# Robust Multicovers with Budgeted Uncertainty 

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#### Abstract

The Min- $q$-Multiset Multicover problem presented in this paper is a special version of the Multiset Multicover problem. For a fixed positive integer $q$, we are given a finite ground set $J$, an integral demand for each element in $J$ and a collection of subsets of $J$. The task is to choose sets of the collection (multiple choices are allowed) such that each element in $J$ is covered at least as many times as specified by the demand of the element. In contrast to Multiset Multicover, in Min- $q$ Multiset Multicover each of the chosen subsets may only cover up to $q$ of its elements with multiple choices being allowed.

Our main focus is a robust version of Min- $q$-Multiset Multicover, called Robust Min- $q$-Multiset Multicover, in which the demand of each element in $J$ may vary in a given interval with an additional budget constraint bounding the sum of the demands. Again, the task is to find a selection of subsets which is feasible for all admissible demands.

We show that the non-robust version is NP-complete for $q$ greater than two, whereas the robust version is strongly NP-hard for any positive $q$. Furthermore, we present two solution approaches based on constraint generation and investigate the corresponding separation problems.

We present computational results using randomly generated instances as well as instances emerging from the problem of locating emergency doctors.


## 1 Introduction

Covering problems arise in many real world applications. Therefore, there has been a lot of research regarding this area of optimization. In the classical Set Cover problem we are given a set $U$, a collection of subsets $\mathcal{S} \subseteq 2^{U}$ and a positive integer $k$. The decision version asks for the existence of a

[^0]subcollection $\mathcal{S}^{\prime} \subseteq \mathcal{S}$ of size at most $k$, such that, for all $u \in U$, there is some $S \in \mathcal{S}^{\prime}$ with $u \in S$. This problem is well-known to be strongly NPcomplete, see [20]. In the Set Multicover problem each element $u \in U$ is given a demand $d_{u} \in \mathbb{Z}_{\geq 0}$ expressing the number of times the element $u$ has to be covered. Finally, considering the Multiset Multicover problem, the subsets in the collection may be multisets, cf. [27]. These two variants are clearly generalizations of Set Cover and therefore remain strongly NPcomplete. Furthermore, the problems above remain strongly NP-complete if only subsets of a fixed size $q \geq 3$ are regarded, cf. [20].

The notion of robustness has gained a lot of attention in operations research. The core idea of robust optimization can be summarized as follows: At the time of computation not all data of the instance may be known exactly. Instead of fixed parameters we are given a set of scenarios $\mathcal{U}$, the uncertainty set, where each scenario defines $n$ fixed parameters for some $n \in \mathbb{N}$. We assume that any of the scenarios contained in $\mathcal{U}$ may actually occur, but we do not know the true scenario in advance. The aim is to find a solution taking into account all scenarios. The work was pioneered in [41] and become a major research area within the optimization community with [5, 6, 7]. A thorough general introduction and overview of robust optimization can be found in [3]. Furthermore, a recent overview is given by [19].

We denote the scenarios by vectors $\xi \in \mathbb{R}^{n}$ where each entry corresponds to some parameter of the instance. Several methods of defining uncertainty sets have been proposed in current literature, cf. [9, 8, 30, 28] for a general overview. One arising concept is that of discrete uncertainty where the uncertainty sets may only contain a finite number of possible scenarios, cf. [30, 29]. Further, when considering interval uncertainty the uncertainty set can be described as

$$
\mathcal{U}=\left\{\xi \in \mathbb{R}^{n}: \xi_{i} \in\left[a_{i}, b_{i}\right], i=1, \ldots, n\right\}
$$

for some $a_{i}, b_{i} \in \mathbb{R}$, cf. [30]. In this paper, we investigate discrete budgeted uncertainty as a combination of the above concepts, i.e.,

$$
\mathcal{U}=\left\{\xi \in \mathbb{R}^{n}: \xi_{i} \in\left[a_{i}, b_{i}\right] \cap \mathbb{Z}, i=1, \ldots, n \text { and } \sum_{i=1}^{n} \xi_{i} \leq \Gamma\right\}
$$

for some $a_{i}, b_{i}, \Gamma \in \mathbb{Z}$, cf. $[9,8]$. Note that this definition of discrete budgeted uncertainty set differs from other settings using the same expression, e.g. $[13,10,35]$. Here we bound the total sum of the uncertainty values. In our paper, the scenarios define parameters of the constraints and do not appear in the objective function. We aim for a solution that fulfills the constraints for all possible scenarios.

In [23], the authors studied robust versions of the classical Set Cover problem where the possible scenarios are given by all demand-subsets of a
certain fixed size. They provide approximation algorithms for robust twostage problems: Some of the sets may be selected in a first stage at lower cost and in a second stage, after the scenario is known, the remaining sets are chosen. Further, an approximation algorithm using the online algorithm for Set Cover $[2,12]$ within an LP-rounding-based algorithm can be found in [16]. The robust Set Cover problem was also studied from a polyhedral point of view in [17], whereas new formulations for robust Set Cover problems were given in [33].

In this paper we introduce for fixed $q \in \mathbb{N}$ the problem $q$-Multiset Multicover, which can be located between Set Cover by q-sets (cf. [20, SP2] for the version with $q=3$ ) and Multiset Multicover [26, 38] as we will see later. In $q$-Multiset Multicover, the subsets have arbitrary size, but the number of elements they may cover is bounded by $q$. In fact it is closely related to Multiset Multicover by q-sets. However, this relation cannot be generalized to the following robust version of this problem constituting the main focus of this paper.

We investigate a robust version of $q$-Multiset Multicover with discrete budgeted uncertainty where the scenarios correspond to demand vectors. After analyzing the complexity of all introduced problems, we present different solution approaches based on constraint generation and give computational results for both random instances and instances inspired by a real world problem.

To that end, we discuss an application of $q$-Multiset Multicover in the healthcare sector which motivated the study of robust multicovers. In the application we are asked to assign emergency doctors to facilities such that occurring emergencies may be handled in a satisfactory manner. The number of emergencies are uncertain and are represented using the proposed discrete budgeted uncertainty set where the total number of occurring emergencies is budgeted to avoid unrealistic situations. This leads to a multicover problem, where the elements are the regions in which the emergencies occur and the sets correspond to the subsets of regions which can be reached within a guaranteed response time from the facility chosen.

The article is organized as follows: In Section 2, we introduce $q$-Multiset Multicover, present possible integer programming formulations and prove NP-completeness for $q \geq 3$. Section 3 deals with the robust version of $q$ Multiset Multicover. We discuss whether the introduced formulations can be transferred and show that the problem is NP-hard for any $q>0$. Solution approaches are presented in Section 4, while Section 5 displays the corresponding computational results.

## 2 Problem definition and classifications

Let $G=(V, E)$ be an undirected graph. We denote by $N_{G}(v)$ the neighborhood of $v \in V$ in $G$, i.e., the set of all vertices adjacent to $v$. For a subset $S \subseteq V, N_{G}(S)$ is the set of all nodes adjacent to some node in $S$. For a directed graph $G=(V, R)$ we indicate by $N_{G}^{+}(v)$ the set of successors of $v \in V$, i.e., the set of vertices $w$ such that there is a directed arc from $v$ to $w$. Analogously, by $N_{G}^{-}(v)$ we denote the set of predecessors of $v \in V$. If the corresponding graph $G$ is clear from the context, we omit the subscript $G$. Now, we may formally define $q$-Multiset Multicover for a fixed integer $q \in \mathbb{Z}_{>0}$ :

Problem 1 ( $q$-Multiset Multicover ( $q$-MSMC)).
Instance: Finite ground set $J$, weights $d_{j} \in \mathbb{Z}_{\geq 0}$ for all $j \in J$, a collection of subsets $\mathcal{J} \subseteq 2^{J}$ and a positive integer $B \in \mathbb{Z}_{>0}$.
Question: Are there integers $x_{A} \in \mathbb{Z}_{\geq 0}$ for $A \in \mathcal{J}$ with $\sum_{A \in \mathcal{J}} x_{A} \leq B$, such that there exist integers $y_{A j} \in \mathbb{Z}_{\geq 0}$ for $A \in \mathcal{J}, j \in J$ satisfying

$$
\sum_{A \in \mathcal{J}: j \in A} y_{A j} \geq d_{j} \quad \forall j \in J \quad \text { and } \sum_{j \in A} y_{A j} \leq q \cdot x_{A} \quad \forall A \in \mathcal{J} ?
$$

The interpretation of the problem is as described in the introduction: Can we choose $B$ subsets, with multiple choices being allowed since $x_{A} \in$ $\mathbb{Z}_{\geq 0}$, such that the demand of each element is covered, when each subset may only cover up to $q$ elements (again multiple choices are allowed since $y_{A j} \in \mathbb{Z}_{\geq 0}$ ). For a fixed subset $A$, the integer $y_{A j}$ in the problem definition models the amount of demand of element $j$ covered by the subset $A$.

Remark 2. If, instead of regarding the subsets $A \in \mathcal{J}$, we regard all multisets of cardinality $q$ of $A$, we get an instance of Multiset Multicover, raising the input size only by a polynomial factor as $q$ is not part of the input. Thereby, $q$-MSMC is in some sense a representation of certain Multiset Multicover instances, having smaller input size. This connection, however, is lost when regarding the robust version of $q$-MSMC, cf. Section 3.

In the sequel, it will be useful to consider the following alternative definition of $q$-MSMC.

Problem 3 ( $q$-Multiset Multicover ( $q$-MSMC) - alternative definition).
Instance: Finite sets $I, J$ with $I \cap J=\emptyset$, weights $d_{j} \in \mathbb{Z}_{\geq 0}$ for all $j \in J$, a bipartite graph $G=(I \cup J, E)$ and a positive integer $B \in \mathbb{Z}_{>0}$.
Question: Are there integers $x_{i} \in \mathbb{Z}_{\geq 0}$ for $i \in I$ with $\sum_{i \in I} x_{i} \leq B$, such that there exist integers $y_{i j} \in \mathbb{Z}_{\geq 0}$ for $i \in I, j \in J$ satisfying

$$
\sum_{i \in N(j)} y_{i j} \geq d_{j} \quad \forall j \in J \quad \text { and } \quad \sum_{j \in N(i)} y_{i j} \leq q \cdot x_{i} \quad \forall i \in I ?
$$

For an instance of $q$-MSMC, we call the set $I$ locations and the set $J$ regions. Further, $d_{j}$ describes the number of clients or the demand in region $j \in J$ and $x_{i}$ denotes the number of suppliers in location $i \in I$. The number $q$ can be interpreted as the number of clients a single supplier may serve. In the optimization version Min- $q$-Multiset Multicover (Min- $q$-MSMC) we aim for a minimum number of suppliers. Identifying each location $i$ with its neighborhood $N_{G}(i)$ yields the equivalence of the two problem definitions.

It can readily be seen that the following integer program models Min- $q$ MSMC for some demand vector $d \in \mathbb{Z}_{\geq 0}^{|J|}$ :

$$
\begin{array}{rll}
(\operatorname{MIP} 1)(d) & \min _{x, y} & \sum_{i \in I} x_{i} \\
\text { s.t. } & \sum_{i \in N(j)} y_{i j} \geq d_{j} \quad & \forall j \in J, \\
& \sum_{j \in N(i)} y_{i j} \leq q \cdot x_{i} & \forall i \in I, \\
y_{i j} & \geq 0 & \forall i \in I, j \in J, \\
& x_{i} \in \mathbb{Z}_{\geq 0} & \forall i \in I . \tag{1e}
\end{array}
$$

Note that the variables $y_{i j}$ are not forced to be integral. Observation 5 argues why this is no restriction. Furthermore, in Lemma 4 we prove that (IP 2)(d) is an alternative formulation to (MIP 1)(d).

$$
\begin{align*}
(\text { IP } 2)(d) \quad \min _{x} & \sum_{i \in I} x_{i}  \tag{2a}\\
\text { s.t. } & \sum_{i \in N(S)} q \cdot x_{i} \geq \sum_{j \in S} d_{j} \quad \forall S \subseteq J,  \tag{2b}\\
& x_{i} \in \mathbb{Z}_{\geq 0} \quad \forall i \in I . \tag{2c}
\end{align*}
$$

Lemma 4. For an instance of Min-q-Multiset Multicover it holds that $x \in$ $\mathbb{Z}_{\geq 0}^{|I|}$ is a feasible solution to (IP 2)(d) if and only if there exists $y \in \mathbb{R}_{\geq 0}^{|I||J|}$ such that $(x, y)$ is a feasible solution for (MIP 1)(d). In this case, the variables $y$ can be chosen to be integral.

Proof. If $(x, y)$ is a feasible solution for (MIP 1$)(d)$ then $x$ is also feasible for (IP 2)(d), as for any $S \subseteq J$ we have:

$$
\begin{aligned}
\sum_{i \in N(S)} q \cdot x_{i} & \geq \sum_{i \in N(S)} \sum_{j \in N(i)} y_{i j}=\sum_{i \in N(S)}\left(\sum_{j \in N(i) \cap S} y_{i j}+\sum_{j \in N(i) \backslash S} y_{i j}\right) \\
& \geq \sum_{i \in N(S)} \sum_{j \in N(i) \cap S} y_{i j}=\sum_{j \in S} \sum_{i \in N(j)} y_{i j} \geq \sum_{j \in S} d_{j} .
\end{aligned}
$$



Figure 1: Flow network for the proof of Lemma 4. All thick arcs have infinite capacity.

Now assume we are given a feasible solution $x$ for (IP 2)(d). Let $G$ be the graph from the instance of Min- $q$-MSMC. We define a directed graph $H=\left(I \cup J \cup\{s\} \cup\{t\}, R \cup R_{s} \cup R_{t}\right)$, where $R$ contains all arcs in $E(G)$ directed from $I$ to $J, R_{s}=\{(s, i): i \in I\}$ and $R_{t}=\{(j, t): j \in J\}$, cf. Fig. 1 . We set the capacities of each arc $r \in R(H)$ as

$$
c(r)= \begin{cases}\infty, & r \in R, \\ q \cdot x_{i}, & r \in R_{s}, \\ d_{j}, & r \in R_{t} .\end{cases}
$$

We claim that the maximum $s$ - $t$-flow in $H$ has flow value $\sum_{j \in J} d_{j}$. Note that given an $s$ - $t$-flow $f$ with flow value $\sum_{j \in J} d_{j}$, we can define a feasible solution to (MIP 1)(d) by $(x, y)$, where $y_{i j}=f(i, j)$ for all $(i, j) \in R$.

Further, the flow value of any $s-t$-flow can never be larger than $\sum_{j \in J} d_{j}$ (consider the $s$ - $t$-cut with $T=\{t\}$ ). Thus, it suffices to show that a maximum $s$ - $t$-flow in $H$ has flow value no less than $\sum_{j \in J} d_{j}$. To this end let $S, T \subseteq V(H)$ be any $s$ - $t$-cut in $H$. Let $J^{\prime}=J \backslash S$, possibly being the empty set. If any location in the neighborhood of $J^{\prime}$ is contained in $S$, the cut contains an arc with infinite capacity, so assume $N_{H}^{-}\left(J^{\prime}\right) \cap S=\emptyset$ so that $N_{H}^{-}\left(J^{\prime}\right) \subseteq T$. Since $x$ is a feasible solution to (IP 2)(d) we obtain for any subset $Q \subseteq J$

$$
\sum_{i \in N_{H}^{-}(Q)} q \cdot x_{i}=\sum_{i \in N_{G}(Q)} q \cdot x_{i} \geq \sum_{j \in Q} d_{j} .
$$

We get

$$
c(S, T) \geq \sum_{j \in J \cap S} d_{j}+\sum_{i \in N_{H}^{-}\left(J^{\prime}\right)} q \cdot x_{i} \geq \sum_{j \in J \cap S} d_{j}+\sum_{j \in J^{\prime}} d_{j}=\sum_{j \in J} d_{j} .
$$

Thus, every $s$ - $t$-cut has capacity larger or equal to $\sum_{j \in J} d_{j}$ and by the Max-Flow-Min-Cut Theorem we obtain the desired result, cf. [1].

Observation 5. Note that the capacities of the $\operatorname{arcs}$ in $R_{s}$ and $R_{t}$ defined in the proof of Lemma 4 are integral. Thus, there exists an integral flow $f$ in $H$ if and only if there exists a continuous flow $f^{\prime}$ in $H$ and the variable $y_{i j}$ can be interpreted as the number of clients in region $j$ taken over by the suppliers in location $i$, cf. [1].

In the following, we state our results on the complexity of $q$-MSMC. Formal proofs of these claims can be found in A.

Observation 6. Min-1-Multiset Multicover is solvable in linear time.
Theorem 7. Min-2-Multiset Multicover can be solved in $O\left(|I|^{5 / 2}|J|^{5 / 2}\right)$.
Theorem 8. For any fixed $q \geq 3, q$-Multiset Multicover is NP-complete in the strong sense.

Observation 9. There is a $\log (q)$-approximation for Min- $q$-Multiset Multicover.

Now, we concentrate on a robust version of $q$-Multiset Multicover.

## 3 Problem definition and classification of the robust version

In this section, we extend the initial problem Min- $q$-Multiset Multicover to include uncertainty in the number of clients $d_{j}$ of each region $j \in J$. We apply concepts of robust optimization such as strict and adjustable robustness and combine interval and budgeted uncertainty as mentioned in the introduction, cf. [3, 4]. For each region $j \in J$, we consider two non-negative integers $a_{j}$ and $b_{j}$ with $a_{j} \leq b_{j}$, which respectively correspond to the minimum and maximum number of clients in that region. Concerning the total amount of clients in all regions, we additionally require this value not to exceed some given constant $\Gamma \in \mathbb{Z}_{\geq 0}$ to prevent the global worst case. Then, a vector $\xi \in \mathbb{Z}^{|J|}$ with $\xi_{j} \in\left[a_{j}, b_{j}\right]$ and $\sum_{j \in J} \xi_{j} \leq \Gamma$ is called a scenario and we denote by $\mathcal{U}$ the set of all scenarios, i.e.,

$$
\mathcal{U}=\left\{\xi \in \mathbb{Z}^{|J|}: \xi_{j} \in\left[a_{j}, b_{j}\right] \forall j \in J, \sum_{j \in J} \xi_{j} \leq \Gamma\right\}
$$

The set $\mathcal{U}$ is also called the uncertainty set. Note that $\mathcal{U}$ is finite since we only consider integral demands in our problem. For a vector $x \in \mathbb{R}^{n}$ and some set $F \subseteq\{1 \ldots n\}$ we use the common notation $x(F)=\sum_{i \in F} x_{i}$.

Assumption 1. In order to obtain a meaningful uncertainty set we assume that $\sum_{j \in J} a_{j} \leq \Gamma \leq \sum_{j \in J} b_{j}$ implying $\mathcal{U} \neq \emptyset$. Moreover, we assume without loss of generality that $\Gamma$ is chosen in such a way that $b_{j}+\sum_{k \neq j} a_{k} \leq \Gamma$ such that, for each region $j \in J$, there exists a scenario $\xi$ with $\xi_{j}=b_{j}$. Otherwise we could decrease the upper bound $b_{j}$ in the corresponding region.

The intuition of the robust version of Min- $q$-MSMC is to choose a minimum number of suppliers, such that in any scenario of $\mathcal{U}$ all clients may be served. In the following, we will see how to incorporate this intuition into the models introduced in Section 2. We begin by robustifying (IP 2)(d).

Each scenario $\xi \in \mathcal{U}$ defines a single problem in the fashion of Problem 3 when denoting the amount of clients by $d_{j}=\xi_{j}$ for all $j$. Therefore, given a fixed scenario $\xi$, we consider the integer linear program (IP 2)( $\xi$ ). In terms of robust optimization, we obtain the uncertain integer linear program:

$$
\begin{equation*}
\left\{\min \left\{\sum_{i \in I} x_{i}: x \text { is feasible for }(\operatorname{IP} 2)(\xi)\right\}: \xi \in \mathcal{U}\right\} . \tag{3}
\end{equation*}
$$

Our aim is to find $x_{i} \in \mathbb{Z}_{\geq 0}$ for all $i \in I$, such that all clients can be served independently of the actually occurring "true" scenario. Therefore, we concentrate on the analysis of the following problem:

Problem 10 (Robust $q$-MSMC, set formulation).
Instance: Set of possible locations $I$, set of regions $J$, non-negative integers $a_{j}, b_{j}$ with $a_{j} \leq b_{j}$ for all $j \in J$, integer $\Gamma$ satisfying Assumption 1, bipartite graph $G=(I \cup J, E)$ and a positive integer $B \in \mathbb{Z}_{>0}$.
Question: Are there $x_{i} \in \mathbb{Z}_{\geq 0}$ for all $i \in I$ such that $\sum_{i \in I} x_{i} \leq B$ and for all subsets $S \subseteq J$ and all scenarios $\xi \in \mathcal{U}$ we have

$$
\sum_{i \in N(S)} q \cdot x_{i} \geq \sum_{j \in S} \xi_{j} ?
$$

The minimization problem corresponding to Robust $q$-MSMC, i.e. Robust Min- $q$-MSMC, can be formulated as the robust counterpart of (3):

$$
\begin{align*}
\text { (IP 4) } \min _{x} & \sum_{i \in I} x_{i}  \tag{4a}\\
\text { s.t. } & \sum_{i \in N(S)} q \cdot x_{i} \geq \sum_{j \in S} \xi_{j} \quad \forall S \subseteq J, \forall \xi \in \mathcal{U},  \tag{4b}\\
& x_{i} \in \mathbb{Z}_{\geq 0} \quad \forall i \in I . \tag{4c}
\end{align*}
$$

Note that the uncertain data only occurs on the right hand side of the above constraints. Thus, this problem can be simplified by computing, for every subset $S \subseteq J$, the maximum of $\sum_{j \in S} \xi_{j}$ over the uncertainty set $\mathcal{U}$. This maximum is given by $\tilde{d}_{S}:=\min \{b(S), \Gamma-a(J \backslash S)\}$ :

- If $b(S)+a(J \backslash S) \leq \Gamma$, then $\max _{\xi \in \mathcal{U}} \sum_{j \in S} \xi_{j}=b(S)$.
- If $b(S)+a(J \backslash S)>\Gamma$, then $\max _{\xi \in \mathcal{U}} \sum_{j \in S} \xi_{j}=\Gamma-a(J \backslash S)$. (Since $b(S)>\Gamma-a(J \backslash S)$ and $a(S) \leq \Gamma-a(J \backslash S)$ a corresponding scenario clearly exists.)

Consequently, we can replace $\sum_{j \in S} \xi_{j}$ in line (4b) of (IP 4) by $\tilde{d}_{S}$ and reformulate the question posed in Problem 10 as follows: Are there $x_{i} \in \mathbb{Z}_{\geq 0}$ for all $i \in I$ such that $\sum_{i \in I} x_{i} \leq B$ and for all subsets $S \subseteq J$ :

$$
\sum_{i \in N(S)} q \cdot x_{i} \geq \tilde{d}_{S} ?
$$

Observe that when using this formulation the problem is independent from the uncertainty set $\mathcal{U}$. But in comparison to the non-robust formulation of Section 2, the value $\tilde{d}_{S}$ cannot be split into a sum of clients over the single regions of $S$ anymore.

As in Section 2, we aim to obtain an equivalent assignment formulation for Problem 10. A first idea is to consider the robust counterpart of the uncertain IP

$$
\begin{equation*}
\left\{\min \left\{\sum_{i \in I} x_{i}:(x, y) \text { is feasible for }(\operatorname{MIP} 1)(\xi)\right\}: \xi \in \mathcal{U}\right\} . \tag{5}
\end{equation*}
$$

Since $(x, y)$ needs to be feasible for (MIP 1$)(\xi)$ for any scenario $\xi$ and since for each region $j \in J$ there exists a scenario with $\xi_{j}=b_{j}$ (cf. Assumption 1 ), the solution vector $(x, y)$ can be computed by solving the mixed integer linear program (MIP 1)(b). In general, $\sum_{j \in S} b_{j}=b(S)=\tilde{d}_{S}$ does not hold for all $S \subseteq J$ and we see that formulation (5) cannot be equivalent to (3). Furthermore, the upper bound on the number of clients $\Gamma$ is not needed in (5). Eliminating the constraint $\sum_{j \in J} \xi_{j} \leq \Gamma$ in the definition of the uncertainty set $\mathcal{U}$ would lead to $\tilde{d}_{S}=b(S)$. Only in this special case, both formulations (5) and (3) are equivalent as shown in Section 2.

Actually, computing a global solution $y$ is far too conservative and applying strict robustness is unrewarding. Moreover, (5) does not match the intuition of Robust Min- $q$-MSMC as we have to fix the $y_{i j}$ before the actual scenario is revealed. Recalling the interpretation of the variables $y_{i j}$ in Observation 5, it is meaningful to fix the variables $y_{i j}$ only after the realization of the true scenario $\xi$ is known. Thus, we only need to settle the decision over the $x_{i}, i \in I$, before the realization becomes apparent, while additionally ensuring the existence of an assignment $y$ of suppliers to clients. Therefore, we apply the concept of adjustable robustness [4] with $x$ containing the "here and now" variables and $y$ corresponding to the "wait
and see" variables. Then, our aim is to find $x_{i} \in \mathbb{Z}_{\geq 0}$ for all $i \in I$, minimizing $\sum_{i \in I} x_{i}$, such that, for every $\xi \in \mathcal{U}$, there exist $y(\xi)$ with $(x, y(\xi))$ being feasible for (MIP 1$)(\xi)$. This approach leads to the adjustable robust counterpart of (5):

$$
\begin{array}{rll}
\text { (MIP 6) } & \min _{x, y} & \sum_{i \in I} x_{i} \\
\text { s.t. } & \sum_{i \in N(j)} y_{i j}(\xi) \geq \xi_{j} & \forall j \in J, \xi \in \mathcal{U}, \\
& \sum_{j \in N(i)} y_{i j}(\xi) \leq q \cdot x_{i} & \forall i \in I, \xi \in \mathcal{U}, \\
& y_{i j}(\xi) \geq 0 & \forall i \in I, \forall j \in J, \xi \in \mathcal{U}, \\
& x_{i} \in \mathbb{Z}_{\geq 0} & \forall i \in I . \tag{6e}
\end{array}
$$

The corresponding decision problem is defined as follows:
Problem 11 (Robust $q$-MSMC, assignment formulation).
Instance: Set of possible locations $I$, set of regions $J$, non-negative integers $a_{j}, b_{j}$ with $a_{j} \leq b_{j}$ for all $j \in J$, integer $\Gamma$ satisfying Assumption 1, bipartite graph $G=(I \cup J, E)$ and a positive integer $B \in \mathbb{Z}_{>0}$.
Question: Are there $x_{i} \in \mathbb{Z}_{\geq 0}$ for all $i \in I$, such that $\sum_{i \in I} x_{i} \leq B$ and for all scenarios $\xi \in \mathcal{U}$ there are $y_{i j}(\xi) \in \mathbb{R}_{\geq 0}$ for all $i \in I, j \in J$, such that

$$
\sum_{i \in N(j)} y_{i j}(\xi) \geq \xi_{j} \forall j \in J \text { and } \sum_{j \in N(i)} y_{i j}(\xi) \leq q \cdot x_{i} \forall i \in I ?
$$

As in the non-robust version, the assignment variables $y_{i j}(\xi)$ can be chosen to be integral whenever there exists a solution $(x, y) \in \mathbb{Z}^{|I|} \times \mathbb{R}^{|I||J| \mid \mathcal{U |}}$ for Problem 11. Thus, we can interpret the variable $y_{i j}(\xi)$ as the number of clients in region $j$ taken over by suppliers in location $i$ in case scenario $\xi$ occurs. A similar problem is analyzed in [18], whereas a general approach for adjustable robustness in the LP-case with right hand side uncertainty is investigated in [34].

Now, we are able to prove the equivalence between the robust set formulation defined in Problem 10 and the adjustable robust assignment formulation defined in Problem 11.

Proposition 12. Problem 10 and Problem 11 are equivalent.
Proof. Since the objective functions are the same, it remains to be shown that any solution $(x, y)$ of (MIP 6) yields a solution $x^{\prime}$ of (IP 4) with the same objective value and vice versa. Thus, let $(x, y)$ be feasible for (6) with $y=\left(y\left(\xi_{1}\right), y\left(\xi_{2}\right), \ldots\right)$. Fix a scenario $\xi \in \mathcal{U}$. Then $(x, y(\xi))$ is feasible for
(MIP 1$)(\xi)$. Due to the equivalence of the formulations in the non-robust version, we get that $x$ fulfills

$$
\sum_{i \in N(S)} q \cdot x_{i} \geq \sum_{j \in S} \xi_{j} \quad \forall S \subseteq J
$$

Since this argument holds true for any fixed scenario $\xi$, we obtain that $x$ is feasible for (IP 4).

On the other hand, given a solution $x$ of (IP 4), for any fixed scenario $\xi$, there exists $y(\xi)$ such that $(x, y(\xi))$ is feasible for (MIP 1$)(\xi)$ due to the results of Section 2. In total, we obtain that $(x, y)$ with $y=\left(y\left(\xi_{1}\right), y\left(\xi_{2}\right), \ldots\right)$ is feasible for (MIP 6).

At this point, we see that our initial link to Multiset Multicover by $q$-sets is lost when including robustness, since the assignment variables $y \in \mathbb{R}^{|I||J| \mid \mathcal{U |}}$ can be specified in a subsequent step when the "true" scenario is already known. In the robust case the value $x_{i}$ has to be specified in advance so that, for each possible scenario $\xi$, there exists a selection of $x_{i}$ sets for each location $i \in I$ satisfying the upcoming demand. Therefore, considering robustness leads to a completely new problem in comparison to Section 2 which we investigate further in the following. Before we concentrate on the complexity of Robust $q$-MSMC, we state some properties of the problem.

Observation 13. Let $z$ be the optimal solution value of Robust Min- $q$ MSMC. Then:
(a) $z \geq\left\lceil\frac{\Gamma}{q}\right\rceil$.
(b) Define $\bar{x} \in \mathbb{Z}^{|I|}$ in the following way: For all $j \in J$, choose $i \in N(j)$ and increase $\bar{x}_{i}$ by $\left[\frac{b_{j}}{q}\right\rceil$. Then, $\bar{x}$ is feasible for (IP 4) and we get $z \leq \sum_{j \in J}\left\lceil\frac{b_{j}}{q}\right\rceil$.
(c) It suffices to consider all scenarios $\xi \in \mathcal{U}$ with $\sum_{j \in J} \xi_{j}=\Gamma$.

From part (a) and part (b) it follows that Robust Min- $q$-MSMC has a finite optimal solution. Furthermore, from now on we restrict the problem to scenarios whose demands sum up to $\Gamma$. We call such a scenario an extreme scenario and denote by $\mathcal{U}^{\prime} \subseteq \mathcal{U}$ the set of all extreme scenarios.

In the following, we utilize the Dominating Set problem [20] to show NPhardness for Robust $q$-MSMC. In the former problem, given an undirected graph $G=(V, E)$ and a positive integer $K \leq|V|$, the question is whether there exists a subset $V^{\prime} \subseteq V$ with $\left|V^{\prime}\right| \leq K$ such that for all $u \in V \backslash V^{\prime}$ there is $v \in V^{\prime}$ for which $\{u, v\} \in E$. This problem is well-known to be NP-complete.

Theorem 14. For fixed $q \in \mathbb{Z}_{>0}$, Robust $q$-MSMC is strongly NP-hard.

Proof. We show that there exists a polynomial time reduction from Dominating Set to Robust $q$-MSMC. To this end, let an undirected graph $G=$ $(V, E)$ with $V=\{1, \ldots, n\}$ and an integer $K \leq n$ be given. To construct an instance of Robust $q$-MSMC we set $I=\{1, \ldots, n\}$ and $J=\{n+1, \ldots, 2 n\}$. For every edge $\{u, v\} \in E$, we add the edge $\{u, n+v\}$ and the edge $\{v, n+u\}$ to the bipartite graph $G^{\prime}$ with vertex set $I \cup J$ and edge set $E^{\prime}$. Additionally, for every $v \in V$, the edge $\{v, n+v\}$ is added to $E^{\prime}$. Moreover, we define $a_{j}=0, b_{j}=1$ for all $j \in J, \Gamma=1$ and $B=K$. Thus, we have $\tilde{d}_{S}=\min \{|S|, 1\}=1$ for any non-empty subset $S \subseteq J$.

Let $V^{\prime} \subseteq V=I$ be a solution of Dominating Set such that $\left|V^{\prime}\right| \leq K$. Then, we set $x_{i}=1$ for all $i \in V^{\prime}$ and zero else so that $\sum_{i \in I} x_{i} \leq B$ already holds true. Fix an arbitrary subset $S \subseteq J, S \neq \emptyset$. We want to show that

$$
\sum_{i \in N(S)} q \cdot x_{i} \geq 1
$$

Thus, we need to prove that at least one value $x_{i}$ for $i \in N(S)$ is set to one, i.e. $N(S) \cap V^{\prime} \neq \emptyset$. Choose an arbitrary element $n+v \in S$ with $v \in\{1, \ldots, n\} . V^{\prime}$ is a dominating set, so we have $v \in V^{\prime}$ or there is $u \in V^{\prime}$ adjacent to $v$ in $G$. In the former case, $v \in N(S)$ since $G^{\prime}$ contains the edge $\{v, n+v\}$. In the latter case, $u \in N(S)$ since $G^{\prime}$ contains the edge $\{u, n+v\}$. Thus, $\sum_{i \in N(S)} q \cdot x_{i} \geq 1$ holds true in any case, so that $x$ is a solution of Robust $q$-MSMC.

Conversely, suppose that $x$ is a solution of Robust $q$-MSMC such that $\sum_{i \in I} x_{i} \leq B$. Since $\tilde{d}_{S}=1$ for all $S \subseteq J, S \neq \emptyset$, we can assume without loss of generality that $x_{i} \leq 1$ for all $i \in I$. The set $V^{\prime}$ is defined to contain all vertices $v \in V$ such that $x_{v}=1$. Clearly, $\left|V^{\prime}\right| \leq B=K$ and we claim that $V^{\prime}$ is a dominating set for $G$. To this end, choose a vertex $u \in V-V^{\prime}$ and consider the set $S=\{n+u\} \subseteq J$. Since $x$ is a feasible solution, there is $v \in N(S)$ with $x_{v}=1$, i.e. $v \in V^{\prime}$. Since $v \in N(S)$, either $u=v$ or the vertices $u$ and $v$ are adjacent in $G$ by construction of $G^{\prime}$ yielding the claim.

Note that we did not prove NP-completeness of Robust $q$-MSMC. In the following section, we see that checking a given vector $x$ for feasibility is co-NP-complete.

## 4 Solving the Robust Min- $q$-Multiset Multicover

In the previous section we have shown that Robust Min- $q$-MSMC is an NP-hard problem. As both formulations of the problem as (mixed) integer linear programs contain a large number of constraints, it is reasonable to apply constraint generation to obtain a solution, cf. [15, 22, 36, 42].

Thus, focusing on the set formulation, at any point in the constraint generation process, a collection of subsets $\mathcal{S} \subseteq 2^{J}$ is given and we solve the
relaxed problem obtained by only considering the constraints corresponding to sets $S \in \mathcal{S}$ in (IP 4). This problem is called restricted master problem. In the separation step, given an optimal solution $\bar{x}$ of the restricted master problem, we are looking for a new subset $S \subseteq J$ such that the constraint induced by $S$ is not fulfilled yet, i.e.,

$$
\sum_{i \in N(S)} q \cdot \bar{x}_{i}<\tilde{d}_{S}
$$

In the next iteration, $\mathcal{S}$ is updated by adding the newly found set $S$ and the restricted master problem is solved once more. If there exists no set $S$ fulfilling the above inequality, we know that $\bar{x}$ is the optimal solution for Robust Min- $q$-MSMC. Initially, $\mathcal{S}$ is the empty set yielding the optimal solution $\bar{x}_{i}=0$ for all locations $i \in I$ in the restricted master problem. Analogously, this procedure can be applied to the assignment formulation (MIP 6) using an (initially empty) set $\mathcal{U}^{\prime \prime} \subseteq \mathcal{U}^{\prime}$ of extreme scenarios. The important step of these methods is an efficient way to solve the occurring separation problems. These can be formulated as follows:
Problem 15 (Separation for Robust $q$-MSMC, set formulation).
Instance: Set of possible locations $I$, non-negative integers $\bar{x}_{i}$ for all $i \in I$, set of regions $J$, non-negative integers $a_{j}, b_{j}$ with $a_{j} \leq b_{j}$ for all $j \in J$, integer $\Gamma$ satisfying Assumption 1, bipartite graph $G=(I \cup J, E)$.
Question: Is there a subset $S \subseteq J$ such that $q \cdot \bar{x}(N(S))<\tilde{d}_{S}$ ?
Problem 16 (Separation for Robust $q$-MSMC, assignment formulation).
Instance: See Problem 15.
Question: Is there an extreme scenario $\xi \in \mathcal{U}^{\prime}$ such that there is no $y \geq 0$ with

$$
\begin{equation*}
\sum_{i \in N(j)} y_{i j} \geq \xi_{j} \quad \forall j \in J \text { and } \sum_{j \in N(i)} y_{i j} \leq q \cdot \bar{x}_{i} \quad \forall i \in I ? \tag{7}
\end{equation*}
$$

In the following, we concentrate on the analysis of these two problems. Using Farkas' Lemma (see e.g., [22, 39]), Problem 16 asks for an extreme scenario $\xi \in \mathcal{U}^{\prime}$ such that there are vectors $\mu, \nu \geq 0$ with $\mu_{i} \geq \nu_{j}$ for all regions $j \in J$ and locations $i \in N(j)$ and

$$
\sum_{i \in I} q \cdot \bar{x}_{i} \cdot \mu_{i}<\sum_{j \in J} \xi_{j} \cdot \nu_{j}
$$

Therefore, we can also solve Problem 16 by asking for an extreme scenario $\xi$ such that the optimal objective value of the problem

$$
\begin{array}{rlr}
(\mathrm{LP} 8) & \min _{\mu, \nu} & \sum_{i \in I} q \cdot \bar{x}_{i} \cdot \mu_{i}-\sum_{j \in J} \xi_{j} \cdot \nu_{j} \\
\text { s.t. } & \mu_{i} \geq \nu_{j} \quad \forall j \in J, i \in N(j) \\
& \mu_{i}, \nu_{j} \geq 0 \quad \forall i \in I, j \in J
\end{array}
$$

is less than zero. Note that the zero vector is feasible here, so that the optimal objective value never exceeds zero.

Definition 17. Let an instance of the separation problem be given. A set $S \subseteq J$ with $q \cdot \bar{x}(N(S))<\tilde{d}_{S}$ is called violating subset. Analogously, an extreme scenario $\xi \in \mathcal{U}^{\prime}$ such that (LP 8) has a solution $(\mu, \nu)$ with objective value less than zero is called violating scenario.

The following Lemma 18 is an easy consequence from the equivalence of the Problems 10 and 11 . We will nevertheless give a constructive proof. This will enable us to find a violating scenario in polynomial time if we are given a violating subset and vice versa.

Lemma 18. Let an instance of the separation problem be given. Then, there exists a violating scenario $\xi \in \mathcal{U}^{\prime}$ if and only if there exists a violating subset $S \subseteq J$.

Proof. Let $\xi \in \mathcal{U}^{\prime}$ and ( $\mu, \nu$ ) such that ( $\mu, \nu$ ) is feasible for (LP 8) with objective value $\sum_{i \in I} q \cdot \bar{x}_{i} \cdot \mu_{i}-\sum_{j \in J} \xi_{j} \cdot \nu_{j}<0$ be given. Since the corresponding constraint matrix is totally unimodular, we can assume $\mu$ and $\nu$ to only contain integral values. First of all, suppose $\nu^{\star}:=\max \left\{\nu_{j}: j \in J\right\}>1$ and consider the index set $J^{\star}:=\left\{j \in J: \nu_{j}=\nu^{\star}\right\}$. Then, $\mu_{i} \geq \nu^{\star}$ for all $i \in N\left(J^{\star}\right)$ and without loss of generality we can assume that even equality holds. Now, we obtain

$$
\sum_{i \in N\left(J^{\star}\right)} q \cdot \bar{x}_{i} \cdot \mu_{i}-\sum_{j \in J^{\star}} \xi_{j} \cdot \nu_{j}=\left(\sum_{i \in N\left(J^{\star}\right)} q \cdot \bar{x}_{i}-\sum_{j \in J^{\star}} \xi_{j}\right) \cdot \nu^{\star}=: A \cdot \nu^{\star} .
$$

If $A<0$ we can choose $S=J^{\star}$ and have found a solution for the separation problem of the set formulation since

$$
\begin{aligned}
0>A & =\sum_{i \in N\left(J^{\star}\right)} q \cdot \bar{x}_{i}-\sum_{j \in J^{\star}} \xi_{j} \geq \sum_{i \in N\left(J^{\star}\right)} q \cdot \bar{x}_{i}-\min \left\{b\left(J^{\star}\right), \Gamma-a\left(J \backslash J^{\star}\right)\right\} \\
& =\sum_{i \in N\left(J^{\star}\right)} q \cdot \bar{x}_{i}-\tilde{d}_{J^{\star}} .
\end{aligned}
$$

Otherwise, if $A \geq 0$, we can decrease $\mu_{i}$ for all $i \in N\left(J^{\star}\right)$ and $\nu_{j}$ for all $j \in J^{\star}$ by one and have found another feasible solution ( $\mu^{\prime}, \nu^{\prime}$ ) with objective value
smaller than 0 since

$$
\begin{aligned}
0 & >\sum_{i \in I} q \cdot \bar{x}_{i} \cdot \mu_{i}-\sum_{j \in J} \xi_{j} \cdot \nu_{j} \\
& =\left(\sum_{i \in N\left(J^{\star}\right)} q \cdot \bar{x}_{i}-\sum_{j \in J^{\star}} \xi_{j}\right) \cdot \nu^{\star}+\sum_{i \notin N\left(J^{\star}\right)} q \cdot \bar{x}_{i} \cdot \mu_{i}-\sum_{j \notin J^{\star}} \xi_{j} \cdot \nu_{j} \\
& \geq\left(\sum_{i \in N\left(J^{\star}\right)} q \cdot \bar{x}_{i}-\sum_{j \in J^{\star}} \xi_{j}\right) \cdot\left(\nu^{\star}-1\right)+\sum_{i \notin N\left(J^{\star}\right)} q \cdot \bar{x}_{i} \cdot \mu_{i}-\sum_{j \notin J^{\star}} \xi_{j} \cdot \nu_{j} \\
& =\sum_{i \in I} q \cdot \bar{x}_{i} \cdot \mu_{i}^{\prime}-\sum_{j \in J} \xi_{j} \cdot \nu_{j}^{\prime} .
\end{aligned}
$$

Thus, repeating this argument, we either find the desired violating set $S$ or we end with a binary solution $\mu_{i}, \nu_{j} \in\{0,1\}$ for all $i, j$. In the latter case set $S=\left\{j \in J: \nu_{j}=1\right\}$. Then, it holds true that $\mu_{i} \geq 1$ for all $i \in N(S)$. With no loss of generality, we can assume $\mu_{i}=1$ for $i \in N(S)$ and $\mu_{i}=0$ else since the objective value only becomes smaller. This yields

$$
0>\sum_{i \in I} q \cdot \bar{x}_{i} \cdot \mu_{i}-\sum_{j \in J} \xi_{j} \cdot \nu_{j}=\sum_{i \in N(S)} q \cdot \bar{x}_{i}-\sum_{j \in S} \xi_{j} \geq \sum_{i \in N(S)} q \cdot \bar{x}_{i}-\tilde{d}_{S}
$$

and we have found the desired set $S$.
On the other hand, given $S \subseteq J$ with $\sum_{i \in N(S)} q \cdot \bar{x}_{i}<\tilde{d}_{S}$, we choose an extreme scenario $\xi \in \mathcal{U}^{\prime}$ with $\sum_{j \in S} \xi_{j}=\tilde{d}_{S}$ :

- If $b(S)+a(J \backslash S) \leq \Gamma$, we set $\xi_{j}=b_{j}$ for all $j \in S$. Since $a(J \backslash S) \leq$ $\Gamma-b(S)$ and $b(J \backslash S) \geq \Gamma-b(S)$ we can choose the demands in the remaining regions $j \in J \backslash S$ so that $\sum_{j \in J} \xi_{j}=\Gamma$.
- If $b(S)+a(J \backslash S)>\Gamma$, we set $\xi_{j}=a_{j}$ for all $j \in J \backslash S$. Since $a(S) \leq \Gamma-a(J \backslash S)$ and $b(S)>\Gamma-a(J \backslash S)$ we can choose the demands in the remaining regions $j \in S$ so that $\sum_{j \in J} \xi_{j}=\Gamma$.

Finally, set $\nu_{j}=1$ for all $j \in S$ as well as $\mu_{i}=1$ for all $i \in N(S)$. All other variables are set to zero. This yields the desired extreme scenario $\xi$ and the solution $(\mu, \nu)$ for the separation problem in the assignment formulation with objective value

$$
\sum_{i \in I} q \cdot \bar{x}_{i} \cdot \mu_{i}-\sum_{j \in J} \xi_{j} \cdot \nu_{j}=\sum_{i \in N(S)} q \cdot \bar{x}_{i}-\tilde{d}_{S}<0
$$

This completes the proof.
Lemma 18 allows to switch between both separation problems as the proof reveals how to construct a violating scenario from a given violating
subset and vice versa. Furthermore, concerning complexity the problems are equally hard since the transformations can be computed in polynomial time. Recall that the polyhedron of feasible solutions corresponding to (LP 8) is integral, even if the constraints $\mu_{i}, \nu_{j} \leq 1 \forall i \in I, j \in J$ are added, since this does not destroy total unimodularity.

In the following, we show that Problem 15 is NP-complete. To prove this fact, we additionally need the definition of the Knapsack problem, cf. [20]. Here, we are given a finite set $U$, for each element $u \in U$ a size $s(u) \in \mathbb{Z}_{>0}$ and a profit $p(u) \in \mathbb{Z}_{>0}$, and two positive integers $B$ and $K$. The question is whether there exists a subset $U^{\prime} \subseteq U$ such that $\sum_{u \in U^{\prime}} s(u) \leq B$ as well as $\sum_{u \in U^{\prime}} p(u) \geq K$. In the following we write $s\left(U^{\prime}\right)$, respectively $p\left(U^{\prime}\right)$, to refer to the sum over the sizes/profits of the single elements in $U^{\prime}$.

Lemma 19. For fixed $q \in \mathbb{Z}_{>0}$, Problem 15 is NP-complete.
Proof. Clearly, Problem 15 is contained in NP since given any subset $S \subseteq J$ we can check in polynomial time whether the stated inequality is satisfied.

We show that Knapsack reduces to Problem 15 in polynomial time. Let an arbitrary instance of Knapsack be given with a set $U=\{1, \ldots, n\}$, sizes $s(u)$ and profits $p(u)$ associated with each element $u \in U$ and two integers $B, K \in \mathbb{Z}_{>0}$. We define a bipartite graph $G=(I \cup J, E)$ with $I=\{1, \ldots, n, 2 n+1\}=U \cup\{2 n+1\}, J=\{n+1, \ldots, 2 n, 2 n+2\}$ and

$$
\begin{aligned}
E= & \{\{u, n+u\},\{2 n+1, n+u\},\{u, 2 n+2\} \text { for } u=1, \ldots, n\} \\
& \cup\{\{2 n+1,2 n+2\}\} .
\end{aligned}
$$

Furthermore, we set $\bar{x}(u)=s(u)$ and $b(n+u)=q \cdot(p(u)+s(u))$ for all $u \in U$. Further, we set $\bar{x}(2 n+1)=K-1, b(2 n+2)=0, \Gamma=q \cdot(B+K)$. Finally, we set $a(j)=0$ for all $j \in J$.

Now given a solution $U^{\prime} \subseteq U \subseteq I$ of Knapsack with $s\left(U^{\prime}\right) \leq B$ and $p\left(U^{\prime}\right) \geq K$, we choose $S=\left\{n+u: u \in U^{\prime}\right\} \subseteq J$. Then, $S$ is nonempty since $U^{\prime} \neq \emptyset$ and we have $\Gamma / q-\bar{x}\left(U^{\prime}\right)=B+K-s\left(U^{\prime}\right) \geq B+K-B=K$ and

$$
\begin{aligned}
\frac{b(S)}{q}-\bar{x}\left(U^{\prime}\right) & =\sum_{u \in U^{\prime}} \frac{b(n+u)}{q}-\sum_{u \in U^{\prime}} \bar{x}(u)=\sum_{u \in U^{\prime}} p(u)+s(u)-\sum_{u \in U^{\prime}} s(u) \\
& =\sum_{u \in U^{\prime}} p(u) \geq K,
\end{aligned}
$$

yielding $\min \{b(S) / q, \Gamma / q\}-\bar{x}\left(U^{\prime}\right) \geq K$. Subtracting $K-1=\bar{x}(2 n+1)$ on both sides we obtain:

$$
\begin{aligned}
& \min \{b(S) / q, \Gamma / q\}-\bar{x}\left(U^{\prime}\right)-\bar{x}(2 n+1) \geq 1>0 \\
\Leftrightarrow \quad & \min \{b(S) / q, \Gamma / q\}-\bar{x}(N(S)) \geq 1>0,
\end{aligned}
$$

i.e. $\min \{b(S), \Gamma\}-q \cdot \bar{x}(N(S))>0$. Thus, $S$ is a solution for Problem 15 .

On the other hand, let $S \subseteq J$ be a solution for Problem 15 with the property $\min \{b(S), \Gamma\}>q \cdot \bar{x}(N(S))$. Then, $S$ must contain an element of the form $n+u$ for some $u \in U$, since if $S=\{2 n+2\}$ the inequality is not fulfilled. Set $S^{\prime}=S \backslash\{2 n+2\}$ and $U^{\prime}=\left\{u: n+u \in S^{\prime}\right\} \subseteq U$. Our aim is to show that $U^{\prime}$ is a solution for Knapsack. We have $S^{\prime} \neq \emptyset$ and $N\left(S^{\prime}\right)=U^{\prime} \cup$ $\{2 n+1\}$. Moreover, it also holds true that $\min \left\{b\left(S^{\prime}\right), \Gamma\right\}>q \cdot \bar{x}\left(N\left(S^{\prime}\right)\right)$, since $b(2 n+2)=0, \bar{x}\left(N\left(S^{\prime}\right)\right) \leq \bar{x}(N(S))$ and $q>0$. Reformulating the right hand side we get $\bar{x}\left(N\left(S^{\prime}\right)\right)=\bar{x}\left(U^{\prime}\right)+\bar{x}(2 n+1)=\bar{x}\left(U^{\prime}\right)+K-1$, so that in total we have $\min \left\{b\left(S^{\prime}\right), \Gamma\right\}-q \cdot \bar{x}\left(U^{\prime}\right)>q \cdot(K-1)$, i.e., $\min \left\{b\left(S^{\prime}\right) / q, \Gamma / q\right\}-\bar{x}\left(U^{\prime}\right) \geq K$. When inserting the above definitions this expression becomes

$$
\begin{equation*}
\min \left\{p\left(U^{\prime}\right)+s\left(U^{\prime}\right), B+K\right\}-s\left(U^{\prime}\right) \geq K \tag{9}
\end{equation*}
$$

Now, we need to differentiate between two cases: If $p\left(U^{\prime}\right)+s\left(U^{\prime}\right) \leq B+K$, (9) yields $p\left(U^{\prime}\right)=p\left(U^{\prime}\right)+s\left(U^{\prime}\right)-s\left(U^{\prime}\right) \geq K$ and $s\left(U^{\prime}\right) \leq B+K-p\left(U^{\prime}\right) \leq$ $B+K-K=B$. If $p\left(U^{\prime}\right)+s\left(U^{\prime}\right)>B+K$, (9) yields $B+K-s\left(U^{\prime}\right) \geq K$, i.e., $s\left(U^{\prime}\right) \leq B$. Furthermore, $p\left(U^{\prime}\right)>B+K-s\left(U^{\prime}\right) \geq B+K-B=K$. Thus, $U^{\prime}$ is a solution for Knapsack.

Corollary 20. Problem 16 is NP-complete.
Proof. Clearly the preceding Lemma 19 and Lemma 18 imply NP-hardness for Problem 16. Furthermore, Problem 16 is contained in NP since we only need to solve a linear program when given a scenario $\xi$ as a certificate, yielding NP-completeness in total.

Moreover, Lemma 19 reveals that checking whether a given vector $\bar{x}$ is feasible for a given instance of Robust $q$-MSMC is co-NP-complete: $\bar{x}$ is feasible if and only if the answer to the separation problem is "no".

Therefore, we start with a BIP formulation for the set formulation to solve the separation problem. Let $z_{j}$ be one if region $j$ is contained in $S$ and zero otherwise. Furthermore, let $y_{i}$ be one if $i \in N(S)$ and zero else. It is easy to see that Problem 15 can be formulated as the following binary program:

$$
\begin{array}{ll}
\min _{y, z, d} & \sum_{i \in I} q \cdot \bar{x}_{i} \cdot y_{i}-d \\
\text { s.t. } & d \leq \sum_{j \in J} b_{j} \cdot z_{j} \\
d & \leq \Gamma-\sum_{j \in J} a_{j}+\sum_{j \in J} a_{j} \cdot z_{j}, \\
y_{i} & \geq z_{j} \quad \forall j \in J, i \in N(j), \\
y_{i}, z_{j} & \in\{0,1\} \quad \forall i \in I, j \in J, \tag{10e}
\end{array}
$$

where $\bar{x}$ is the given fixed vector which we wish to test for feasibility. This program can be simplified by setting $\Gamma^{\prime}:=\Gamma-\sum_{j \in J} a_{j}$ and letting $k$ be the
maximum number of regions a location can serve, i.e., the maximum degree among the vertices $I$ in the bipartite graph $G=(I \cup J, E)$. Then, the above formulation can be rewritten:

$$
\begin{array}{cl}
\min _{y, z, d} & \sum_{i \in I} q \cdot \bar{x}_{i} \cdot y_{i}-d \\
\text { s.t. } & d \leq \sum_{j \in J} b_{j} \cdot z_{j}, \\
& d \leq \Gamma^{\prime}+\sum_{j \in J} a_{j} \cdot z_{j}, \\
& k \cdot y_{i} \geq \sum_{j \in N(i)} z_{j} \quad \forall i \in I, \\
& y_{i}, z_{j} \in\{0,1\} \quad \forall i \in I, j \in J . \tag{11e}
\end{array}
$$

Concerning the assignment formulation, we consider (MIP 12) obtained from (LP 8) by including the constraints on $\xi$. Note that the objective becomes now a quadratic and non-convex function since $\xi_{j}$ has changed into a variable.

$$
\begin{align*}
\text { (MIP 12) } \min _{\mu, \nu, \xi} & \sum_{i \in I} q \cdot \bar{x}_{i} \cdot \mu_{i}-\sum_{j \in J} \xi_{j} \cdot \nu_{j}  \tag{12a}\\
\text { s.t. } & \mu_{i} \geq \nu_{j} \quad \forall i \in I, j \in N(i),  \tag{12b}\\
& a_{j} \leq \xi_{j} \leq b_{j} \quad \forall j \in J,  \tag{12c}\\
& \sum_{j \in J} \xi_{j}=\Gamma,  \tag{12d}\\
& \mu_{i}, \nu_{j} \geq 0 \quad \forall i \in I, j \in J,  \tag{12e}\\
& \xi_{j} \in \mathbb{Z} \quad \forall j \in J . \tag{12f}
\end{align*}
$$

When forcing $\nu_{j} \in\{0,1\}$ (which we can do without loss of generality, see the proof of Lemma 18), we can use the Big-M method to regain a linear objective. Once $\nu_{j} \in\{0,1\}$, the variables $\mu_{i}$ can be assumed to be in $\{0,1\}$ without loss of generality. Then, the interpretation of the variables $\mu_{i}$ and $\nu_{j}$ equals that of $y_{i}$ and $z_{j}$ and constraint (12b) can be simplified in the
same manner as before. In total we obtain:
(IP 13) $\min _{\mu, \nu, \omega, \xi} \sum_{i \in I} q \cdot \bar{x}_{i} \cdot \mu_{i}-\sum_{j \in J} \omega_{j}$
s.t. $\quad k \cdot \mu_{i} \geq \sum_{j \in N(i)} \nu_{j} \quad \forall i \in I$,
$\omega_{j} \leq \xi_{j} \quad \forall j \in J$,
$\omega_{j} \leq \Gamma \cdot \nu_{j} \quad \forall j \in J$,
$a_{j} \leq \xi_{j} \leq b_{j} \quad \forall j \in J$,
$\sum_{j \in J} \xi_{j}=\Gamma$,

$$
\begin{array}{cl}
\mu_{i}, \nu_{j} \in\{0,1\} & \forall i \in I, j \in J  \tag{13~g}\\
\xi_{j} \in \mathbb{Z} & \forall j \in J .
\end{array}
$$

By Lemma 18 it suffices to solve (IP 13) since the optimal objective function value of (MIP 12) is less than zero if and only if that of (IP 13) is less than zero. In the optimal solution of (IP 13), the variables $\xi_{j}$ will be chosen in order to maximize the term $\sum_{j \in J} \omega_{j}$ which is the same as $\tilde{d}_{S}$ using the new interpretation of $\nu_{j}$. Thus, the optimal objective function values of (MIP 11) and (IP 13) coincide and we can also use the proof of Lemma 18 to solve (IP 13): In the first step we solve (MIP 11) yielding the optimal solution $(y, z, d)$ with objective function value $\Theta$. If $\Theta=0$, we obtain the optimal solution of (IP 13) by setting $\mu=\nu=0$ and choosing an arbitrary extreme scenario $\xi \in \mathcal{U}^{\prime}$ (which is also the optimal solution of (MIP 12)). If $\Theta<0$, we set $\mu=y$ and $\nu=z$ and choose the extreme scenario $\xi$ depending on whether $b(S)+a(J \backslash S)$ exceeds $\Gamma$ or not where $S=\left\{j \in J: z_{j}=1\right\}$ (cf. proof of Lemma 18). On the other hand, it is easy to see that a solution $(\mu, \nu, \xi)$ of (IP 13) translates to a solution ( $y, z, d$ ) of (MIP 11).

## 5 Computational results

After having analyzed Robust Min- $q$-Multiset Multicover theoretically, in this section we present some computational results with $q$ being fixed to three exemplarily. The results consist of two main parts: In the first part randomly created instances are considered and analyzed, whereas in the second part we model a real world problem as Robust Min-3-MSMC and display computational results based on real world data. For all instances of Robust Min-3-MSMC, we apply both solution approaches based on constraint generation as introduced in Section 4. The solution approach based on the assignment formulation, cf. (MIP 6), is referred to as asf while the approach using the set formulation, cf. (IP 4), is referred to as setf. For the separation of asf we used (IP 13) and for the separation of setf we
used (MIP 11). We also tested combining the separation approaches using Lemma 18, but the difference of the running times was neglectably small.

To investigate the price of robustness, we compare the objective value of a solution of Robust Min-3-MSMC to the objective value of an average solution. We obtain the objective value of the average solution by randomly choosing a fixed number of possible scenarios with total demand $\Gamma$, solving the corresponding non-robust version and determining the median of all these objective values. Furthermore, we compare the robust solution to the solution in which the worst case in all regions is assumed, i.e., the number of clients in each region is set to the upper bound. We refer to this solution by worst case solution. The objective value of the robust version will be significantly smaller for most instances than the one of the worst case solution. Although this comparison may seem irrelevant as the worst case scenario does not even exist in the robust concept, the comparison does indeed give insight: Fix for the moment one location $i \in I$. If the total demand of its neighborhood $N(i)$ is at most $\Gamma$, there exists a scenario (that needs to be covered) in which the demand of each region in $N(i)$ is at its upper bound. If this situation holds for all locations $i$, it is possible that the worst case is in some sense simulated by the constraints arising from such scenarios. As an extreme example regard the case where the neighborhoods of all locations are disjoint. In this case it holds true that the robust solution value is the same as the worst case solution value, whereas the average case solution value may be significantly smaller. Additionally, we analyze the running times of both approaches asf and setf.

To solve the integer linear and mixed integer linear programs, the Gurobi Optimizer 8.01 [24] with the Python Interface (Python Software Foundation, https://www.python.org) was used. All computations were done on a machine with an $\operatorname{Intel}(\mathrm{R}) \mathrm{Xeon}(\mathrm{R}) \mathrm{CPU} \mathrm{E} 5-26900$ @ $2.90 \mathrm{GHz}, 16$ cores and 192 GB main memory. The operation system is Ubuntu 64-Bit. We permitted each solution approach to use four threads. Further, for the realworld instances, we aborted (if not finished) the computation after 15 min wall-clock time and for the random instances after 2 min . Computation times are all processing times given in seconds. In the following two sections, we explain the creation of the instances for the random case and the real world case. Furthermore, we present and interpret some interesting results.

### 5.1 Random instances

Our random instances are created as follows: We fix the number of regions to 100 and choose the number of locations from the set $\{10,20,30\}$. The bipartite graph of the instance with edge probability $p$ is then generated in two steps. To avoid infeasible instances we first randomly choose a location for each region, so that the demand of each region can be covered. In a second step, we add each possible remaining edge independently with

| $\|I\|$ | $10,20,30$ |
| :---: | :--- |
| $p$ | $0.1,0.2,0.3$ |
| $\left(k_{1}, k_{2}\right)$ | $(0,1),(10,10),(10,50),(10,100),(50,50),(100,100)$ |
| $d$ | $0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1$ |

Table 1: Choices of the parameters for the random instances.
probability $\frac{|I| p-1}{|I|-1}$ such that the expected number of edges in the created graph is exactly $|I||J| p$. The lower bound $a_{j}$ for the number of clients in each region is picked uniformly at random from the fixed discrete interval $\left[0, k_{1}\right]$. To obtain the corresponding upper bound $b_{j}$ a random integer taken from the interval $\left[1, k_{2}\right]$ is added to $a_{j}$ in a second step. Furthermore, we define the bound $\Gamma$ to be

$$
\Gamma=\sum_{j \in J} a_{j}+\left\lfloor d \cdot\left(\sum_{j \in J} b_{j}-\sum_{j \in J} a_{j}\right)\right\rceil
$$

for $d \in\{i / 10,0 \leq i \leq 10\}$. Thus, for $d=0$, the robust solution corresponds to the best case solution since $\mathcal{U}=\left\{\left(a_{1}, \ldots, a_{|J|}\right)\right\}$ and, for $d=1$, it corresponds to the worst case solution $\left(\mathcal{U}=\left\{\left(b_{1}, \ldots, b_{|J|}\right)\right\}\right)$. In the sequel, we refer to $d$ as gamma factor.

Table 1 summarizes all choices of the parameters. Thus, for $k:=\left(k_{1}, k_{2}\right)$, each combination of $|I|, p, k, d$ defines the structure of an instance for which we create 50 representatives $I_{1}(|I|, p, k, d), \ldots, I_{50}(|I|, p, k, d)$. To be able to compare the impact of differing demand ranges or gamma factors, the underlying graph of instance $I_{r}(|I|, p, k, d)$ equals that of $I_{r}\left(|I|, p, k^{\prime}, d^{\prime}\right)$ for every $r$ and fixed values for $|I|$ and $p$. Furthermore, we are interested in the relative gap between the robust solution value and the worst case solution value, respectively average case solution value, to analyze the extra cost of robustness. For the random instances we choose ten extreme scenarios uniformly at random to compute the average case solution as explained above. Interestingly, the chosen median solution value of each instance $I_{r}(|I|, p, k, d)$ is close to $\lceil\Gamma / 3\rceil$ so that almost every doctor covers three demand points. Thus, the average case solution value is always very close to the average over the trivial lower bounds $\lceil\Gamma / 3\rceil$.

In the following, we present some interesting findings during our analysis of these random instances. Fig. 2 depicts the logarithmic (to the base of 10) average relative gap between the objective values of the worst case solutions $\left(w c s v_{r}\right)$ and the robust solutions $\left(r s v_{r}\right)$, i. e., $\log _{10}\left(1 / 50 \sum_{r=1}^{50} w c s v_{r} / r s v_{r}\right)$ (black), as well as the logarithmic average relative gap between the objective values of the average case solutions and the robust solutions (cyan). As a reference point the logarithmic average relative gap between the robust solution and itself is drawn as a dashed horizontal line (magenta). Moreover,


Figure 2: Logarithmic average relative gaps between the worst case and the robust solution as well as between the average case solution and the robust solution for 50 random instances with $|J|=100, k_{1}=10$ and varying other parameters.
the average processing times (in seconds) of asf and setf are displayed in Fig. 3, where we average only over the processing times for instances that were actually solved to optimality. The markers in Fig. 3 are given an alpha value determining their transparency where the alpha is computed by dividing the number of solved instances by 50 .

Focusing on Fig. 2 we first observe that for gamma factor 1.0 or 0.0 the objective values of the robust solution and the average case solution coincide as in these two cases there only exists one unique extreme scenario. Further, we can see that with rising gamma factor the objective value of the worst case solution gets closer to the objective value of the robust solution which is also expected.

Regarding the upper three plots in Fig. 2 we can see that increasing the density of the graph decreases the relative gap between the average case solution values and the robust solution values and increases the relative gap between the worst case solution values and the robust solution values. This can be explained by the fact that in dense graphs suppliers have more possibilities to serve clients and can therefore act more flexible than in sparse graphs. An extreme example would be a location adjacent to all regions. We could then simply put all suppliers in that location and get a solution that
is as high as the trivial lower bound. Furthermore, the relative gap between the worst case solution values and the robust solution values becomes larger with increasing density. Looking at the data we can see that the worst case solution and the average case solution do not change too much with increasing density, it is in fact the robust solution that becomes cheaper. We can conclude that in dense graphs we get robustness almost for free, whereas in sparse graphs we have to pay quite a bit for turning the solution into a robust one. Note that increasing the number of locations while keeping $p$ fixed improves the robust solution in a similar manner.

In the second row of Fig. 2 we see the impact of increasing the range for the upper demand of the regions. With increasing $k_{2}$ the relative gap between both the worst case solution values and the robust solution values as well as the average case solution values and the robust solution values becomes larger. This behavior does not solely depend on the increased value of $k_{2}$ but rather on the relative difference of $k_{1}$ and $k_{2}$. For example, on instances with $k_{1}=k_{2}=100, p=0.1$ and $|I|=30$ we can see that the robust solution values and the average solution values coincide in most cases. We conclude that the price of robustness is especially cheap if $k_{1} \geq k_{2}$ and the graph is dense enough. A reason for this behavior could be that if all regions have high lower bounds on their demand (in comparison to their upper demand) the number of suppliers positioned in locations adjacent to some fixed region is quite large in any average case solution. Therefore, there are more possibilities for allocating them to the clients.

Overall the price of robustness is very low if the graph is dense or the relative gap between $k_{1}$ and $k_{2}$ is low. For sparse graphs the price of robustness can become very large. For example in the instances with $|I|=10$ and $p=0.1$ the robust solution value coincides with the worst case solution value quite often. Note that following the random graph construction given above, in all these instances every region is adjacent to exactly one location.

Fig. 3 depicts the average running times of the two solution approaches for 50 instances with $|J|=100, k_{1}=10$ and varying other parameters. If the plot does not contain a point for some gamma factor it means the corresponding solution approach did not finish computation in the given time window of two minutes. We can see, that in most cases setf seems to be the better choice. Taking a closer look, it becomes clear that asf performs especially well when the robust solution value coincides with the average case solution value. In these cases a very small number of extreme scenarios (often even just one) needs to be added in order to get a robust solution. With increasing number of required scenarios the running time for asf explodes. The running time of setf does not seem to depend on this too much. The only major impact on the running time of setf seems to be the number of regions and locations.


Figure 3: Average processing times in seconds of asf and setf for at most 50 random instances with $|J|=100, k_{1}=10$ and varying other parameters. The transparency of the markers reflect the amount of solved instances.

### 5.2 Real world example: placing emergency doctors

As a real world application, we regard the problem of placing as few emergency doctors as possible into given facilities such that the emergencies happening in one shift can still be handled in a satisfactory way. In this context, the uncertain demand of each region reflects the unknown number of emergencies happening in that region during the considered shift. Thus, the proposed discrete budgeted uncertainty set allows, for each region, variations in a given interval but the total number of emergencies is bounded. The proposed model seems fitting for the application since all realistic scenarios should be covered equally well.

Using map data from OpenStreetMap [37], we construct a graph modeling the street network of some fixed part of the map. In our computations, we considered the street network inside a bounding box enclosed in the federal state Rhineland-Palatinate in Germany. The size of the bounding box is roughly $300 \mathrm{~km}^{2}$ mostly consisting of rural areas. The GPS coordinates of the south-west corner of the box are $(7.2606,49.1703)$ and the GPS coordinates of the north-east corner of the box are (8.3890, 49.9537). Inside the box there are currently 38 emergency facilities, cf. [11], which we choose as locations of the instance denoted by the set $I$.

In the street network the edge weight corresponds to the time needed for a doctor to travel along this particular edge based on the maximum speed allowed on the associated piece of road. To obtain the regions, for each street node, we compute a list of locations from which the street node can be reached within 15 min . In Rhineland-Palatinate, 15 min is the time at which the first responder must be present at the emergency after he left the facility. Now, we define all street nodes with the same list of facilities to be in the same region and denote the set of all regions by $J$. With our bounding box this results in a set of 426 regions and gives a straight forward way to define the graph of the instance: Simply add edges between each region and all locations of its list.

We further set $a_{j}=0$ and $b_{j}=1$ for all $j \in J$. The interpretation of these bounds is that in any region there might or might not occur an accident during the regarded shift. We do not allow more than one emergency in a given region as our regions are rather small and we therefore deem the case of more than one occurring emergency during one shift to be unrealistic. The total number of emergencies $\Gamma$ is then set to different values for the tests. We assume that an emergency doctor is able to handle up to three emergencies in one shift, i.e., $q$ is fixed to three as in the random case above. Clearly, this assumption is not exact since emergencies may overlap and one doctor may not be able to attend to even two emergencies if they overlap. Nevertheless, we are convinced that this approach is reasonable since adding uncertainty to the value $q$ as well would lead to overlapping uncertainties rapidly increasing the conservativeness of the solution.

Note that our model does not forbid local worst cases: For any subset of the regions of size at most $\Gamma$ there is a scenario in which the demand of each region in the subset is set to one. Thus, if the regarded part of the map is too large, implicitly raising the assumed maximum number of emergencies $\Gamma$, the solution to our model also covers scenarios in which emergencies massively occur in very small parts of the entire map. Thus, when using Robust Min-3-MSMC for this application, the regarded size of the map should be reasonable.

Fig. 4 shows the comparison of the objective value of the robust model to the median objective value of the average case. Due to larger input data compared to the random instances, we choose five random extreme scenarios for the average case solution. Fig. 5 depicts the running times of setf and asf. Test runs were made for each $\Gamma \in\{3 i: i=0, \ldots, 142\}$.

Taking a closer look at Fig. 4, we can see that the objective value of the average case solution is always close to the trivial lower bound $\lceil\Gamma / 3\rceil$. Thus, almost all doctors cover three emergencies in the fixed average scenarios. The behavior of the robust solution is somehow expected. The greatest absolute deviation from the average solution is attained for $\Gamma$ between 54 and 156 , so roughly for $\Gamma$ attaining a value between $1 / 10$ and $1 / 40$ of the


Figure 4: On the left, the number of doctors needed in the real world instances of the robust model, the average case and the worst case. On the right, the logarithmic (to base 10) relative gaps between the average case solution and the robust solution as well as between the worst case solution and the robust solution.
total sum of the demands 426. The relative distance between the average case solution value and the robust solution value decreases linearly. In the regarded map area, fixing the number of emergencies between 20 and 40 seems adequate. The price of robustness for the application in this area does seem quite low, given the fact that in the average case solutions not even all regions have to be reachable by some doctor.

The running time of setf is acceptable for all regarded demand bounds $\Gamma$. On the other hand, asf was not able to solve all instances within the time limit of 15 min . Furthermore, the variance in the running time is much higher than for setf. Interestingly, in some cases asf outperforms setf significantly. Especially in the area where $\Gamma$ is around 150. Thus, it seems worthwhile to use both approaches in practice parallelly. If one is looking for only one solution approach setf should be preferred because it seems more reliable as the variance in the running times is smaller. Surprisingly, for random instances created as described in the previous section with parameters similar to this application $\left(|I|=38,|J|=426, a_{j}=0, b_{j}=1\right.$, $p=0.07$ ), the running times of both solution approaches increase rapidly, where the given $p$ roughly models the density of the graph arising from our application. For example, setf took roughly 20 h of computation for solving instances with $\Gamma=45$. For larger $\Gamma$, it did not finish solving the problem after 48 h . asf was not able to finish any instance within the time limit of 48 hours. We think that this behavior is due to the planar-like structure of the graph arising from the application. For future research, it might be interesting to work on complexity results for instances with these planar-like


Figure 5: Processing times in seconds of setf and asf for different demand bounds.
structures.
We conclude that for map areas of the tested size Robust Min-3-MSMC is of interest for the given practical application. Though, we think that for a larger regarded map area, a model including some local condition on the scenarios to prevent the described massive occurrences of emergencies in comparably small map areas might suit the application even better. This could be a direction of future research on the topic.

## 6 Conclusion

We have presented a novel problem called $q$-Multiset Multicover which we have identified to be a special case of Multiset Multicover. We have shown that it is NP-complete for fixed values of $q \in \mathbb{N}$ with $q \geq 3$ but polynomial time solvable for $q=1,2$. The main focus of this paper was the robust version of $q$-Multiset Multicover, which we proved to be strongly NP-hard for all $q \in \mathbb{N}$. Further we have given two different integer programming formulations of the problem and discussed their up- and downsides. We presented strategies for solving the problems based on constraint generation. Our computational results based on random instances and instances corresponding to a real world application are quite promising. They show that the model and the robust approach can be of great use for practical problems since it is able to hedge against uncertainty with fewer resources compared to an all worst-case approach.

## 7 Acknowledgments

Map data copyrighted OpenStreetMap contributors and available from https://www. openstreetmap.org.
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## A Complexity Results for $q$-MSMC

At this point we will give formal proofs for the statements concerning the complexity of Min- $q$-MSMC in Section 2.

Observation 21. Min-1-Multiset Multicover is solvable in linear time.
Proof. If $q=1$, given an instance for Min- $q$-MSMC, in any solution each client needs to be assigned a unique supplier. This means, for each client in some region $j \in J$ we may put a single supplier in some location $i \in N(j)$. This will yield a feasible solution with $\sum_{i \in I} d_{i}$ suppliers, which is clearly optimal. We can find this solution in time linear in $|I|+|J|$.

For $q=2$ we can still solve Min- $q$-MSMC in polynomial time. We show how to compute an optimal solution using an algorithm for the Edge Cover problem. Recall that for a graph $G=(V, E)$ an edge cover is a subset of the edges $E^{\prime} \subseteq E$, such that each vertex $v \in V$ is incident to at least one edge $e \in E^{\prime}$. Regard the following procedure: For a given instance of the problem, duplicate each region $j \in J$ exactly $d_{j}$ times yielding a set $V_{j}$ for each $j \in J$. We see that we may bound any $d_{j}$ by $|I|$ in the proof of Theorem 7. Regard the graph $G=(V, E)$ with vertex set $V=\bigcup_{j \in J} V_{j}$ where the edge $(u, v)$ for $u \in V_{j_{1}}, v \in V_{j_{2}}$ are in $E$ if $N\left(j_{1}\right) \cap N\left(j_{2}\right) \neq \emptyset$. Note that this implies that the graph induced by some set $V_{j}$ is the complete graph. Next compute a minimum edge cover $E^{\prime}$ in $G$ and initially set $x_{i}=0$ for all $i \in I$. For each edge $(u, v) \in E^{\prime}$, with $u \in V_{j_{1}}, v \in V_{j_{2}}$ we increase $x_{i}$ by 1 for some $i \in N\left(j_{1}\right) \cap N\left(j_{2}\right)$, meaning we add a supplier in location $i$ who covers one demand point in region $j_{1}$ and one in region $j_{2}$.

Theorem 22. The above procedure solves Min-2-Multiset Multicover and can be implemented to run in time $O\left(|I|^{5 / 2}|J|^{5 / 2}\right)$.

Proof. We first prove the correctness of the procedure. Let $G=(V, E)$ be the graph defined in the procedure. Let $x$ be as defined by the procedure above and let $E^{\prime}$ be the minimum edge cover from the procedure. As we have a node in $G$ for every client and the nodes corresponding to the clients are covered by the edges in $E^{\prime}$ it is clear that $x$ defines a feasible solution for Min- $q$-MSMC. It remains to show, that given a solution $x$ to Min- $q$-MSMC, there is an edge cover with $\sum_{i \in I} x_{i}$ edges. By the equivalence of (IP 2) and (MIP 1) we can find $y_{i j} \in \mathbb{Z}_{\geq 0}$ for all $i \in I, j \in J$ fulfilling

$$
\sum_{i \in N(j)} y_{i j} \geq d_{j} \forall j \in J \text { and } \sum_{j \in N(i)} y_{i j} \leq 2 x_{i} \forall i \in I .
$$

Clearly, we may assume equality in the second set of equations and can thereby determine for each supplier the two clients he serves. We initially set $E^{\prime}$ to the empty set. If, for each supplier, we now select the edge between
the two clients he serves and add it to $E^{\prime}$, we get an edge cover of $G$ with $\sum_{i \in I} x_{i}$ edges. This proves the correctness of the procedure.

To see the running time, first note that we may bound the number of clients $d_{j}$ in any region by the number of suppliers: Assume $d_{j} \geq|I|+1$ for some $j \in J$ in some instance. Then, in any solution of (MIP 1) there is some $i \in N(j)$ such that $y_{i j} \geq 2$. Thus, given an optimal solution, choose $i$ such that $y_{i j} \geq 2$. Removing one supplier from $i$ now yields an optimal solution to the same instance with the demand of region $j$ being $d_{j}-2$. As a consequence we may also solve this instance and then afterwards add a supplier to any location connected to $j$ to get an optimal solution of the original problem. We can therefore decrease the demands of all $j$ with $d_{j} \geq|I|+1$ to $|I|$, respectively $|I|-1$ by adding $\left\lceil 1 / 2\left(d_{j}-|I|\right)\right\rceil$ doctors to any location connected to $j$. This can be done in constant time for any region $j \in J$. With this observation, it can readily be seen that the constructed graph has at most $N:=|I| \cdot|J|$ vertices whereas the number $M$ of edges is upper bounded by $O\left(|I|^{2}|J|^{2}\right)$. A minimum edge cover in a graph with $N$ vertices and $M$ edges can be obtained by first solving a maximum matching problem in time $O\left(\sqrt{N} M \log _{N}\left(N^{2} / M\right)\right)$ [21] and then using $O(M)$ time to augment the matching [40, 32, 20]. This gives the claimed running time.

It is fairly easy to see that $q$-Multiset Multicover is a generalization of Set Cover by $q$-sets, i.e., the restriction of Set Cover where all sets are of size exactly $q$. Since this is an NP-complete problem, the next result is not surprising. For the sake of completeness we will nevertheless give a formal proof.

Theorem 23. For any fixed $q \geq 3$, $q$-Multiset Multicover is NP-complete in the strong sense.

Proof. Let $q \in \mathbb{N}$ with $q \geq 3$. As a consequence of Lemma 4, for a given instance of $q$-MSMC, we may test if a given solution $x$ is feasible by one Max-Flow computation. Therefore, $q$-MSMC is contained in NP.

To see that the problem is NP-hard in the strong sense we illustrate a reduction from Exact Cover by 3-sets, which is known to be NP-hard in the strong sense, cf. [20]. Let $X$ be a set and $\mathcal{S}$ be a collection of subsets of $X$ where $|S|=3$ for all $S \in \mathcal{S}$. We create an instance of $q$-MSMC in the following way. Due to legibility, assume the subsets $S \in \mathcal{S}$ have unique indices $i_{S}$. Let $I:=\left\{i_{S}: S \in \mathcal{S}\right\}, J:=X$ and define the graph of the instance via $N\left(i_{S}\right)=S$ for all $S \in \mathcal{S}$. Further, let $d_{j}=1$ for all $j \in X$ and $B=|X| / 3$. Now, let $\mathcal{S}^{\prime}$ be a solution to the instance of Exact Cover by 3-sets. Clearly, setting $x_{S}$ to one if and only if $S \in \mathcal{S}^{\prime}$ and zero else yields a feasible solution to $q$-MSMC with $\sum_{S \in \mathcal{S}} x_{i_{S}}=B$. On the other hand, note that in any solution $x$ to $q$-MSMC $x_{i_{S}} \leq 1$ for all $S \in \mathcal{S}$. Furthermore, since
$\left|N\left(i_{S}\right)\right|=3$ the actual value of $q$ is of no further interest as long as $q \geq 3$. Thus, $\mathcal{S}^{\prime}=\left\{S: x_{S}>0\right\}$ is a solution to Exact Cover by 3-sets.

Remark 2 reveals $q$-MSMC to be a special case of Multiset Multicover. It is well known that Multiset Multicover can be approximated within a factor of $\log (s)$ where $s$ is the size of the largest multiset of an instance, see e.g. [25, 31, 14]. If we regard Min- $q$-MSMC as Multiset Multicover problem, all multisets have fixed size $q$. We therefore automatically get a $\log (q)$ approximation for Min- $q$-MSMC:

Observation 24. There is a $\log (q)$ approximation for Min- $q$-Multiset Multicover.


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