Computing near-optimal Value-at-Risk portfolios using Integer Programming techniques

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Abstract

Value-at-Risk (VaR) is one of the main regulatory tools used for risk management purposes. However, it is difficult to compute optimal VaR portfolios; that is, an optimal risk-reward portfolio allocation using VaR as the risk measure. This is due to VaR being non-convex and of combinatorial nature. In particular, it is well-known that the VaR portfolio problem can be formulated as a mixed-integer linear program (MILP) that is difficult to solve with current MILP solvers for medium to large-scale instances of the problem. Here, we present an algorithm to compute near-optimal VaR portfolios that takes advantage of this MILP formulation and provides a guarantee of the solution's near-optimality. As a byproduct, we obtain an algorithm to compute tight lower bounds on the VaR portfolio problem that outperform related algorithms proposed in the literature for this purpose. The near-optimality guarantee provided by the proposed algorithm is obtained thanks to the relation between minimum risk portfolios satisfying a reward benchmark and the corresponding maximum reward portfolios satisfying a risk benchmark. These alternate formulations of the portfolio allocation problem have been frequently studied in the case of convex risk measures and concave reward functions. Here, this relationship is considered for general risk measures and reward functions. To illustrate the efficiency of the presented algorithm, numerical results are presented using historical asset returns from the US financial market.

Keywords: Value-at-Risk; Portfolio Allocation; Integer Programming Relaxations; Minimizing Risk vs Maximizing Reward equivalence

1 Introduction

In the context of portfolio risk and asset liability management, Value-at-Risk (VaR) measures the exposure of a portfolio to high losses. VaR is prominent in current regulatory frameworks

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for banks (see, e.g., the Basel II and Basel III Accords), as well as for insurance companies (see, e.g., the Solvency II Directive). Thus, VaR is an important and popular tool for risk management in the modern financial and risk management literature (see, e.g., Jorion, 2001; Wozabal, 2012). Accordingly, the development of risk management methods based on VaR has been the focus of extensive research work (see, e.g., Alexander et al., 2006; Bazak and Shapiro, 2001; Benati and Rizzi, 2007; Darbha, 2001; El Ghaoui et al., 2003; Gaivoronski and Pflug, 2004; Glasserman et al., 2000; Gneiting, 2011a; Kaplanski and Koll, 2002; Wozabal et al., 2010).

Although VaR is widely used to measure the risk of a given portfolio of assets, it is not commonly used as a risk measure in the context of computing optimal VaR portfolios; that is, an optimal risk-reward portfolio allocation using VaR as the risk measure. Instead, other risk measures such as the portfolio return's Variance (cf., Markowitz, 1952), and the portfolio loss' Conditional Value-at-Risk (CVaR) (cf., Rockafellar and Uryasev, 2000) are more commonly used. This is because, in contrast with the above mentioned risk measures, VaR is non-convex and of combinatorial nature (cf., Gaivoronski and Pflug, 2004). As a result, the VaR portfolio problem is inherently difficult solve (see, e.g., Natarajan et al., 2009).

VaR does not (in general) satisfies the commonly accepted axioms of *coherent* risk measures (cf., Artzner et al., 1999; Rockafellar et al., 2004)). On the other hand, VaR satisfies the so-called *natural risk statistic* axioms (Heyde et al., 2006). More importantly, it has been recently shown in Gneiting (2011b) that VaR is an *elicitable* risk measure (cf., Bellini and Bignozzi, 2015). Loosely speaking, elicitability is related to how accurately a risk measure can be forecasted. More precisely, as discussed in Bellini and Bignozzi (2015), it has been shown that while CVaR is generally considered a better risk measure from a mathematical point of view, it requires a higher number of samples for an accurate estimation (see Daníelsson, 2011) and it is less robust than VaR (see Cont et al., 2010).

Because of the computational difficulties of optimally solving general instances of the VaR portfolio problem, different heuristics have been proposed in the literature. In particular, consider the work of Çetinkaya and Thiele (2015); Gaivoronski and Pflug (2004); Larsen et al. (2002); Verma and Coleman (2005). Also, given that the VaR portfolio problem belongs to the general class of *chance constraint* optimization problems (cf., Campi and Calafiore, 2005), other approximation approaches that can be used are based on relaxations of the VaR quantile constraint for which probabilistic guarantees can be obtained. In particular, consider the work of Campi and Calafiore (2005); De Farias and Van Roy (2004); Erdogan and Iyengar (2006).

When the standard sampling approach (cf., Rockafellar and Uryasev, 2000) is used to model the uncertain asset returns, it is well known (see, e.g., Benati and Rizzi, 2007; Feng et al., 2015) that the (resulting) VaR portfolio problem can be solved to optimality by formulating the problem as a mixed-integer linear program (MILP). However, this formulation is difficult to solve with current MILP solvers for instances with medium to large number of assets in the portfolio (see, e.g., Benati and Rizzi, 2007). Recently, improvements in the solution of this MILP formulation have been obtained in Feng et al. (2015), by tailoring special branch-and-cut techniques to solve the problem, as well as improving the big-M values used on its MILP formulation. Although these improvements allow for the solution of VaR portfolio problem instances where thousands of scenarios are used to model the uncertain

asset returns, the number of assets that are considered in the portfolio is of the order of 25 assets, similar to Benati and Rizzi (2007). Moreover, their solution approach is useful only when the common total wealth constraint is not considered (Feng et al., 2015, Sec. 5).

We present an algorithm to compute near-optimal VaR portfolios that takes advantage of the VaR portfolio problem MILP formulation and provides a guarantee of the near-optimality of the solution. The algorithm makes a straight-forward use of current state-of-the-art MILP solvers (e.g., CPLEX and Gurobi). Furthermore, this algorithm can be used to obtain guaranteed near-optimal solutions for instances of the VaR problem with up to a hundred assets and thousands of samples to model the uncertain asset returns. In particular, this allows the use of VaR for strategic asset allocation instead of only tactical asset allocation (e.g., by industry sectors). As a byproduct, we obtain an algorithm to compute tight lower bounds on the VaR portfolio problem that outperforms the algorithms for this purpose recently proposed by Larsen et al. (2002). These algorithms aim at approximating the optimal solution of the VaR portfolio problem by iteratively constructing appropriate instances of the Conditional Value-at-Risk portfolio problem.

The main contribution of the article in relation to the current VaR portfolio allocation literature is to provide a performance-guaranteed heuristic solution approach for the problem which can be used to address the solution of medium to large-scale instances of the problem. The near-optimal guarantee provided by the proposed algorithm is based on the relation between two alternate formulations of the portfolio problem; that is, between minimum risk portfolios satisfying a reward benchmark and the corresponding maximum reward portfolios satisfying a risk benchmark. It is well-known that these alternate formulations of the portfolio problem are equivalent for the mean-variance portfolio model of Markowitz (1952). Recently, Krokhmal et al. (2002, Thm. 3) have shown that this equivalence holds for general convex risk measures and concave reward functions. We also study the relationship between the alternate risk-reward and reward-risk formulations of the portfolio problem for more general risk measures and reward functions. Besides providing the foundation for the proposed algorithm to find near-optimal solutions for the VaR portfolio problem, these results extend the characterization provided by Krokhmal et al. (2002, Thm. 3), and rectify some incorrect statements made in Lin (2009) about alternate formulations of the VaR portfolio problem.

The rest of the article is organized as follows. In Section 2, the MILP formulation of the VaR portfolio problem is presented. In Section 3, the relationship between the alternate formulations of the portfolio problem is studied for general risk measures and reward functions. These results are used in Section 4 to develop the proposed algorithm to find near-optimal solutions for the VaR portfolio problem. In Section 5, we illustrate the efficiency of the proposed algorithm by presenting numerical results on instances of the VaR portfolio problem constructed using historical asset returns from the US financial market.

2 The MILP formulation of the VaR portfolio problem

The Value-at-Risk (VaR) of a portfolio measures its exposure to high losses. Specifically, for a given $\alpha \in (0,1)$ (typically $0.01 \le \alpha \le 0.05$), the VaR of a portfolio is defined as the $1-\alpha$ quantile of the portfolio's losses (cf., Rockafellar and Uryasev, 2000); or equivalently as the α quantile of the portfolio's returns. Here, we follow the latter definition (as in Gaivoronski

and Pflug, 2004).

We begin by formally stating the VaR portfolio (allocation) problem; which aims at minimizing the exposure of the portfolio to high losses while maintaining a minimum expected level of profit. Consider n risky assets that can be chosen by an investor in the financial market. Let $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n)^{\mathsf{T}}$ be a random variable in \mathbb{R}^n representing the uncertain returns of the n risky assets from the current time t = 0 to a fixed future time t = T. Let $x = (x_1, \dots, x_n)^{\mathsf{T}} \in \mathbb{R}^n_+$ denote a portfolio on these assets; that is, the percentage of the available funds to be allocated in each of the n risky assets. Following Gaivoronski and Pflug (2004), given $\alpha \in (0, 1)$, the α -level VaR of the portfolio is defined as follows:

$$VaR_{\alpha}(x^{\mathsf{T}}\boldsymbol{\xi}) = -\mathbb{Q}_{\alpha}(x^{\mathsf{T}}\boldsymbol{\xi}),\tag{1}$$

where $\mathbb{Q}_{\alpha}(x^{\mathsf{T}}\boldsymbol{\xi})$ is the α quantile of the portfolio's return distribution; that is, $\mathbb{Q}_{\alpha}(x^{\mathsf{T}}\boldsymbol{\xi}) = \inf\{v : \mathbb{P}r(x^{\mathsf{T}}\boldsymbol{\xi} \leq v) > \alpha\}$. Also, the expected portfolio return from t = 0 to t = T is given by $\mathbb{E}(x^{\mathsf{T}}\boldsymbol{\xi})$. Above, $\mathbb{P}r(\cdot)$ and $\mathbb{E}(\cdot)$ respectively indicate probability and expectation.

A (single-period) VaR portfolio problem aims at finding the portfolio $x \in \mathbb{R}^n_+$ to be constructed at t = 0, in order to minimize the portfolio's VaR_{α} , subject to the portfolio having a given minimum expected return μ_0 , and possibly satisfying some linear diversification constraints. Formally, the VaR portfolio problem is:

min
$$-\operatorname{VaR}_{\alpha}(x^{\mathsf{T}}\boldsymbol{\xi})$$

s.t. $\mathbb{E}(x^{\mathsf{T}}\boldsymbol{\xi}) \ge \mu_0$ (2)
 $x^{\mathsf{T}}e = 1$
 $x \in \mathcal{X} \subseteq \mathbb{R}^n_+,$

where $e \in \mathbb{R}^n$ is the vector of all-ones, $\mu_0 \in \mathbb{R}$ is the given target minimum expected portfolio return, and $\mathcal{X} \subseteq \mathbb{R}^n$ is a given set defined by linear constraints; which are typically used to enforce certain diversification constraints on the portfolio $x \in \mathbb{R}^n_+$. For the moment, it is assumed that no short positions are allowed in the portfolio; which is the most common situation in practice (cf., Michaud, 1998).

As discussed in Gaivoronski and Pflug (2004), there are two main approaches to solve (2): the parametric approach, in which it is assumed that the asset returns are governed by a known distribution (see, e.g., Lobo (2000), where asset returns are assumed to be normally distributed); and the sampling approach, which uses a finite number of samples $\xi^1, \ldots, \xi^m \in \mathbb{R}^n$ of the asset returns (see, e.g., Gaivoronski and Pflug (2004)), that are typically obtained from historical data, simulations, or a combination of both. The latter approach is used in the well-known Conditional Value-at-Risk (CVaR) portfolio model (cf., Rockafellar and Uryasev, 2000). Here, we adopt the sampling approach, which following Gaivoronski and Pflug (2004, Section 2.1) leads to the VaR portfolio problem (2) being written as:

$$z_{\text{VaR}} := \min -\nu$$
s.t.
$$\nu = \min^{\lfloor \alpha m \rfloor + 1} \{ x^{\intercal} \xi^{1}, \dots, x^{\intercal} \xi^{m} \}$$

$$x^{\intercal} \mu \geq \mu_{0}$$

$$x^{\intercal} e = 1$$

$$x \in \mathcal{X} \subseteq \mathbb{R}^{n}_{+}, \nu \in \mathbb{R},$$

$$(3)$$

where ν represents the $\operatorname{VaR}_{\alpha}(x^{\intercal}\xi)$, the vector of mean return estimates is, for simplicity, considered to be given by $\mu := (1/m) \sum_{j=1}^{m} \xi^{j}$. However, our results are independent of this choice, and a variety of other estimation methods can be used (see, e.g., Black and Litterman, 2001; Meucci, 2009). Also, for $k \in \{1, \ldots, m\}$, and $u^{j} \in \mathbb{R}$, $j = 1, \ldots, m$, the k-th smallest element in $\{u^{1}, \ldots, u^{m}\}$ is denoted by $\min^{k}\{u^{1}, \ldots, u^{m}\}$ (i.e., $\min^{k}\{u^{1}, \ldots, u^{m}\}$ is the k-th order statistic $u^{(k)}$ in $\{u^{1}, \ldots, u^{m}\}$).

Problem (3) is equivalent (see, e.g., Benati and Rizzi, 2007; Feng et al., 2015) to the following mixed-integer linear program (MILP):

$$z_{\mathrm{VaR}} = \max \ \nu$$

s.t.
$$\sum_{j=1}^{m} y_{j} = \lfloor \alpha m \rfloor$$

$$My_{j} \geq \nu - x^{\mathsf{T}} \xi^{j}, \qquad j = 1, \dots, m$$

$$x^{\mathsf{T}} \mu \geq \mu_{0}$$

$$x^{\mathsf{T}} e = 1$$

$$x \in \mathcal{X} \subseteq \mathbb{R}^{n}_{+}, \nu \in \mathbb{R}$$

$$y_{j} \in \{0, 1\}, \qquad j = 1, \dots, m,$$

$$(4)$$

where M is a big enough constant (i.e., $M > 2 \max\{|\xi_i^j| : i \in \{1, ..., n\}, j \in \{1, ..., m\}\})$, and as in (3), ν represents the VaR of the portfolio $x \in \mathbb{R}^n_+$. The extra binary variable y_j denotes whether ν is to the right $(y_j = 1)$ or to the left $(y_j = 0)$ of the sample portfolio return $x^{\mathsf{T}}\xi^j$, for j = 1, ..., m.

In the literature, it is common to consider two alternate formulations of the portfolio allocation problem. That is, besides the portfolio allocation formulation in which one seeks to obtain the minimum risk portfolio satisfying a reward benchmark (as in eq. (2) above), the alternate formulation in which one seeks to obtain the maximum reward portfolio satisfying a risk benchmark is commonly considered. It is well-known that these alternate formulations of the portfolio problem are equivalent for the classical mean-variance portfolio model of Markowitz (1952) (see, e.g., Krokhmal et al., 2002). Due to the non-convexity of the VaR risk measure, it is not surprising that this equivalence does not hold in general for the VaR portfolio problem considered here. However, the relationship between these two alternate formulations of the VaR portfolio problem is fundamental to develop the algorithm presented here to address the solution of this problem. Below, we formally present the alternate maximum reward portfolio satisfying a minimum VaR risk benchmark $\tilde{\nu} \in \mathbb{R}$.

$$\max \quad \mathbb{E}(x^{\mathsf{T}}\boldsymbol{\xi})$$
s.t.
$$-\operatorname{VaR}_{\alpha}(x^{\mathsf{T}}\boldsymbol{\xi}) \leq \tilde{\nu} .$$

$$x^{\mathsf{T}}e = 1$$

$$x \in \mathcal{X} \subseteq \mathbb{R}^{n}_{+}.$$

$$(5)$$

Using the sampling approach, and similar to (2), problem (5) can be reformulated as the

following MILP:

$$z_{\text{VaR}}^* = \max \quad x^{\mathsf{T}} \mu$$
s.t.
$$\sum_{j \in I} y_j \leq \lfloor \alpha m \rfloor$$

$$M y_j \geq \tilde{\nu} - x^{\mathsf{T}} \xi^j, \qquad j = 1, \dots, m$$

$$x^{\mathsf{T}} e = 1$$

$$x \in \mathcal{X} \subseteq \mathbb{R}_+^n, v \in \mathbb{R}$$

$$y_j \in \{0, 1\}, \qquad j = 1, \dots, m.$$

$$(6)$$

The relationship between the two alternative formulations of the VaR portfolio problems (4) and (6) will be analyzed in the next section. Moreover, in Section 4, this relationship is exploited to obtain approximate solutions of the VaR portfolio problem (4) with a near-optimallity guarantee.

3 On alternate portfolio allocation problem formulations

In portfolio allocation problems one seeks to find the portfolio with minimum risk subject to a constraint on the minimum level of the portfolio's reward. Alternatively, the portfolio allocation problem is also formulated as the problem of obtaining the portfolio with maximum reward subject to a constraint on the maximum level of the portfolio's risk. Similar to Krokhmal et al. (2002), these two problems can be formally and respectively stated as follows:

$$\beta(a) = \min_{\text{s.t.}} \phi(x)$$
s.t. $R(x) \ge a$

$$x \in \mathcal{X}.$$

$$(7)$$

$$\alpha(b) = \max_{\text{s.t.}} R(x)$$
s.t. $\phi(x) \le b$
 $x \in \mathcal{X}$, (8)

where $x \in \mathbb{R}^n$ represents the portfolio of assets, $\phi(x) : \mathbb{R}^n \to \mathbb{R}$ measures the portfolio's risk, $R(x) : \mathbb{R}^n \to \mathbb{R}$ measures the portfolio's reward, and $\mathcal{X} \in \mathbb{R}^n$ represents the set of admissible portfolios (e.g., \mathcal{X} could contain long only positions constraints or benchmark constraints). Also, $a, b \in \mathbb{R}$, respectively represent the minimum required reward, and the maximum allowed risk. Throughout, we assume that the set $\mathcal{X} \in \mathbb{R}^n$ is compact (any position on an asset is typically constrained to be within certain lower and upper bounds), and use the usual convention $\beta(a) = +\infty$ (resp. $\alpha(b) = -\infty$) if problem (7) (resp. (8)) is infeasible.

For the classical mean-variance portfolio allocation model introduced by Markowitz (1952), it is well known that there is a one-to-one correspondence between the optimal portfolios obtained from these two models (i.e., (7) and (8)). In more generality, it has been recently shown by Krokhmal et al. (2002, Thm. 3) that this type of one-to-one relationship will hold in more generality whenever the risk measure $\phi(x)$ is convex and the reward measure R(x) is concave.

Not surprisingly, when the risk measure $\phi(x)$ is defined by the portfolio's VaR, there is not a one-to-one correspondence between the portfolio allocation models (7) and (8). However, as it will be illustrated therein, when using VaR as a risk measure, relaxations of both these problems are useful in addressing the solution of (7). Given this, and the fact that is has been erroneously reported in Lin (2009) that there is a one-to-one correspondence between the portfolio allocation problems (4) and (6), it is worth to study below the relationship between these two problems in a general setting when the risk measure $\phi(x)$ is not necessarily convex and/or the measure of reward $\mathbb{R}(x)$ is not necessarily concave; extending Krokhmal et al. (2002, Thm. 3) to provide both sufficient and necessary conditions for both (7) and (8) to have a one-to-one correspondence. These results are formally stated in the remainder of this section.

We define (recall that by assumption $\mathcal{X} \subseteq \mathbb{R}^n$ is compact) the minimum risk and maximum reward that any portfolio in the admissible set $\mathcal{X} \subseteq \mathbb{R}^n$ can have as:

$$\underline{b} = \min_{\text{s.t.}} \phi(x) \quad \text{and} \quad \overline{a} = \max_{\text{s.t.}} R(x) \\
\text{s.t.} \quad x \in \mathcal{X} \quad \text{s.t.} \quad x \in \mathcal{X}.$$
(9)

Theorem 1 below, provides sufficient and necessary conditions for a one-to-one correspondence between the portfolio allocation problems (7) and (8).

Theorem 1. Let $I \subseteq [\alpha(\underline{b}), \overline{a}]$ be an interval. The relation $a = \alpha(\beta(a))$ holds for any $a \in I$ if and only if $\beta(a)$ is strictly increasing for all $a \in I$.

Proof. First notice that $\beta(a)$ is non-decreasing as a function of a. Now, assume that there exists $a_1 \in I$ such that $a_1 < \alpha(\beta(a_1)) =: a_2$. Then $\beta(a_1) \le \beta(a_2)$ as $\beta(\cdot)$ is non-decreasing, and $\beta(a_2) = \beta(\alpha(\beta(a_1))) \le \beta(a_1)$ as $\beta(\alpha(b)) \le b$ for all b. Therefore, $\beta(a_1) = \beta(a_2)$. To prove the other direction, assume $\beta(a)$ is not strictly increasing in I. Then, there exist $a_1, a_2 \in I$ with $a_1 < a_2$ such that $\beta(a_1) = \beta(a_2)$. Then, using that $\alpha(\beta(a)) \ge a$ for all a, we obtain $\alpha(\beta(a_1)) = \alpha(\beta(a_2)) \ge a_2 > a_1$. (See Figure 1 for an illustration of the proof.)

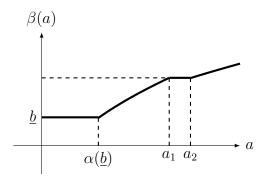


Figure 1: Illustration of Theorem 1.

As mentioned before, it is shown in (Krokhmal et al., 2002, Thm. 3) that convexity in the risk measure, and concavity in the reward, provides sufficient conditions for Theorem 1 to hold. This result can be obtained as a corollary of Theorem 1.

Corollary 1. Let $\phi(x) : \mathbb{R}^n \to \mathbb{R}$ be convex and $R(x) : \mathbb{R}^n \to \mathbb{R}$ be concave. Assume $\{a \in [\alpha(\underline{b}), \bar{a}] : \beta(a) > \underline{b}\}$ is non-empty and let $\underline{a} = \inf\{a \in [\alpha(\underline{b}), \bar{a}] : \beta(a) > \underline{b}\}$ Then $a = \alpha(\beta(a))$ for any $a \in [\underline{a}, \bar{a}]$.

Proof. From Theorem 1 is enough to show that $\beta(\cdot)$ is strictly increasing on $(\underline{a}, \overline{a}]$. For sake of contradiction, let $\underline{a} < a_1 < a_2 \le \overline{a}$ be such that $\beta(a_1) = \beta(a_2)$. Let $x_i := \operatorname{argmin}\{\phi(x) : R(x) \ge a_i, x \in \mathcal{X}\}$ for i = 1, 2. Thus $\phi(x_1) = \phi(x_2)$. Let \hat{x} be the optimal min-risk portfolio, i.e. $\phi(\hat{x}) = \underline{b}$ and $R(\hat{x}) = \alpha(\underline{b})$. Let $\epsilon := \frac{a_2 - a_1}{2a_2 - a_1 - \alpha(\underline{b})}$. Then $0 < \epsilon < 1$. Let $x' = \epsilon \hat{x} + \epsilon x_1 + (1 - 2\epsilon)x_2$. From the convexity of ϕ , we get that

$$\phi(x') = \phi(\epsilon \hat{x} + \epsilon x_1 + (1 - 2\epsilon)x_2) \le \epsilon \phi(\hat{x}) + \epsilon \phi(x_1) + (1 - 2\epsilon)\phi(x_2) = \epsilon \underline{b} + (1 - \epsilon)\phi(x_1) < \phi(x_1).$$

Also, by the concavity of R(x), we get that

$$R(x') \ge \epsilon R(\hat{x}) + \epsilon R(x_1) + (1 - 2\epsilon)R(x_2) \ge \epsilon \alpha(\underline{b}) + \epsilon a_1 + (1 - 2\epsilon)a_2 = a_1.$$

Thus, $x_1 \neq \operatorname{argmin}\{\phi(x) : R(x) \geq a_1, x \in \mathcal{X}\}\$, a contradiction.

Krokhmal et al. (2002, Thm. 3) assume a regularity condition for each value of the pair (a, b). In contrast, in Corollary 1, the ranges of a and b for which the one-to-one correspondence between the alternative formulations holds is precisely characterized.

Although sufficient, the convexity condition in Corollary 1 is not necessary to have the one-to-one correspondence between the portfolio allocation problems (7) and (8). To illustrate this, we consider the following simple example in which the risk measure $\phi(x)$ is related to the well-known Huber's function (see, e.g., Huber and Ronchetti, 2009) that commonly appears in robust statistics.

Example 1. Let $\kappa > 1$ be given. Let the functions $\phi : \mathbb{R} \to \mathbb{R}$ and $R : \mathbb{R} \to \mathbb{R}$ be given by

$$\phi(x) = \begin{cases} x^2 & \text{if } |x| \le \kappa \\ x + \kappa(\kappa - 1) & \text{if } |x| \ge \kappa \end{cases},$$

and R(x) = x. Also, let the set $\mathcal{X} = [-2\kappa, 2\kappa]$. The function $\phi(x)$ is not convex, as $2\phi(\kappa) = 2\kappa^2 > (\kappa - 1)^2 + \kappa + 1 + \kappa(\kappa - 1) = \phi(k - 1) + \phi(\kappa + 1)$ (see Figure 2 (left)). Thus the conditions of Corollary 1 are not satisfied. However, it is easy to see that the function $\beta(a)$ is strictly increasing in the domain $a \geq \alpha(\underline{b}) = 0$ (see Figure 2 (right)). Note that by changing the domain $\mathcal{X} = \mathbb{R}_+$ and $R(x) = x^2$ one has a similar example where $\beta(a)$ is strictly increasing but now neither $\phi(x)$ is convex nor R(x) is concave.

As mentioned earlier, when the risk measure $\phi(x)$ is defined by the portfolio's return VaR, there is in general not a one-to-one correspondence between the portfolio allocation problems (7) and (8). This is formally stated in the next remark, which corrects the erroneous characterization between problems (7) and (8) given in Lin (2009).

Remark 1. When the risk measure $\phi(x)$ in (7) is defined by the portfolio's return Valueat-Risk (VaR) $\beta(a)$ is not in general strictly increasing (in the domain $a \geq \alpha(\underline{b})$). This is illustrated with the numerical example below.

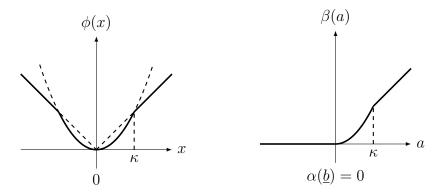


Figure 2: Illustration of Example 1. Function $\phi(x)$ (left), and corresponding $\beta(a)$ (right).

Example 2. Consider the instance of problem (7) in which $x \in \mathbb{R}^2$ represents the percentage of money invested in the two assets Microsoft (MSFT) and 3M (MMM). Let $\mathcal{X} = \{x \in \mathbb{R}^2_+ : x_1 + x_2 = 1\}$. Also, let $\phi(x)$ and R(x) respectively be the estimates of the portfolio's return $VaR_{5\%}$ and expected portfolio return based on a sample monthly returns of (MSFT) and (MMM), from April 1986 to December 2006 (source Wharton Research Data Services (WRDS)). After computing $\beta(a)$ in (7) one obtains Figure 3.

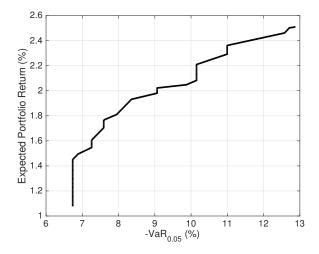


Figure 3: Instance of $\beta(a)$ (cf., (7)) not being strictly increasing when the portfolio's risk measure is the VaR of the portfolio returns.

Note that the areas of Figure 3 in which the risk remains constant while the expected portfolio return increases show that the VaR is not strictly increasing as a function of the expected portfolio return.

We finish this section by showing that one can take advantage of the alternative formulations (7) and (8) to obtain a measure of the closeness to optimality of a feasible solution of (7) when an appropriate bound on the optimal value of (8) can be obtained.

Lemma 1. Let $a \leq \bar{a}$. If $\alpha(b) < a$, then $\beta(a) \geq b$.

Proof. Notice that if $b < \underline{b}$ then by definition $\beta(a) \ge \underline{b} > b$. Thus we assume $b \ge \underline{b}$. For the sake of contradiction assume $\beta(a) < b$. Then there exists $x \in \mathcal{X}$ such that $R(x) \ge a$ and $\phi(x) < b$. Thus x is a feasible solution for (8), which implies $\alpha(b) \ge a$, a contradiction. \square

Proposition 1. Given $a \leq \bar{a}$. Let $\tilde{x} \in \mathbb{R}^n$ be a feasible solution of (7) and $\delta \geq 0$ be a given tolerance. If $\alpha(\phi(\tilde{x}) - \delta) < a$. Then $\phi(\tilde{x}) - \delta \leq \beta(a) \leq \phi(\tilde{x})$.

In what follows, we use Proposition 1 to provide an algorithm to address the solution of the VaR portfolio problem. As mentioned earlier, in this case, solving the associated minimum risk portfolio problem (7) or the maximum return portfolio problem (8) to optimality is inherently difficult.

4 The algorithm

In this section, we provide an algorithm to obtain approximate solutions for the MILP formulation of the VaR portfolio problem (4). First, the goal of the algorithm is to find a near-optimal feasible solution for (4) (cf., Section 4.1). Next, the goal is to provide a near-optimality guarantee for this feasible solution (cf., Section 4.2).

4.1 Lower bound for optimal VaR

Let us denote $[m] := \{1, \dots, m\}$. Now, given $J \subseteq [m]$, let $J^c := [m] \setminus J$, and consider the following problem:

$$\underline{z}_I := \max \ \nu$$

s.t.
$$\sum_{j \in J} y_j = \lfloor \alpha m \rfloor$$

$$My_j \ge \nu - x^{\mathsf{T}} \xi^j, \qquad j \in J$$

$$0 \ge \nu - x^{\mathsf{T}} \xi^j, \qquad j \in J^c$$

$$x^{\mathsf{T}} \mu \ge \mu_0$$

$$x^{\mathsf{T}} e = 1$$

$$x \in \mathcal{X} \subseteq \mathbb{R}^n_+, v \in \mathbb{R}$$

$$y_j \in \{0, 1\}, \qquad j \in J.$$

$$(10)$$

Note that (10) is the optimization problem obtained from (4) by setting $y_j = 0$ for all $j \in J^c$. Hence $\underline{z}_J \leq z_{\text{VaR}}$ for all $J \subseteq [m]$. In Algorithm A below, the formulation (10), together with an appropriate update of the set J, is used iteratively to obtain near-optimal feasible solutions to (4). Specifically, after setting an initial set $J = J_0 \subset [m]$ problem (10) is solved. Let $y^J \in \{0,1\}^{|J|}$ be the optimal value of the binary variables of (10). These values are used to construct the linear program below obtained by fixing the binary variables $y \in \{0,1\}^m$ in (4) such that $y_J = y^J$, and $y_j = 0$, for all $j \in J^c$.

$$\max \nu$$

s.t.
$$My_j^J \ge \nu - x^{\mathsf{T}}\xi^j$$
, $j = 1, \dots, m$
 $x^{\mathsf{T}}\mu \ge \mu_0$
 $x^{\mathsf{T}}e = 1$
 $x \in \mathcal{X} \subseteq \mathbb{R}^n, \nu \in \mathbb{R}$. (11)

After solving (11), the shadow prices associated with its big-M constraints (the first set of constraints in (11)) are used to update the set $J \subseteq [m]$. That is, the set J is augmented by the samples' indices whose corresponding big-M constraints in (11) have a positive shadow price. This type of update is similar to the one used when solving MILPs using branch and price techniques (see, e.g., Mehrotra and Trick, 2007). As described in Algorithm A, this procedure is applied iteratively until no further improvement in the lower bound of (4) can be obtained. The VaR of the portfolio obtained at the end of the algorithm serves as a lower bound for the optimal VaR portfolio problem.

Algorithm A Lower bound of optimal VaR.

```
1: procedure Lower bound(\alpha, \mu_0, J_0 \subseteq [m], |J_0| \ge |\alpha m|)
 2:
           J \leftarrow J_0
           J^{\text{old}} \leftarrow [m]
 3:
           while J^{\text{old}} \cap J \neq J^{\text{old}} do
 4:
                y^J \leftarrow \arg_y(\underline{P}_J)
 5:
                d \leftarrow shadow prices of the big-M constraints in (10)
 6:
 7:
                J \leftarrow \{i \in J : y_i^J = 1\} \cup \{i \in J^c : d_i > 0\}
 8:
           end while
 9:
                                                                                                              ▶ feasible portfolio for (4)
10:
           return \tilde{x} \leftarrow \arg_x (10)
                                                                                                                    \triangleright lower bound for (4)
11:
           return \tilde{\nu} \leftarrow \arg_v(10)
12:
           return \tilde{y} \leftarrow \arg_{y}(10), I_0 \leftarrow \{i \in [m] : \tilde{y}_i = 1\}
                                                                                     ▶ to initialize Algorithm B in Section 4.2
13: end procedure
```

As it will be shown in Section 5, Algorithm A provides a tighter lower bound $\tilde{\nu} = \underline{z}_J$, for the VaR portfolio problem (4) than those obtained using the CVaR-based algorithms proposed by Larsen et al. (2002) in a comparable running time. More importantly, Algorithm A provides a feasible solution $\tilde{x}, \tilde{\nu}$, for the VaR portfolio problem (4) whose near-optimality can be guaranteed using Algorithm B, which is presented in the next section.

4.2 Upper bound for optimal return

In this section, the aim is to obtain a measure of the closeness to optimality of the feasible solution \tilde{x}, \tilde{v} , for the VaR portfolio problem obtained by Algorithm A. For this purpose, we first apply Proposition 1 to the alternative formulations of the VaR portfolio problem (4) and (6). Specifically, let $\delta > 0$ be a specified tolerance, and $\tilde{x} \in \mathbb{R}^n_+$ be a feasible portfolio for (4), with associated VaR $\tilde{\nu}$; that is, $\tilde{\nu} = \min^{\lfloor \alpha m \rfloor + 1} \{\tilde{x}^{\mathsf{T}}\xi^1, \dots, \tilde{x}^{\mathsf{T}}\xi^m\}$. Then, from Proposition 1, it follows that if the optimal value of (6) satisfies

$$z_{\text{VaR}}^* < \mu_0 \Rightarrow \tilde{\nu} - \delta \le z_{\text{VaR}} \le \tilde{\nu}.$$
 (12)

That is, the near-optimality of the feasible portfolio $\tilde{x} \in \mathbb{R}^n_+$ to the optimal portfolio corresponding to the VaR portfolio problem (4), can be measured in terms of $\delta \in \mathbb{R}_{++}$.

Clearly, directly solving (6) to check whether condition $z_{\text{VaR}}^* < \mu_0$ in (12) holds for a feasible portfolio $\tilde{x} \in \mathbb{R}_+^n$ of (4) is as computationally inefficient as directly solving (4). Therefore, we present an appropriate upper bound for the alternative formulation of the

VaR portfolio problem (6) that allows to efficiently guarantee the near-optimality of the feasible solutions of the VaR problem obtained after using Algorithm A. Specifically, given $I \subseteq [m]$ with $|I| \ge \lfloor \alpha m \rfloor$ and $\tilde{\nu}$, a lower bound (4) (i.e, $\tilde{\nu} \le z_{\text{VaR}}$), consider the problem

$$\overline{\mu}_{I} := \max \quad x^{\mathsf{T}} \mu$$
s.t.
$$\sum_{j \in I} y_{j} \leq \lfloor \alpha m \rfloor$$

$$M y_{j} \geq \tilde{\nu} - x^{\mathsf{T}} \xi^{j}, \quad j \in I$$

$$x^{\mathsf{T}} e = 1$$

$$x \in \mathcal{X} \subseteq \mathbb{R}_{+}^{n},$$

$$y_{j} \in \{0, 1\}, \qquad j \in I.$$
(13)

Notice that $\overline{\mu}_{[m]} = z_{\text{VaR}}^*$ (cf., (6)). Next, we show that (13) is a relaxation of (6).

Proposition 2. Let $I \subseteq [m]$ with $|I| \ge \lfloor \alpha m \rfloor$. Then problem (13) is a relaxation of (6). That is, $\overline{\mu}_I \ge z_{\text{VaR}}^*$.

Proof. Let $x \in \mathbb{R}^n_+, y \in \{0,1\}^m$ be feasible for (6), then we have that $x^{\mathsf{T}}e = 1$, and $x \in \mathcal{X}$. Moreover, the fact that there exist $y \in \{0,1\}^m$ such that $\sum_{j \in [m]} y_j \leq \lfloor \alpha m \rfloor$, and $My_j \geq \tilde{\nu} - x^{\mathsf{T}}\xi^j$, for all $j \in [m]$, implies that $\tilde{\nu} \leq \min^{\lfloor \alpha m \rfloor + 1} \{x^{\mathsf{T}}\xi^j : j \in [m]\} \leq \min^{\lfloor \alpha m \rfloor + 1} \{x^{\mathsf{T}}\xi^j : j \in I\}$. Thus, $y_I \in \{0,1\}^{|I|}$ satisfies $\sum_{j \in I} y_j \leq \lfloor \alpha m \rfloor$, and $My_j \geq \tilde{\nu} - x^{\mathsf{T}}\xi^j$, for all $j \in I$. Thus, (x, y_I) is a feasible solution for (13) with objective value $x^{\mathsf{T}}\mu$.

Notice that from Proposition 2, it follows that

$$\overline{\mu}_I < \mu_0 \Rightarrow z_{\text{VaR}}^* < \mu_0.$$

In Algorithm B below, we take advantage of this fact by iteratively using the formulation (13), together with an appropriate update of the set I, with the aim of showing the near-optimality of the feasible solution of the VaR portfolio problem (4) obtained from Algorithm A. The set I is updated by heuristically adding samples from the set $[m] \setminus I$ (see, Algorithm B for details) until condition (12) is satisfied.

5 Numerical Results

In this section, we present numerical results to compare the performace of Algorithm A against the CVaR-based algorithms proposed by Larsen et al. (2002) to obtain lower bounds on the VaR portfolio problem (4). Moreover, we compare the performance of Algorithm A and Algorithm B to obtain guaranteed near-optimal solutions for the VaR portfolio problem (4), against directly solving (4) using state-of-the-art mixed integer linear programming (MILP) solvers.

To carry out the experiments we use the daily returns data from Kenneth R. French's website http://mba.tuck.dartmouth.edu/pages/faculty/ken.french on 100 portfolios formed on size and book-to-market (10 x 10). The data can be downloaded at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/ftp/100_Portfolios_10x10_Daily_TXT.zip. From

Algorithm B Upper bound for optimal return

```
1: procedure Upper bound(\alpha, \beta, \mu_0, \delta, Iter<sup>max</sup>, and \tilde{x}, \tilde{\nu}, I_0 from Algorithm (A))
             m' \leftarrow |(\beta \alpha m)|
 2:
             I \leftarrow I_0,
 3:
             \nu \leftarrow \tilde{\nu} + \delta
 4:
             \overline{\mu}_I \leftarrow \tilde{x}^{\mathsf{T}} \mu
 5:
             while \overline{\mu}_I \geq \mu_0, I \subset [m], and Iter \leq Iter<sup>max</sup> do
 6:
                   \overline{\mu}_I \leftarrow \text{objective value of } (13)
                                                                                                                                 \triangleright +\infty if (13) is infeasible
 7:
                   x \leftarrow \arg_x (13)
                                                                                                                              ▷ optimal portfolio for (13)
 8:
                   I^{\mathrm{old}} \leftarrow I
 9:
                   \nu' \leftarrow \min^{m'+1} \{ x^{\mathsf{T}} \xi^j : j \in [m] \setminus I_0 \}
10:
                   I \leftarrow I^{\text{old}} \cup \{j \in [m] \setminus I^{\text{old}} : x^{\mathsf{T}} \xi^j \leq \nu'\}
11:
12:
             end while
             if Iter < Iter^{max} then
13:
14:
                   return The \delta near-optimality of \tilde{x}, \tilde{\nu} is proven
15:
                   return The \delta near-optimality of \tilde{x}, \tilde{\nu} could not be verified
16:
17:
             end if
18: end procedure
```

this data, instances of the VaR portfolio problem (4) having number of assets $n \in [30, 90]$, number of samples $m \in [1000, 3500]$ (for every value of n), and expected profit $\mu_0 \in \{\mu^- + \frac{i}{k+1}(\mu^+ - \mu^-) : i = 1, \dots, k\}$, with k = 6 and μ^+ (resp. μ^-) is the largest (lowest) asset mean return. Similar to Benati and Rizzi (2007); Feng et al. (2015), the parameter $\alpha \in (0,1)$ (cf., (1)) is set to the popular value used in practice of $\alpha = 0.01$.

All the code necessary to create the instances of the optimization problems discussed throughout the article is implemented using Matlab 2016a and the modelling language YALMIP, which is available at users.isy.liu.se/johanl/yalmip/. Gurobi 6.5.0 is used to obtain the solution of all the necessary linear programs and MILPs on a Intel(R) Core (TM) i3-2310M CPU @ 2.10 GHz, 4GB RAM machine.

5.1 Lower bound for portfolio's VaR

We compare the performace of Algorithm A against the CVaR-based algorithms proposed by Larsen et al. (2002) to obtain lower bounds on the VaR portfolio problem (4).

In all instances, Algorithm A is initialized by setting J_0 as the first $m_0 = \lceil 2\alpha m \rceil$ samples of the instance. Also, the algorithms being compared are allowed to run for a maximum time of up 3600 seconds.

The lower bound results on the VaR portfolio problem (4) obtained by the three (3) algorithms are summarized in Table 1, Figure 4, 5, and 6. In Table 1, for every combination of the number of assets (n) and the number of samples (m), an average is taken over the instances with different values of μ_0 , between the values μ_0^{\min} and μ_0^{\max} . For each algorithm, the column "gap", indicates the relative percentage error between the lower bound on the VaR portfolio problem (4) and its optimal solution provided by solving the MILP (4). In the few instances when the MILP (4) cannot be solved within the maximum allowed time (of 3600)

s.), the optimal solution of (4) is replaced by the best (higher) lower bound obtained from the lower bound algorithms. Thus, the gap columns in Table 1 cleary show that Algorithm A provides tighter lower bounds on the VaR portfolio problem (4), than the ones provided by Algorithm 1 and Algorithm 2 (cf., Larsen et al., 2002). Also, it is clear that the percentage by which Algorithm A provides tighter bounds than Algorithms 1 and 2 is substantial and ranges between 1% - 7% on average. A more granular evidence of this result is shown in Figures 4 and 5. In these figures, for each algorithm, the relative gap with respect to the optimal value of the VaR portfolio problem (4) for each of the instances considered is plotted in the y-axis, while the x-axis labels indicate the values of the number of samples (m), and expected return (μ_0) of the instance. Also, the number of assets (n) is indicated in each of the plots. In the next section, the tightness of the bounds provided by Algorithm A will be key to be able to guarantee the near-optimality of the solutions for the VaR portfolio problem (4) provided by Algorithm A.

As shown in Table 1, the tighter bounds obtained by Algorithm A, in comparison with Algorithm 1 and Algorithm 2 in Larsen et al. (2002), are obtained in comparable running times. As mentioned earlier, in Table 1, for every combination of the number of assets (n) and the number of samples (m), an average is taken over the instances with different values of μ_0 , between the values μ_0^{\min} and μ_0^{\max} . For each algorithm, the column " T/T^* ", indicates the average (over instances with different values of μ_0 , and equal n, m) of the ratio between the time taken by each of the algorithms and the minimum of these times on an instance with a particular $\mu_0 \in [\mu_0^{\max}, \mu_0^{\min}]$. From these results it is clear that the average times of the three algorithms are mostly comparable. In Figure 6, the average running time information of the algorithms is provided. It is clear from this figure that most of the time, the average time taken by the three algorithms is similar, and that even when there are significant differences between the times, such differences are not of significant practical relevance, since the times required by the algorithms are in the range of at most hundreds of seconds.

5.2 Near-optimal VaR portfolio

In this section, we show that by using Algorithm A and Algorithm B, one can efficiently compute guaranteed near-optimal solutions for the VaR portfolio problem (4). For that purpose, to obtain the results described below, we first run Algorithm A with J_0 being the first $m_0 = \lceil 2\alpha m \rceil$ samples of the instance. The resulting portfolio $\tilde{x} \in \mathbb{R}^n_+$, VaR lower bound $\tilde{\nu} \in \mathbb{R}$, and the set I_0 (cf., end of Algorithm A) are then used to initialize Algorithm B, Also, we set $\beta = 0.1$, and $\delta = 0.01\tilde{\nu}$. That is, we run Algorithm B seeking to provide a 1% near-optimality guarantee for the portfolio $\tilde{x} \in \mathbb{R}^n_+$. In Table 2 and Figure 7, the results of finding a near optimal solution to the VaR portfolio problem using Algorithms A and B versus directly solving the MILP formulation (4) is compared. For the purpose of brevity, of all the instances considered, Table 2 shows, for a particular number of assets nand samples m, the instances in which the MILP solver finds the optimal solution of the VaR problem in the shortest and longest time (depending on the value of μ_0). By comparing the columns "VaR*" and "VaR" in Table 2, it is clear that the lower bound on the VaR portfolio problem (4) it's equal or very close to the optimal value of the VaR portfolio problem (4) (as illustrated also in Figures 4 and 5). Note that these lower bounds are well within the 1% desired tolerance. In Table 2, T^* is the time taken to solve the MILP formulation (4)

Size		μ_0		Alg	g. A	Alg	g. 1	Alg. 2		
\overline{n}	\overline{m}	min	max	gap	T/T^*	gap	T/T^*	gap	T/T^*	
30	1000	0.019	0.058	0.04	1.1	1.23	1.4	0.94	1.2	
30	1500	0.043	0.071	0.08	1.4	1.37	1.2	1.48	1.0	
30	2000	-0.018	0.056	0.29	1.1	3.02	1.1	2.32	1.0	
30	2500	0.012	0.046	0.00	1.1	2.85	1.2	2.56	1.0	
50	1000	0.015	0.069	0.00	1.0	2.23	1.2	1.13	1.1	
50	1500	0.062	0.075	0.14	1.3	1.59	1.1	1.59	1.0	
50	2000	-0.018	0.056	0.29	1.1	2.87	1.1	2.45	1.0	
50	2500	0.012	0.068	0.00	1.1	3.04	1.0	3.54	1.0	
50	3000	0.005	0.054	0.00	1.4	7.00	1.0	6.22	1.1	
50	3500	0.012	0.058	0.00	1.1	1.14	1.0	1.14	1.0	
70	1000	0.015	0.069	0.00	1.0	1.50	2.6	2.71	4.3	
70	1500	0.039	0.075	0.06	1.3	0.84	1.5	1.30	1.2	
70	2000	-0.017	0.061	0.00	1.2	1.96	1.9	1.80	1.8	
70	2500	0.012	0.068	0.00	2.1	5.82	2.6	4.78	2.4	
70	3000	0.005	0.018	0.00	6.0	4.33	9.3	2.32	1.0	
70	3500	0.012	0.069	0.12	2.1	2.94	1.0	2.73	1.1	
90	1000	0.007	0.048	0.00	1.1	7.55	1.1	2.37	1.1	
90	1500	0.063	0.084	0.00	1.5	0.94	1.0	0.94	1.0	
90	2000	0.061	0.061	0.02	1.5	1.68	1.0	0.28	1.0	
90	2500	0.012	0.068	0.28	3.4	7.49	1.1	6.28	1.0	
90	3000	0.005	0.068	0.00	2.9	2.38	1.5	2.09	1.0	
90	3500	0.012	0.069	0.00	3.5	3.12	1.0	3.43	1.1	

Table 1: Performance of Algorithm A vs. Algorithm 1 and Algorithm 2 in (Larsen et al., 2002) to compute lower bounds on the VaR portfolio allocation problem (4). The column gap indicates the VaR lower bound % gap to the optimal VaR. The column T/T^* , is the ratio between the time required to obtain the lower bound T against the minimum time needed by the three algorithms T^* .

of the VaR portfolio problem, and T is the time that is taken to obtain a guaranteed near optimal solution for the VaR portfolio problem using Algorithms A and B. Thus, it is clear from the T^*/T columns in Table 2 that the latter approach is between 1.2 to 46 times faster than directly solving the MILP formulation. On average, the speed up provided by using Algorithms A and B is approximately of 14 times. Given the time is takes to solve some of the instances of the VaR portfolio, this speed up would be crucial to solve practical instances of the VaR portfolio problem. The effect of the speed up provided by Algorithms A and B can be seen graphically in Figure 7, where the time required by Algorithms A and B, versus the time required to solve the MILP formulation of the VaR optimization problem instances in Table 2, is shown in a semilogarithmic plot.

6 Final Remarks

Thus far, we have only considered portfolio allocation problems in which no short positions are allowed (i.e., $\mathcal{X} \subseteq \mathbb{R}^n_+$ in (2)). In practice, none of the main characteristics of the MILP

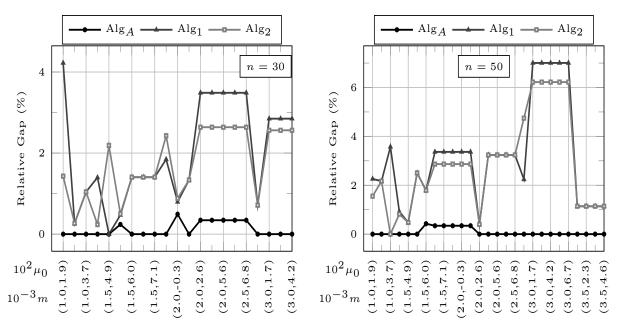


Figure 4: Comparison of the relative gap between the optimal value of the VaR portfolio problem (4) and the lower bounds for (4) provided by Algorithm A, and Algorithms 1 and 2 by Larsen et al. (2002).

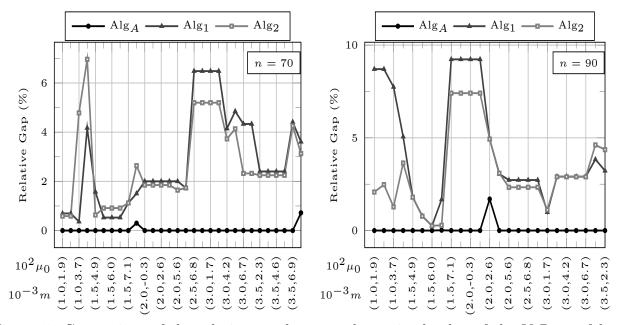


Figure 5: Comparison of the relative gap between the optimal value of the VaR portfolio problem (4) and the lower bounds for (4) provided by Algorithm A, and Algorithms 1 and 2 by Larsen et al. (2002).

formulation (4) of the VaR portfolio problem change when considering portfolios were short positions are allowed (i.e., $\mathcal{X} \subseteq \mathbb{R}^n$ in (2)). Clearly, only the choice of the Big-M constant M is affected by allowing short positions. However, under the practical assumption that there is $U \in \mathbb{R}_+$ (e.g., due to liquidity) such that $U \geq \max\{i \in \{1, \ldots, n\} : |x_i|\}$, then the M

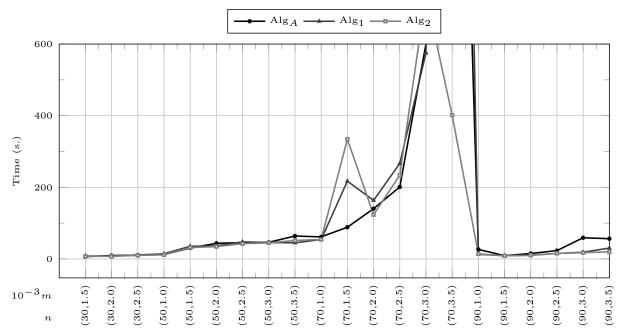


Figure 6: Comparison of average time in seconds needed to run Algorithm A, and Algorithm 1 and 2 in (Larsen et al., 2002) to obtain lower bounds for the VaR portfolio problem (4), for instances with given n, m and different values of μ_0 (cf. Table 1).

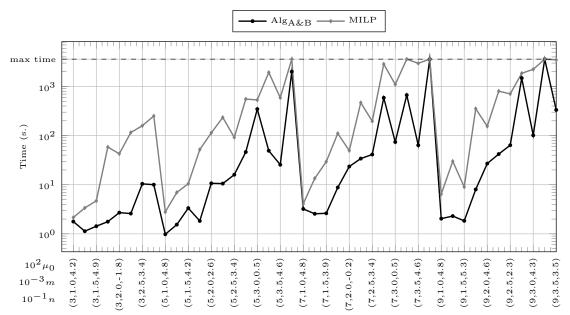


Figure 7: Comparison of the time taken by Algorithm (A) and (B) vs directly solving the MILP formulation of the VaR portfolio problem (4) for instances with different values of n, m, and μ_0 .

in (4) can be set to $M > 2U \max\{|\xi_i^j| : i \in \{1, ..., n\}, j \in \{1, ..., m\}\}.$

With that said, in this paper, we studied the VaR portfolio selection problem, which is of high relevance in practice, and even in theory, thanks to development of the so-called *natural risk statistic* axioms (Heyde et al., 2006) and the introduction of the concept of *elic-*

			Full MILP		Alg. A & B					Full MILP		Alg. A & B			
n	m	μ_0	VaR*	T^*	VaR	Т	T^*/T	n	m	μ_0	VaR*	T^*	VaR	Т	T^*/T
30	1000	0.058	-2.554	2.2	-2.560	1.78	1.2	70	1500	0.075	-2.002	29.5	-2.008	2.6	11.2
30	1000	0.042	-1.982	3.4	-1.982	1.13	3.0	70	1500	0.039	-1.684	110.8	-1.684	8.8	12.6
30	1500	0.071	-1.968	4.7	-1.968	1.43	3.3	70	2000	0.061	-1.836	49.4	-1.836	23.5	2.1
30	1500	0.049	-1.702	58.4	-1.702	1.77	32.9	70	2000	-0.002	-1.703	467.6	-1.703	34.0	13.7
30	2000	0.026	-1.721	42.7	-1.727	2.7	15.8	70	2500	0.068	-2.121	196.4	-2.121	41.3	4.7
30	2000	-0.018	-1.721	115.8	-1.727	2.59	44.8	70	2500	0.034	-1.876	2860.1	-1.876	589.6	4.8
30	2500	0.057	-1.951	158.7	-1.951	10.42	15.2	70	3000	0.068	-2.024	1101.1	-2.024	74.1	14.8
30	2500	0.034	-1.951	250.6	-1.951	10.01	25.1	70	3000	0.005	*	*	-1.851	667.7	*
50	1000	0.069	-2.449	2.8	-2.449	0.98	2.8	70	3500	0.069	-1.943	2960.4	-1.957	63.7	46.5
50	1000	0.048	-1.973	7.0	-1.973	1.54	4.5	70	3500	0.046	*	*	*	*	*
50	1500	0.075	-2.041	10.5	-2.050	3.34	3.1	90	1000	0.076	-2.325	6.5	-2.339	2.0	3.1
50	1500	0.042	-1.699	52.0	-1.699	1.83	28.5	90	1000	0.048	-1.906	30.0	-1.906	2.3	13.0
50	2000	0.011	-1.721	113.6	-1.727	10.68	10.7	90	1500	0.084	-2.313	9.0	-2.313	1.8	4.8
50	2000	0.026	-1.721	234.5	-1.727	10.55	22.2	90	1500	0.053	-1.669	353.0	-1.669	8.0	44.0
50	2500	0.068	-2.164	91.9	-2.164	15.99	5.8	90	2000	0.061	-1.825	154.8	-1.825	27.0	5.7
50	2500	0.034	-1.920	554.3	-1.920	46.33	12.0	90	2000	0.046	-1.696	803.4	-1.696	42.1	19.1
50	3000	0.067	-2.028	531.7	-2.004	346.59	1.5	90	2500	0.068	-2.115	706.5	-2.115	63.8	11.1
50	3000	0.005	-1.887	1933.9	-1.887	49.15	39.4	90	2500	0.023	-1.826	1855.4	-1.826	1483.7	1.2
50	3500	0.069	-1.982	589.4	-1.984	25.53	23.1	90	3000	0.068	-2.024	2237.0	-2.024	101.4	22.1
50	3500	0.046	*	*	-1.849	2003.1	*	90	3000	0.043	*	*	*	*	*
70	1000	0.069	-2.296	4.1	-2.296	3.21	1.3	90	3500	0.069	-1.931	3479.2	-1.931	333.1	10.4
70	1000	0.048	-1.901	13.5	-1.901	2.56	5.3	90	3500	0.035	*	*	*	*	*
Av	Average Speed Up 14.					14.35	Average Speed Up						13.64		

Table 2: Comparison of VaR values and running times of full MILP formulation (4) vs. Algorithms A & B, for instances of the VaR portfolio problem (4) with different no. of assets (n), no. of samples (m), and expected return μ_0 . The column T^*/T shows the speed up obtained with Algorithms A & B over solving the full MILP formulation. Instances with "***" where not solved within the 3600s. limit time. The " \star " in column T^*/T that ratio cannot be computed.

itability (cf., Bellini and Bignozzi, 2015) to classify risk measures. To address the inherently difficult task of solving the VaR portfolio problem, here we propose a tandem of approximation algorithms to produce near-optimal solutions to the VaR portfolio problem. This is done by first using Algorithm A to obtain a good feasible solution for the VaR portfolio problem, and as such, provide a lower bound for the optimal VaR associated with (4). This algorithm is shown to outperform recent algorithms proposed for this purpose by (Larsen et al., 2002), based on the iterative solution of appropriate CVaR portfolio problems. Then, in Algorithm B, one aims to prove a %1 optimallity guarantee for the feasible solution obtained at the end of Algorithm A. The results obtained here, show that using both Algorithm A and B allows to more efficiently solve VaR portfolio problems with up to a hundred assets and thousands of samples, compared to solving the VaR portfolio problem directly with a MILP solver. This results clearly improve the recent results of (Larsen et al., 2002) on lower bounds for the VaR portfolio problem, and the recent results of (Feng et al., 2015, Sec. 5)

on solving VaR portfolio problems for 25 assets without taking into account the total wealth constraint. Moreover, the proposed algorithms are funded on a study of the alternative formulations of the risk-reward portfolio allocation problem that extends the work done in this area recently by Krokhmal et al. (2002, Thm. 3)

Finally, we believe that the proposed algorithms can be also applied to solve the broader group of chance constrained optimization problems (cf., Sarykalin et al., 2008).

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