A multi-stage stochastic optimization model of a pastoral dairy farm

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Abstract

Pastoral dairy farmers make sequential decisions in the face of long-term environmental uncertainty and price volatility. Decisions made early in the season, such as the number of cows to stock per hectare, can have significant effects later in the season if the farmer is forced to import additional feed to meet the cows' energy demands during a drought. In this paper, we present POWDer: the milk Production Optimizer incorporating Weather Dynamics. POWDer is a novel multi-stage stochastic program that divides the dairy farming season into weeks and links these weeks by a system of linear dynamics. By applying POWDer to a case farm in New Zealand, we demonstrate POWDer's promise as a tool that can help participants in the New Zealand dairy industry understand and plan for the challenge of farming in a stochastic world.

Keywords: OR in agriculture, stochastic dual dynamic program, multi-stage, stochastic programming, dairy

1. Introduction

Uncertainty in the agricultural supply sector has received increasing attention from researchers and practitioners in recent years (Borodin et al., 2016). This is not surprising, given the inherent uncertainty in all steps of the agricultural supply chain. In this paper, we focus on the first step of the supply chain for dairy products; namely the dairy farmer. Dairy farmers face supply uncertainty from bad weather and price uncertainty from international markets. Ignoring these uncertain effects can lead to poor farming decisions. In addition, farms may be located in disparate geographic locations, and even farms that are located in the same local area may have

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differences in topography and soil type. Therefore, it is necessary to model conditions on an individual-farm basis in order to produce meaningful results for the farmer. Although our models are generally applicable to a wide-class of outdoor, pastoral farms, for this paper we calibrate and test them in a New Zealand setting.

In New Zealand, dairy farming forms a large part of the economy. In the 2016 season dairy farming was responsible for 29% of New Zealand's export earnings (down from 35% in 2014) (Ballingal & Pambudi, 2017). In addition, despite producing only 3% of the total global production volume, New Zealand exports 96% of its domestic production. As a consequence, it is the largest player in the international dairy commodity market with around 28% of international exports (Shadbolt & Apparo, 2016). These factors make the operation of New Zealand dairy farms a critical component in the global dairy supply chain.

Established farmers in New Zealand largely rely on experience, low-debt, efficiency, and a financial buffer to help them survive economic downturns (Singh, 2015). However, as dairy markets liberalize, and new environmentally focused domestic regulations (such as a proposed charge on the commercial use of water (Cann, 2017)) are introduced, relying on previous experience may result in sub-optimal decision-making.

There have been a number of efforts to develop decision-support models for dairy farmers. In addition to "rule of thumb" recommendations, such as those by Dairy NZ (2012b), considerable work has been undertaken to develop simulation models of dairy cows and whole-farm ecosystems in order to answer "what-if" type questions. Review papers by Bryant & Snow (2008), Feola et al. (2012), and Snow et al. (2014) give a comprehensive overview of the current mathematical simulation models. Only a few of the models explored in the reviews incorporated optimization, and those that did usually implemented some form of an evolutionary search algorithm (Hart et al., 1998; Neal et al., 2005; Aryal et al., 2008; Groot et al., 2012).

One notable exception to this is the Integrated Dairy Enterprise Analysis (IDEA) (Doole et al., 2013), a nonlinear, non-convex optimization model of a New Zealand dairy farm, which we consider to be the current state-of-the-art. The model divides the dairy farming season into fortnightly blocks and optimizes operating profit. Decision variables in IDEA include a number of operational decisions (such as the type and quantity of supplementary feed to buy each fortnight), as well as some strategic decisions such as stocking rate (the number of cows per hectare). A detailed description of the model is given in Doole et al. (2012). IDEA has been used to investigate various management strategies on New Zealand dairy farms including supplementary feeding (Doole, 2014a), reducing greenhouse gas emissions (Doole, 2014b; Adler et al., 2015), stocking rates (Doole & Romera, 2013; Romera & Doole, 2015, 2016), and general profitability (Doole, 2015). Although IDEA has been demonstrated to be a useful tool to investigate optimal management strategies, the use of non-convex optimization makes it difficult to claim that the solutions it produces are globally optimal. Furthermore, it is not able to capture the stochastic nature of dairy farming, in that it does not incorporate weather or price risk.

IDEA shows how we can model the farming season as a sequential decision-making process. When uncertainty is incorporated, such problems are known as multi-stage stochastic programs. (We direct the reader to the following works for an introduction to this class of problems: (Puterman, 1994; Shapiro et al., 2009; Powell, 2011).) In a recent review paper, Borodin et al. (2016) lay out the state-of-the-art in handling uncertainty in agricultural supply chain management. They conducted a comprehensive survey of papers and a number of stochastic programming approaches were identified in the literature. Most were two-stage models and only a few were concerned with dairy (Flaten & Lien, 2007; Heikkinen & Pietola, 2009; Guan & Philpott, 2011; Relund Nielsen et al., 2011). The paper of Guan & Philpott (2011) is notable in that it formulates, and solves, a large multi-stage stochastic optimization problem. However, it concerns the operations of a large dairy processing company, rather than an individual farmer. To the best of our knowledge, our work is the first detailed multi-stage stochastic optimization model of a single dairy farm.

Borodin et al. (2016) note that “multi-stage stochastic programs, in general, are intractable”, and that “even if the use of [multi-stage stochastic programming] is showing increasing promise, there are still very few real-world implementations ... in the agricultural area.” However, multi-stage stochastic programming has been widely applied to real-world problems in the electricity sector with considerable success (for example, in Brazil (Maceira et al., 2008)). A key contributor to this has been the development (Pereira & Pinto, 1991; Philpott & Guan, 2008) and improvement (Philpott & de Matos, 2012; de Matos et al., 2015) of the *stochastic dual dynamic programming* (SDDP) algorithm.

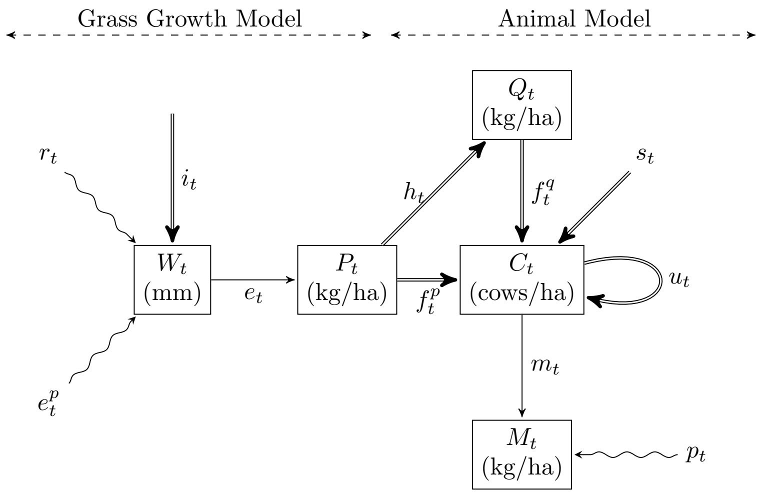
SDDP is a dynamic programming-inspired algorithm. It decomposes the multi-stage stochastic optimization problem in time into a sequence of subproblems. Each subproblem (i.e. each set of decisions to be made in some interval in time) is an optimization problem that chooses an action in order to minimize the cost associated with the current decision plus the *cost-to-go* of the remaining stages given the action taken. Dynamic programming (Bellman, 1957) estimates the cost-to-go function (also called the *Bellman function*) at a set of discretized points. However, because of this discretization, the method is limited to low-dimensional problems (dynamic programming's "curse of dimensionality"). Instead of evaluating the function at a set of discretized points, SDDP approximates the Bellman function with a set of piecewise linear functions called *cuts*. When the problem instance has a specific form (stagewise independence of the noise, convexity of the Bellman function, and continuous state variables), the SDDP algorithm can efficiently find an approximately optimal policy. In this paper we focus on the model and the results, rather than the solution method. Readers are directed to the following works for a more in-depth discussion of SDDP: (Pereira & Pinto, 1991; Philpott & Guan, 2008; Shapiro, 2011).

The key contributions of this paper are to develop a multi-stage stochastic programming model of a pastoral dairy farm that incorporates environmental and economic uncertainty and to solve it using SDDP. Such a model represents a significant step forward in the ability to gain insight into the interrelationships that weather and price uncertainty have on the decision-making of dairy farmers. We call the resulting model POWDer – the milk Production Optimizer incorporating Weather Dynamics.

2. Model Description

In this section we formulate POWDer as a discrete-time stochastic optimal control problem over 52 weekly stages, $t = 1, 2, \ldots, 52$. We call the sequence of 52 weeks a *season*. POWDer is a combination of three separate models: a grass growth model, an animal model, and a milk price model. Before we detail the specifics of the model, we provide an overall schematic of the system in Figure 1 to aid the reader's understanding of the various interactions between the model components in one week. States (denoted below by uppercase letters) are in square boxes. Random inputs are shown by wavy arcs. Actions that can be chosen by the farmer are denoted by double-lines with a bold arrow head. Response variables that cannot be chosen by the farmer are denoted by straight arrows with a single line. In the following description of the model, all of the variables we refer to are contained in Figure 1.

The reader should note that without loss of generality, we normalize all values in the following to a per hectare figure. We now describe the model in detail, beginning with the grass growth



r_t: rainfall (mm)
i_t: irrigation (mm)
e_t: evapotranspiration (mm)
h_t: quantity of pasture to harvest (kg/ha)
f_t^p: grass from pasture to feed (kg/ha)
C_t: number of cows lactating (cows/ha)
u_t: cows to dry-off (cows/ha)
M_t: milk solids produced (kg/ha)
e_t^p: potential evapotranspiration (mm)
W_t: soil moisture (mm)
P_t: pasture cover (kg/ha)
Q_t: quantity of grass in storage (kg/ha)
f_t^q: grass from storage to feed (kg/ha)
s_t: palm kernel to feed (kg/ha)
m_t: energy for milk production (MJ/ha)
p_t: forecast end-of-season milk price (\$/kg)

Figure 1: Schematic of interactions during one week (stage *t*) of the POWDer model.

model.

2.1. Grass Growth Model

The basis for any whole-farm model in a pastoral farming context is a model for grass growth. For New Zealand conditions, there is a large literature on different modelling approaches (Rickard et al., 1986; Moir et al., 2000; Bryant & Snow, 2008; Johnson et al., 2012; FARMAX, 2016). However, many of these models are highly detailed and nonlinear. Therefore, the technical requirements of SDDP prevent us from using these models. In this section, we introduce two previously published models of grass growth. The first relates grass growth to the total quantity of grass on the farm (since if there is no grass, none can grow), but excludes any weather effects such as drought. The second relates grass growth to the weather (via evapotranspiration[1]) but excludes the total quantity of grass on the farm. Therefore, we propose a new model that assumes that the two models act independently to limit grass growth.

The first model (derived from (Garcia, 2000; Baudracco et al., 2012)) assumes that grass growth is related to the current pasture cover (kilograms[2] of grass per hectare). The model uses a logistic growth function so that:

$$P_{t+1} - P_t = 4\frac{\Delta P_{\max}}{P_{\max}} \times P_t\left(1 - \frac{P_t}{P_{\max}}\right), \tag{1}$$

where P_t is the pasture cover (kg/ha) at the start of week t, the constant $P_{\max}$ is the maximum possible pasture cover (the point at which the senescence rate approaches the growth rate (Baudracco et al., 2012)), and the constant $\Delta P_{\max}$ is the maximum possible rate of pasture growth.

In contrast to the logistic growth function, Moir et al. (Moir et al., 2000) propose that grass growth is proportional to the evapotranspiration rate such that:

$$P_{t+1} - P_t = \kappa_t \times e_t, \tag{2}$$

where P_t is the pasture cover (kg/ha) at the start of week t, e_t is the evapotranspiration (mm) during week t and κ_t is a constant (kg/mm) that can be interpreted as an index of soil fertility. One weakness of this model is that it ignores the impact that the current pasture cover has on the growth rate. Evapotranspiration depends on a number of factors. Firstly, there must be sufficient water in the soil for evapotranspiration to occur. Secondly, more water will evaporate and transpire on a hot, sunny day than a cold, cloudy day. Therefore, the evapotranspiration rate, e_t, during week t is:

$$e_t = \min\left\{e_t^p,\ W_t + r_t\right\}, \tag{3}$$

where e_t^p is the potential evapotranspiration (Priestley & Taylor, 1972) during week t (mm) as determined by the weather during week t, W_t is the plant-available water stored in the root zone of the soil at the start of week t (mm) and r_t is the incident rainfall during week t (mm). Both the evapotranspiration rate e_t^p and the incident rainfall r_t are random variables in week t.

[1]Evapotranspiration is the sum of both direct evaporation (from soil to atmosphere) and plant transpiration (where the water moves from the soil through the plant and evaporates from the parts exposed to the atmosphere such as the leaves).

[2]When discussing pasture cover, we mean kilograms of dry-matter (i.e. pasture with the water content removed). Other authors denote this as kgDM. However, for simplicity, we shall just refer to it as kg.

In reality, farmers sub-divide their farms into a number of individual fenced units called paddocks. The herd is rotated around the paddocks at a rate of one to two paddocks per day, depending on growth rate of the grass. This leaves each paddock with a different pasture cover (which therefore grow at different rates). In this paper, we make the simplifying assumption that all the paddocks are identical. Therefore, P_t represents the average pasture cover across the entire farm.

One extra feature of pasture is that it can be harvested to form silage[3] or hay. This can be stored until later in the season when the pasture cover is low. We denote the quantity of grass that is in storage at the start of week t by Q_t (kg/ha). Therefore:

$$Q_{t+1} = Q_t + \beta h_t - f_t^q, \tag{4}$$

where h_t is the quantity of pasture harvested in week t (kg/ha) and f_t^q is the quantity (in kg/ha) of the grass from storage fed to the herd in week t (discussed in Section 2.2). β is a conversion factor between pasture and feed in storage. This accounts for spoilage and wastage.

To incorporate the two models of pasture growth (Eq. 1 and Eq. 2), we assume that both models act independently to limit grass growth. Thus, our final dynamical model for pasture cover is:

$$P_{t+1} = P_t + \min\left\{\kappa \times e_t,\ 4\frac{\Delta P_{\max}}{P_{\max}} \times P_t\left(1 - \frac{P_t}{P_{\max}}\right)\right\} - h_t - f_t^p, \tag{5}$$

where f_t^p is the quantity (kg/ha) of grass from pasture fed to the herd in week t. We discuss this further in Section 2.2.

We choose this model so that when the actual evapotranspiration e_t is high, grass growth is limited by the current pasture cover P_t, and when evapotranspiration is low, grass growth is limited by the actual evapotranspiration. A plot showing how the grass growth rate is determined using this model is shown in Figure 2.

In addition to proposing that grass growth is proportional to evapotranspiration, Moir et al. (2000) also give some simple water balance equations to describe how the water in the soil changes over time. First, there is some maximum quantity of water that can be stored in the soil (and available to the plant):

$$W_t \leq \bar{W}, \tag{6}$$

[3] A fermented, high-moisture alternative to hay.

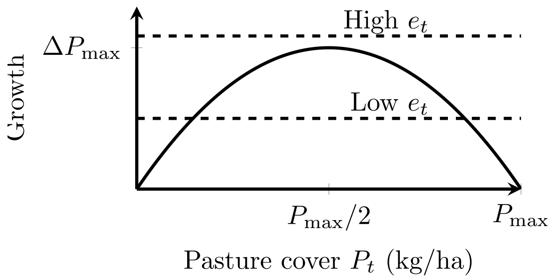


Figure 2: Plot of grass growth against the pasture cover for low and high levels of actual evapotranspiration e_t. Solid line is the logistic growth model given by Eq. (1). Dotted lines are the evapotranspiration growth model given by Eq. (2) for two rates of actual evapotranspiration. Actual grass growth is the minimum of the solid and dotted lines given a pasture cover P_t and actual evapotranspiration rate e_t.

where $\bar{W}$ is a constant equal to the maximum available water content of the soil. Any excess water is assumed to drain away. Second, the change in soil moisture W_t from the start of week t to the start of week $t+1$ is limited by the quantity of rainfall r_t, plus irrigation i_t, less evapotranspiration e_t:

$$W_{t+1} \leq W_t + r_t - e_t + i_t. \tag{7}$$

There are also some domain constraints on the state and control variables:

$$P_t, Q_t, W_t, f_t^p, f_t^q, h_t, i_t \geq 0, \ \forall t. \tag{8}$$

Stochastic Process. For simplicity, we assume that the rainfall r_t, and potential evapotranspiration e_t^p, can be modelled by a stagewise-independent joint stochastic process that empirically samples with replacement from historical readings in week t. We call a single sequence of 52 observations (one for each week) of the random weather variables, r_t and e_t^p, a *weather scenario*. The stochastic process can include some stagewise dependence with an increase in complexity of the solution process. (See Morton & Infanger (1996) and Shapiro (2011) for different approaches to this.) However, we choose to model the process using stagewise independence as it is a commonly used assumption in the SDDP literature (Pereira & Pinto, 1991; Philpott, 2017).

2.2. Animal Model

Animal models are another critical component of any whole-farm model and are well studied in the literature (Baudracco et al., 2011; Gregorini et al., 2013; Johnson, 2013). However, like the grass growth models mentioned earlier, these are complicated nonlinear models that account for a large number of variables. Due to a desire to maintain tractability of the model with the inclusion of stochasticity, we seek a simplified version of these models.

The first decision a farmer faces is their stocking rate (number of cows per hectare). In this model, we assume this is fixed *a priori* over the season and denote it by $\bar{C}$. The season begins with the cow giving birth to a calf and starting lactation. We assume that the cost of raising the calves is a fixed cost and so it is not modelled directly in POWDer. At some point during the season, the farmer can choose to *dry-off* a cow (stop its lactation). This reduces the required energy intake for the cow (since it is no longer producing milk). However, once the cow is dried-off, it cannot restart its lactation. Therefore, the farmer faces a stopping problem, where they have to trade-off the decision to keep milking the cow (and earning money) against managing feed reserves for the next season. In New Zealand, the average lactation length is around 276 days (Dairy NZ & LIC, 2016). We denote the number of lactating cows per hectare in week t as C_t. Each week, the farmer can choose to dry-off u_t cows/ha, so that:

$$C_{t+1} = C_t - u_t. \tag{9}$$

In addition, $C_t \geq 0$, $u_t \geq 0$, and $C_1 = \bar{C}$.

Now that we have the concept of a herd (as measured by the stocking rate) and the idea that cows must be in one of two states (lactating or dried-off), we create a simple model of a single cow. To do this, we consider an energy balance: energy consumed by a cow in the form of pasture, or supplement such as hay, silage, or corn, must equal the energy spent on maintenance[4], changes in body mass, pregnancy, and milk production. To simplify the model, we ignore other factors like nutrition.

Energy Requirements. To calculate the energy required by the cow for maintenance, pregnancy, and changes in body mass, we draw from multiple published models in the literature. It is sufficient for general readers to understand that, as a result of the model, we can calculate the net energy required during week t by a cow (excluding milk production) that is lactating (which we denote by $\varepsilon_t^{\text{lac}}$) and dried-off (which we denote by $\varepsilon_t^{\text{dry}}$). These are set as constants in POWDer. For more detail, we refer readers to Appendix A.

[4]Maintenance is the energy required to run the cow's core bodily functions. This can be considered a fixed cost of keeping the cow alive.

Milk Production. If a lactating cow consumes more energy than the total required $\varepsilon_t^{\mathrm{lac}}$, then milk will be produced. Dairy NZ (2012b) assume that there is a linear relationship between this extra energy intake, which we denote m_t, and the quantity of milk produced. However, cows have a biological maximum milk production capacity that varies over the season. Therefore, following the approach taken by Baudracco et al. (2011), we use the alveolar model of Vetharaniam et al. (2003) to approximate the maximum production (in MJ/week), as well as the net energy content of milk as a function of the fat and protein composition based on the work of Freer et al. (2007). However, for simplicity, we retain the linear relationship between energy input and milk production. This is then scaled by the number of cows lactating so that the quantity of milk produced by the herd in week t (kg/week)[5] from the m_t MJ of extra energy intake is m_t/η_t^m kg, where η_t^m is the net energy content of milk (MJ/kg) that is set as a constant in the model. In addition, we impose bounds on the total energy m_t that can be used by the herd for milk production so that:

$$\underline{\nu}\, C_t \leq m_t \leq \nu_t\, C_t, \tag{10}$$

where ν_t is the maximum biological rate of milk production (MJ/week) for a single lactating cow that is set as a constant in the model and $\underline{\nu}$ is the minimum biological rate of milk production (MJ/week) for a single lactating cow that is also set as a constant in the model. A rate of milk production lower than $\underline{\nu}$ can lead to lower milk quality and poor animal health (Dairy NZ, 2012a), and so this is not allowed.

Supplementation. Palm kernel is a by-product of the production of palm oil (Dairy NZ, 2008). It is commonly used as a supplementary feed (food that is fed in addition to grass) on New Zealand farms since it has a high energy content, is easily obtained (typically arriving on-farm within a day or two of ordering), and is relatively inexpensive. In this model we assume that the farmer orders a quantity s_t (kg/ha) of palm kernel on the spot-market and feeds it to their cows during week t. For simplicity, we ignore the ability of the farmer to store palm kernel and engage in forward contracting programs. Moreover, we do not consider other supplementary feeds such as maize.

[5]When discussing the mass of milk, we mean the mass of milksolids (milkfat + protein), not the mass of liquid milk. Milksolids are usually denoted kgMS; however, we drop this notation to avoid confusing readers unfamiliar with the dairy industry.

Energy Intake. Recall that in week t, the farmer feeds their herd f_t^p kg/ha of grass from pasture and f_t^q kg/ha of grass from storage. In addition, let η^p be the metabolizable energy content of pasture (both consumed from the paddock directly and stored), and let η^s be the metabolizable energy content of palm kernel. The total energy consumed by the herd in week t is therefore $\eta^p \times \left(f_t^p + f_t^q\right) + \eta^s \times s_t$.

Finally, we enforce an energy balance constraint for the entire herd so that:

$$\underbrace{\eta^p \times \left(f_t^p + f_t^q\right) + \eta^s \times s_t}_{\text{Energy Intake}} = \underbrace{C_t \times \varepsilon_t^{\text{lac}}}_{\text{Lactating Requirements}} + \underbrace{(\bar{C} - C_t) \times \varepsilon_t^{\text{dry}}}_{\text{Dry Requirements}} + \underbrace{m_t}_{\text{Milk Energy}}. \tag{11}$$

In the energy balance equation, the actions of the farmer are the quantities (in kg/ha) of grass from pasture f_t^p, grass from storage f_t^q, and palm kernel s_t to feed. In response, the cows apportion m_t MJ to milk production. The reader should note that as a combination of Eq. 10 and Eq. 11, the actions of the farmer are constrained so that they must provide sufficient energy input to meet the minimum demand of the cow ($\varepsilon_t^{\text{dry}}$ if the cow is dry, $\varepsilon_t^{\text{lac}} + \underline{\nu}$ if the cow is lactating), but no more than the maximum energy demand of the cow ($\varepsilon_t^{\text{dry}}$ if the cow is dry, $\varepsilon_t^{\text{lac}} + \nu_t$ if the cow is lactating).

In the next section we describe the milk price model within POWDer.

2.3. Milk Price Model

The contemporary New Zealand milk processing sector is dominated by Fonterra, a large processing co-operative that collects 84% of the milk produced in New Zealand (Taylor & Atherfold, 2017). Members of the co-operative are dairy farmers who share in the co-operative profits based on the number of kilograms of milk supplied during the season. The milk price is determined *ex-post* at the end of each year-long season. Therefore, during the season, farmers are uncertain about the price they will receive for milk they produce. Fonterra publish a forecast of the milk price at least every quarter, but these forecasts have large uncertainties of up to ±50% near the beginning of the season, with these forecasts becoming more accurate as the season progresses (Woodford, 2016b,a). We model this in POWDer as follows.

At the start of each week $t = 1, 2, \ldots, 52$, farmers observe a *forecast* milk price p_t, provided by Fonterra, of the *end-of-season* milk price p_{53}. We assume that the first week's forecast is p_1 = \$6/kg (the long-run average end-of-season milk price (Fonterra, 2018)). In the week after the end of the season (i.e. week 53), each farmer is paid the end-of-season milk price p_{53} for each kg of milk that they produced during the season. For simplicity, we shall model the sequence of forecast milk prices using nine equally likely scenarios. In each scenario, the forecast does not change until the start of week 26, when we assume new information emerges leading to an update. No further updates are made until the final week when the end-of-season milk price p_{53} is realized. This process can be represented by a scenario tree which branches prior to the beginning of weeks 26 and 53 as shown in Figure 3. This approach was chosen for two main reasons. First, this reduces the computational complexity of the model. Secondly, other approaches (like modelling the price process as an auto-regressive process) violate some of the technical assumptions required by SDDP. We call a single sequence of 53 observations of p_t (one forecast for each week of the season, plus the end-of-season milk price) a *price scenario*.

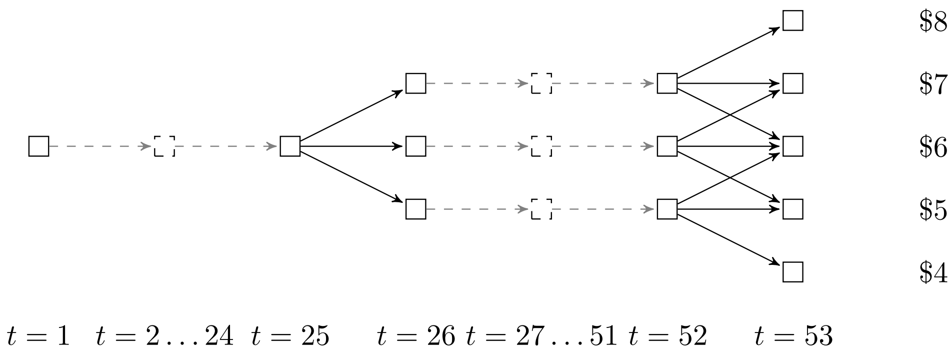


Figure 3: Scenario tree of the stochastic process for milk price. $t = 1$ corresponds to week 1.

As discussed above, we assume that the farmer sells all their milk in the final stage at the price associated with the leaf node in the scenario tree. This necessitates the introduction of a new variable to keep track of the quantity of milk produced to date:

$$M_{t+1} = M_t + m_t/\eta_t^m, \tag{12}$$

where M_t is the quantity of milk (kg/ha) produced to date at the start of week t.

We now describe the formulation of the multi-stage optimization model that utilizes the three sub-models previously described.

2.4. Optimization Model

We formulate POWDer as a finite-horizon, discrete-time, stochastic optimal control problem over a single year (season). We decompose the season into a sequence of 53 weekly blocks (called stages) and link these blocks together by state variables (information that is necessary to communicate the status of the farm between stages).

Each weekly block $t = 1, 2, \ldots, 52$ has two inputs (the value of the state variables at the start of the week and a realization of the random variables) and an output (the value of the state variables at the end of the week). There are five state variables in the model: the soil moisture W_t (mm), the pasture cover P_t (kg/ha), the quantity of grass in storage Q_t (kg/ha), the number of cows milking C_t (cows/ha), and the quantity of milk produced to date M_t (kg/ha). These state variables are illustrated by the five rectangular boxes in Figure 1. The transition of the state variables from their value at the start of the week to their value the end of the week is governed by the series of constraints and relationships that we have detailed in the above section (i.e., Eqs. (3)–(12)). We briefly summarize the main points below.

At the beginning of each week $t = 1, 2, \ldots, 52$, the farmer observes the potential evapotranspiration e^p_t (mm), the quantity of rainfall r_t (mm), and the milk price forecast p_t (\$/kg). This makes the model a *Hazard-Decision* or *Wait-and-See* model. We assume that the farmer can observe these random variables at the start of the week, which loses little generality since short-term weather forecasts are reasonably accurate. The random variables are illustrated by the three wavy lines in Figure 1.

After the farmer has observed the state of the farm and the current realization of the random variables, the farmer needs to decide the quantity of irrigation to apply i_t (mm/ha), the quantity of pasture to harvest h_t (kg/ha), the number of cows to dry-off u_t (cows/ha), and the quantities (kg/ha) of grass (from pasture f^p_t and storage f^q_t) and palm kernel s_t to feed the cows. These control variables are illustrated by the double-lines with a bold arrow head in Figure 1.

There are also two auxiliary variables that are consequences of the states, controls, and random variables. These are the actual evapotranspiration e_t (mm/week) (which differs from the *potential* evapotranspiration e^p_t), and the quantity of milk m_t (kg/ha) produced by the herd during week t. These variables are illustrated by the single lines in Figure 1.

After the end-of-season milk price is realized at the start of week $t = 53$, the farmer sells the total quantity of milk produced over the season M_{53} at the end-of-season milk price p_{53}.

Objective. In POWDer, we assume the farmer is risk-neutral. The objective of the farmer is to choose actions at each stage to maximize, in expectation, the total profit from selling milk, less the cost of purchasing palm kernel, harvesting pasture, and applying irrigation:

$$\max \quad \mathbb{E}_{\omega}\left[p_{53}^{\omega} M_{53}^{\omega} - \sum_{t=1}^{52}\left(c^{s} s_{t}^{\omega} + c^{h} h_{t}^{\omega} + c^{i} i_{t}^{\omega}\right)\right], \tag{13}$$

where the sample space of ω is the Cartesian product over stages of the weather and price scenarios. Variables with the superscript ω refer to the value of that variable in stage t and scenario ω. There are three costs which appear as constants in the model: c^s is the cost of palm kernel (\$/kg), c^h is the cost of harvesting (\$/kg/ha), and c^i is the cost of irrigation (\$/mm). This optimization problem can be formulated as a dynamic programming recursion where the value-to-go in stage t, V_t, is the maximum expected future profit from stage $t+1$ until the final stage, assuming that the farmer takes the optimal action in each stage (given the current state). For $t = 1, 2, \ldots, 52$, V_t can be expressed as follows:

$$\begin{aligned} V_t(x_t, e_t^p, r_t, p_t) = \max \quad & \underbrace{\mathbb{E}_{e_{t+1}^p,\, r_{t+1},\, p_{t+1}}\left[V_{t+1}(x_{t+1}, e_{t+1}^p, r_{t+1}, p_{t+1})\right]}_{\text{Future Profit}} - \underbrace{\left(c^s s_t + c^h h_t + c^i i_t\right)}_{\text{Week } t \text{ cost}} \\ \text{s.t.} \quad & Eq.\ (3) - (12) \end{aligned}$$

where x_t denotes the vector of state variables (W_t, P_t, Q_t, C_t, M_t) evaluated at the start of stage t.

In the final stage $t = 53$, the formulation is different since the season has ended, and all that occurs is the final payment of milk:

$$V_{53}(x_{53}, e_{53}^p, r_{53}, p_{53}) = \underbrace{p_{53} M_{53}}_{\text{Milk Payment}}.$$

The objective of the farmer is to maximize the expectation of the first-stage objective:

$$\max \quad \mathbb{E}_{e_1^p,\, r_1}[V_1(x_1, e_1^p, r_1, p_1)], \tag{14}$$

where x_1 specifies the initial state of the farm, and $p_1 = \$6/\text{kg}$ as defined earlier. Below, we outline two further enhancements to the model to improve its realism.

Pasture Cover Penalty. Although POWDer is a single-season model, dairy farming is a multi-year endeavor. Therefore, we need to ensure that the farm is in an equal (or better) condition at the end of the season than it was at the start, in order to ensure that the next season's performance is not negatively affected by this season's actions. Therefore, we convert the final stage problem to an optimization problem and add an artificial penalty variable $\Delta^p \geq 0$ to measure the amount by which the final pasture cover is less than the initial pasture cover:

$$P_{53} + \Delta^p \geq P_1.$$

Then we penalize Δ^p by a large coefficient (arbitrarily chosen to be 1000) in the objective function of the final stage:

$$\begin{aligned} V_{53}(x_{53}, p_{53}) = \max \quad & p_{53} M_{53} - 1000\Delta^p \\ \text{s.t.} \quad & P_{53} + \Delta^p \geq P_1. \end{aligned}$$

Fat Evaluation Index Penalty. Feeding palm kernel as a supplement to lactating cows changes the amount and the ratio of fatty acids in the milkfat produced by the cows. When fed palm kernel at high levels, the milkfat produced by a cow becomes difficult to process and meet customer requirements for products (Dairy NZ, 2017b). To quantify this, a Fat Evaluation Index has been developed by a collaboration of industry groups. Beginning June 1, 2018, New Zealand dairy farmers will have their milk graded into four Fat Evaluation Index grades: *A*, *B*, *C*, and *D*. An *A* grade is given to milk that is acceptable for processing. A *B* grade is given to milk that is approaching the acceptable limit for processing. A *C* grade is given to milk that exceeds the acceptable limit by a small amount. A *D* grade is given to milk that exceeds the limit by a large amount. Milk that is graded as a *C* or *D* will incur (currently unspecified) financial penalties. Current information released by the industry suggests that as a rule of thumb, three kilograms of palm kernel per cow per day is approximately the upper limit for maintaining an *A* or *B* grade (Dairy NZ, 2017b). To model the penalties associated with the excess intake of palm kernel, we introduce a penalty based on the quantity of palm kernel fed per cow per day (Figure 4). If the farmer feeds less than 3 kg/cow/day, the penalty is zero. Between 3 and 4 kg/cow/day, the penalty increases at a rate of \$0.25/kg/cow/day. Between 4 and 5 kg/cow/day, the penalty increases at a rate of \$0.5/kg/cow/day. Above 5 kg/cow/day, the penalty increases at a rate of \$1/kg/cow/day.

At the time of this research penalties for exceeding the Fat Evaluation Index threshold have not yet been decided and are likely to be based on the total milk produced, rather than palm kernel

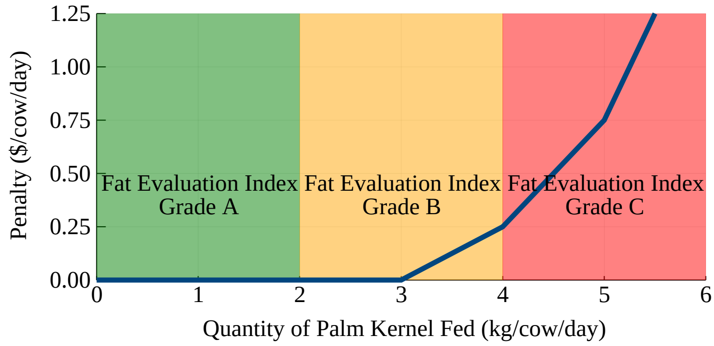


Figure 4: Fat Evaluation Index Penalty for differing levels of intake. Shaded vertical bands represent different Fat Evaluation Index grades: A in green, B in orange, and C in red (D not shown).

fed. Our approximation as a convex cost on palm kernel fed, preserves the convexity necessary for optimization using SDDP.

Summary. POWDer models the decision process as follows. (The reader may find it helpful to view Figure 1 in conjunction with this summary.) At the beginning of week t, the farmer measures the five state variables in the model: the soil moisture W_t (mm), the pasture cover P_t (kg/ha), the quantity of feed in storage Q_t (kg/ha), the number of cows milking C_t (cows/ha), and the quantity of milk produced to date M_t (kg/ha). Before choosing an action, the farmer observes the realization of the three random variables: the potential evapotranspiration e_t^p (mm), the quantity of rainfall r_t (mm), and the forecast milk price p_t ($/kg). Taking into account the current state and the observation of the random variables, the farmer decides the quantity of irrigation to apply i_t (mm), the quantity of pasture to harvest h_t (kg/ha), the number of cows per hectare to dry-off u_t (cows/ha), and the quantities (kg/ha) of grass (from pasture f_t^p and storage f_t^q) and palm kernel s_t to feed the herd. As a result of these actions, the actual evapotranspiration e_t (mm) and energy apportioned for milk production m_t (MJ/ha) are observed, and the system transitions to a new state that will serve as the incoming state in the next week. In addition, the farmer will incur the cost of purchasing palm kernel, harvesting pasture, and applying irrigation. In the 53$^{\text{rd}}$ week, the farmer sells the milk produced during the season M_{53} at the end-of-season milk price p_{53}.

3. Case Study

In this section we investigate the application of POWDer to a working dairy farm in the Bay of Plenty region of New Zealand. In the following, we refer to it as the *case farm*.

3.1. Calibration

We begin by describing how POWDer was calibrated to the case farm. The case farm is not irrigated, so i_t was set to 0 for all stages. In addition, the farmer estimated the cost of harvesting c^h at \$275/t, and the metabolizable energy content of pasture η^p and palm kernel η^s to both be 11 MJ/kg. Furthermore, a maximum lactation length of 44 weeks was set (so that $C_t = 0,\ t = 45, \ldots, 52$) as the farmer considered lactation lengths longer than this to be impractical. The minimum quantity of energy for milk production $\underline{v}$ was set to 500 MJ/cow/week based on the value reported by (Dairy NZ, 2012a). We also scaled the net energy content of milk η_t^m predicted by the model of Freer et al. (2007) by 1.2 so that it matched the value (≈80 MJ/kg) reported by Dairy NZ (2012a). The farmer confirmed that, given the quantity of feed consumed by the herd, the simulated milk production in the results below was representative of the case farm.

DairyBase. DairyBase (Dairy NZ, 2017a) is a voluntary database of financial and physical key performance indicators for New Zealand dairy farmers. It provides a standardized reporting mechanism that can be used to benchmark a farm's performance. The owner of the case farm provided us with access to their DairyBase data for the 2013/14 and 2014/15 production seasons. The data included information such as the total milk production per hectare, the quantity of pasture and palm kernel eaten, and the lactation length. It also included a detailed break-down of operating expenditure. We provide a summary of the data in Table 1. We draw the reader's attention to two items: first, the palm kernel cost is not the price per tonne from the supplier (typically \$275/t – \$350/t for palm kernel (Dairy NZ, 2016)) but includes the additional costs of storage, as well as wastage and spoilage; and second, the line item *Fixed Expenses* accounts for all costs excluding the cost of palm kernel. This allows us to standardize the operating profit per hectare to account for changes in milk price and palm kernel costs.

Based on the DairyBase data, the stocking rate $\bar{C}$ was set at 3 cows/ha and the cost of palm kernel c^s was set at \$500/t.

	2013/14	2014/15	Average
Stocking Rate (cows/ha)	3	3	3
Milk Price ($/kg)	8.40	4.40	6.40
Milk Production (kg/ha)	1240	1146	1193
Lactation Length (days)	275	256	266
Pasture Consumption (t/ha)	12.6	11.7	12.15
Palm Kernel Consumption (t/ha)	2.8	2.9	2.85
Palm Kernel Cost ($/t)	473	419	446
Fixed expenses ($/ha)	3512	3560	3536

Table 1: Summary data for the case farm over two seasons, and the average of these.

Weather Data. Historical data were obtained from the NIWA CliFlo database (NIWA, 2017) for the incident rainfall r_t and evapotranspiration potential e^p_t from 1 January 1997 to 31 December 2016 for the three closest stations to the farm (Wharawhara Water Stn, Tauranga Aero Stn, and Whakamaramara). Then, for each week in the dataset, we averaged the data from all three stations. This resulted in 20 historical readings for each week of the year for both the evapotranspiration potential e^p_t and incident rainfall r_t. Plots showing the historical distributions of rainfall and evapotranspiration potential by week are given in Figure 5.

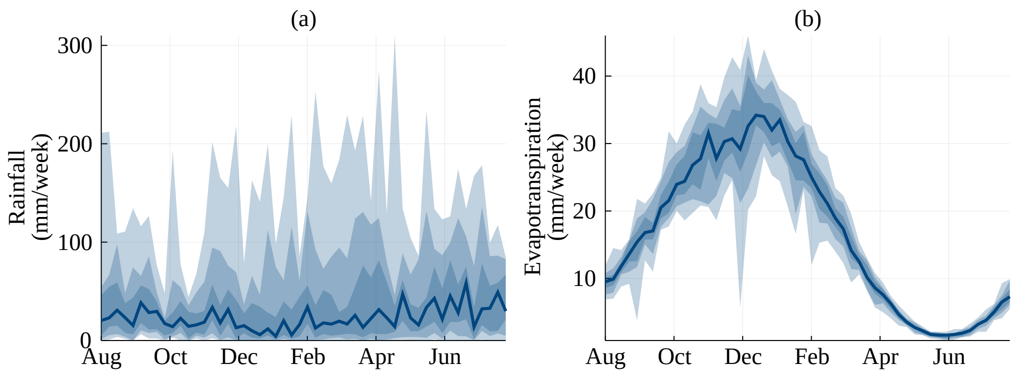


Figure 5: Historical distributions for rainfall (a) and evapotranspiration (b). In order of increasing darkness, shaded bands correspond to 0-100, 10-90, and 25-75 percentiles. The dark single line represents the 50$^{\text{th}}$ percentile for each week.

We construct a stagewise-independent stochastic process for the weather by empirically sampling with replacement from historical readings that occurred during that week. There are 20 readings in each week. This gives a total of 20^{52} possible different weather scenarios for the stochastic weather process. When coupled with the nine possible price scenarios from the milk price model, there are a total of 9×20^{52} possible outcomes in the sample space ω in Eq. 13.

Grass Model Parameters. The maximum rate of pasture growth $\Delta P_{\max}$ was assumed to be 65 kg/ha/day, and the maximum pasture cover $P_{\max}$ was assumed to be 3500 kg/ha. These values are regional averages obtained from Dairy NZ (2012b) and from an interview with the owner of the case farm. The farmer was only able to supply cumulative annual growth data. Therefore, it was difficult to calibrate the soil-fertility index κ. Instead, we calibrated κ, by dividing the monthly average pasture growth of a nearby farm (chosen by the farmer from Dairy NZ (2012b)) with the average weekly evapotranspiration readings for the case farm to obtain a weekly estimate for κ_t. These weekly estimates were then scaled so that when κ_t was multiplied by the historical potential evapotranspiration data for the 2013/14 and 2014/15 seasons, the annual pasture growth matched the total grown on the farm during the 2013/14 and 2014/15 seasons. The farmer confirmed that the predicted growth rates were representative of the case farm.

3.2. Solution Method

The POWDer model was implemented in the Julia (Bezanson et al., 2017) language using the SDDP.jl (Dowson & Kapelevich, 2017) and JuMP (Lubin & Dunning, 2015) packages. Gurobi (Gurobi Optimization, 2016) was used to solve the linear subproblems. The model was solved for 1000 SDDP iterations to form an approximately optimal policy using the initial conditions $(W_1, P_1, Q_1, C_1, M_1) = (150, 2500, 0, \bar{C}, 0)$. The model converged to a first stage objective value (Eq. 14) of \$2226/ha. Then, 1000 Monte Carlo replications were conducted using the policy and summary statistics were collected for each of these 1000 scenario realizations. Using a single thread of an Intel i7-4770 CPU with 16GB of memory, the solution time is approximately eight minutes. In total, the solution process involved solving over 2.3 million linear programs.

3.3. Results

In Table 2, we present a summarized set of statistics to compare the simulated results against the historical average of what happened on the case farm during the 2013/14 and 2014/15 seasons. In all simulations, the model consumed more palm kernel than the case-study farm. This suggests that for the case farm, even at a cost of \$500/t for palm kernel (which is larger than the average cost reported in the DairyBase data), feeding a low constant amount of 3 kg/cow/day of palm kernel is profitable. In addition, due to the extra feed imports, the average lactation length (i.e., the mean number of days milking for a cow in the herd) in almost all simulations was longer than that of the case farm (a median of 44 weeks compared to 38.6 weeks). The extra feed and lactation length increased the milk production per hectare from the historical average of 1193 kg/ha to a median value of 1360 kg/ha in the simulations (+14% increase). In turn, this increased operating profit from \$2197/ha to a median value of \$2542/ha. However, in the worst simulated case the operating profit was −\$1608/ha. This coincided with a scenario with low rainfall and low price.

In Figure 6 and Figure 7, we visualize 1000 Monte Carlo simulations across a range of different dimensions. Both figures sampled the same weather scenarios but different price scenarios. In Figure 6, we plot simulations that sampled the low milk price state during the second half of the year (i.e. p_{26} = \$5/kg), whereas in Figure 7, we plot simulations that sampled the high milk price state during the second half of the year (i.e. p_{26} = \$7/kg).

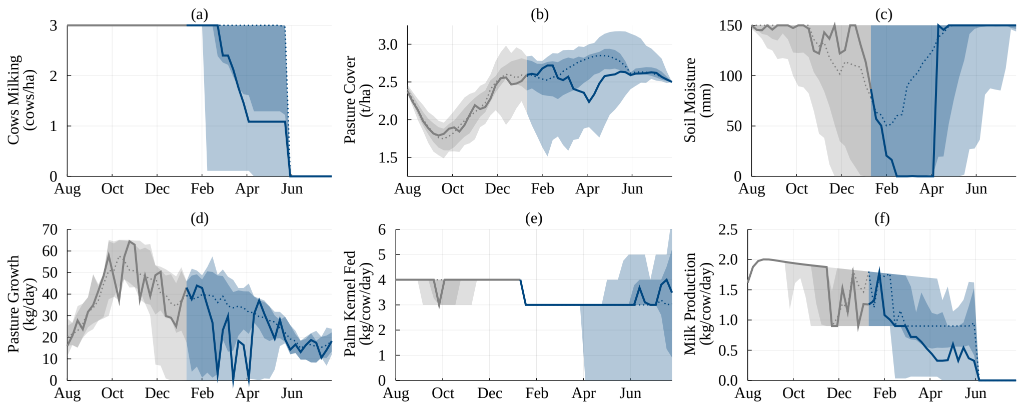


Figure 6: Simulated seasons that sample a low milk price during the second half of the season using the optimal policy.

In each of the subplots, we plot the 0–100 percentiles of the distribution of the plotted variable as a light shaded band. The dark shaded bands correspond to the 10–90$^{\text{th}}$ percentiles. The dotted line corresponds to the 50$^{\text{th}}$ percentile. In addition, we highlight one randomly chosen scenario out of the 1000 simulations with a thick solid line. The same weather scenario was chosen in each plot to highlight how the farmer's weekly decisions vary with different price forecasts.

The first half of each subplot is plotted in gray because both figures visualize the same 1000

	Simulation Percentiles					Historical
	0	25	50	75	100	Avg.
Milk Price ($/kg)	4	5	6	7	8	–
Palm Kernel Cost ($/t)	500	500	500	500	500	–
Avg. Lactation Length (weeks)	31.1	44.0	44.0	44.0	44.0	38.6
Milk Production (kg)						
per Hectare	1006	1302	1360	1407	1511	1193
per Cow	335	434	453	469	504	398
Milk Revenue ($/ha)	4024	6827	8171	9600	11920	7158
Feed Consumed (t/ha)						
Pasture	9.64	11.83	12.16	12.41	13.14	12.15
Palm Kernel	3.58	3.86	4.32	4.41	4.66	2.85
% Feed Imported	22.1	24.5	26.0	26.8	30.6	19
Palm Kernel Expense ($/ha)	1790	1929	2162	2203	2330	1425
Fixed Expense ($/ha)	3536	3536	3536	3536	3536	3536
Operating Profit ($/ha)	−1608	1243	2542	3873	6157	2197
Fat Evaluation Index Penalty ($/ha)	96	146	264	297	427	–

Table 2: Summary results of 1000 simulations of one SDDP policy. Percentiles are calculated independently between rows. Therefore, the simulation with the minimum average lactation length may not coincide with the scenario with the minimum milk production. Historical operating profit was calculated using the average milk price and palm kernel cost from the simulation. All other historical values are derived from the average of the case farm's DairyBase data for the 2013/14 and 2014/15 seasons. Operating profit for the simulation percentiles excludes the Fat Evaluation Index penalty for comparison with historical average.

weather scenarios, and the new price information is not revealed until week 25. Therefore, the simulations for the first half of the year are identical between Figure 6 and Figure 7. All the cows are kept milking (Figure 6a and Figure 7a), and they are fed 4 kg/cow/day of palm kernel (Figure 6e and 7e). This level of feeding is at the upper end of the acceptable Fat Evaluation Index grade scale. This suggests that the small penalty ($0.25/kg/cow/day) that is incurred is less than the value of the additional milk that is produced. However, differences in rainfall and evapotranspiration between individual simulated scenarios lead to different trajectories for soil moisture (Figure 6c and Figure 7c) and pasture growth (Figure 6d and Figure 7d). Despite these

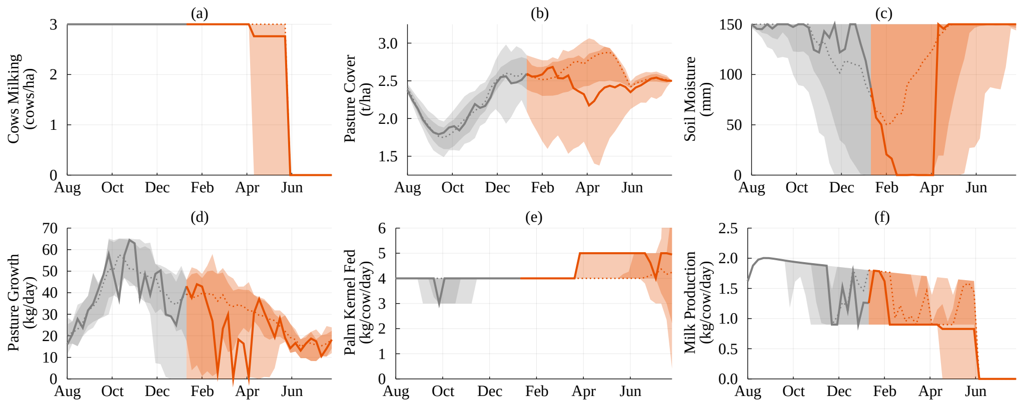


Figure 7: Simulated seasons that sample a high milk price during the second half of the season using the optimal policy.

differences, the pasture cover follows a clearly observable trend. For the first two months, the rate of grass growth is less than that fed to the cows. This causes the pasture cover to decrease. However, in late September, the pasture growth begins to exceed the rate of consumption and the pasture cover increases. Pasture growth peaks in late October before gradually declining over the remainder of the season.

During the second half of the year, the optimal controls diverge depending on the forecast milk price that is observed. In the high milk-price states (Figure 7), the optimal management strategy is to keep the entire herd milking for as long as possible (Figure 7a) in almost all weather scenarios. This is achieved by increasing the quantity of palm kernel fed to 5 kg/cow/day in April (Figure 7e). This incurs an additional Fat Evaluation Index penalty. However, based on the current price forecast, the expected value of the milk exceeds the cost of the Fat Evaluation Index penalty.

In contrast, depending on the observed rainfall and evapotranspiration, the trajectories in the low milk price states (Figure 6) begin drying off the herd in February. However, if weather conditions are favourable for growing grass, the dry-off decision is delayed. In the trajectory highlighted by the solid blue line, a drought during late February and March (Figure 6c) causes the model to begin drying off cows during February (Figure 6a). As the drought progressively worsens, more cows are dried off, and the total pasture cover (Figure 6b) declines. By April, the drought breaks, and the pasture cover begins to increase. However, due to the low forecast price, the model does not increase the rate of palm kernel intake (Figure 6e) to extend lactation.

Instead, the rate of palm kernel intake is increased during June to avoid the penalty arising from the final pasture cover (Figure 6b) being lower than the initial pasture cover.

The quantity of milk produced per day (Figure 6f and Figure 7f) tends to switch between the maximum limit and the minimum limit (Eq. 10). This bang-bang behavior is a well-known artifact of linear optimal control problems. This feature can also be observed in Figure 6e and Figure 7e where, in most cases, the optimal quantity of palm kernel to feed is at one of the break-points in the Fat Evaluation Index penalty function (Figure 4).

3.4. A Lower Stocking Rate

The 2015/16 season was characterized by very low international milk prices. As a result, the end-of-season milk price was \$3.90/kg. In order to reduce costs, the case farm reduced its stocking rate from 3 cows/ha to 2.7 cows/ha. To investigate this decision, we re-solved the POWDer model with $\bar{C}$ set to 2.7 cows/ha. All other parameters were kept the same.

In Table 3, we present the same set of summarized statistics as Table 2. Compared with the case where the stocking rate was 3 cows/ha, the median operating profit per hectare increases from \$2542/ha to \$2846/ha (+12%). This is achieved by increasing per cow production from a median of 453 kg/year to 502 kg/year despite the quantity of palm kernel imported decreasing from a median of 4.32 t/ha to 3.62 t/ha (−16%) and a similar quantity of pasture being grown (12.16 t/ha compared with 12.15 t/ha). One explanation for this is that by decreasing the stocking rate, the quantity of feed needed to account for fixed costs (such as maintenance and pregnancy) decreases. Therefore, for a similar quantity of feed, more milk can be produced (provided the cows are not at their biological maximum).

In Figure 8 we plot the two distributions in operating profit for both stocking rates. The five peaks in the density functions correspond to the five different milk prices that can be observed. Variations around each peak correspond to the uncertainty associated with the weather. Under every scenario, the simulated operating profit with 2.7 cows/ha was greater than the simulated operating profit with 3 cows/ha, and the expected operating profit (Eq. 14) increased from \$2270/ha to \$2646/ha (+17%).

Reducing the stocking rate has additional benefits for the farmer that the model does not capture. Costs that are incurred on a per cow basis (which we amortized into a per hectare figure) such as animal health and young stock management (animals that are too young to produce milk and are grazed off-farm) will decrease. In addition, the environmental impact of the farming

	Simulation Percentiles					Historical
	0	25	50	75	100	Avg.
Milk Price ($/kg)	4	5	6	7	8	–
Palm Kernel Cost ($/t)	500	500	500	500	500	–
Avg. Lactation Length (weeks)	34.6	44.0	44.0	44.0	44.0	38.6
Milk Production (kg)						
per Hectare	1021	1297	1355	1398	1472	1193
per Cow	378	480	502	518	545	398
Milk Revenue ($/ha)	4103	6798	8128	9556	11702	7158
Feed Consumed (t/ha)						
Pasture	9.64	11.81	12.15	12.39	12.94	12.15
Palm Kernel	2.98	3.23	3.62	3.71	3.98	2.85
% Feed Imported	19.0	21.4	22.8	23.6	28.0	19
Palm Kernel Expense ($/ha)	1489	1617	1812	1855	1989	1425
Fixed Expense ($/ha)	3536	3536	3536	3536	3536	3536
Operating Profit ($/ha)	−1275	1551	2846	4162	6312	2197
Fat Evaluation Index Penalty ($/ha)	22	72	173	203	349	–

Table 3: Summary results of 1000 simulations of one SDDP policy with stocking rate of 2.7 cows/ha. Percentiles are calculated independently between rows. Therefore, the simulation with the minimum average lactation length may not coincide with the scenario with the minimum milk production. Historical operating profit was calculated using the average milk price and palm kernel cost from the simulation. All other historical values are derived from the average of the case farm's DairyBase data for the 2013/14 and 2014/15 seasons. Operating profit for the simulation percentiles excludes the Fat Evaluation Index penalty for comparison with historical average.

operation may decrease since there are fewer animals to produce greenhouse gas emissions and nitrate leeching. Moreover, labour costs will decrease as the time spent performing manual tasks such as milking will decrease. This suggests that we may have underestimated the financial benefit of the reduced stocking rate since we assumed that the fixed cost per hectare was the same for the two configurations.

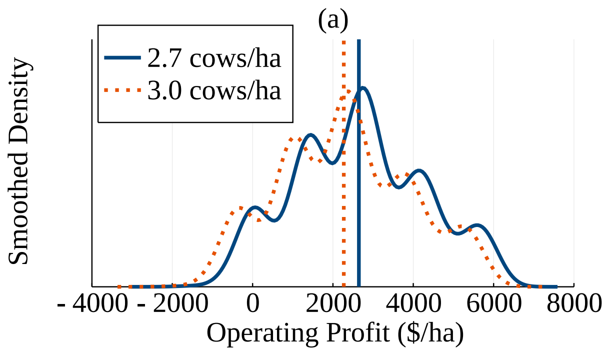


Figure 8: Smoothed density of operating profits ($/ha) under differing stocking rates. Vertical lines correspond to the mean of each distribution. The five peaks in each distribution correspond to the five end-of-season milk prices p_{53}: $4/kg, $5/kg, $6/kg, $7/kg, and $8/kg.

4. Discussion

Our current model only takes eight minutes to solve. This allows scope for a more detailed model. In particular, the stochastic price model used in POWDer is a simplistic approximation for the true price process. For simplicity, we have also chosen to ignore recent developments such as the launch of the NZX[6] Milk Price Futures (NZX, 2018) and more detailed milk price forecast models that utilize the bi-monthly Global Dairy Trade auctions (AgriHQ, 2016). A more detailed model could be created by extending the scenario tree approach taken in this paper and allowing the farmer to sell milk within the season rather than just at the end.

One of the largest critiques of this model is the way in which we calculate the net energy requirements of the cow (detailed in Appendix A). Readers familiar with cow biology may question the assumption that the cow follows a predetermined body condition trajectory, with any net energy contributing linearly to milk production. In reality, the cow will partition the energy between lipid deposition and milk production. Attempts to model this partitioning have resulted in non-convex relationships (e.g. (Baudracco et al., 2011)). This would create a non-convex Bellman function, precluding the use of SDDP as a solution technique. We justify the approach taken in this model by observing that variations from the typical body condition trajectory typically represent poor health outcomes for the animal (Roche et al., 2009).

[6] New Zealand Stock Exchange

A second critique is that to simplify the model, we have assumed that the cows will consume all of the energy provided by the farmer in the form of pasture and palm kernel. This ignores consumption limits, as well as the substitution effect that supplementary feeding has upon pasture intakes (Doole et al., 2012). However, when compared with the DairyBase data from the case farm, POWDer produced realistic results that could be meaningfully interpreted.

A third critique is our approximation of the stochastic processes for rainfall and evapotranspiration potential. In this paper, we assume that they are stagewise-independent and drawn from historical observations. Future work should be conducted to validate these assumptions and explore alternative processes, such as an auto-regressive process.

Finally, in the introduction we outlined the IDEA (Integrated Dairy Enterprise Analysis) model, which we consider to be the current state-of-the-art for a multi-stage optimization model of a dairy farm. Although similar in many respects, IDEA and POWDer answer different questions. IDEA focuses on optimizing a detailed model of a farm under deterministic conditions. It models many variables that POWDer does not, including, for example, the impact of stocking rate on pasture utilization. In contrast, POWDer solves a simplified model of a farm that incorporates stochasticity. This trade-off is necessary in order to maintain tractability of the model. In the future, it would be interesting to incorporate some of the features of IDEA into POWDer, such as a heterogeneous herd. It would also be interesting to simulate the dry-off and feeding decisions that arise from POWDer in IDEA, which is a higher fidelity model.

5. Conclusion

In this paper we have described a multi-stage stochastic linear optimization program of a New Zealand dairy farm. In comparison to existing deterministic models, POWDer relies upon a simpler farm-level model to provide insight into farm management practices under uncertainty.

We have shown that optimal management strategies for a case farm in the Bay of Plenty region of New Zealand differ based on the combination of economic and weather uncertainties. In particular, the model is able to decide the quantity of palm kernel to feed and when to dry cows off, based on the forecast milk price and current pasture cover. We used the model to analyze the impact of a reduction in stocking rate for the case farm. This found that the operating profit improved in every scenario, even if we exclude the additional environmental and economic benefits associated with a reduction in stocking numbers.

Overall, this paper demonstrates that large multi-stage stochastic programs in an agricultural context can be solved efficiently to generate meaningful insights for practitioners. We hope that POWDer serves as a foundation upon which improvements, particularly around the milk price process, can be made.

Supplementary Materials. All of Julia code to implement POWDer in `SDDP.jl`, as well as all of the data needed to run the case-study and replicate the results in this paper are available as supplementary materials at `https://github.com/odow/MilkPOWDER`.

Acknowledgments

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Appendix A. Description of Animal Model

As we outlined in Section 2.2, we model the cow by a simple energy balance. In addition to milk production, we assume that the cow requires energy for three functions: maintenance, pregnancy, and to change body condition. In this appendix, we provide the reader with more detail about the literature we draw from to estimate these requirements and highlight some subtle differences that we have made.

Body Condition. The Body Condition Score (BCS) is an assessment of the proportion of body fat in a cow and is used by farmers as an important indicator of animal health (Roche et al., 2009). In New Zealand, a ten-point scale is used, where 1 represents a severely emaciated cow and 10 represents a severely obese cow. Over a season, cows typically follow a genetically driven cycle in their body condition score (although under or over feeding can cause the cow to deviate from the typical trajectory). The cycle begins after the cow calves and starts with the mobilization of body lipid into energy to contribute to the energetic cost of early lactation. This causes the BCS to decrease. A few weeks after conception for the next season (100 – 120 days into the season), the cow reverses this trend and begins to deposit body lipid in preparation for the next calving. This causes the BCS to increase (Friggens et al., 2004).

In this model, we assume that the BCS trajectory is known *a priori*, and that the cow modifies its milk production to satisfy the energy balance, rather than a combination of both milk production and BCS change. This is a reasonable assumption as deviating from the trajectory typically represents a poor health outcome for the cow. However, it also reduces the number of actions the farmer can take as underfeeding the cow is no longer feasible.

For the case study in this paper, we assume that the cow follows the body condition trajectory as calculated by the Friggens model[7] (Friggens et al., 2004) assuming an initial liveweight of 450kg and an initial BCS of 5.3 (in NZ Units (Roche et al., 2009)).

Next, we need to relate the change in BCS to the net energy requirements of the cow during a week. To do this, we draw from Dairy NZ (2012b), who assume that a one unit change in BCS corresponds to a change of 6.58% of the cow's liveweight. Therefore, a one unit change in BCS for a 450 kg cow is approximately a 30 kg change in liveweight. However, the energy required

[7]The "adjusted" parameter set (as it is called by Friggens et al. (2004)) was used as it demonstrated a slightly better fit for their historical data.

to increase liveweight by 1 kg is greater than the energy released by decreasing liveweight by 1 kg, and this amount of energy depends upon whether the cow is lactating or dry (see Table A.4).

	Lactating Cows	Dry Cows
Liveweight Gain	50	72
Liveweight Loss	-37	-30

Table A.4: Energy required per change in liveweight (MJ/kg) for cows that are lactating and dried-off (Dairy NZ, 2012b).

However, with this information we can calculate *a priori*, the contribution to the cow's energy balance due to changes in BCS for each week.

Maintenance. Maintenance is the energy required to run the cow's core bodily functions. This can be considered a fixed cost of keeping the cow alive. In order to meet this requirement when the cow is underfed, energy is mobilized from body lipids (decreasing the BCS) and milk production is reduced. In this model, we consider the energy for maintenance to be constant, and the farmer must provide sufficient energy input to meet this requirement. For a 450 kg cow, Dairy NZ (2012b) assume the maintenance requirement to be 54 MJ/day.

Pregnancy. Dairy NZ (2012b) give very approximate energy requirements for pregnancy at the 2, 4, 8, and 12 weeks before calving, as well as an annual total. To obtain a better estimate, we use the energy required for pregnancy equations of SCA (1990) (which were also used in Baudracco et al. (2011)). However, it is our belief that this equation is over-fitted[8]. If we assume $W = 47$ kg, we can fit, using simple linear regression, the exponential function:

$$\text{energy for pregnancy } = 0.2278e^{0.01989d}, \tag{A.1}$$

where d is the number of days since conception. The greatest absolute error between the fitted curve and the original equation is 0.46 MJ/day and, over a 284-day gestation, the simplified model predicts that the cow will require 6.5 MJ in additional energy for pregnancy. That is less than 1 kg of pasture over a season. In our view this small discrepancy does not justify the additional terms in the equation of SCA (1990). This model also gives similar results to the data

[8] The full equation is given (in MJ/day) as $9.663 \times 10^{-5} \times W^2 \times e^{-5.76\times10^{-5}d} \times 10^{151.665-151.64\times e^{-5.75\times10^{-5}d}}$ where d is the number of days since conception and W is the birth weight of the calf (kg).

given in Dairy NZ (2012b), which suggests that a Jersey-Friesian cross needs 3240 MJ/year for pregnancy (compared to 3272 MJ/year from Eq. A.1) and 21 MJ/day eight weeks before calving (compared to 21.2 MJ/day from Eq. A.1). Using Eq. A.1, we can calculate the energy (MJ/day) required by the cow for a given week t.

Total Energy Requirements. If Figure A.9, we plot the energy required by the cow over time when it is lactating and dried-off. Of the total required energy, the fixed cost (54 MJ/day) of maintenance represents a large proportion. Towards the end of the season, the energy required for pregnancy also increases. The orange lines represent the net energy requirements for changes in body condition (BCS). At the start of the season, the cow loses body condition, and so the required energy is negative. After reaching nadir around conception, the cow begins gaining body condition and the requirement is positive. Note that dry cows require more energy for gaining body condition than lactating cows due to the values in Table A.4.

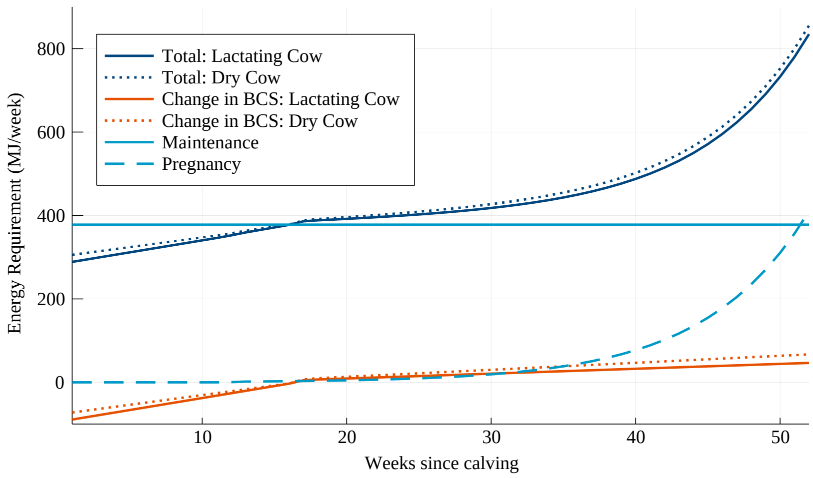

Figure A.9: Energy required per week for lactating and dry cows.