corollary.

Corollary 1 (Threshold properties). *If $b/\alpha > -c_d$, then the optimal thresholds $S_a^\star$, $S_d^\star$, and $S_m^\star$ can be ordered as follows:*

Case i) If $c_d \leq -c_m \leq c_{dua}$, then $S_d^\star \leq S_m^\star \leq S_a^\star$.

Case ii) If $c_d \leq c_{dua} \leq -c_m$, then $S_d^\star \leq S_a^\star \leq S_m^\star$.

Case iii) If $c_{dua} \leq -c_m \leq c_d$, then $S_a^\star \leq S_m^\star \leq S_d^\star$.

Case iv) If $-c_m \leq c_{dua} \leq c_d$, then $S_m^\star \leq S_a^\star \leq S_d^\star$.

Case v) If $-c_m \leq c_d \leq c_{dua}$, then $S_m^\star \leq S_d^\star \leq S_a^\star$.

Case vi) If $c_{dua} \leq c_d \leq -c_m$, then $S_a^\star \leq S_d^\star \leq S_m^\star$.

Figure 3 illustrates numerically the optimal policy for case iv. The values of the optimal policy parameters are computed by a value iteration algorithm.

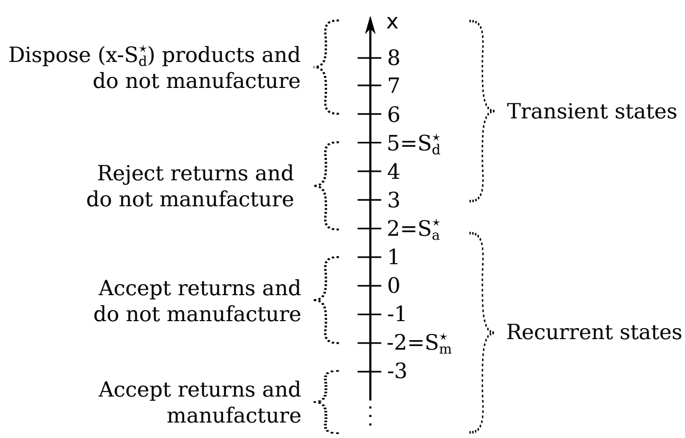


Figure 3: Optimal policy when $\{\alpha = 0.001, \lambda = \mu = \delta = c_a = 1, h = b = c_e = 2, c_m = 15, c_d = 4\}$.

We can additionnaly prove that in most situations either the DUA option or the SD option is sufficient to achieve optimality. In cases iii and iv, the inventory level will never exceed $S_d^\star$, except at time 0 if the initial inventory is larger than $S_d^\star$. In these cases, the SD option will never be used except at time 0. In cases i, ii and v, the DUA option is useless, as the SD option can be used at any time at a lower cost ($c_d \leq c_{dua}$). Finally it is optimal to manufacture all the time, when the revenue obtained via the SD option is higher than the cost for manufacturing a product ($c_m + c_d \leq 0$).

Theorem 2.

1. *In cases i, ii and v ($c_d \leq c_{dua}$), there exists an optimal policy that never uses the DUA option.*

2. *In cases iii and iv ($c_d \geq c_{dua}$ and $c_m + c_d \geq 0$), there exists an optimal policy that never uses the SD option (except at time $t = 0$).*

3. *In cases i, ii and vi ($c_m + c_d \leq 0$), there exists an optimal policy that manufactures new products all the time.*

Case vi is the only situation where it can be advisable to use both options when $t > 0$, as illustrated in Figure 4. In the rest of the paper, $NoDUA$ (resp. $NoSD$) will denote the best policy that never uses the DUA option (resp. SD option). In Figure 4, we clearly observe for this numerical example that the optimal policy outperforms $NoDUA$ and $NoSD$.

Note that in all cases, it might be advisable to use the SD option to dispose products at time $t = 0$, as illustrated in Figure 5 which shows the effect of the initial inventory $X(0)$ on the policy costs. For this numerical example, the optimal SD threshold $S_d^\star$ is equal to 8 and we observe that *NoDispose* is not optimal when $X(0) > S_d^\star = 8$.

Situations where $c_m + c_d \leq 0$ are not very realistic, as we assume that we can dispose an arbitrary number of products. It would mean that we can sell on a secondary market an infinite number of newly manufactured products, with

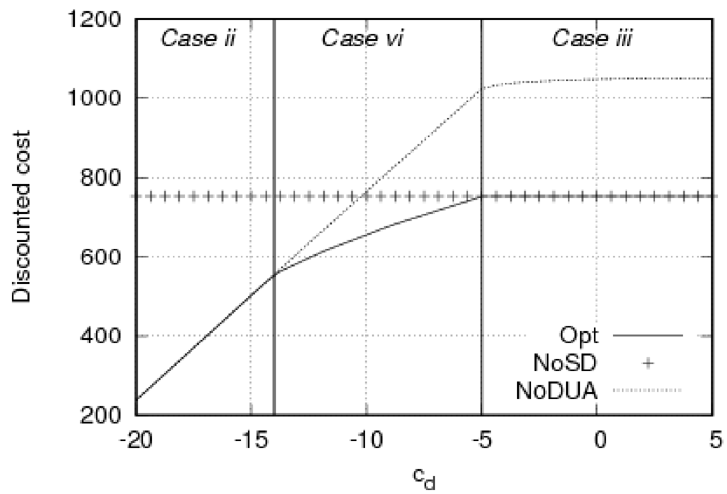


Figure 4: Effect of the disposal cost c_d on the discounted cost when $\{\delta = 0.5,\ \mu = 1.05,\ \lambda = 1,\ h = 1,\ b = 10\ ,\ c_m = 5,\ c_a = 4,\ c_e = -10,\ \alpha = 0.01,\ X(0) = 0\}$.

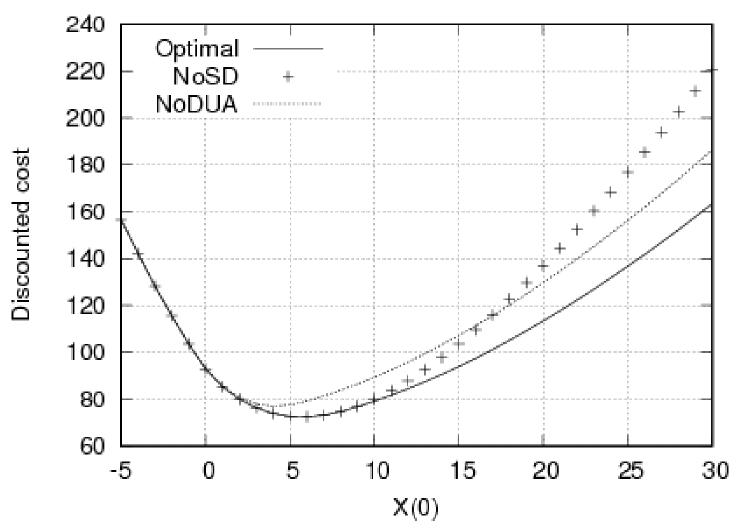


Figure 5: Effect of the initial quantity of products in stock on the discounted cost when $\{\delta = 0.5,\ \mu = 1.05,\ \lambda = 1,\ h = 1,\ b = 2\ ,\ c_m = 10,\ c_a = 5,\ c_e = 2,\ c_d = 2,\ \alpha = 0.1\}$ (case iv).

a positive marginal benefit. Hence we will focus on cases i, ii and vi in Section 5. Section 6 will investigate an extension of our model where the demand from the secondary market is limited.

Average cost. Theorem 1 and its corollaries also hold for the average cost criterion as the optimal average cost policy is the limit of the optimal discounted cost policy, when the discount rate α goes to 0 (Weber and Stidham, 1987). Hence the assumptions $b/\alpha > -c_d$ and $b/\alpha > \min(-c_{dua}, c_m)$ can be removed from Theorem 1 for the average cost criterion. In addition, we can remove in Theorem 2 the exception at time $t = 0$, as the transient behavior does not matter in average cost.

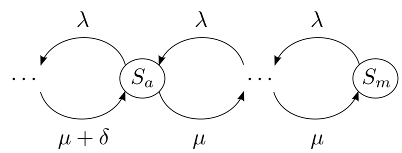
Service time with general distribution. If we consider more general service time distributions, the structure of the optimal policy is more complex and Theorem 1 does not hold anymore. When the service time is not exponential, the optimal policy must take into account the time elapsed from the last production start. For instance, if the manufacturing of a new product has just begun, a return could be accepted while it would not have been accepted if the manufacturing was close to end.

The proof of the first and third part of Theorem 2 did not use the exponential assumption and hence extends to general service times. We conjecture that the second part of Theorem 2 also extends to more general service times. In cases iii and iv, the only reason to dispose a serviceable product would be due to an increase in the work-in-process. As we can preempt manufacturing at not cost, it would be then possible to preempt manufacturing instead of disposing a serviceable product.

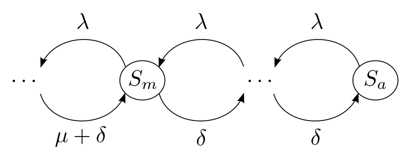
5. Optimal policy parameters

In this section, we derive additional results for the average cost problem in cases iii, iv, and v. The other cases are not investigated as they are less realistic. Note that in cases i and ii, the optimal policy reduces to a single disposal parameter $S_d^\star = \left\lfloor \ln \frac{h}{h+b} / \ln \rho_m \right\rfloor$, see e.g. Veatch and Wein (1996).

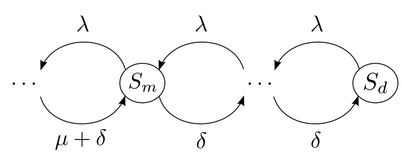
The Markov chain induced by a threshold policy (S_a, S_m, S_d) is given in Figure 6, which shows the recurrent states $\mathcal{R} = \{x \in \mathbb{Z} : x \leq S_d$ and $x \leq \max(S_a, S_m)\}$.



(a) Case iii: $S_a \leq S_m \leq S_d$



(b) Case iv: $S_m \leq S_a \leq S_d$



(c) Case v: $S_m \leq S_d \leq S_a$

Figure 6: Recurrent region of Markov chain induced by policy (S_a, S_m, S_d).

Case iii. In case iii, the optimal policy is such that $S_a^\star \leq S_m^\star \leq S_d^\star$. Consider a policy (S_a, S_m, S_d) with the same properties, i.e. $S_a \leq S_m \leq S_d$. The inventory level does not exceed S_m in steady state and the SD option is never used.

Let $q = S_m - S_a$ be the threshold difference and $N_q = S_m - X$. The index q in N_q emphasizes that the probability distribution of N_q depends on q. The average cost of policy (S_a, S_m, S_d) can then be expressed as

$$\begin{aligned}
C(S_a, S_m) =& hE[X]^+ + bE[X]^- \\
& + c_m \mu P(X < S_m) + \delta[c_a P(X < S_a) + c_e P(X \geq S_a)], \\
=& hE[S_m - N_q]^+ + bE[S_m - N_q]^- \\
& + c_m \mu P(N_q > 0) + \delta[c_a P(N_q > q) + c_e P(N_q \leq q)].
\end{aligned}$$

Since $c_m \mu P(N_q > 0) + \delta[c_a P(N_q > q) + c_e P(N_q \leq q)]$ is independent of S_m for a given q, minimizing $C(S_a, S_m)$ is equivalent to finding the value of S_m that minimizes

$$hE(S_m - N_q)^+ + bE(S_m - N_q)^-.$$

This problem is again a newsvendor problem where the base-stock level is S_m, the demand is N_q, the shortage cost is b, and the holding cost is h.

Let $S_m(q)$ and $S_a(q) = S_m(q) - q$ and $C(q)$ be respectively the optimal thresholds and average cost when q is given. Then we have the classical newsvendor results (see e.g. Porteus (2002)):

$$S_m(q) = \min \left\{ x | F_{N_q}(x) \geq \frac{b}{b+h} \right\}, \tag{13}$$

where F_{N_q} is the cumulative distribution of N_q. As N_q follows a birth death process, its distribution can be easily computed:

$$P(N_q = 0) = \begin{cases} \dfrac{(1-\rho)(1-\rho_m)}{1-\rho-{\rho_m}^q(\rho_m-\rho)} & \text{if } \rho_m \neq 1, \\ \dfrac{(1-\rho)}{1+q(1-\rho)} & \text{if } \rho_m = 1, \end{cases} \tag{14}$$

$$P(N_q = n) = \begin{cases} {\rho_m}^n \; P(N_q = 0) & \text{if } 0 \leq n \leq q, \\ \rho^{n-q} {\rho_m}^q P(N_q = 0) & \text{if } n \geq q, \end{cases} \tag{15}$$

and

$$\begin{aligned}
F_{N_q}(x) &= \begin{cases} \sum_{n=0}^{x} P(N_q = n) & \text{if } x \leq q, \\ \sum_{n=0}^{q-1} P(N_q = n) + \sum_{n=q}^{x} P(N_q = n) & \text{if } x \geq q, \end{cases} \\
&= \begin{cases} \dfrac{1-{\rho_m}^{x+1}}{1-\rho_m} P(N_q = 0) & \text{if } x \leq q, \\ \left[\dfrac{1-{\rho_m}^q}{1-\rho_m} + \dfrac{1-\rho^{x-q+1}}{1-\rho} {\rho_m}^q \right] P(N_q = 0) & \text{if } x \geq q. \end{cases}
\end{aligned} \tag{16}$$

We need to distinguish two cases. If $F_{N_q}(q) \geq \frac{b}{h+b}$, then $S_m(q) \leq q$ and $S_a(q) \leq 0$. If $F_{N_q}(q) \leq \frac{b}{h+b}$, then $S_m(q) \geq q$ and $S_a(q) \geq 0$. Equations (13), (14) and (16) give the optimal acceptance threshold $S_m(q)$, for a given difference $q = S_m - S_a \geq 0$:

$$S_m(q) = \begin{cases} \left\lfloor \dfrac{\ln\left(\dfrac{h\rho^q}{h+b}\left[1 + \dfrac{(1-\rho)(1-\rho_m{}^q)}{\rho_m{}^q(1-\rho_m)}\right]\right)}{\ln(\rho)} \right\rfloor & \text{if } F_{N_q}(q) \leq \frac{b}{h+b}, \\ \left\lfloor \dfrac{\ln\left(\dfrac{h}{h+b} + \dfrac{b}{h+b}\dfrac{\rho_m^q(\rho_m-\rho)}{1-\rho}\right)}{\ln(\rho_m)} \right\rfloor & \text{if } F_{N_q}(q) \geq \frac{b}{h+b}. \end{cases} \tag{17}$$

These results are consistent with those of Bradley (2005) who considers a dual-source M/M/1 make-stock queue.

So, we have transformed a 3-variables optimization problem into the problem of minimizing

$$\begin{aligned} C(q) = hE[S_m(q) - N_q]^+ + bE[S_m(q) - N_q]^- \\ + c_m\mu P(N_q > 0) + \delta[-c_{dua}P(N_q > q) + c_e], \end{aligned} \tag{18}$$

where q is a non negative integer and $S_m(q)$ is given by (17). We provide in Appendix D a closed-form expression for $C(q)$. In extensive numerical tests, we have observed that $C(q)$ is unimodal. However, $C(q)$ is not always convex (see Figure 7 for a counter example).

Cases iv and v. By symmetry with case iii, similar formulas for the optimal base-stock levels can be derived for cases iv and v. For instance, cases iii and iv have the same Markov chain if we simply exchange μ with δ and S_a with S_m.

6. Extensions where disposal options are complementary

We now present several extensions of our basic model where neither $NoSD$, nor $NoDUA$ is optimal. We remind that $NoDUA$ (resp. $NoSD$) denotes the

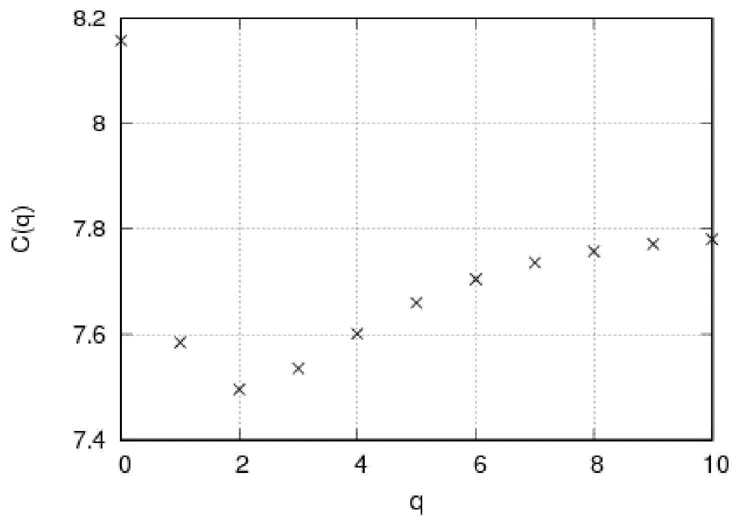


Figure 7: Effect of q on the average cost $C(q)$ when $\{\delta = 1,\ \mu = 2,\ \lambda = 1.1,\ h = 1,\ b = 2,\ c_m = 10,\ c_a = 8,\ c_e = c_d = -5,\ \alpha = 0\}$ (case iii).

best policy that never uses the DUA option (resp. SD option). Each extension relaxes a single assumption.

Limited secondary market

In this extension, we assume that the SD option concerns a secondary market with a demand that occurs according to a Poisson process with rate λ'. When the serviceable inventory is not empty, a demand from the secondary market can be served with cost c_d (revenue $-c_d$) or rejected. When the serviceable inventory is empty, a demand from the secondary market is lost.

The MDP formulation of Section 4 must be adapted as follows. Redefine τ as $\alpha + \lambda + \mu + \delta + \lambda'$. Let (v_k) the sequence of value functions defined as

$$v_k(x) = \frac{1}{\tau} \left[\begin{array}{c} C(x) + \lambda T_a v_{k-1}(x) + \mu T_m v_{k-1}(x) \\ + \delta T_r v_{k-1}(x) + \lambda' T'_d v_{k-1}(x) \end{array} \right]$$

with

$$T'_d f(x) = \begin{cases} \min\{f(x), f(x-1) + c_d\} & \text{if } x > 0, \\ f(x) & \text{otherwise,} \end{cases}$$

and operators T_a, T_m, T_r defined as in Section 3

Using the same arguments as in Section 4 we can prove that when $b/\alpha > -c_d$, operators T_a, T_m, T_r, T_d' preserve convexity (Koole, 1998), implying that the optimal value function is convex.

It follows that the optimal policy is a modified (S_a, S_m, S_d) policy (see Definition 1) where a demand from the secondary market is satisfied if and only if $x > S_d$. Part 2 of Theorem 2 still holds, with the same arguments. However parts 1 and 3 of Theorem 2 do not hold when considering a limited secondary market, as illustrated in what follows.

Figure 8 plots the effect of λ' on the average cost of three policies (optimal, $NoSD$ and $NoDUA$), with cost parameters corresponding to case i of Corollary 1. When λ' goes to infinity, the secondary market becomes unlimited and $NoDUA$ is optimal, as shown in Theorem 2, part 1. When $\lambda' < 0.8$, we observe that $NoDUA$ is not optimal, as returned products can not be disposed quickly enough to the secondary market. When λ' goes to 0, the SD option becomes useless, as the demand from the secondary market vanishes.

We also observe that the DUA and SD options are complementary for intermediate values of λ'. On the one hand, the DUA option is preferable to dispose returned products directly without waiting for the secondary market demand. On the other hand, the SD option is useful to dispose some products with higher margin.

Start-up cost

In this second extension, we include a start-up cost K when the manufacturing facility starts manufacturing. The MDP formulation is detailed in Appendix E. To the best of our knowledge, the problem of characterizing the optimal policy is still open in the simpler case of an M/M/1 make-to-stock queue without returns.

In Figure 9, we observe that the two disposal options are complementary when $K \geq 200$. On the one hand, $NoDUA$ is not optimal as it is less costly to dispose a return upon arrival ($c_{dua} = -2.5 < c_d = 4$). On the other hand,

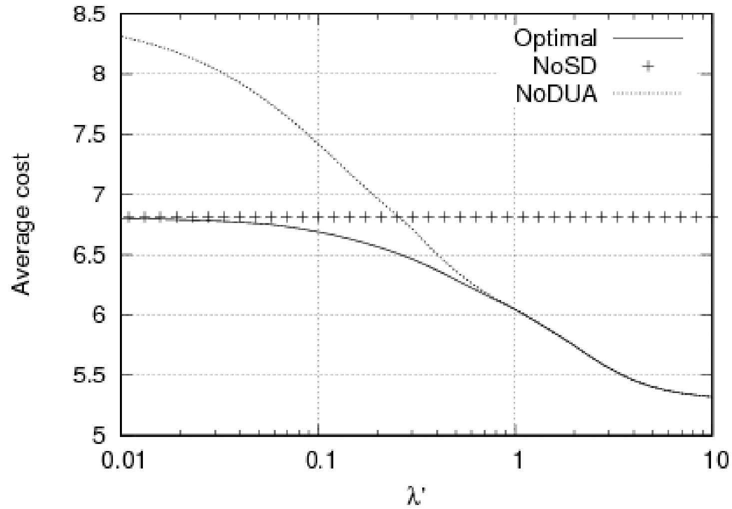


Figure 8: Effect of the disposal option rate λ' on the average cost when $\{\lambda = 1,\ \mu = 1.05,\ \delta = 0.8,\ h = 1,\ b = 2,\ c_m = 6,\ c_a = 4,\ c_e = 1,\ c_d = -7,\ \alpha = 0\}$ (case i).

when K is large enough, the optimal policy prefers to manufacture all the time and dispose products using the SD option, rather than stop and restart the manufacturing facility at a large start-up cost.

We obtain similar results (not reported) if we include a start-up time exponentially distributed with rate β when the manufacturing facility starts manufacturing. For large values of $1/\beta$, it is preferable to manufacture all the time and to dispose some manufactured products.

Markov modulated demand

In this extension, we consider a Markov modulated demand. The MDP formulation is detailed in Appendix E. The demand rate λ_e depends on an exogenous state variable e which evolves according to the continuous-time Markov chain represented in Figure 10.

Figure 11 illustrates the effect of the switching rate $\gamma_1 = \gamma_2$ on the average costs. We observe that the SD and DUA options are complementary when the switching rate is small enough. When the demand rate is high ($\lambda_2 = 2$), the optimal policy builds stock in order to avoid backlog costs. When the demand

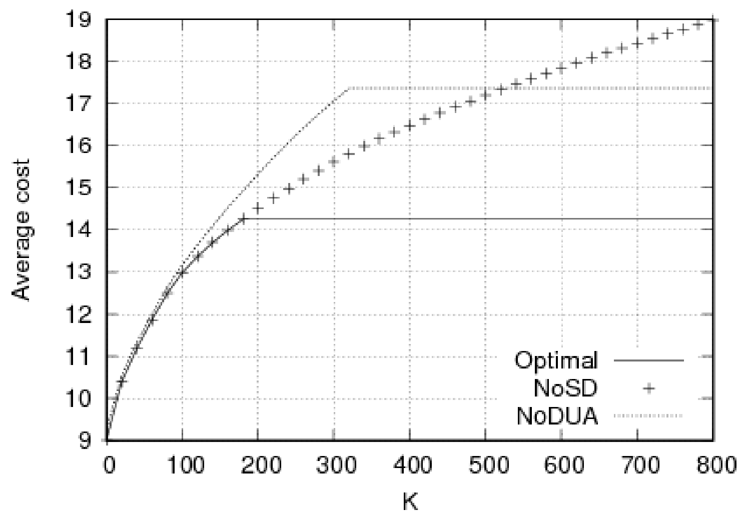


Figure 9: Effect of the start-up cost K on the average cost when $\{\lambda = 1,\ \mu = 1.05,\ \delta = 0.7,\ h = 1,\ b = 5,\ c_m = 8,\ c_a = c_d = 4,\ c_e = 1.5,\ \alpha = 0\}$.

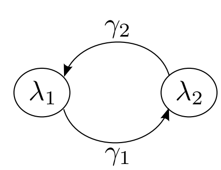


Figure 10: Time varying demand rate.

rate switches to $\lambda_1 = 0$, it is profitable to dispose some products by using the SD option. The DUA option is always useful, as it is less costly to dispose a return upon arrival than later using the SD option ($c_{dua} = 0 < c_d = 6$).

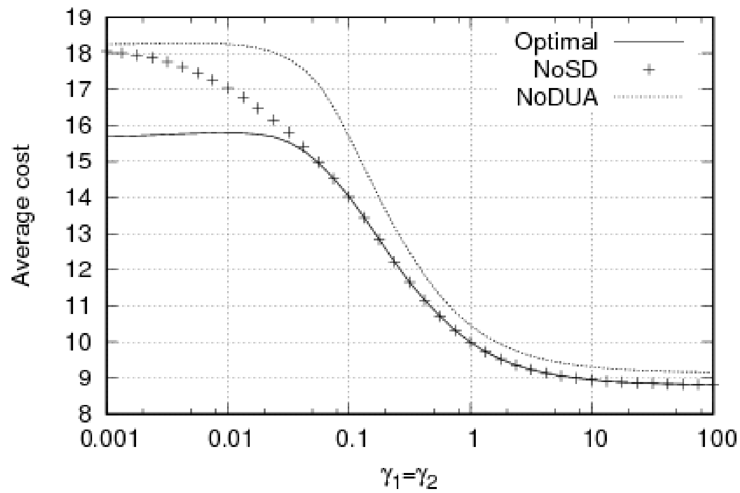


Figure 11: Effect of the switching rate γ_i on the average cost when $\{\lambda_1 = 0,\ \lambda_2 = 2,\ \mu = 1.5,\ \delta = 0.8,\ h = 1,\ b = 5,\ c_m = 10,\ c_a = c_e = 4,\ c_d = 6,\ \alpha = 0\}$.

Non-zero remanufaturing lead time

In this last extension, accepted returns are remanufactured one by one by a separate controllable remanufacturing facility (see Figure 12). The remanufacturing lead-time is exponentially distributed with rate μ_r and we denote by $\rho_r = \delta/\mu_r$ the load of the remanufacturing facility when no returns are disposed before remanufacturing.

The acceptance cost for one accepted return is c'_a (e.g. representing handling, administration cost) and the cost related to the actual remanufacturing of one return is c_r. Products in the remanufacturable inventory incur a unit holding cost h_r per unit of time. After remanufacturing, products are placed in the serviceable inventory with unit holding cost h per unit of time.

When $h_r \geq h$, it is optimal to remanufacture as soon as possible (push policy). Otherwise, the optimal control of the remanufacturing facility is more

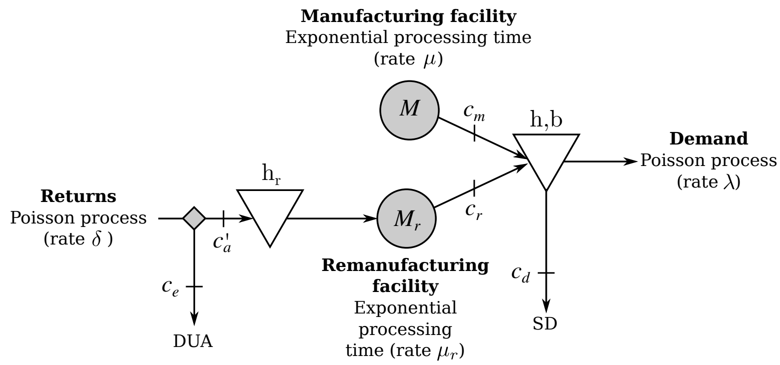


Figure 12: System with non-zero remanufacturing lead time and two disposal options.

complex and depends on the state of the two inventory levels.

Figure 13 shows the effect of μ_r on the average costs. On the one hand, we observe that $NoDUA$ performs very badly when μ_r is close to λ, due to a congestion in front of the remanufacturing facility. Note that the average remanufacturing inventory is bounded below by $\rho_r/(1-\rho_r)$ when $\rho_r < 1$ and is infinite otherwise. On the other hand, we observe that $NoSD$ is not optimal for larger values of μ_r, as the SD option is cheaper ($c_d = 1 < c_e - c_a' - c_r = 28$). Finally, we observe that the DUA and SD options are complementary for intermediate values of μ_r.

When accepted returns have to wait for remanufacturing, the state of the system may meanwhile change such that it may no longer be optimal to remanufacture all accepted returns. Therefore, in addition to the DUA and the SD options, we may also consider a Remanufacturable Disposal (RD) option which allows to dispose at any time a remanufacturable product before being remanufactured. The RD option is more general than the DUA option as it allows to dispose a remanufacturable product at any time. However, we have observed in a number of numerical experiments that the DUA option performs almost as well as the RD option.

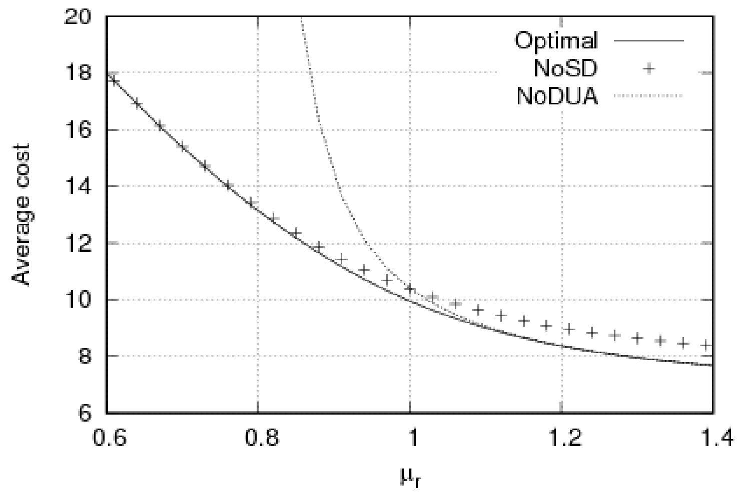


Figure 13: Effect of remanufacturing rate μ_r on the average cost when $\{\lambda = 1,\ \delta = 0.8,\ h = 1,$ $h_r = 0.95,\ b = 3,\ \mu = 10,\ c_m = 10,\ c'_a = 1,\ c_r = 1\ c_e = 30,\ c_d = 1,\ \alpha = 0\}$.

7. Conclusion

In this paper, we showed in a queuing framework that the DUA and SD options can be complementary (or not) for a variety of situations.

For an M/M/1 make-to-stock queue with product returns, we proved that the optimal policy is a threshold policy with three policy parameters that can be ordered with respect to manufacturing and disposal costs. We also derived closed-form results for the optimal thresholds and costs. We established that in most situations either the SD option or the DUA option is sufficient to achieve optimality. For two situations (secondary market with a high profit margin, discounted cost with a large initial inventory level), we provided numerical examples where both options are complementary.

Next, we explored numerically four extensions by considering a limited secondary market, a manufacturing start-up cost, a Markov modulated demand and a positive remanufacturing lead time. For these extensions, we showed that there are situations where the two disposal options are complementary.

As avenues for future research, it would be interesting to investigate the structure of the optimal policies for the extensions considered in the numerical experiments. It would also be of interest to test our insights via case studies in practice.

Appendix A. Proof of Lemma 1

Let f be a convex value function. Convexity implies that $\Delta f(x) \geq c_d$ if and only if $x \geq S := \min[x : \Delta f(x) \geq c_d]$. Thus $\tilde{T}_d$ can be rewritten as

$$\tilde{T}_d f(x) = \begin{cases} f(S) + (x - S)c_d, & \text{if } x \geq S, \\ f(x), & \text{if } x \leq S. \end{cases}$$

It follows that

$$\begin{aligned} \Delta \tilde{T}_d f(x) &= \tilde{T}_d f(x+1) - \tilde{T}_d f(x), \\ &= \begin{cases} c_d & \text{if } x \geq S, \\ \Delta f(x) < c_d & \text{if } x < S. \end{cases} \end{aligned}$$

We conclude that $\Delta \tilde{T}_d f(x)$ is non-decreasing in x. Hence $\tilde{T}_d f$ is convex and operator $\tilde{T}_d$ preserves convexity.

It is also well known that operators T_a, T_m, and T_r preserves convexity (Koole, 1998). Hence operator $\mathcal{T}$, as a convex combination of T_a, T_m and T_r, also preserves convexity.

Appendix B. Proof of Lemma 3

Assume that $b/\alpha + c_d > 0$. We shall prove by contradiction that $\tilde{S}_d^\star \geq 0$. Assume that $\tilde{S}_d^\star < 0$. We will show that the optimal policy $\tilde{\pi}^\star = (\tilde{S}_a^\star, \tilde{S}_m^\star, \tilde{S}_d^\star)$ has a strictly larger cost than policy $\phi = (\tilde{S}_a^\star + 1, \tilde{S}_m^\star + 1, \tilde{S}_d^\star + 1)$. Let $\tilde{v}^\star(x)$ and $\tilde{v}^\phi(x)$ be the expected discounted costs respectively under policy $\tilde{\pi}^\star$ and ϕ, when the initial state is x.

Let $\tilde{X}^\star(t)$ and $\tilde{X}^\phi(t)$ be the inventory levels at time t under respectively policy $\tilde{\pi}^\star$ and ϕ. If $\tilde{X}^\phi(0) = \tilde{S}_d^\star + 1$ and $\tilde{X}^\star(0) = \tilde{S}_d^\star$, then we have $\tilde{X}^\phi(t) = \tilde{X}^\star(t) + 1$ for all $t \geq 0$. It implies that

$$\tilde{v}^\star(\tilde{S}_d^\star) = \tilde{v}^\phi(\tilde{S}_d^\star + 1) + b/\alpha. \tag{B.1}$$

As $\tilde{S}_d^\star = \min[x : \tilde{v}^\star(x+1) - \tilde{v}^\star(x) \geq c_d]$, we have the inequality

$$\tilde{v}^\star(\tilde{S}_d^\star + 1) \geq \tilde{v}^\star(\tilde{S}_d^\star) + c_d. \tag{B.2}$$

(B.1), (B.2) and assumption $b/\alpha + c_d > 0$ give together:

$$\begin{aligned} \tilde{v}^\star(\tilde{S}_d^\star + 1) &\geq \tilde{v}^\phi(\tilde{S}_d^\star + 1) + b/\alpha + c_d, \\ &> \tilde{v}^\phi(\tilde{S}_d^\star + 1). \end{aligned}$$

This is contradicting. Hence, the initial assumption $\tilde{S}_d^\star < 0$ is false. We conclude that $\tilde{S}_d^\star \geq 0$.

Appendix C. Proof of Lemma 4

Assume that $\tilde{S}_d^\star \geq 0$ and $b/\alpha > \min(-c_{dua}, c_m)$. We adopt the same notations as in Appendix B. We distinguish two cases.

Case 1: $\min(-c_{dua}, c_m) = c_m$

In this case, $\tilde{S}_m^\star \geq \tilde{S}_a^\star$. We shall prove by contradiction that $\tilde{S}_m^\star \geq 0$. Assume that $\tilde{S}_m^\star < 0$. As we have also assumed that $\tilde{S}_d^\star \geq 0$, the recurrent region is the set of states $\{x : x \leq \tilde{S}_m^\star\}$.

If $\tilde{X}^\phi(0) = \tilde{S}_m^\star + 1$ and $\tilde{X}^\star(0) = \tilde{S}_m^\star$, then $\tilde{X}^\phi(t) = \tilde{X}^\star(t) + 1$ for all $t \geq 0$. It implies that

$$\tilde{v}^\star(\tilde{S}_m^\star) = \tilde{v}^\phi(\tilde{S}_m^\star + 1) + b/\alpha, \tag{C.1}$$

As $\tilde{S}_m^\star = \min[x : \tilde{v}^\star(x+1) - \tilde{v}^\star(x) \geq -c_m]$, we have the inequality

$$\tilde{v}^\star(\tilde{S}_m^\star + 1) \geq \tilde{v}^\star(\tilde{S}_m^\star) + c_m. \tag{C.2}$$

(C.1), (C.2) and assumption $b/\alpha > \min(-c_{dua}, c_m) = c_m$ give together:

$$\begin{aligned}\tilde{v}^\star(\tilde{S}_m^\star + 1) &\geq \tilde{v}^\phi(\tilde{S}_m^\star + 1) + b/\alpha - c_m,\\ &> \tilde{v}^\phi(\tilde{S}_m^\star + 1).\end{aligned}$$

This in contradiction with the optimality of policy $\tilde{\pi}^\star$. Hence, the assumption $\tilde{S}_m^\star < 0$ is false and we conclude that $\tilde{S}_m^\star \geq 0$.

Case 2: $\min(-c_{dua}, c_m) = -c_{dua}$

In this case, we have $\tilde{S}_a^\star \geq \tilde{S}_m^\star$. The proof is exactly the same as in case 1 by replacing $\tilde{S}_m^\star$ by $\tilde{S}_a^\star$ and c_m by $-c_{dua}$. We obtain that $\tilde{S}_a^\star \geq 0$.

In the end, $\max(\tilde{S}_a^\star, \tilde{S}_m^\star) \geq 0$.

Appendix D. Average cost

This section provides a closed-form expression for the average cost $C(q)$ in case iii. Similar results can be obtained in other cases.

To simplify notations, let $P_{n,q} = P(N_q = n)$. Closed-form expressions for $P_{n,q}$ come from (14) and (15). From (18), we have

$$\begin{aligned}C(q) &= hE[S_m(q) - N_q]^+ + bE[S_m - N_q]^- + c_m\mu P(N_q > 0) + \delta[-c_{dua}P(N_q > q) + c_e],\\ &= h\left(S_m(q)\sum_{n=0}^{S_m(q)} P_{n,q} - \sum_{n=0}^{S_m(q)} nP_{n,q}\right)\\ &\quad + b\left(\sum_{n=S_m(q)}^{\infty} nP_{n,q} - S_m(q)\sum_{n=S_m(q)}^{\infty} P_{n,q}\right)\\ &\quad + c_m\mu\left[1 - P_{0,q}\right] + \delta\left[-c_{dua}\rho^{-q}{\rho_m}^q P_{0,q}\sum_{n=q+1}^{\infty}\rho^n + c_e\right].\end{aligned}$$

If $S_m(q) \geq q$, then

$$\begin{aligned}C(q) &= P_{0,q}\left(\begin{array}{c} h\left[\ S_m(q)\sum_{n=0}^{q-1}{\rho_m}^n - \sum_{n=0}^{q-1} n{\rho_m}^n\ \right]\\ +h\rho^{-q}{\rho_m}^q\left[\ S_m(q)\sum_{n=q}^{S_m(q)}\rho^n - \sum_{n=q}^{S_m(q)} n\rho^n\ \right]\\ +b\rho^{-q}{\rho_m}^q\left[\ \sum_{n=S_m(q)}^{\infty} n\rho^n - S_m(q)\sum_{n=S_m(q)}^{\infty}\rho^n\ \right]\\ -c_m\mu - \delta c_{dua}\rho^{-q}{\rho_m}^q\sum_{n=q+1}^{\infty}\rho^n \end{array}\right)\\ &\quad + c_m\mu + \delta c_e.\end{aligned}$$

If $S_m(q) \leq 0$, then

$$\begin{array}{rcl} C(q) & = & P_{0,q} \left(\begin{array}{c} h\left[\ S_m(q)\sum_{n=0}^{S_m(q)} {\rho_m}^n - \sum_{n=0}^{S_m(q)} n{\rho_m}^n \ \right] \\ +b\left[\ \sum_{n=S_m(q)}^{q-1} n{\rho_m}^n - S_m(q)\sum_{n=S_m(q)}^{q-1} n{\rho_m}^n \ \right] \\ +b\rho^{-q}{\rho_m}^q\left[\ \sum_{n=q}^{\infty} n\rho^n - S_m(q)\sum_{n=q}^{\infty}\rho^n \ \right] \\ -c_m\mu - \delta c_{dua}\rho^{-q}{\rho_m}^q\sum_{n=q+1}^{\infty}\rho^n \end{array} \right) \\ & & +c_m\mu + \delta c_e. \end{array}$$

The sums in $C(q)$ can be computed with the following formulas

$$\sum_a^b \rho^x = \frac{\rho^a - \rho^{b+1}}{1-\rho},$$

$$\sum_a^b x\rho^x = \frac{\rho}{(1-\rho)^2}(b\rho^{b+1} - (b+1)\rho^b - (a-1)\rho^a + a\rho^{a-1}).$$

Appendix E. Extensions

In the following, the sequences of value functions v_k and w_k converge to the optimal value functions $v^\star$ and $w^\star$.

Start-up cost

- $\mathbf{x} = (x_1, x_2) \in \mathbb{Z} \times \{0,1\}$, $\mathbf{e}_1 = (1,0)$, $\mathbf{e}_2 = (0,1)$
- $\mathsf{T}_d f(\mathbf{x}) = \min_{n \in \{0,\ldots,x_1^+\}}\{f(\mathbf{x} - n\mathbf{e}_1) + nc_d\}$,
- $\mathsf{T}_a f(\mathbf{x}) = f(\mathbf{x} - \mathbf{e}_1)$,
- $\mathsf{T}_r f(\mathbf{x}) = \min\{f(\mathbf{x}) + c_e, f(\mathbf{x} + \mathbf{e}_1) + c_a\}$,
- $\mathsf{T}'_m f(\mathbf{x}) = \begin{cases} f(\mathbf{x}) & \text{if } x_2 = 0, \\ f(\mathbf{x} + \mathbf{e}_1) + c_m & \text{if } x_2 = 1, \end{cases}$
- $\mathsf{T}_K f(\mathbf{x}) = \begin{cases} \min\{f(\mathbf{x}), f(\mathbf{x} + \mathbf{e}_2) + K\} & \text{if } x_2 = 0, \\ \min\{f(\mathbf{x}), f(\mathbf{x} - \mathbf{e}_2) & \text{if } x_2 = 1, \end{cases}$
- $\tau = \alpha + \lambda + \mu + \delta$

MDP formulation:

$$\begin{aligned} v_k(\mathbf{x}) &= \mathsf{T}_d w_k(\mathbf{x}), \\ w_k(\mathbf{x}) &= \frac{1}{\tau}\left[\begin{array}{l} C(\mathbf{x}) + \lambda \mathsf{T}_K \mathsf{T}_a v_{k-1}(\mathbf{x}) \\ +\mu \mathsf{T}_K \mathsf{T}'_m v_{k-1}(\mathbf{x}) + \delta \mathsf{T}_K \mathsf{T}_r v_{k-1}(\mathbf{x}) \end{array}\right]. \end{aligned}$$

Markov modulated demand

- $\mathsf{T}_m f(\mathbf{x}) = \min\{f(\mathbf{x}), f(\mathbf{x}+\mathbf{e}_1) + c_m\}$,
- $\mathsf{T}_{switch} f(\mathbf{x}) = \begin{cases} \gamma_1 f(\mathbf{x}+\mathbf{e}_2) + \gamma_2 f(\mathbf{x}) & \text{if } x_2 = 0, \\ \gamma_1 f(\mathbf{x}) + \gamma_2 f(\mathbf{x}-\mathbf{e}_2) & \text{if } x_2 = 1, \end{cases}$
- $\mathsf{T}_{var} f(\mathbf{x}) = \begin{cases} \lambda_1 f(\mathbf{x}-\mathbf{e}_1) + \lambda_2 f(\mathbf{x}) & \text{if } x_2 = 0, \\ \lambda_1 f(\mathbf{x}) + \lambda_2 f(\mathbf{x}-\mathbf{e}_1) & \text{if } x_2 = 1, \end{cases}$
- $\tau = \alpha + \lambda_1 + \lambda_2 + \mu + \delta + \gamma_1 + \gamma_2$

MDP formulation:

$$\begin{aligned} v_k(\mathbf{x}) &= \mathsf{T}_d w_k(\mathbf{x}), \\ w_k(\mathbf{x}) &= \frac{1}{\tau}\left[\begin{array}{l} C(\mathbf{x}) + \mu \mathsf{T}_m v_{k-1}(\mathbf{x}) + \mathsf{T}_{var} v_{k-1}(\mathbf{x}) \\ +\delta \mathsf{T}_r v_{k-1}(\mathbf{x}) + \mathsf{T}_{switch} v_{k-1}(\mathbf{x}) \end{array}\right]. \end{aligned}$$

Non-zero remanufacturing lead time

- $\mathsf{T}_{dis} f(\mathbf{x}) = \min_{n_1,n_2 \in \{0,\ldots,x_1^+\}\times\{0,\ldots,x_2^+\}} \{f(\mathbf{x} - n_1\mathbf{e}_1 - n_2\mathbf{e}_1) + n_1 c_e + n_2 c_d\}$,
- $\mathsf{T}_{arr} f(\mathbf{x}) = f(\mathbf{x}-\mathbf{e}_2)$,
- $\mathsf{T}_{ret} f(\mathbf{x}) = \min\{f(\mathbf{x}) + c_e, f(\mathbf{x}+\mathbf{e}_1)\}$,
- $\mathsf{T}_{man} f(\mathbf{x}) = \min\{f(\mathbf{x}), f(\mathbf{x}+\mathbf{e}_2) + c_m\}$,
- $\mathsf{T}_{rem} f(\mathbf{x}) = \begin{cases} \min\{f(\mathbf{x}), f(\mathbf{x}+\mathbf{e}_2) + c_a\} & \text{if } x_1 > 0, \\ f(\mathbf{x}) & \text{otherwise.} \end{cases}$
- $\tau = \alpha + \lambda + \mu + \mu_r + \delta$

MDP formulation:

$$\begin{aligned} v_k(\mathbf{x}) &= T_{dis} w_k(\mathbf{x}) \\ w_k(\mathbf{x}) &= \frac{1}{\tau} \left[\begin{array}{l} C(\mathbf{x}) + \lambda T_{arr} v_{k-1}(\mathbf{x}) + \mu T_{man} v_{k-1}(\mathbf{x}) \\ + \mu_r T_{rem} v_{k-1}(\mathbf{x}) + \delta T_{ret} v_{k-1}(\mathbf{x}) \end{array} \right] \end{aligned}$$

References

Akçalı, E., Çetinkaya, S., 2011. Quantitative models for inventory and production planning in closed-loop supply chains. International Journal of Production Research 49 (8), 2373–2407.

Angelus, A., 2011. A multiechelon inventory problem with secondary market sales. Management Science 57 (12), 2145–2162, 00009.

Aras, N., Verter, V., Boyaci, T., 2006. Coordination and priority decisions in hybrid manufacturing/remanufacturing systems. Production and Operations Management 15 (4), 528–543.

Benjaafar, S., ElHafsi, M., Lee, C.-Y., Zhou, W., 2011. Technical note-optimal control of an assembly system with multiple stages and multiple demand classes. Operations Research 59 (2), 522–529.

Bradley, J. R., 2005. Optimal control of a dual service rate m/m/1 production-inventory model. European Journal of Operational Research 161 (3), 812–837.

Buzacott, J., Shanthikumar, J., 1993. Stochastic Models of Manufacturing Systems. Prentice Hall.

de Véricourt, F., Karaesmen, F., Dallery, Y., 2002. Optimal stock allocation for a capacitated supply system. Management Science 48 (11), 1486–1501.

DeCroix, G., 2006. Optimal policy for a multiechelon inventory system with remanufacturing. Operations Research 54, 532–543.

Fathi, M., Zandi, F., Jouini, O., 2015. Modeling the merging capacity for two streams of product returns in remanufacturing systems. Journal of Manufacturing Systems 37, 265–276.

Gayon, J.-P., Benjaafar, S., de Véricourt, F., 2009. Using imperfect demand information in production-inventory systems with multiple demand classes. Manufacturing and Service Operations Management 11 (1), 128–143.

Germs, R., Van Foreest, N. D., Kilic, O. A., 2016. Optimal policies for production-clearing systems under continuous-review. European Journal of Operational Research 255 (3), 747–757.

Ilgin, M., Gupta, S., 2010. Environmentally conscious manufacturing and product recovery (ECMPRO): a review of the state of the art. Journal of Environmental Management 91 (3), 563–591.

Inderfurth, K., 1997. Simple optimal replenishment and disposal policies for a product recovery system with leadtimes. OR Spektrum 19 (2), 111–122.

Inderfurth, K., 2005. Impact of uncertainties on recovery behavior in a remanufacturing environment: a numerical analysis. International Journal of Physical Distribution & Logistics Management 35 (5), 318–336.

Ketzenberg, M., 2009. The value of information in a capacitated closed loop supply chain. European Journal of Operational Research 198 (2), 491–503.

Kim, E., Saghafian, S., van Oyen, M. P., 2013. Joint control of production, remanufacturing, and disposal activities in a hybrid manufacturing-remanufacturing system. European Journal of Operational Research 231 (2), 337–348.

Koole, G., 1998. Structural results for the control of queueing systems using event-based dynamic programming. Queueing Systems 30 (3), 323–339.

Li, Y., Zhang, J., Chen, J., Cai, X., 2010. Optimal solution structure for multi-period production planning with returned products remanufacturing. Asia-Pacific Journal of Operational Research 27 (05), 629–648.

Morse, P., 1958. Queues, Inventories and Maintenance. John Wiley and Sons, New York.

Porteus, E. L., 2002. Foundations of stochastic inventory theory. Stanford University Press.

Puterman, M. L., 1994. Markov Decision Processes, Discrete stochastic, Dynamic programming. ed. John Wiley & Sons, Inc.

Simpson, V., 1978. Optimum solution structure for a repairable inventory problem. Operations Research 26 (2), 270–281.

van der Laan, E., Dekker, R., Salomon, M., Ridder, A., 1996. An (s, q) inventory model with remanufacturing and disposal. International Journal of Production Economics 46-47, 339–350.

van der Laan, E., Salomon, M., 1997. Production planning and inventory control with remanufacturing and disposal. European Journal of Operational Research 102 (2), 264–278.

Veatch, M. H., Wein, L. M., 1996. Scheduling a Make-To-Stock queue: Index policies and hedging points. Operations Research 44 (4), 634–647.

Vercraene, S., Gayon, J.-P., 2013. Optimal control for a production/inventory system with products returns. International Journal of Production Economics 142, 302–310.

Vercraene, S., Gayon, J.-P., Flapper, S. D. P., 2014. Coordination of manufacturing, remanufacturing and returns acceptance in hybrid manufacturing/remanufacturing systems. International Journal of Production Economics 148, 62–70.

Weber, R. R., Stidham, S., 1987. Optimal control of service rates in networks of queues. Advances in Applied Probability 19 (1), 202–218.

Wein, L., 1992. Dynamic scheduling of a multiclass make-to-stock queue. Operations Research 40 (40), 724–735.

Zikopoulos, C., Tagaras, G., 2007. Impact of uncertainty in the quality of returns on the profitability of a single-period refurbishing operation. European Journal of Operational Research 182 (1), 205–225.

Zikopoulos, C., Tagaras, G., 2008. On the attractiveness of sorting before disassembly in remanufacturing. IIE Transactions 40 (3), 313–323.

Zipkin, P., 2000. Foundations of inventory management. McGraw-Hill.