Fair elimination-type competitions[☆]

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Abstract

We study the impact of two basic principles of fairness on the structure of elimination-type competitions and perform our analysis by focusing on sports competitions. The first principle states that stronger players should have a larger chance of winning than weaker players, while the second principle provides equally strong players the same chances of being the final winner. We apply these requirements to different kinds of knockout competitions, and characterise the structures satisfying them. In our results, a new competition structure that we call an *antler* is found to play a referential role.

Keywords:
OR in sports; Fairness; Graphs; Seeding rules; Tournament design

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1. Introduction

Numerous choice problems require the selection of an alternative from a set of options on the basis of the information obtained from pairwise comparisons among the available alternatives. Examples of these problems can be found in voting theory (cf. Brams and Fishburn 2002, Laslier 1997, Levin and Nalebuff 1995, Moulin 1986), multi-criteria decision making (Larichev 2001, Olson 1996), and promotion mechanisms implemented in firms (Rosen 1986). However, the most popular problem of this type is probably that of selecting a winner in a sports competition, where the alternatives are the competing "players" and the pairwise comparisons take the form of "matches".

In this work, we present an axiomatic approach to the fairness aspect of these kinds of selection problems when the competition is of an eliminative nature. Our analysis is potentially applicable in different situations, but we frame the study in the field of sports competitions, which by itself is highly relevant due the enormous economic and social relevance that the sports industry has nowadays.

Among other results, we identify a particular competition structure, which we call an *antler*, that can be used as a reference for clarifying the discussion as to the fairness of different kinds of competitions, which is usually guided by intuition. In particular, we prove that any elimination-type structure containing an antler as a substructure may give weaker players greater probabilities of being the final winners than stronger players (Theorem 1). As a matter of fact, the US National Football League (NFL) playoffs have an antler structure and the North American National Basketball Association (NBA) playoffs contain an antler, which means that they suffer from the said drawback, while the postseason playoffs of the two leagues in the North American Major League Baseball (MLB) do not contain an antler and thus have no such problem. This result has implications for the usual knockout tournaments involving 2^q players in q rounds. A consequence of Theorem 1 is that no such competitions guarantee that stronger players have a higher chance of winning if $q \geq 3$.

Every sports competition needs a well-defined and pre-established set of basic rules that determines the "competition system": who plays against whom and at which stage of the competition. Different objectives can be plausibly considered when designing the competition system, such as the intensity of the matches, suspense, attracting the interest of the spectators, optimising organisational costs, and so on. However, fairness is in general within the goals of any competition designer.

Discussions about whether one or another system is more or less fair than another are often made at an intuitive and informal level. In our work, we provide a structured analysis of such discussions and formally define two neat principles of fairness that respond to what is commonly pursued in real practice and translate to our context the Aristotelian Justice Principle of "treating equals equally and unequals unequally". In our framework, these principles require on the one hand that the competition system should favour stronger players (we call this "monotonicity in strength"), and on the other hand that equally strong players should have the same chances of being the final winner ("equal treatment"). We then study to what extent different elimination-type competition formats perform in relation to these principles, trying to give formal support to such informal debates. For example, the typical kind of problem that we want to solve is how to fairly seed the tennis players in a professional tennis tournament based on their ATP ranking, but also to discuss about the fairness of alternative systems of competition.

In elimination competitions, which are also called "knockout" tournaments, the competition is organised in rounds or "stages". Losers are eliminated and players progress as they win their corresponding matches in the round, being paired off in the next round, so that the final winner is the player who wins all the rounds. Typical examples are the playoff formats found in North American professional sports. These competitions can be represented by binary trees, as exemplified in Figure 1.

Elimination competitions, such as the one displayed in Figure 1, are called *balanced* because all players are required to win the same number of

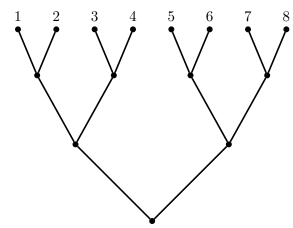


Figure 1: A balanced elimination competition.

matches to become the final winner. In some cases some players have the right of "byes"; meaning that, on the basis of a previous qualification rating, they have the privilege to skip the initial round (or rounds) without the need of playing. In fact, "byes" become necessary if the number of players is not a power of 2. A special type of elimination competition with byes has a so-called "stepladder" structure (see Figure 2). This system and its variants are used in ten-pin bowling and squash, for example.

The design of an elimination competition requires a solution to the problem of "seeding"; that is, of assigning players' names to the "leaves" of the competition's tree. This involves deciding the pairing in the initial matches and, if that is the case, which player(s) deserve(s) the byes. Clearly, the seeding will have a crucial impact on the chances for a player to become the final winner.

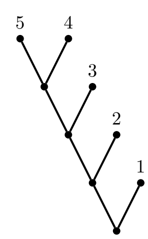


Figure 2: A stepladder competition.

Each format has its pros and cons, related for instance with the number of matches needed to have a final winner, organisational costs, profitability for the organiser, or the manipulability by the players. While admitting the importance of all of these issues, in this work we exclusively concentrate on the analysis of elimination-type competitions from the point of view of the fairness that is intrinsically associated to their structure according to our axiomatic constraints.

1.1. Literature overview

The literature about fairness in elimination competition systems is rather disseminated. Related works usually study particular competition systems and fairness aspects related with their specificities. Most of the attention in this respect has been paid to the performance of alternative seeding procedures in *balanced* elimination-type competitions with a limited number of players. The closest works to ours are found in the fields of management mathematics and operations research: Horen and Riezman (1985) provide results about fair seeding for four-player and eight-player balanced elimination-type competitions; Prince et al. (2013) provide some computational results for the eight-player and 16-player cases using an alternative notion of fairness. Karpov (2016, 2018) axiomatically studies particular seeding rules in balanced competitions under certain restrictions of the probability domain. Dagaev and Suzdaltsev (2018) solve a discrete optimisation problem to analyse under which conditions certain seedings in balanced competitions maximise spectator interest when they care about competitive intensity. Groh et al. (2012) studies a property similar to monotonicity in strength for the case of four-players balanced competitions where the players may exert different effort depending on heterogeneous valuations of winning. Less closely related are the works by Geenens (2014), which analyses how decisive a game is with respect to the final victory, or Aronshtam et al. (2017), which studies the computational complexity of manipulability of knockout competitions in order to favour particular players.

Hwang (1982) and Schwenk (2000) pay attention to random seeding and re-seeding methods. Baumann et al. (2010) addresses the disadvantages of these kinds of methods, such as the increase in travel costs, the reduction in gambling demand and spectator interest, and incentive compatibility problems. In this work, we assume that the seeding is deterministic from the very beginning, taking the aforementioned drawbacks into consideration and appealing to real practice, which includes the most prestigious elimination-type competitions, such as the football World Cup, the ATP and WTA tennis tournaments and the playoff stages of the main North American sports.

Fairness in sports has also been recently analysed from other different perspectives, which are worth mentioning: Kendall and Lenten (2017) provide a comprehensive review of changes in sporting rules which have led to unexpected unfair consequences; Csató (2018, 2019d) and Dagaev and Sonin (2018) show that a player can be strictly better off with a weaker performance in some tournaments; Fornwagner (2018), Lenten (2016), Lenten et al. (2018) analyse similar situations in the case of leagues with drafts. The principle that equally skilled players should have the same probability of winning has been studied in the case of tie-breaking mechanisms in football and tennis (Apesteguía and Palacios-Huerta 2010, Brams and Ismail 2018, Che and Hendershott 2008, Cohen-Zada et al. 2018, Palacios-Huerta 2012), the schedule in sequential round-robin tournaments (Durán et al. 2017, Krumer and Lechner 2017, Sahm 2019), the kick-off time of the matches (Krumer 2019), the determination of the brackets in a multi-stage tournament on the basis of the first stage results (Guyon 2018) and the rules of seeding the FIFA (Féderation Internationale de Football Association) World Cup groups (Guyon 2015, Cea et al. 2019 and Laliena and López 2019).[1]

There is a remarkable line of research that introduces explicitly exertion of effort as a strategic variable and studies the possibility for manipulation by players (cf. Brown and Minor 2014, Rosen 1986, Groh et al. 2012, Krumer et al. 2017, Pauly 2014 and Vong 2017). The analysis has always been made for a small number of players (usually four) because it is generally accepted that the extension to a larger number of players involves an excessive complexity due to the highly complicated combinatorial structure of the problem.

[1] We thank an anonymous referee for drawing our attention to these references.

As to the comparison among different competition systems, most studies apply statistical simulation techniques to check the fulfillment of particular properties, or how the competition systems perform according to particular metrics (cf. Appleton 1995, McGarry and Schutz 1997, Scarf et al. 2009, Ryvkin 2010 and Ryvkin and Ortmann 2008). There is also a considerable body of literature in operations research related to sports that considers many aspects other than fairness which lie outside the scope of this work. The interested reader is referred to Wright (2014) for a survey in the field.

1.2. Our contribution

We formalise the mentioned basic ideas of fairness by means of two simple axioms that have a "rank-preserving" flavour: the first one is a "monotonicity in strength" condition requiring that "stronger players should have a higher probability of being final winners", and an "equal treatment" condition which states that "equally strong players should have the same probability of being the final winner".

Our results include characterisations of the competition formats satisfying these fairness properties. We also specify the class of seeding rules that let the structures satisfy the axioms. Generally speaking, *equal treatment* leads to balanced competitions in which every player participates in the same number of matches (Theorem 2 and Theorem 3), while when *monotonicity in strength* is under consideration, all elimination competitions fulfilling it should not contain a special substructure, which we call an "*antler*" (Theorem 1). This structure combines the characteristics of balanced elimination competitions with the "byes" spirit of stepladders. Moreover, we show that the seeding rule for which an antler-free competition satisfies monotonicity in strength is unique. As already pointed out above, it is possible to find major competitions that are antler-free (the postseason playoffs of the two MLB leagues – National and American – or any stepladder) and others that are not (the NFL playoffs or any balanced elimination competition with more than four players).

The rest of the work is organised as follows. Section 2 presents the basic elements of the formal model and introduces the two fairness axioms. Section 3 is devoted to the characterisation results. Section 4 concludes and addresses possible extensions of the model. All proofs are relegated to the Appendix.

2. The model

The main ingredients of our model are the graph representation of a competition system (Subsection 2.1), the ordinal information about each player's relative strength, represented by a binary relation R and the set of "winning probability matrices" that are consistent with R (Subsection 2.2), the notion of a seeding rule, and the probability of each player being the final winner as a consequence of all the previous elements (Subsection 2.3). In Subsection 2.4 we present the two fairness axioms by making use of all the previous formal elements.

2.1. Graph representation of competition systems

We assume that matches always take place between two players in such a way that ties are not possible and each match is represented by an *elementary binary tree*; that is, a graph with three nodes $\{a, b, w\}$ and two links $\{aw, bw\}$ with, let us say, player i being assigned to node a, player j being assigned to node b, and the winner of the match between i and j being assigned to node w. In this case, we say that i *is matched with* j and that the winner of this match *reaches* node w. Elimination competitions can then be represented by a finite *binary tree* connecting in a specific way such elementary binary trees, like those in Figures 1 and 2.

Given a binary tree t, we denote by $V(t)$ the set of its nodes (or *vertices*). The set of leaves (or *terminal nodes*) of t is denoted by $\Lambda(t)$, $\Lambda(t) \subset V(t)$. The distance between two nodes of $t \in G$ is defined by the minimal number of edges that are necessary to connect them. The *level* $\ell(v)$ of a node $v \in V(t)$ is the distance between it and the root of the binary tree t. The k-th level of a tree t is the set of all its nodes of level k. In our context, the level of a tree is interpreted as a *round*.[2] The *height* $h(t)$ of a binary tree t is the maximal level of its leaves, $h(t) = max_{\lambda \in \Lambda(t)}\{\ell(\lambda)\}$. By $\Lambda^k(t)$ we denote the set of leaves of t whose level is k. We say that a binary tree $t \in G$ is *balanced* (or that it *represents a balanced competition*) if the level of all of its leaves is the same. Notice that a *stepladder* competition is represented by a binary tree t with two leaves at level $h(t)$ and a unique leaf at each level ℓ for all $\ell < h(t)$.

2.2. Players' strength and winning probabilities

Real competitions often use some objective strength ordering to decide the seeding in an elimination tournament, or how teams are slotted into different groups in a two-stage competition. Those orderings typically depend on past performance. Prominent examples are the Association of Tennis Professionals (ATP) or the Women's Tennis Association (WTA) rankings, used in the major professional tennis competitions; the FIFA World Ranking, used for the FIFA World Cup; and the NBA playoffs, the seeding of which is determined by a team's ranking in the regular season.

In what follows, we consider a finite set N of competing players and assume that the elements of N are completely ordered according to a binary relation R of *strength* so that, for all $i, j \in N$, iRj is interpreted as *"player i is at least as strong as player j"*. The corresponding asymmetric and symmetric factors of R are denoted, respectively, by P and I, so that iPj reads *"i is strictly stronger than j"* and iIj reads *"i and j are equally strong"*.

We attach a probabilistic meaning to R in the sense that iRj means that "the probability that player i defeats in a match player j is greater than or equal to 0.5". We denote this fact by $p_{ij} \geq 0.5$. Given that R is complete, we have that iPj is accordingly interpreted as $p_{ij} > 0.5$ and iIj is

[2] Rounds are usually numbered in inverse terms; that is, the last level of the tree constitutes the first round of the competition, the second last level constitutes the second round and so on.

interpreted as $p_{ij} = 0.5$. Also, given that R is transitive, for all $i, j, k \in N$, $p_{ij} \geq 0.5$ and $p_{jk} \geq 0.5$ implies $p_{ik} \geq 0.5$. Throughout the next sections, we take the non-deterministic view that $0 < p_{ij} < 1$ holds for all $i, j \in N$. This assumption is taken simply to show that the presence of deterministic values is not what makes the different theorems and lemmas hold in a trivial way. It is easy to check that all results also hold for the case of $p_{ij} \in [0, 1]$ for all $i, j \in N$. We adopt the convention that the players in N are ordered according to R; that is, if iPj then $i < j$ (if iIj then either $i < j$ or $j < i$).

According to this interpretation, every binary relation of strength R induces a set of *winning probability matrices* defined on $N \times N$ that *support* (or are compatible with) R. More precisely, we denote by $\mathcal{P}_R$ the set of all probability matrices such that, for $\mathbf{p} \in \mathcal{P}_R$, we have that $p_{ij} \geq 0.5$ if and only if iRj. We assume that R is known. However, it is not necessary to know the particular values of $\mathbf{p}$ because all of the fairness properties under analysis are required to be fulfilled for every probability matrix $\mathbf{p} \in \mathcal{P}_R$ given a strength binary relation R.

We assume that each $\mathbf{p} \in \mathcal{P}_R$ satisfies the following two conditions:

$$\forall i, j \in N,\ p_{ij} + p_{ji} = 1. \tag{1}$$

$$\forall i, j \in N,\ p_{ij} \geq 0.5 \text{ implies } p_{ik} \geq p_{jk} \text{ for each } k \in N \setminus \{i, j\}. \tag{2}$$

Note that condition (1), together with the convention that iPj implies $i < j$ and the fact that $p_{ij} \geq 0.5$ if and only if iRj has the consequence that $i < j$ implies $p_{ij} \geq 0.5$.

Conditions (1) and (2) follow related models such as David (1963), Hwang (1982), Horen and Riezman (1985), and Schwenk (2000). The interpretation of (1) is straightforward. Condition (2) simply expresses the fact that any player defeats with higher probability a weaker player than a stronger player. It also implies that if two players are equally strong ($p_{ij} = 0.5$), then they should defeat with equal probability any third player.

Conditions (1) and (2) are equivalent to what is sometimes referred as "strong stochastic transitivity" of the representing probability matrix (cf. David 1963). If players are displayed in the matrix according to their strength, then strongly stochastically transitive matrices are nondecreasing in rows, nonincreasing in columns and, whenever $p_{ij} = 0.5$, the corresponding rows and columns of i and j are equal. Given a binary relation R, our fairness properties are required to be fulfilled for each probability matrix $\mathbf{p} \in \mathcal{P}_R$, so that the particular details of $\mathbf{p}$ are not needed for the results.

As the reader can easily see, if a probability matrix $\mathbf{p} \in \mathcal{P}_R$ satisfies the above two conditions (as we assume), then the binary relation R is transitive. Moreover, for $a, b, c, d \in N$ we have that

$$aRbRcRd \text{ implies } p_{ad} \geq p_{bc}. \tag{3}$$

This fact is frequently used in the proofs to follow.

2.3. Seeding and the probability of being the final winner

Given a finite set N of competing players and a binary tree t, a *seeding* is a function $s : \Lambda(t) \to N$ that assigns each leaf of t to a player of N. When $s(\lambda) = i$ holds for $\lambda \in \Lambda(t)$ and $i \in N$, we say that *"player i is assigned, or "seeded", to leaf λ"*. We say that s is a *feasible* seeding for t when each player in N is seeded to exactly one leaf of t and $|\Lambda(t)| = |N|$.

Definition 1 *An* elimination-type competition *is a pair (t, N) such that t is a binary tree representing the structure of the competition and N is the set of players to be seeded to the leaves of t with $|\Lambda(t)| = |N|$.*

Definition 2 *An elimination-type competition, (t, N), is* balanced *if t is a balanced binary tree.*

Given an elimination-type competition (t, N), the set of all feasible seedings for (t, N) will be denoted by $\mathcal{S}^{(t,N)}$. A seeding $s \in \mathcal{S}^{(t,N)}$ determines the set of potential matches that can be played at each round. Moreover, if a probability matrix $\mathbf{p}$ is given, the set of potential matches at each round is endowed with a probability distribution. Then, given (t, N), $\mathbf{p} \in \mathcal{P}_R$, and $s, s' \in \mathcal{S}^{(t,N)}$, we say that s and s' are *equivalent* if the probability distribution associated with the set of potential matches at each level for s and for

s' is the same. For instance, Figure 3 represents, for a balanced binary tree of height 2, a situation where the two left seedings are equivalent but none of these two seedings is equivalent to the right one.

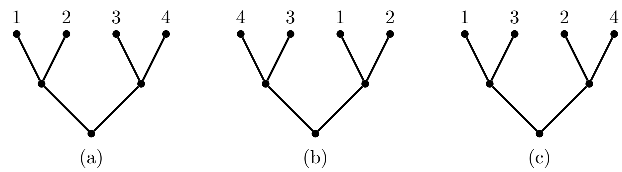


Figure 3: The seedings in (a) and (b) are equivalent, while those in (a) and (c), and in (b) and (c) are not.

Given an elimination-type competition (t, N), we say that the player who reaches the root of t is the *winner of the competition* (t, N). Also, given an elimination-type competition (t, N), a seeding $s \in \mathcal{S}^{(t,N)}$ and a probability matrix $\mathbf{p} \in \mathcal{P}_R$, we denote by $\varphi_i(t, s, \mathbf{p})$ the probability that player $i \in N$ wins the competition. Notice that, if s and s' are equivalent seedings, then $\varphi_i(t, s, \mathbf{p}) = \varphi_i(t, s', \mathbf{p})$ holds for each $i \in N$.

2.4. Fairness axioms

In order to formally state the two fairness principles discussed in the Introduction, we assume that a binary relation R of strength is defined on the player set N.

Monotonicity in Strength (MS) An elimination-type competition (t, N) satisfies MS if there exists $s \in \mathcal{S}^{(t,N)}$ such that, for all $i, j \in N$ and for all $\mathbf{p} \in \mathcal{P}_R$ such that $p_{ij} > 0.5$, $\varphi_i(t, s, \mathbf{p}) > \varphi_j(t, s, \mathbf{p})$ holds.

Equal Treatment (ET) An elimination-type competition (t, N) satisfies ET if there exists $s \in \mathcal{S}^{(t,N)}$ such that, for all $i, j \in N$ and for all $\mathbf{p} \in \mathcal{P}_R$ such that $p_{ij} = 0.5$, $\varphi_i(t, s, \mathbf{p}) = \varphi_j(t, s, \mathbf{p})$ holds.

MS requires that the competition should benefit stronger players under any of the possible probability matrices compatible with the strength of the players. In fact, many competitions are precisely designed to avoid that worse teams win by luck: for example, stepladder competitions seem to be precisely aimed to benefit better players, best players are matched with the worst ones in knockout competitions or, sometimes, matches with multiple legs give a significant home advantage (like in basketball).[3] Analogously, ET requires that the competition should treat equally every pair of equally strong players, independently of the remaining values in the probability matrix.

Alternative approaches to MS and ET could consist of imposing the corresponding property *for at least one* probability matrix, or considering particular probability matrices. The fact that the axioms are stated *for all* probability matrices is consistent with our view that the seeder, in general, has limited information and might not know the exact numerical probabilities in the matrix. In such an environment MS and ET *guarantee* that stronger players will have a greater probability of winning and equally strong players will have the same probability of winning. This is also important from a practical perspective. In competitions consisting of a regular season followed by a knockout stage, the rules used for seeding in the knockout stage are usually proxies of R obtained on the basis of end-of-season standings. This is the case of the most popular North American professional sports. Thus, it is relevant to know whether a competition structure, and a seeding in that competition, make the competition satisfy the axioms regardless of the precise numerical scores reflecting the strength of the teams at the end of the regular season.

3. Characterisation results

This section is divided in two subsections. Subsection 3.1 contains several definitions and preliminary results that end with a characterisation of the class of elimination competitions that satisfy monotonicity in strength. Sub-

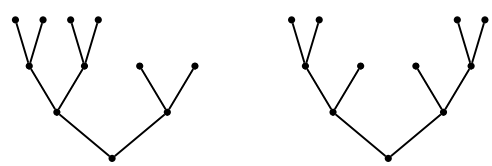
[3] Csató (2020) checks this property in the qualification for the UEFA European Championship. This is an example that this axiom is relevant not only theoretically but also for researchers modelling sports tournaments and for administrators in charge of tournament design.

section 3.2 includes a characterisation of the competitions satisfying equal treatment.

3.1. Antler-free competitions and monotonicity in strength

In order to characterise the class of elimination competitions satisfying MS, a special type of binary trees (antlers) needs to be introduced.

Definition 3. *A binary tree t is an* antler *if $h(t) = 3$; $|\Lambda(t)| = 6$; $\left|\Lambda^3(t)\right| = 4$ and $\left|\Lambda^2(t)\right| = 2$. We say that an antler is* asymmetric *if the leaves in $\Lambda^2(t)$ have a common immediate predecessor and is* symmetric *if the leaves in $\Lambda^2(t)$ have distinct immediate predecessors.*



(a) (b)

Figure 4: The binary tree in (a) is an asymmetric antler, while the one in (b) is a symmetric antler.

Antlers combine the characteristics of balanced elimination trees with the "byes" spirit of stepladders and are a natural way of organising a 6-player knockout competition.

The most prominent example of symmetric antlers is the US National Football (NFL) Conference playoffs. It was also applied in the final playoffs of the Australian National Soccer League (NSL) until 2004 and in some popular e-sports such as the Superliga Orange of the Spanish League of Professional Videogames (LVP) and the World Finals of the Clash Royal League (CRL).

As for asymmetric antlers, it is a structure followed, for example, by the Final Playoffs of the Australian A-League soccer championship (the successor of the aforementioned NSL) or the Pro Kabaddi Final Playoffs (a popular sport in South Asia). The British rugby's Super League Playoffs followed, from 2002 to 2008, a slight modification of an asymmetric antler, where the loser of the match between the two teams with a bye had a second opportunity by playing the winner of the two rounds between the no-bye teams. A similar structure is followed by the League of Legend's (LOL) Season playoffs, another popular worldwide professional e-sport.

Antlers are of a definite theoretical interest. As later proved in Theorem 1, they constitute the minimal competitions violating MS in the sense that removing any match from an antler results in a system that satisfies MS and any tree that contains an antler lets the competition system violate MS.

This naturally leads to the definition of an *antler-free* tree as a binary tree that does not contain any (symmetric or asymmetric) antler as a subgraph.

With respect to their graph structure, antler-free trees can be characterised as binary trees having particular features. Lemma 1 in this section provides such a characterisation with the aim of facilitating the definition of the increasingly balanced rule as well as the statement and the proof of Theorem 1. Let us consider first the following preliminary definitions.

A *root-to-leaf path* connects the root of t with a leaf of t. By $\gamma(t)$ we denote a root-to-leaf path of length $h(t)$ (i.e., $\gamma(t)$ is a *maximal root-to-leaf path* in t) and by $V_{\gamma(t)}$ we denote the set of nodes of $\gamma(t)$. For $v \in V_{\gamma(t)}$, t_v denotes the subtree of t with root v and $\Lambda_{-\gamma}(t_v)$ the set of leaves of t_v for which there is a shortest path to v *not including any other node* from $V_{\gamma(t)}$. Finally, we denote by $h_{-\gamma}(t_v)$ the maximal distance between v and the leaves in $\Lambda_{-\gamma}(t_v)$.

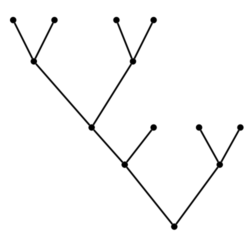
Definition 4. *A binary tree t is an* extended stepladder of degree x*,* $x \in \{1, \ldots, h(t)\}$*, if* $\max_{v \in V_{\gamma(t)}} h_{-\gamma}(t_v) = x$.

That is, in an extended stepladder of degree x, x is the maximal distance between a node of $\gamma(t)$ and a leaf that is not in $\gamma(t)$.[4] Any binary tree is in fact an extended stepladder of some degree. For example, balanced elimi-

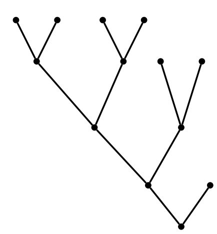
[4]Clearly, by t being a binary tree, there are at least two maximal root-to-leaf paths in t. Despite this fact, it can be easily shown that the degree of an extended stepladder is *robust* with respect to the selection of the maximal root-to-leaf path.

nation competitions with four players and asymmetric antlers are extended stepladders of degree 2, balanced elimination competitions with eight players and symmetric antlers are extended stepladders of degree 3, while standard stepladders and elementary binary trees are extended stepladders of degree 1.

We denote by ES_x the set of extended stepladders of degree *at most* x (note that $ES_x \subseteq ES_{x'}$ for $x' \geq x$). Furthermore, we use ES_2^* to denote the subclass of ES_2 defined as follows. An extended stepladder t of degree at most 2 belongs to ES_2^* only if there exists a maximal root-to-leaf path $\gamma(t)$ such that for all $v, v' \in V_{\gamma(t)}$ with $|\ell(v) - \ell(v')| = 1$, we have that $h_{-\gamma}(t_v) = 2$ implies $h_{-\gamma}(t_{v'}) = 1$. Clearly, $ES_1 \subseteq ES_2^*$ but not every extended stepladder of degree 2 belongs to ES_2^*. Figure 5 exemplifies two extended stepladders of degree 2 with only one of them belonging to ES_2^*. Notice further that asymmetric antlers do belong to ES_2 but not to ES_2^*, while symmetric antlers do even not belong to ES_2 because they are extended stepladders of degree 3. Another interesting example of a competition belonging to ES_2^* can be found in the MLB, whose two leagues postseason playoffs – American and National – correspond exactly to Figure 6.



(a)



(b)

Figure 5: Extended stepladders of degree 2. Only the one displayed in (a) belongs to ES_2^*.

The next lemma characterises antler-free binary trees.

Lemma 1 *A binary tree belongs to ES_2^* if and only if it is antler-free.*

Let us now introduce a seeding rule, which we call "*increasingly balanced*". This rule takes into account the following two characteristics of antler-free binary trees: (1) they allow for four players to be involved in

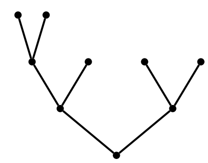


Figure 6: Structure of the two leagues' brackets in the MLB Postseason playoffs.

a balanced elimination competition, and (2) they also incorporate byes at different levels.

We formally define the increasingly balanced seeding for binary trees in ES_2 and, therefore, by $ES_2^* \subset ES_2$ and Lemma 1, for antler-free trees as well. Since the rule makes use of the notion of a balanced seeding for balanced elimination competitions with four players, we first introduce this type of seeding. Given an admissible elimination competition (t, N) with $|N| = 4$ and a binary relation of strength R, we say that a seeding $s : \Lambda(t) \to N$ is *balanced* (and we denote it by s_{4^*}) if there are players $i, j \in N$ who are initially playing against each other under s such that iRk and kRj holds for each $k \in N \setminus \{i, j\}$. Thus, a seeding that matches 1 with 4 and 2 with 3 is always balanced. But a seeding that matches 1 with 3 and 2 with 4 would also be balanced if (and only if) $1I2$ or $3I4$. Similarly, a seeding that matches 1 with 2 and 3 with 4 would also be balanced if (and only if) $2I3I4$.

We are now prepared to define the increasingly balanced seeding.

Definition 5. *Let (t, N) be an elimination competition with $t \in ES_2$. Given a binary relation of strength R, we say that a seeding $s : \Lambda(t) \to N$ is* increasingly balanced *(and we denote it by s_{ib}) if the following three conditions hold:*

(1) For all $\lambda, \lambda' \in \Lambda(t)$, $\lambda \in \Lambda^{\ell}(t)$ and $\lambda' \in \Lambda^{\ell'}(t)$ with $\ell > \ell'$ implies $s(\lambda')Rs(\lambda)$;

(2) For all $\ell \in \{1, \ldots, h(t) - 1\}$, $\Lambda^{\ell}(t) = \{\lambda, \lambda', \lambda''\}$ with λ' and λ'' having a common intermediate predecessor implies $s(\lambda)Rs(\lambda''')$ for each $\lambda''' \in \{\lambda', \lambda''\}$;

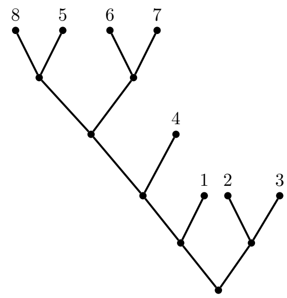


Figure 7: An increasingly balanced seeding in an extended stepladder of degree 2.

(3) $\left|\Lambda^{h(t)}(t)\right| = 4$ *implies that: (a)* iRj *holds for each* $i \in N$ *with* $\ell(s^{-1}(i)) < h(t)$ *and* $j \in N$ *with* $\ell(s^{-1}(j)) = h(t)$*, and (b)* $s(\lambda) = s_{4^*}(\lambda)$ *for each* $\lambda \in \Lambda^{h(t)}(t)$.

In other words, s_{ib} assigns players to leaves in such a way that weaker players are seeded to higher levels in the tree (condition (1) of the definition). When more than one player is seeded at the same level, then the rule distinguishes between two possibilities: (a) if the level is not maximal then the best player among those seeded at that level is seeded to the leaf, guaranteeing that he/she will play against the survivor of the previous elimination process (condition (2); this would be the case of players 1, 2 and 3 in Figure 7); and (b) if the level is the maximal one and there are four leaves at it, then among the weakest four players, the weakest one is matched with the fourth weakest and the other two are matched together (condition (3); this would be the case of players 5 to 8 in Figure 7).

Note also that there are two cases in which s_{ib} is silent. The first case is when there are three leaves of t at the same level and, therefore, the two weakest players among the three seeded at that level play their initial match. Clearly, in such a case, the two possible seedings of these players are equivalent (in Figure 7 this means that by switching players 2 and 3 we obtain an equivalent seeding). The second case is when there are only two leaves of t at level $h(t)$. In this case, the two seedings of the two weakest players are equivalent (this would correspond to switching the two players that play the first round in Figure 6).

It should also be noted that, due to the structure of the extended stepladder competition of degree 2, if there are three or four leaves at a certain level, then four players are playing a balanced elimination sub-competition. The key feature of s_{ib} is that it ensures that the strongest of the newly seeded players at that level will play against the survivor of the previous elimination process who, by the construction of s_{ib}, is necessarily weaker than any of the newly seeded players. In other words, s_{ib} ensures that in any balanced elimination sub-competition played by four players, the strongest player is matched with the weakest player.

Theorem 1 not only characterises the set of elimination competitions that satisfy MS as those displayed by an antler-free binary tree but also uniquely specifies the increasingly balanced seeding as the one for which MS is satisfied.

Theorem 1 *An elimination-type competition* (t, N) *satisfies MS with respect to* $s \in \mathcal{S}^{(t,N)}$ *if and only if* t *is antler-free and* $s = s_{ib}$.

The proof of Theorem 1 is quite complex and is relegated to the Appendix. The following example is just a brief outline of part of the intuition behind the theorem. Consider an asymmetric antler competition (cf. Figure 4(a)) and let us briefly show that it violates MS even if the seeding rule is s_{ib}. The key point here is Lemma 3 in the Appendix, which connects to part (1) in the definition of s_{ib} and shows that for a competition to satisfy MS, better players should not be seeded to leaves that are further away from the root. In other words, if $1R2R3R4R5R6$, players 3 to 6 should be seeded at level 3 and play a balanced subcompetition, while players 1 and 2 will be seeded at level 2. Consider then a probability matrix which is compatible with R in such a way that players 1, 2, and 3 are almost equally strong and much stronger than the other three players. Then the probability for player 3 to reach the root will be significantly higher than that of player 2 (initially playing against the almost equally strong player 1). This contradicts MS since player 2 is stronger than player 3. Such contradictions do not hold for antler-free competitions such as stepladders, even if they are arbitrarily large. However, similar reasoning can be applied to other basic non-antler-free trees and, by means of inductive techniques, the argument can be extended to larger structures containing them.

As already mentioned in the Introduction, Theorem 1 has implications in many real competitions where for instance 2^q players participate in balanced elimination competitions. A remarkable corollary of Theorem 1 is that no balanced competition satisfies MS if $q \geq 3$, that is, if eight or more players compete, then it cannot be guaranteed that stronger players have a higher probability of being the final winners. This includes the case of the widespread balanced seeding, confirming the folk wisdom of, for example, many NCAA basketball followers (see Baumann et al. 2010).[5] Obviously, the result also concerns any other seeding like the *equal gap* and the *close* seedings (Dagaev and Suzdaltsev 2018 and Karpov 2016, 2018). The reason for this is that the binary trees that represent those competitions contain an antler.

3.2. Balanced competitions and equal treatment

The following theorem has the flavour of an impossibility result. It shows that the only elimination competition that satisfies ET is a degenerate competition consisting of a unique match between two players.

Theorem 2 *An elimination-type competition (t, N) satisfies ET if and only if t is an elementary binary tree.*

Theorem 2 shows that the equal treatment requirement turns out to be too restrictive when applied to elimination competitions. We explore next the implications of considering a weaker version of ET.

[5] In the case of eight players the "balanced" seeding would consist of matching 1 with 8 and 4 with 5 in one branch, and 2 with 7 and 3 with 6 in the other branch of the binary tree. Horen and Riezman (1985) provide a formal proof of this type of conclusion for the case of 8-player balanced knockout tournaments by using deterministic probability matrices.

Weak Equal Treatment (WET) An elimination-type competition (t, N) satisfies WET if there exists $s \in \mathcal{S}^{(t,N)}$ such that iIj for all $i, j \in N$ implies $\varphi_i(t, s, \mathbf{p}) = \varphi_j(t, s, \mathbf{p})$ for all $i, j \in N$.

WET expresses the idea that, as for the final probability of winning, the competition should not be biased towards any particular player if all of them are equally skilled. WET is weaker than ET in the sense that it only applies when all players are equally strong. This weakening makes it possible to obtain a characterisation theorem that connects equal treatment with the class of balanced competitions.

Theorem 3 *An elimination-type competition* (t, N) *satisfies WET if and only if* t *is balanced.*

4. Concluding remarks and further research

The results of this work enable the evaluation and comparison of different types of elimination competitions on the basis of two reasonable principles of fairness. Our model connects with the specific line of research devoted to the study of seeding procedures in the case of balanced elimination competitions (cf. Dagaev and Suzdaltsev 2018, Horen and Riezman 1985, Hwang 1982, Karpov 2016, 2018, Prince et al. 2013, and Schwenk 2000). We see as especially remarkable the way in which monotonicity leads to a singular structure, which we have called an *antler* and which was found to play a referential role in our analysis, helping to clarify the discussion about the fairness of different real elimination competitions. Only antler-free competitions guarantee MS and, for that purpose, the increasingly balanced seeding, s_{ib}, is needed.

To the best of our knowledge three types of antler-free competition are played in practice: stepladder competitions; four-player balanced competitions and the MLB leagues postseason playoffs (Figure 6).

In stepladders, s_{ib} means that the stronger the player is the later it enters the competition. In four-player balanced competitions, s_{ib} requires he pairings to be (1,4) and (2,3). In Figure 6 (the MLB playoffs) s_{ib} means that 1 plays against the winner out of 4 and 5, and the winner out of (1, (4,5)) plays against the winner of the match between 2 and 3.

For all three types of competition referred to above, real antler-free competitions follow the increasingly balanced seeding (including the MLB post-season playoffs).[6] This means that Theorem 1 concurs with the intuition of the competition designers in those cases and that the increasingly balanced seeding can be extended to more complex structures with an arbitrary high number of players and byes whenever they are antler-free.

In general, our results show that there are limited numbers of competition systems that are *fair* in the sense of satisfying both types of fairness we consider: The combination of ET and MS results in balanced competitions with only two players.[7] When a weaker version of equal treatment, WET, is considered instead, the result is just extended to balanced competitions with two or four players.

For possible extensions of our model, we note that the stochastic transitivity condition assumed with respect to the probability matrices is sufficient but not necessary for the associated binary relation of strength to be transitive. Notice that any weakening of this condition would result in a larger number of probability matrices satisfying it and, thus, in even smaller class of competitions fulfilling the corresponding fairness axioms.

The fact that MS is required to hold *for all* probability matrices compatible with R is one of the reasons why so few fair competitions are obtained. As already explained, we have founded this approach, for example, on the informational basis that could be reasonably assumed for the competition

[6] A recent counterexample is the Spanish Football Supercopa held in January, 2020 in Saudi Arabia between Barcelona, Real Madrid, Valencia and Atlético de Madrid with a four-players knockout competition format, where the pairings were decided randomly.

[7] As noted by a referee, ET and MS together have the same flavour as the Self-consistency axiom in the context of aggregation of paired comparisons (Chebotarev and Shamis 1997, 1998), which leads to some impossibility results when trying to implement scoring procedures (Csató 2019b, 2019c). See also Csató 2019a for a closely related result in the context of journal ranking.

designers. In fact, in order to fluently run the proofs of the theorems we have often used "pathological" probability matrices which enable us to rule out different kinds of competition types as fair. However, it would be of great interest to know whether the class of fair competition structures can be enlarged by considering "almost fair" competitions, in the sense that they would only be unfair under artificious or extreme cases. We believe that our results pave the way for such research.

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Appendix to Fair elimination-type competitions

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1. Appendix: Proofs

We start by noting that in elimination competitions, provided that two players i and j are equally strong ($p_{ij} = 0.5$), it is always possible to exchange the leaves they have been assigned by some seeding without affecting the probabilities of winning of any player. The reason for this is simple and it is based on condition (2), which implies that $p_{ik} = p_{jk}$ holds for each $k \in N \setminus \{i, j\}$. Thus, we are generally allowed to fix a particular player from those who are equally strong in a given situation without loss of generality.

Proof of Lemma 1. The proof consists of the following tree steps.

Step 1 If t is symmetric-antler-free, then $t \in ES_2$.

Proof. Let t be a symmetric-antler-free binary tree and suppose that $t \notin ES_2$. The latter implies that t is an extended stepladder of degree $x \geq 3$ with respect to a maximal root-to-leaf path $\gamma(t)$, as defined previously. Therefore, there exists a node $v \in V_{\gamma(t)}$ and a leaf $\lambda \in \Lambda_{-\gamma}(t_v)$ such that the distance between v and λ is x. Denote by π the path connecting v and λ. Let y be the immediate successor of v in π, y' the immediate successor of y in π and y'' the immediate successor of y' in π (notice that such nodes exist because

$x \geq 3$). Because t is a binary tree, y has another immediate successor $z' \neq y'$ and y' has another immediate successor $z' \neq y''$. Meanwhile, given that $v \in V_{\gamma(t)}$ and that $\gamma(t)$ is a maximal root-to-leaf path, there are at least three consecutive successor nodes x, x' and x'' that belong to $V_{\gamma(t)}$ (otherwise, there would be a longer root-to-leaf path connecting the root with λ). Again, since t is a binary tree, x has another immediate successor $w' \neq x'$ and x' has another immediate successor $w'' \neq y''$. Now, notice that the set of nodes $\{v, x, y, x', y', x'', y'', z', w', z'', w''\}$ and the corresponding edges form a symmetric antler, which is a contradiction.

Step 2 If t is antler-free, then $t \in ES_2^*$.

Proof. Notice first that if t is antler-free then t is both symmetric-antler free and asymmetric-antler free. Given that $ES_2^* \subset ES_2$ and in view of Step 1 it suffices to show that if $t \in ES_2$ does not contain an asymmetric antler, then $t \in ES_2^*$. Suppose not and let $\gamma(t)$ be a maximal root-to-leaf path in t. If $t \in ES_2 \setminus ES_2^*$, then there are two nodes $v, v' \in V_{\gamma(t)}$ with $\ell(v') = \ell(v) + 1$ and $h_{-\gamma}(t_v) = h_{-\gamma}(t_{v'}) = 2$.

Moreover, given that $\gamma(t)$ is a maximal root-to-leaf path, v' has at least two consecutive successors x and x' belonging to $V_{\gamma(t)}$, and given that t is a binary tree, x has another immediate successor $y \neq x'$. Consider then the set of nodes consisting of v, v', x, x', y, the immediate successors of v and v', as well as the leaves in $\Lambda_{-\gamma}(t_v) \cup \Lambda_{-\gamma}(t_{v'})$. Note that this set of nodes together with the corresponding edges form an asymmetric antler, which is a contradiction.

Step 3 If $t \in ES_2^*$, then t is antler-free.

Proof. Note first that if $t \in ES_1 \subseteq ES_2^*$, then it is antler-free. Suppose then that t is an extended stepladder of degree 2 belonging to ES_2^*. Clearly, t does not contain a symmetric antler t' because each symmetric antler is an extended stepladder of degree 3 and, thus, t containing t' implies that t should be an extended stepladder of degree at least 3, which is a contradiction. Let us show now that $t \in ES_2^*$ implies that t does not contain an asymmetric antler.

Suppose that, to the contrary, t contains an asymmetric antler t^A. Let $\gamma(t)$ and $\gamma'(t^A)$ be maximal root-to-leaf paths in t and t^A, respectively. There are two possibilities:

(i) $V_{\gamma'(t^A)} \cap V_{\gamma(t)} = \emptyset$. Consider the root v_0^A of t^A and the closest predecessor v of v_0^A such that $v \in V_{\gamma(t)}$. Let d be the distance between v_0^A and v. Then we have that $h_{-\gamma}(t_v) > d + 3$ in contradiction to t being an extended stepladder of degree 2.

(ii) $V_{\gamma'(t^A)} \cap V_{\gamma(t)} \neq \emptyset$. Let $V_{\gamma'(t^A)} = \{v_0^A, v_1^A, v_2^A, v_3^A\}$ be such that, for all $i \in \{1, 2, 3\}$, v_i^A is the immediate successor of v_{i-1}^A and v_0^A is the root of t^A. Given that $V_{\gamma'(t^A)} \cap V_{\gamma(t)} \neq \emptyset$ there exists $v_i^A \in V_{\gamma'(t^A)} \cap V_{\gamma(t)}$. Note that $v_i^A \in V_{\gamma(t)}$ implies $v_j^A \in V_{\gamma(t)}$ for all $j < i$. Therefore $v_0^A \in V_{\gamma(t)}$. We distinguish then two cases: either $v_0^A \in V_{\gamma(t)}$ and $v_1^A \notin V_{\gamma(t)}$ or $v_0^A, v_1^A \in V_{\gamma(t)}$. If $v_0^A \in V_{\gamma(t)}$ and $v_1^A \notin V_{\gamma(t)}$, then $h_{-\gamma}(t_{v_0^A}) \geq 3$ in contradiction to t being an extended stepladder of degree 2. If $v_0^A, v_1^A \in V_{\gamma(t)}$, then by the structure of an asymmetric antler, and given that t is an extended stepladder of degree 2, we know that $h_{-\gamma}(t_{v_0^A}) = h_{-\gamma}(t_{v_1^A}) = 2$, which is a contradiction to $t \in ES_2^*$.
■

Proof of Theorem 1. We start with two additional lemmas. Lemma 2 states that a four-player balanced elimination competition satisfies MS only for the balanced seeding as defined in Section 3, while Lemma 3 shows that, for a competition to satisfy MS, better players should not be seeded to leaves that are further away from the root of the tree. We then proceed with proof of Theorem 1.

Lemma 2 *A balanced elimination competition* (t, N) *with* $h(t) = 2$ *satisfies MS with respect to a seeding* $s \in \mathcal{S}^{(t,N)}$ *if and only if* $s = s_{4^*}$.

Proof. Let (t, N) be as above with $N = \{1, 2, 3, 4\}$ and recall that $1R2R3R4$ holds. Assume, w.l.o.g., that s_{4^*} is such that player 1 is matched with player 4 and player 2 is matched with player 3. Consider then the seeding rule $s = s_{4^*}$ and fix *any* probability matrix $\mathbf{p} \in \mathcal{P}_R$. We have to show that $p_{ij} > 0.5$ for some $i, j \in N$ implies $\varphi_i(t, s_{4^*}, \mathbf{p}) > \varphi_j(t, s_{4^*}, \mathbf{p})$. There are six possible cases.

Case 1 ($i = 1$ and $j = 2$). We have $\varphi_1(t, s_{4^*}, \mathbf{p}) = (p_{14} \cdot p_{23} \cdot p_{12}) + (p_{14} \cdot p_{32} \cdot p_{13}) > (p_{23} \cdot p_{14} \cdot p_{21}) + (p_{23} \cdot p_{41} \cdot p_{24}) = \varphi_2(t, s_{4^*}, \mathbf{p})$, where the inequality follows from $p_{12} > p_{21}$ (due to $p_{12} > 0.5$) and from $p_{13} \geq p_{23}$, $p_{14} \geq p_{24}$ and $p_{32} \geq p_{41}$ (due to (3)).

Case 2 ($i = 1$ and $j = 3$). We have $\varphi_1(t, s_{4^*}, \mathbf{p}) = (p_{14} \cdot p_{23} \cdot p_{12}) + (p_{14} \cdot p_{32} \cdot p_{13}) > (p_{32} \cdot p_{14} \cdot p_{31}) + (p_{32} \cdot p_{41} \cdot p_{34}) = \varphi_3(t, s_{4^*}, \mathbf{p})$, where the inequality follows from $p_{13} > p_{31}$ (due to $p_{13} > 0.5$), $p_{23} \geq p_{32}$ (by $p_{23} \geq 0$), $p_{12} \geq p_{41}$ (by $p_{12} \geq 0.5$ and $p_{14} \geq 0.5$), and from $p_{14} \geq p_{34}$ (due to $p_{13} > 0.5$ and condition (2)).

Case 3 ($i = 1$ and $j = 4$). We have $\varphi_1(t, s_{4^*}, \mathbf{p}) = (p_{14} \cdot p_{23} \cdot p_{12}) + (p_{14} \cdot p_{32} \cdot p_{13}) > (p_{41} \cdot p_{23} \cdot p_{42}) + (p_{41} \cdot p_{32} \cdot p_{43}) = \varphi_4(t, s_{4^*}, \mathbf{p})$, where the inequality follows from $p_{14} > p_{41}$ (due to $p_{14} > 0.5$), $p_{12}, p_{13} \geq 0.5$, and $p_{42}, p_{43} \leq 0.5$.

Case 4 ($i = 2$ and $j = 3$). We have $\varphi_2(t, s_{4^*}, \mathbf{p}) = (p_{23} \cdot p_{14} \cdot p_{21}) + (p_{23} \cdot p_{41} \cdot p_{24}) > (p_{32} \cdot p_{14} \cdot p_{31}) + (p_{32} \cdot p_{41} \cdot p_{34}) = \varphi_3(t, s_{4^*}, \mathbf{p})$, where the inequality follows from $p_{23} > p_{32}$ (due to $p_{23} > 0.5$), $p_{21} \geq p_{31}$ and $p_{24} \geq p_{34}$ (due to $p_{23} > 0.5$ and condition (2)).

Case 5 ($i = 2$ and $j = 4$). We have $\varphi_2(t, s_{4^*}, \mathbf{p}) = (p_{23} \cdot p_{14} \cdot p_{21}) + (p_{23} \cdot p_{41} \cdot p_{24}) > (p_{41} \cdot p_{23} \cdot p_{42}) + (p_{41} \cdot p_{32} \cdot p_{43}) = \varphi_4(t, s_{4^*}, \mathbf{p})$, where the inequality follows from $p_{24} > p_{42}$ (due to $p_{24} > 0.5$), $p_{21} \geq p_{41}$ (due to $p_{24} > 0.5$ and condition (2)), and from $p_{14}, p_{23} \geq 0.5$ and $p_{32}, p_{43} \leq 0.5$ (due to $\mathbf{p}$ being compatible with R).

Case 6 ($i = 3$ and $j = 4$). We have in this last case $\varphi_3(t, s_{4^*}, \mathbf{p}) = (p_{32} \cdot p_{14} \cdot p_{31})$
$+(p_{32} \cdot p_{41} \cdot p_{34}) > (p_{41} \cdot p_{23} \cdot p_{42}) + (p_{41} \cdot p_{32} \cdot p_{43}) = \varphi_4(t, s_{4^*}, \mathbf{p})$, where the inequality follows from $p_{34} > p_{43}$ (due to $p_{34} > 0$), $p_{14} \geq p_{23}$ (by (3)), and from $p_{31} \geq p_{41}$ and $p_{32} \geq p_{42}$ (due to $p_{34} > 0.5$ and condition (2)).

We conclude that (t, N) satisfies MS with respect to s_{4^*}. Notice that it was necessary to prove the six cases because, for example, if $p_{12} > 0.5$ implies $\varphi_1(t, s_{4^*}, \mathbf{p}) > \varphi_2(t, s_{4^*}, \mathbf{p})$, this does not necessarily mean that $p_{13} > 0.5$ implies $\varphi_1(t, s_{4^*}, \mathbf{p}) > \varphi_3(t, s_{4^*}, \mathbf{p})$ in the case $1I2P3$.

Let us now consider a seeding rule $s \in \mathcal{S}^{(t,N)}$ which differs from $s \neq s_{4^*}$. Let $\varepsilon > 0$ be arbitrarily small and $\mathbf{p} \in \mathcal{P}_R$ be defined as follows:

$$\text{p=}\begin{pmatrix} 0.5 & 0.5+\varepsilon & 0.5+2\varepsilon & 1-\varepsilon \\ & 0.5 & 0.5+\varepsilon & 1-2\varepsilon \\ & & 0.5 & 1-3\varepsilon \\ & & & 0.5 \end{pmatrix}$$

There are two possible cases with respect to the seeding produced by s.

Case 1 (the initial matches are between 1 and 2, and 3 and 4, respectively). We have in this case:
$\varphi_3(t,s,\mathbf{p}) = (p_{34} \cdot p_{12} \cdot p_{31}) + (p_{34} \cdot p_{21} \cdot p_{32}) \approx (1 \cdot 0.5 \cdot 0.5) + (1 \cdot 0.5 \cdot 0.5) \approx 0.5$ and
$\varphi_2(t,s,\mathbf{p}) = (p_{21} \cdot p_{34} \cdot p_{23}) + (p_{21} \cdot p_{43} \cdot p_{24}) \approx (0.5 \cdot 1 \cdot 0.5) + (0.5 \cdot 0 \cdot 1) \approx 0.25$, in contradiction to $p_{23} > 0.5$ and (t,N) satisfying MS.

Case 2 (the initial matches are between 1 and 3, and 2 and 4, respectively). Considering again the probability matrix shown previously, we have $\varphi_1(t,s,\mathbf{p}) = (p_{13} \cdot p_{24} \cdot p_{12}) + (p_{13} \cdot p_{42} \cdot p_{14}) \approx (0.5 \cdot 1 \cdot 0.5) + (0.5 \cdot 0 \cdot 1) \approx 0.25$ and $\varphi_2(t,s,\mathbf{p}) = (p_{24} \cdot p_{13} \cdot p_{21}) + (p_{24} \cdot p_{31} \cdot p_{23}) \approx (1 \cdot 0.5 \cdot 0.5) + (1 \cdot 0.5 \cdot 0.5) \approx 0.5$, which is in contradiction to $p_{12} > 0.5$ and (t,N) satisfying MS. ■

Lemma 3 *Let R be a strength relation defined on N, (t,N) an elimination competition, and $s \in \mathcal{S}^{(t,N)}$. If (t,N) satisfies MS with respect to s, then $\ell(\lambda) > \ell(\lambda')$ for $\lambda, \lambda' \in \Lambda(t)$ implies $s(\lambda')Rs(\lambda)$.*

Proof. Suppose that the implication is false. That is, given R, let (t,N) satisfy MS with respect to s such that $s(\lambda)Ps(\lambda')$ holds for some $\lambda, \lambda' \in \Lambda(t)$ with $\ell(\lambda) > \ell(\lambda')$. For (t,N) to satisfy MS, it is necessary that $\varphi_{s(\lambda)}(t,s,\mathbf{p}') > \varphi_{s(\lambda')}(t,s,\mathbf{p}')$ for all probability matrices $\mathbf{p}' \in \mathcal{P}_R$ such that $p'_{s(\lambda),s(\lambda')} > 0.5$. Let us consider a probability matrix $\mathbf{p} \in \mathcal{P}_R$ such that, for all $i,j \in N$, $p_{ij} \approx 0.5$ with $p_{s(\lambda),s(\lambda')} > 0.5$. Then $\varphi_{s(\lambda)}(t,s,\mathbf{p}) \approx 0.5^{\ell(\lambda)}$ and $\varphi_{s(\lambda')}(t,s,\mathbf{p}) \approx 0.5^{\ell(\lambda')}$. Because $\ell(\lambda) > \ell(\lambda')$, $\varphi_{s(\lambda)}(t,s,\mathbf{p}) < \varphi_{s(\lambda')}(t,s,\mathbf{p})$. Taking into account that $p_{s(\lambda),s(\lambda')} > 0.5$, the latter inequality implies that (t,N) violates MS with respect to s, which is a contradiction. ■

Before moving to the proof of the sufficiency and necessity parts of Theorem 2, let us introduce the following additional concept. We say that a binary tree t with $h(t) = 3$ is a *one-bye antler*, if $|\Lambda(t)| = 7$ with $\left|\Lambda^3(t)\right| = 6$ and $\left|\Lambda^2(t)\right| = 1$. Clearly, any one-bye antler is an extended stepladder of degree 3. Further, for t and t' being binary trees, we say that (1) t' is an *extension from the leaves* of t if t' and t have the same root and $\Lambda(t) \subseteq \Lambda(t')$; (2) t' is an *extension from the root* of t if t is a subtree of t'; (3) t' is a *limited extension from the root* of t, if t is a subtree of t' and $\Lambda^{h(t)}(t) \subseteq \Lambda^{h(t')}(t')$. Thus, a limited extension from the root of a tree t never has leaves at a height that is greater than the height of any of the leaves of t.

Proof of Theorem 1 (Sufficiency). Given a strength relation R defined on the player set N, we have to prove that an elimination competition (t, N) with t being antler-free satisfies MS with respect to s_{ib}.

Let (t, N) be such that t is antler-free and $s = s_{ib}$. By Lemma 1, $t \in ES_2^*$. Take a maximal root-to-leaf path $\gamma(t)$ and note that $t \in ES_2^*$ irrespective of the choice of $\gamma(t)$. For $s \in \mathcal{S}^{(t,N)}$, $v \in V_{\gamma(t)}$, and any probability matrix $\mathbf{p}$, we denote by $p_i^v(s)$ the probability that player $i \in N$ reaches v under a given seeding s and by v_h the unique leaf in $V_{\gamma(t)}$. Moreover, we collect in the set $S_v^1(s)$ all players whose first match in the competition is against a player who has already reached some $v' \in V_{\gamma(t)}$ with $\ell(v') > \ell(v)$; correspondingly, $S_v^2(s)$ stands for the set of all players who had to play an initial match before having the possibility to meet a player who has already reached some node from $V_{\gamma(t)}$ at a higher level than v. Note that for each $i \in N$ we have that, due to $t \in ES_2^*$, either $i = s(v_h)$ or $i \in S_v^1(s) \cup S_v^2(s)$ holds for some $v \in V_{\gamma(t)}$.

We denote by v^x the closest predecessor belonging to $V_{\gamma(t)}$ of $x = s(\lambda)$ for some $\lambda \in \Lambda(t)$. Note that, for each $v \in V_{\gamma(t)}$, any probability matrix $\mathbf{p}$, and any two players $k, j \in N$ with $p_{kj} > 0.5$ and $s^{-1}(k), s^{-1}(j) \in \Lambda(t_v)$, we have that $p_k^v(s) > p_j^v(s)$ implies $\varphi_k(t, s, \mathbf{p}) > \varphi_j(t, s, \mathbf{p})$. The reason is that for each $i \in N$ with $s^{-1}(i) \in \Lambda(t_v)$ we have

$$\varphi_i(t, s, \mathbf{p}) = p_i^v(s) \cdot \prod_{x \in S_{v'}^1(s): \ell(v') < \ell(v)} p_{ix} \cdot \prod_{y,z \in S_{v'}^2(s): \ell(v') < \ell(v)} (p_{iy} p_{yz} + p_{iz} p_{zy}).$$

Hence, $\varphi_k(t, s, \mathbf{p}) > \varphi_j(t, s, \mathbf{p})$ is implied by $p_k^v(s) > p_j^v(s)$, $p_{kx} \geq p_{jx}$ for each $x \in N$ following from condition (2), and p_{yz} (p_{zy}) being independent of any other parameter in the respective formulae for k and j.

Thus, to prove the sufficiency part of Theorem 1, let us now consider the increasingly balanced rule s_{ib}. We have to show that (t, N) satisfies MS with respect to s_{ib}. In view of the argument that was just explained, assuming that $p_{kj} > 0.5$, then it is enough to find a node $v \in V_{\gamma(t)}$ with $s_{ib}^{-1}(k), s_{ib}^{-1}(j) \in \Lambda(t)$ and $p_k^v(s_{ib}) > p_j^v(s_{ib})$. We distinguish the following three possible cases:

(i) $\ell(s_{ib}^{-1}(k)) < \ell(s_{ib}^{-1}(j))$ and there is no $m \in N$ with $\ell(s_{ib}^{-1}(m)) = \ell(s_{ib}^{-1}(k))$. Clearly, player k does not need to win any match to reach $v^k \in V_{\gamma(t)}$. Therefore, given that s_{ib} seeds worse players to higher levels, $p_k^{v^k}(s_{ib}) > 0.5$ and because j has to defeat k to reach v^k, $p_j^{v^k}(s_{ib}) < 0.5$.

(ii) $\ell(s_{ib}^{-1}(k)) < \ell(s_{ib}^{-1}(j))$ and there exists $m \in N$ with $\ell(s_{ib}^{-1}(m)) = \ell(s_{ib}^{-1}(k))$. In this case, player k is involved in a balanced sub-competition of four players. Let v^* be the root of the sub-competition (note that $v^* \in V_{\gamma(t)}$ with $p_k^{v^*}(s_{ib})$ being the probability for player k to win the sub-competition). For player j, $p_j^{v^*}(s_{ib})$ is the product of two probabilities: the probability to reach the sub-competition, that is, to reach the node $v \in V_{\gamma(t)}$ such that $\ell(v) = \ell(s_{ib}^{-1}(k))$; and, the probability to win the sub-competition. Given that the sub-competition is played under a balanced seeding, we know from the proof of Lemma 2 that for any probability matrix with $p_{kj} > 0.5$, then the probability for k to win the sub-competition is strictly greater than the one for j. We conclude that $p_k^{v^*}(s_{ib}) > p_j^{v^*}(s_{ib})$ should hold.

(iii) $\ell(s_{ib}^{-1}(k)) = \ell(s_{ib}^{-1}(j))$. Also in this case, players k and j are involved in a balanced sub-competition of four players. Following the same reasoning as in (ii), we obtain $p_k^{v^*}(s_{ib}) > p_j^{v^*}(s_{ib})$.

Proof of Theorem 1 (Necessity). We have to prove that if (t, N) satisfies MS with respect to some seeding s, then t is antler-free and $s = s_{ib}$. To prove that t is antler-free in such a case, we will show that if t contains an antler, then the competition (t, N) violates MS. More precisely, in Steps 1 to 9 of the proof, we show progressively and in an exhaustive way that all of the different types of structures that can contain an antler violate MS. In Step 10 we finally prove that the seeding s with respect to which (t,N) satisfies MS is necessarily $s = s_{ib}$.

Step 1 Let (t,N) be an elimination competition with t being a symmetric antler. Then (t,N) violates MS.

Proof. Note that $N = \{1,\ldots,6\}$ holds in this case. Let λ_ℓ^2 and λ_r^2 be the two leaves of t that are at level 2 of its left and right branch, respectively. Similarly, let λ_ℓ^{3a} and λ_ℓ^{3b} be the two leaves at level 3 of t's left branch, while λ_r^{3a} and λ_r^{3b} be the two leaves at level 3 of t's right branch. We proceed by reduction to the absurd; that is, we assume that (t,N) satisfies MS and then prove that we reach a contradiction. By Lemma 3, any $s \in \mathcal{S}^{(t,N)}$ with respect to which (t,N) satisfies MS should be such that the two strongest players are seeded to λ_ℓ^2 and λ_r^2. Assume w.l.o.g. that these players are 1 and 2, and that $s(\lambda_\ell^2) = 1$ and $s(\lambda_r^2) = 2$. There are then six possible non-equivalent seedings for the remaining players:
(i) $s(\lambda_\ell^{3a}) = 3$, $s(\lambda_\ell^{3b}) = 4$, $s(\lambda_r^{3a}) = 5$, $s(\lambda_r^{3b}) = 6$.
(ii) $s(\lambda_\ell^{3a}) = 3$, $s(\lambda_\ell^{3b}) = 5$, $s(\lambda_r^{3a}) = 4$, $s(\lambda_r^{3b}) = 6$.
(iii) $s(\lambda_\ell^{3a}) = 3$, $s(\lambda_\ell^{3b}) = 6$, $s(\lambda_r^{3a}) = 4$, $s(\lambda_r^{3b}) = 5$.
(iv) $s(\lambda_\ell^{3a}) = 4$, $s(\lambda_\ell^{3b}) = 5$, $s(\lambda_r^{3a}) = 3$, $s(\lambda_r^{3b}) = 6$.
(v) $s(\lambda_\ell^{3a}) = 4$, $s(\lambda_\ell^{3b}) = 6$, $s(\lambda_r^{3a}) = 3$, $s(\lambda_r^{3b}) = 5$.
(vi) $s(\lambda_\ell^{3a}) = 5$, $s(\lambda_\ell^{3b}) = 6$, $s(\lambda_r^{3a}) = 3$, $s(\lambda_r^{3b}) = 4$.

To prove that (t,N) violates MS, we next show that for each of the six possible seedings we can find a probability matrix $\mathbf{p} \in \mathcal{P}_R$ defined on N such that there exists $i \in N$ with $p_{i-1,i} > 0.5$ (and, therefore, $(i-1)Pi$) and $\varphi_i(t,s,\mathbf{p}) > \varphi_{i-1}(t,s,\mathbf{p})$.

(i) Take $\mathbf{p}$ as follows: $p_{jk} > 0.5$ if $j < k$; $p_{j6} \approx 1$ for all $j < 6$, and $p_{jk} \approx 0.5$ for all $j,k < 6$. We have then $\varphi_5(t,s,\mathbf{p}) \approx 0.25 > 0.125 \approx \varphi_4(t,s,\mathbf{p})$ while $p_{45} > 0.5$.

(ii) Consider the same probability matrix $\mathbf{p}$ as in case (i), then $\varphi_4(t,s,\mathbf{p}) \approx 0.25 > 0.125 \approx \varphi_3(t,s,\mathbf{p})$ while $p_{34} > 0.5$.

(iii) Let $\mathbf{p}$ be such that $p_{jk} > 0.5$ if $j < k$; $p_{jk} \approx 1$ if $j \in \{1, 2\}$ and $k \in \{4, 5, 6\}$, and $p_{jk} \approx 0.5$, otherwise. Then $\varphi_2(t, s, \mathbf{p}) \approx 0.5 > 0.375 \approx \varphi_1(t, s, \mathbf{p})$ while $p_{12} > 0.5$.

(iv) Take $\mathbf{p}$ as follows: $p_{jk} > 0.5$ if $j < k$; $p_{jk} \approx 1$ if $j \in \{1, 2\}$ and $k = 6$, and $p_{jk} \approx 0.5$, otherwise. Then $\varphi_2(t, s, \mathbf{p}) \approx 0.375 > 0.25 \approx \varphi_1(t, s, \mathbf{p})$ while $p_{12} > 0.5$.

(v) Let $\mathbf{p}$ be as follows: $p_{jk} > 0.5$ if $j < k$; $1 \approx p_{15} \approx p_{16} \approx p_{25} \approx p_{26} \approx p_{36} \approx p_{46}$, and $p_{jk} \approx 0.5$, otherwise. Then $\varphi_2(t, s, \mathbf{p}) \approx 0.375 > 0.25 \approx \varphi_1(t, s, \mathbf{p})$ while $p_{12} > 0.5$.

(vi) Consider the same probability matrix $\mathbf{p}$ as in cases (i) and (ii), then $\varphi_5(t, s, \mathbf{p}) \approx 0.25 > 0.125 \approx \varphi_4(t, s, \mathbf{p})$ while $p_{45} > 0.5$.

For later steps in the proof, it is important to remark that, according to the probability matrices shown above and the one shown in the proof of Lemma 3, whatever seeding $s \in \mathcal{S}^{(t,N)}$ we consider in a symmetric antler t, not only exists a probability matrix $\mathbf{p}$ and $i \in N$ such that $p_{i-1,i} > 0.5$ and $\varphi_i(t, s, \mathbf{p}) > \varphi_{i-1}(t, s, \mathbf{p})$ but it also holds that it is possible to find such a matrix $\mathbf{p}$ where $p_{i-1,i} \approx 0.5$ and $p_{ik} \approx p_{i-1,k}$ for all $k \in N \setminus \{i-1, i\}$.

Step 2 Let (t, N) be an elimination competition with t being an asymmetric antler, then (t, N) violates MS.

Proof. Clearly $N = \{1, \dots, 6\}$ holds also in this case. Assume w.l.o.g. that t's left branch has four leaves at level $h(t) = 3$, and denote them (from left to right) by λ_ℓ^{3a}, λ_ℓ^{3b}, λ_ℓ^{3c} and λ_ℓ^{3d}. Clearly, t's right branch has two leaves (λ_r^{2a} and λ_r^{2b}) at level 2. Let v_1 be the node in the left branch of t which is an immediate successor of the root of t. Note that $\{\lambda_\ell^{3a}, \lambda_\ell^{3b}, \lambda_\ell^{3c}, \lambda_\ell^{3d}\}$ are the leaves of the balanced subtree t_1 of t whose root is v_1. We proceed again by reduction to the absurd. Assume that (t, N) satisfies MS. By Lemma 3, the two strongest players should be seeded to the two leaves at level 2. Assume w.l.o.g. that these players are 1 and 2 and that $s(\lambda_\ell^2) = 1$ and $s(\lambda_r^2) = 2$. We then fix a seeding $s' : \Lambda(t_1) \to \{3, 4, 5, 6\}$, note that $s' \in \mathcal{S}^{(t_1, N \setminus \{1,2\})}$, and consider the following two possibilities.

Case 1 ($s' \neq s'_{4*}$). Consider the matrix used in the proof of Lemma 2 and apply it to the set of players $\{3,4,5,6\}$. According to the proof, if $s \neq s_{4^*}$, then there exists a pair of players $i, i-1 \in \{3,4,5,6\}$ and a probability matrix $\mathbf{p}' \in \mathcal{P}_{R_{|\{3,4,5,6\}}}$ such that $p'_{i-1,i} > 0.5$ and $p_i'^{v_1} > p_{i-1}'^{v_1}$. Moreover $p'_{i-1,i} \approx 0.5$ and $p'_{i-1,k} \approx p'_{i,k}$ also holds for all $k \in \{3,4,5,6\}$. Let $\mathbf{p}$ be defined on N such that $p_{jk} = p'_{jk}$ for all $j, k > 2$; $p_{13} = p_{23} = p_{12} = 0.5$ and, therefore, $p_{jk} = p_{3k}$ for all $j < 3$ and all $k \in N$. By $p_{i3} \approx p_{i-1,3}$ and by $\mathbf{p}$ satisfying condition (2) we have $p_{i2} \approx p_{i-1,2}$ and $p_{i1} \approx p_{i-1,1}$. Thus, $p_{i-1,k} \approx p_{i,k}$ for all $k \in N$.

Note then that $\varphi_i(t,s,\mathbf{p}) = p_i'^{v_1} \cdot (p_{12}p_{i1} + p_{21}p_{i2})$ and $\varphi_{i-1}(t,s,\mathbf{p}) = p_{i-1}'^{v_1} \cdot (p_{12}p_{i-1,1} + p_{21}p_{i-1,2})$. By $p_i'^{v_1} > p_{i-1}'^{v_1}$, $p_{i1} \approx p_{i-1,1}$ and $p_{i2} \approx p_{i-1,2}$, we have $\varphi_i(t,s,\mathbf{p}) > \varphi_{i-1}(t,s,\mathbf{p})$ in contradiction to (t,N) satisfying MS with respect to s.

Case 2 ($s' = s'_{4^*}$). Consider the following probability matrix $\mathbf{p} \in \mathcal{P}_R$:

$$\text{p}=\begin{pmatrix} 0.5 & 0.5 & 0.5+\varepsilon & 1-2\varepsilon & 1-\varepsilon & 1-\varepsilon \\ & 0.5 & 0.5+\varepsilon & 1-2\varepsilon & 1-\varepsilon & 1-\varepsilon \\ & & 0.5 & 1-3\varepsilon & 1-2\varepsilon & 1-2\varepsilon \\ & & & 0.5 & 1-3\varepsilon & 1-3\varepsilon \\ & & & & 0.5 & 0.5 \\ & & & & & 0.5 \end{pmatrix}$$

According to $\mathbf{p}$, s'_{4^*} matches either 3 with 6 and 4 with 5, or it matches 3 with 5 and 4 with 6. In either case, we have $p_{23} > 0.5$ and, after making the necessary computations, $\varphi_3(t,s,\mathbf{p}) \approx 0.5 > 0.25 \approx \varphi_2(t,s,\mathbf{p})$. Thus, (t,N) violates MS with respect to s.

As in Step 1, it is important to remark that, according to the probability matrix shown previously and the ones shown in the proofs of Lemma 2 and Lemma 3, whatever seeding $s \in \mathcal{S}^{(t,N)}$ we consider in an asymmetric antler t, not only exists a probability matrix $\mathbf{p}$ and $i \in N$ such that $p_{i-1,i} > 0.5$ and $\varphi_i(t,s,\mathbf{p}) > \varphi_{i-1}(t,s,\mathbf{p})$ but it also holds that it is possible to find such a matrix $\mathbf{p}$ where $p_{i-1,i} \approx 0.5$ and $p_{ik} \approx p_{i-1,k}$ for all $k \in N \setminus \{i-1, i\}$.

Step 3 Let (t,N) be an elimination competition system with t being a one-bye antler. Then (t,N) violates MS.

Proof. We proceed again by reduction to the absurd assuming that (t,N)

violates MS. Note that $N = \{1, \ldots, 7\}$ holds in this case. Assume w.l.o.g. that t's left branch has four leaves at level $h(t) = 3$, and denote them (from left to right) by λ_ℓ^{3a}, λ_ℓ^{3b}, λ_ℓ^{3c} and λ_ℓ^{3d}. Clearly, t's right branch has two leaves (λ_r^{3a} and λ_r^{3b}) at that same level and one leaf (λ_r^2) at level 2. By Lemma 3, the best player should be seeded to λ_r^2. Assume w.l.o.g. that $s(\lambda_r^2) = 1$. We distinguish now two possibilities depending on the leaf player 7 has been seeded at.

Case 1 ($s(\lambda) = 7$ for some λ of t's right branch). There are two subcases: (i) $s(\lambda_r^{4a}) = 6$ and $s(\lambda_r^{4b}) = 7$ (or vice versa w.l.o.g.) and (ii): $s(\lambda_r^{4a}) = x$ and $s(\lambda_r^{4b}) = 7$ (or vice versa) with $x < 6$.

(i) If $s(\lambda_r^{4a}) = 6$ and $s(\lambda_r^{4b}) = 7$, then let $\mathbf{p} \in \mathcal{P}_R$ be a probability matrix such that $p_{jk} > 0.5$ for all $j, k \in N$ with $j < k$, $p_{j7} \approx 1$ for all $j \in N \setminus \{7\}$, and $p_{jk} \approx 0.5$ for all $j, k \in N \setminus \{7\}$. By making the necessary calculations, we obtain $\varphi_6(t, s, \mathbf{p}) \approx 0.25 > 0.125 \approx \varphi_5(t, s, \mathbf{p})$ while $p_{67} > 0.5$, which is a contradiction to (t, N) satisfying MS.

(ii) If $s(\lambda_r^{4a}) = x$ and $s(\lambda_r^{4b}) = 7$, then let $\mathbf{p} \in \mathcal{P}_R$ be a probability matrix such that $p_{jk} > 0.5$ for all $j, k \in N$ with $j < k$, $p_{jk} \approx 0.5$ for all $j, k \in N \setminus \{1\}$, and $p_{1k} = 0.7$ for all $k \in N \setminus \{1\}$. By making the necessary calculations, we obtain $\varphi_6(t, s, \mathbf{p}) \approx 0.09 > 0.075 \approx \varphi_x(t, s, \mathbf{p})$ while $p_{x6} > 0.5$, reaching again a contradiction.

Case 2 ($s(\lambda) = 7$ for some λ of t's left branch). Let x be the player whose initial match is against player 7 and suppose w.l.o.g., that $s(\lambda_\ell^{4c}) = 7$ and $s(\lambda_\ell^{4d}) = x$. Then remove from t the nodes λ_ℓ^{4c} and λ_ℓ^{4d}, and also the corresponding edges to their immediate predecessor v_λ. Note that the remaining subgraph t^A of t is a symmetric antler with v_λ being now a leaf of t^A. Consider the seeding $s' : \Lambda(t^A) \to \{1, \ldots, 6\}$ defined as follows: $s'(v_\lambda) = x$ and $s'(\lambda) = s(\lambda)$ for each $\lambda \in \Lambda(t^A) \setminus \{v_\lambda\}$, and note that $s' \in \mathcal{S}^{(t^A, N \setminus \{7\})}$. In other words, s' can be interpreted as a situation in which x wins his or her match against 7 and the remaining matches are not yet played.

By Step 1, the competition $(t^A, N \setminus \{7\})$ violates MS. That is, there exists a probability matrix $\mathbf{p}' \in \mathcal{P}_{R_{|N \setminus \{7\}}}$ such that, for some $i \in N \setminus \{7\}$,

$p'_{i-1,i} > 0.5$ and $\varphi'_i(t^A, s', \mathbf{p}') > \varphi'_{i-1}(t^A, s', \mathbf{p}')$. Moreover, we know that $\mathbf{p}'$ can be constructed in such a way that $p'_{i-1,i} \approx 0.5$ and $p'_{ik} \approx p'_{i-1,k}$ holds for each $k \in N \setminus \{i-1, i, 7\}$.

Consider now the probability matrix $\mathbf{p} \in \mathcal{P}_R$ such that $p_{jk} = p'_{jk}$ for all $j, k \in N \setminus \{7\}$, and $p_{k7} \approx 1$ for all $k \in N \setminus \{7\}$. For the final winning probabilities of each $k < 7$ we have by construction that $\varphi_k(t, s, \mathbf{p}) \approx \varphi'_k(t^A, s', \mathbf{p}')$. By hypothesis, $p'_{i-1,i} > 0.5$ and $\varphi'_i(t^A, s', \mathbf{p}') > \varphi'_{i-1}(t^A, s', \mathbf{p}')$ holds and, thus, $p_{i-1,i} > 0.5$ and $\varphi_i(t, s, \mathbf{p}) > \varphi_{i-1}(t, s, \mathbf{p})$ holds as well. Hence, (t, N) also violates MS in this case. Moreover, as in Step 1, it is interesting to remark for the later steps in the proof that, for any seeding in a one-bye antler t, we can always find a probability matrix $\mathbf{p} \in \mathcal{P}_R$ that makes the competition (t, N) violate MS and such that $p_{i-1,i} \approx 0.5$ and $p_{ik} \approx p_{i-1,k}$ holding for some $i \in N$ and all $k \in N \setminus \{i-1, i\}$.

Step 4 Let (t, N) be an elimination competition system with $h(t) = 3$ and t being balanced. Then (t, N) violates MS.

Proof. Note that $N = \{1, \ldots, 8\}$ holds in this case. Let x be the player whose initial match is against player 8, and remove from t the nodes $s^{-1}(8)$ and $s^{-1}(x)$ together with the corresponding edges to their immediate predecessor, v_λ. Note that the remaining subgraph t^A of t is a one-bye antler with v_λ being now a leaf of t^A. Consider then the seeding $s' : \Lambda(t^A) \to \{1, \ldots, 7\}$ defined as follows: $s'(v_\lambda) = x$ and $s'(\lambda) = s(\lambda)$ for each $\lambda \in \Lambda(t_A) \setminus \{v_\lambda\}$, and notice that $s' \in \mathcal{S}^{(t^A, N \setminus \{8\})}$.

By Step 3, $(t^A, N \setminus \{8\})$ violates MS. That is, there exists a probability matrix $\mathbf{p}' \in \mathcal{P}_{R_{|N \setminus \{8\}}}$ such that $p'_{ij} > 0.5$ and $\varphi'_j(t^A, s', \mathbf{p}') > \varphi'_i(t^A, s', \mathbf{p}')$ for the corresponding final winning probabilities of some $i, j \in N \setminus \{8\}$. Moreover, we know that $\mathbf{p}'$ can be constructed in such a way that $p'_{ij} \approx 0.5$ and $p'_{ik} \approx p'_{jk}$ holds for each $k \in N \setminus \{8\}$.

Consider now the probability matrix $\mathbf{p}$ defined on N such that $p_{jk} \approx p'_{jk}$ holds for all $j, k < 8$ and $p_{k8} \approx 1$ holds for all $k < 8$. For the final winning probabilities of each $k < 8$ we have by construction that $\varphi_k(t, s, \mathbf{p}) \approx \varphi'_k(t^A, s', \mathbf{p}')$. Therefore, $p_{ij} > 0.5$ and $\varphi_j(t, s, \mathbf{p}) > \varphi_i(t, s, \mathbf{p})$ as it is required to prove that the competition (t, N) violates MS. Moreover, by construction, $\mathbf{p}$ is such that $p_{ij} \approx 0.5$ and $p_{ik} \approx p_{jk}$ for each $k \in N$.

Step 5 Let (t, N) be an elimination competition system with t being a limited extension from the root of an antler. Then (t, N) violates MS.

Proof. Let t^A be the (symmetric or asymmetric) antler contained in t and fix *any* $s \in \mathcal{S}^{(t,N)}$. Because s is arbitrary, to show that (t, N) violates MS, it suffices to show that the violation holds with respect to s. Let $N'_{t_A}(s)$ be the set of players seeded by s to a leaf of t^A. For notational convenience, when $i \in N'_{t^A}(s)$ we will label this player as i'.

By Lemma 3, Step 1 in the case of symmetric antlers, and Step 2 in the case of asymmetric antlers, we know that for any seeding in t^A we can find a probability matrix $\mathbf{p} \in \mathcal{P}_{R_{|N'_{t^A}(s)}}$ that makes $\left(t^A, N\right)$ violate MS. In particular, for $s' = s_{|\Lambda(t^A)}$, there exists a matrix $\mathbf{p}'$ and players $i', h' \in N'_{t^A}(s)$ with $p'_{h',i'} > 0.5$ and $\varphi_{i'}(t^A, s', \mathbf{p}') > \varphi_{h'}(t^A, s', \mathbf{p}')$. Moreover, we know that $\mathbf{p}'$ can be constructed in such a way that $p'_{h',i'} \approx 0.5$ and $p'_{h',k'} \approx p'_{i',k'}$ holding for each $k' \in N'_{t^A}(s)$.

Now, for all $k \in N \setminus N'_{t^A}(s)$ let $sup(k) = min\{x' \in N'_{t^A}(s) \text{ such that } x' > k\}$ and $inf(k) = max\{x' \in N'_{t^A}(s) \text{ such that } x' < k\}$.

Let us define a probability matrix $\mathbf{p}$ on N such that (1) for all $x', y' \in N'_{t^A}(s)$, $p_{x'y'} = p'_{x'y'}$; (2) for all $k \in N \setminus N'_{t^A}(s)$ such that $sup(k)$ exists, $p_{k,sup(k)} = 0.5$ (and $p_{kw} = p_{sup(k),w}$ for each $w \in N$); (3) for all $k \in N'_{t^A}(s)$ such that $sup(k)$ does not exist, $p_{k,inf(k)} = 0.5$ (and $p_{kw} = p_{inf(k),w}$ for each $w \in N$).

In other words, $\mathbf{p}$ restricted to the elements of $N'_{t^A}(s)$ is equal to $\mathbf{p}'$, and all the players that are not seeded to t^A are assimilated as equally strong as his or her immediately weaker player in $N'_{t^A}(s)$. Moreover, if for some element k not seeded to t^A there is no weaker player in $N'_{t^A}(s)$, then k is considered as equally strong as its immediately stronger player in $N'_{t^A}(s)$. Thus, by construction, $\mathbf{p} \in \mathcal{P}_R$.

Note that, by $p'_{h'w'} \approx p'_{i'w'}$ for each $w' \in N'_{t^A}(s)$, we have by construction that $p_{h'w} \approx p_{i'w}$ holds for each $w \in N$.

Now, let V_{v_0,v_0^A} be the set of nodes of the shortest path between the root v_0 of t and the root v_0^A of t^A. Note that, due to t being a binary tree, for each $v \in V_{v_0,v_0^A}$ with $\ell(v) \geq 1$ there always exists a unique node $v' \notin V_{v_0,v_0^A}$ with $\ell(v') = \ell(v)$ at distance 2 from v. That is, the two players having reached these two nodes play against each other to arrive at their common immediate predecessor $v'' \in V_{v_0,v_0^A}$ with $\ell(v'') = \ell(v) - 1$. By letting $N_{v'}$ be the set of all players seeded by s to some leaf of the subtree of t rooted at v', we get $\varphi_i(t, s, \mathbf{p}) = \varphi_{i'}(t^A, s', \mathbf{p}') \cdot \prod_{v \in V_{v_0,v_0^A}, \ell(v) \geq 1} \sum_{k \in N_{v'}} p_{i'k} p_k^{v'}$ and $\varphi_h(t, s, \mathbf{p}) = \varphi_{h'}(t^A, s', \mathbf{p}') \cdot \prod_{v \in V_{v_0,v_0^A}, \ell(v) \geq 1} \sum_{k \in N_{v'}} p_{h'k} p_k^{v'}$.

Recall that $p_{i',w} \approx p_{h',w}$ holds for each $w \in N$. Moreover, $p_k^{v'}$ is independent of whether i' or h' have reached node $v \in V_{v_0,v_0^A}$. Therefore, $\varphi_{i'}(t^A, s', \mathbf{p}') > \varphi_{h'}(t^A, s', \mathbf{p}')$ implies $\varphi_i(t, s, \mathbf{p}) > \varphi_h(t, s, \mathbf{p})$. Given that $p_{hi} > 0.5$ by the construction of $\mathbf{p}$, the competition (t, N) violates MS.

Step 6 Let (t, N) be an elimination competition with t being a limited extension from the root of a one-bye antler. Then (t, N) violates MS.

The proof is analogous to the proof of Step 5.

Step 7 Let (t, N) be an elimination competition with t being a limited extension from the root of a balanced tree of height 3. Then, (t, N) violates MS.

Again, the proof is analogous to that of Step 5.

Step 8 Let t^* be a limited extension from the root of an antler t^A and (t, N) be an elimination competition with t being an extension from the leaves of t^*. Then, (t, N) violates MS.

Proof. For the proof of the statement of Step 8, we will need the following additional notation.

Let $d(v_0, v_0^A)$ stand for the geodesic distance between the root v_0 of t and the root v_0^A of t^A. For $x \in \{0, \ldots, h(t) - d(v_0, v_0^A)\}$, we denote by t_0^x the subgraph of t consisting of all nodes $v \in V(t)$ with $\ell(v) \leq d(v_0, v_0^A) + x$ and the corresponding edges of t connecting them. That is, t_0^x is just the tree t

when being truncated at level $d(v_0, v_0^A) + x$. Clearly, $x = h(t^A) = 3$ implies $t_0^x = t^*$.

We denote by M^x the set of matches at level $d(v_0, v_0^A)+x$ of t_0^x (with m^x being a typical element of M^x), and by T_k^x the set of subgraphs of t_0^x that can be obtained from t_0^x by removing a number k of matches at level $d(v_0, v_0^A)+x$ (with t_k^x being a typical element of T_k^x). Clearly, $T_{|M^x|}^x = t_{|M^x|}^x = t_0^{x-1}$.

Moreover, for any tree $t_k^x \in T_k^x$ we consider a set of players $N_k^x = \{1, \dots, n_k^x\}$ that makes competition (t_k^x, N_k^x) feasible; that is, a set of players whose cardinality is $n_k^x = |\Lambda(t_k^x)|$.

Consider now, for any $k \leq |M^4|$, any tree $t_k^4 \in T_k^4$ and the corresponding set of players N_k^4 that makes (t_k^4, N_k^4) feasible. Let R be the ordering of strength defined on N_k^4. Assume that (t_k^4, N_k^4) satisfies MS. By Lemma 3 we know that, for t_k^4 to satisfy MS with respect to some seeding $s \in \mathcal{S}^{\left(t_k^4, N_k^4\right)}$, the worst player according to R should be seeded to some leaf of t_k^4 that belongs to some match in M^4. If the worst player is not unique, then assume w.l.o.g. that the selected player is n_k^4. Let us denote by m'^4 the match to which n_k^4 is seeded, by $(\lambda_a'^4)$ and $(\lambda_b'^4)$ its two leaves, and by $(\overline{n}_k^4) \neq n_k^4$ the second player seeded to m'^4; that is, the opponent of n_k^4. Now, let $(t_{k+1}'^4, N_{k+1}^4)$ be the competition in which $t_{k+1}'^4$ has been obtained from t_k^4 by removing the match m'^4 and N_{k+1}^4 is a set of $n_k - 1$ players. Clearly, the common immediate predecessor w of $\lambda_a'^4$ and $\lambda_b'^4$ becomes now a leaf of $t_{k+1}'^4$ to be denoted by λ_w. Hence, $\Lambda(t_{k+1}'^4) = \Lambda(t_k^4) \cup \{\lambda_w\} \setminus \{\lambda_a'^4, \lambda_b'^4\}$.

The inductive reasoning starts by proving that, roughly speaking, if the competition (t_k^4, N_k^4) satisfies MS and the match where the worst player is seeded at is removed, then the remaining structure also satisfies MS.

Claim Let (t_k^4, N_k^4) and $(t_{k+1}'^4, N_{k+1}^4)$ be as above. If (t_k^4, N_k^4) satisfies MS, then $(t_{k+1}'^4, N_{k+1}^4)$ also satisfies MS.

Proof of the Claim. Assume that (t_k^4, N_k^4) satisfies MS but $(t_{k+1}'^4, N_{k+1}^4)$ does not. Let R' be defined on N_{k+1}^4 such that $R' = R_{|N_{k+1}^4 \setminus \{n_k^4\}}$. Consider the seeding $s' : \Lambda(t_{k+1}'^4) \to N_{k+1}^4$ defined as follows: for each $\lambda \in \Lambda(t_{k+1}'^4) \setminus \{\lambda_w\}$, $s'(\lambda) = s(\lambda)$ and $s(\lambda_w) = \overline{n}_k^4$ (note that $N_{k+1}^4 = N_k^4 \setminus \{n_k^4\}$ and that

$\overline{n}_k^4 \in N_{k+1}^4$). That is, s' can be interpreted as a situation in which $\overline{n}_k^4$ wins his match against n_k^4 and the remaining matches are not yet played. By hypothesis $(t_{k+1}'^4, N_{k+1}^4)$ violates MS. This implies that for the seeding s' there exists some probability matrix $\mathbf{p}' \in \mathcal{P}_{R'}$ defined on N_{k+1}^4 such that $p_{ij}' > 0.5$ and $\varphi_j(t_{k+1}'^4, s', \mathbf{p}') > \varphi_i(t_{k+1}'^4, s', \mathbf{p}')$ holds for some $i, j \in N_{k+1}^4$.

Let $\mathbf{p}$ be a probability matrix on N_k^4, which is defined as follows: $p_{ij} = p_{ij}'$ for all $i, j \in N_k^4 \setminus \{n_k^4\}$, and $p_{i,n_k^4} \approx 1$ for each $i \in N_k^4 \setminus \{n_k^4\}$. Note that $\mathbf{p} \in \mathcal{P}_{R_{|N_k^4}}$ by construction. Also by construction, $\varphi_i(t_{k+1}'^4, s', \mathbf{p}') \approx \varphi_i(t_k^4, s, \mathbf{p})$ holds for each $i \in N_k^4 \setminus \{n_k^4\}$. Therefore, $p_{ij} > 0.5$ and $\varphi_j(t_k^4, s, \mathbf{p}) > \varphi_i(t_k^4, s, \mathbf{p})$ holds for some $i, j \in N_k^4$. Hence, we have a contradiction to the hypothesis that (t_k^4, N_k^4) satisfies MS, which completes the proof of the claim.

Note that this claim holds also for $k+1 = |M^4|$. In this particular case, $t_{k+1} = t_{|M^4|}^4 = t_0^3 = t^*$ which leaves us with the following three possibilities:

(i) There are two leaves at distance 2 from v_0^A; that is, there is no extension from any leaf at distance 2 from v_0^A and, therefore, t^* is a limited extension from the root of a (symmetric or asymmetric) antler.

(ii) There is a unique leaf at distance 2 from v_0^A. In this case, t^* is a limited extension from the root of a one-bye antler.

(iii) There are no leaves at distance 2 from v_0^A. In this case, t^* is a limited extension from the root of a balanced tree of height 3.

For each of these three possible cases, we have proven in the previous steps that no competition whose graph is $t^* = t_{|M^4|}^4$ does satisfy MS.

Now, for any $k \in \{0, \ldots, |M^4| - 1\}$, take any competition (t_k^4, N_k^4) with $t_k^4 \in T_k^4$. Note that from (t_k^4, N_k^4) it is always possible to define a sequence (t_k^4, N_k^4), $(t_{k+1}^4, N_{k+1}^4), \ldots, (t_{|M^4|}^4, N_{|M^4|}^4)$ by removing the match at level $d(v_0, v_0^A)+4$ where the corresponding worst player n_k^4, n_{k+1}^4, …, $n_{|M^4|-1}^4$ has been seeded. Given that $(t_{|M^4|}^4, N_{|M^4|}^4)$ violates MS, and considering the claim, an inductive argument also allows to prove that (t_k^4, N_k^4) violates MS. Therefore, in particular, (t_0^4, N_0^4) violates MS. Recalling that $t_{|M^5|}^5 = t_0^4$, we can recursively replicate the inductive argument at level $d(v_0, v_0^A) + 5$ to conclude that for all $k \in \{0, 1, \ldots, |M^5| - 1\}$, every competition (t_k^5, N_k^5) with $t_k^5 \in T_k^5$ violates MS. The reasoning can be successively applied when t has been truncated at higher levels, until we reach the tree $t = t_0^{h(t)-d(v_0, v_0^A)}$, which proves that the competition system (t, N) violates MS.

Step 9 Let (t, N) be an elimination competition with t containing an antler. Then, (t, N) violates MS.

Proof. The statement follows from the fact that if a tree t contains an antler, then, clearly, it is some form of extension from the leaves of a limited extension from the root of an antler and by Step 8.

Step 10 Let (t, N) be an elimination competition with t being antler-free. Then, (t, N) satisfies MS with respect to $s \in \mathcal{S}^{(t,N)}$ only if $s = s_{ib}$.

Proof. We proceed by reduction to the absurd. We assume that (t, N) satisfies MS, $s \neq s_{ib}$, and show that this leads to a contradiction. In other words, we show that, given a strength ordering R defined on N, it is possible to find a probability matrix $\mathbf{p} \in \mathcal{P}_R$ such that if $s \neq s_{ib}$ then there exist $i, j \in N$ such that $p_{ij} > 0.5$ but $\varphi_i(t, s, \mathbf{p}) < \varphi_j(t, s, \mathbf{p})$.

We know from Lemma 3 that for (t, N) to satisfy MS, it should be the case that for any probability matrix $\mathbf{p}$, and any $\lambda, \lambda' \in \Lambda(t)$, $\ell(\lambda) < \ell(\lambda')$ implies $p_{s(\lambda), s(\lambda')} \geq 0.5$. Moreover, by Lemma 1, $t \in ES_2^*$. Take now a maximal root-to-leaf path $\gamma(t)$ and note that for any probability matrix $\mathbf{p}$, $s \neq s_{ib}$ implies either that

(i) There exist leaves $\lambda_a, \lambda_b \in \Lambda(t)$ with $\ell(\lambda_a) = \ell(\lambda_b) < h(t)$ such that: (1) only λ_a has an immediate predecessor belonging to $V_{\gamma(t)}$ and (2) $p_{s(\lambda_b), s(\lambda_a)} > 0.5$, or that

(ii) $|\Lambda^{h(t)}(t)| = 4$ with the players in $\{s(\lambda) : \lambda \in \Lambda^{h(t)}(t)\}$ not being seeded in a balanced way.

We proceed by showing that in both cases we reach a contradiction.

Case (i) Let k be the number of players seeded by s to leaves at higher level than $\ell(\lambda_a) = \ell(\lambda_b)$. We construct the desired $\mathbf{p}$ in three steps.

First, we set $p_{n-k, n-k+1} > 0.5$ and $p_{n-k+1, z} \approx 1$ to hold for each $z > n - k + 1$. By Lemma 3, the set of players seeded to the leaves at higher level than $\ell(\lambda_a)$ is $\{n, n-1, \ldots, n-k+1\}$. Thus, the probability $p_{n-k+1}^{v^{s(\lambda_a)}}$ with which player $n-k+1$ reaches node $v^{s(\lambda_a)} \in V_{\gamma(t)}$ is arbitrarily close to 1.

Second, let x_1, x_2, and x_3 be the three players seeded to the three leaves at level $\ell(\lambda_a)$ and set $p_{x_1x_2} \geq 0.5$ and $p_{x_2x_3} \geq 0.5$. By construction, $p_{x_3,(n-k+1)} > 0.5$. Note also that, with a probability arbitrary close to 1, the players $(n-k+1)$, x_1, x_2, and x_3 play a balanced sub-competition at level $\ell(\lambda_a)$ with the root of the sub-competition being $v \in V_{\gamma(t)}$ with $\ell(v) = \ell(\lambda_a) - 2$. Moreover, by hypothesis, $n-k+1$ plays a match against some player x_i ($i \in \{2,3\}$) such that $p_{x_1x_i} > 0.5$. Therefore, by Lemma 2, it is possible to define a probability matrix $\mathbf{p}'$ on the player set $\{n-k+1, x_1, x_2, x_3\}$ such that there are players $i, j \in \{n-k+1, x_1, x_2, x_3\}$ with $p'_{ij} > 0.5$ and $p_i'^v < p_j'^v$. Moreover, we know by the proof of Lemma 2 that $\mathbf{p}'$ can always be constructed in such a way that $p'_{iw} \approx p'_{jw}$ holds for each $w \in \{n-k+1, x_1, x_2, x_3\}$. We then take $p_{xy} = p'_{xy}$ to hold for all $x, y \in \{n-k+1, x_1, x_2, x_3\}$. This implies that, according to $\mathbf{p}$, $p_i^v < p_j^v$. It also implies $p_{iw} \approx p_{jw}$ for each $w \in \{n-k+1, x_1, x_2, x_3\}$.

Third, we take $p_{zx_1} = 0.5$ to hold for each $z \in N$ who is seeded at a lower level than $\ell(\lambda_a)$. That is, every player who is seeded at a lower level than $\ell(\lambda_a)$ is considered as being equally strong as the strongest player at level $\ell(\lambda_a)$. Note that the latter fact together with $p_{n-k+1,z} \approx 1$ for each $z > n-k+1$ implies $p_{iw} \approx p_{jw}$ for each $w \in N$.

In summary, the constructed probability matrix $\mathbf{p}$ is as follows: the restriction of p on the player set $\{n-k+1, x_1, x_2, x_3\}$ is $\mathbf{p}'$; x_1 is equally strong as every player who is seeded at lower levels; and, each player in $\{n-k+1, x_1, x_2, x_3\}$ wins with a probability close to 1 the match against any player being seeded at higher levels and is different from $n-k+1$. Now, by using the notation of Step 1 and recalling that $\ell(v) = \ell(\lambda_a) - 2$, we have

$\varphi_i(t, s, \mathbf{p}) = p_i^v(s) \cdot \prod_{x \in S^1_{v'}(s), \ell(v') < \ell(v)} p_{ix} \cdot \prod_{y,z \in S^2_{v'}(s), \ell(v') < \ell(v)} (p_{iy}p_{yz} + p_{iz}p_{zy})$

and

$\varphi_j(t, s, \mathbf{p}) = p_j^v(s) \cdot \prod_{x \in S^1_{v'}(s), \ell(v') < \ell(v)} p_{jx} \cdot \prod_{y,z \in S^2_{v'}(s), \ell(v') < \ell(v)} (p_{jy}p_{yz} +$

$p_{jz}p_{zy})$.

Then, we have that $p_j^v > p_i^v$, $p_{ix} \approx p_{jx}$ for each $x \in N$, and because p_{yz} (p_{zy}) is independent of any other parameter in the respective formulae for i and j, we conclude that $\varphi_j(t, s, \mathbf{p}) > \varphi_i(t, s, \mathbf{p})$ should also hold, which is in contradiction to (t, N) satisfying MS with respect to s in this case.

Case (ii) Note that in this case $\left|\Lambda^{h(t)}\right| = 4$ implies that the node $v \in V_{\gamma(t)}$ with $\ell(v) = h(t) - 2$ is the root of a balanced subtree of t. We construct the desired $\mathbf{p}$ in two steps.

First, let $\{a, b, c, d\} \subseteq N$ be the set of players seeded to the leaves in $\Lambda^{h(t)}$. It follows then from Lemma 2 that, for each of the two possible non-balanced seedings of the players in $\{a, b, c, d\}$ to the leaves in $\Lambda^{h(t)}$, there exists a probability matrix $\mathbf{p}'$ on $\{a, b, c, d\}$ such that $p'_{ij} > 0.5$ and $p'^v_j > p'^v_i$ holds for some $i, j \in \{a, b, c, d\}$. Moreover, it follows from the proof of Lemma 2 that $\mathbf{p}'$ can be constructed in such a way that $p'_{ij} \approx 0.5$ and $p'_{iw} \approx p'_{jw}$ for each $w \in \{a, b, c, d\}$. Thus, we take $\mathbf{p}$ to be such that $p_{xy} = p'_{xy}$ for all $x, y \in \{a, b, c, d\}$. Let a be a strongest player and d a weakest player among those in $\{a, b, c, d\}$. By Lemma 3, for all players $x \in N \setminus \{a, b, c, d\}$ and $i \in \{a, b, c, d\}$, $p_{xi} \geq 0.5$.

Second, we set $p_{zw} = 0.5$ to hold for all $z, w \in N \setminus \{b, c, d\}$.

Thus, $\mathbf{p}$ is as follows: the restriction of $\mathbf{p}$ on the player set $\{a, b, c, d\}$ is $\mathbf{p}'$, while each of the remaining players (who are seeded to leaves at lower levels that $h(t)$ in the tree) is considered as equally strong as the strongest player in $\{a, b, c, d\}$. Moreover, $p'_{iw} \approx p'_{jw}$ holding for each $w \in \{a, b, c, d\}$ implies by construction that $p_{iw} \approx p_{jw}$ is also true for each $w \in N$.

By using the notation of Step 1 and recalling that $\ell(v) = h(t) - 2$, we have

$\varphi_i(t, s, \mathbf{p}) = p_i^v(s) \cdot \prod_{x \in S^1_{v'}(s): \ell(v') < \ell(v)} p_{ix} \cdot \prod_{y,z \in S^2_{v'}(s): \ell(v') < \ell(v)} (p_{iy}p_{yz} + p_{iz}p_{zy})$

and

$\varphi_j(t, s, \mathbf{p}) = p_j^v(s) \cdot \prod_{x \in S^1_{v'}(s): \ell(v') < \ell(v)} p_{jx} \cdot \prod_{y,z \in S^2_{v'}(s): \ell(v') < \ell(v)} (p_{jy}p_{yz} + p_{jz}p_{zy})$.

Again we have that $p_j^v > p_i^v$, $p_{ix} \approx p_{jx}$ for each $x \in N$, and because p_{yz} (p_{zy}) is independent of any other parameter in the respective formulae for i and j, we conclude that $\varphi_j(t, s, \mathbf{p}) > \varphi_i(t, s, \mathbf{p})$ holds. Thus, we have again a contradiction to (t, N) satisfying MS with respect to s. This completes the proof of Theorem 1. ■

Next we prove Theorems 2 and 3 for axioms ET and WET. The presentation of the proofs turns out to be more efficient if Theorem 3 is proved before Theorem 2.

Proof of Theorem 3. Let (t, N) be an elimination competition with t being balanced and let $s \in \mathcal{S}^{(t,N)}$ be an arbitrary but fixed seeding. Given the balancedness of t, any of its leaves has the same level coinciding with $h(t)$. Then, by $p_{ij} = 0.5$ for all $i, j \in N$, $\varphi_i(t, s, \mathbf{p}) = (0.5)^{h(t)}$ holds for each $i \in N$. Thus, (t, N) satisfies WET.

Suppose now that (t, N) is an elimination competition satisfying WET but such that t is not balanced. Suppose that iIj holds for all $i, j \in N$ and thus, $p_{ij} = 0.5$ for all $i, j \in N$. Let $s \in \mathcal{S}^{(G,N)}$ be the seeding with respect to which (t, N) satisfies WET. Given that t is not balanced there are leaves $\lambda, \lambda' \in \Lambda(t)$ with $\ell(\lambda) \neq \ell(\lambda')$. We have then $\varphi_{s(\lambda)}(t, s, \mathbf{p}) = (0.5)^{\ell(\lambda)} \neq (0.5)^{\ell(\lambda')} = \varphi_{s(\lambda')}(t, s, \mathbf{p})$ in contradiction to (t, N) satisfying WET. ■

Proof of Theorem 2. The proof for (t, N) satisfying ET when t is an elementary binary tree (and thus $|N| = 2$) is immediate.

Let us now prove that if (t, N) satisfies ET, then t is an elementary binary tree. Note first that the structure of the proof of Theorem 3 de facto proves that if t is not a balanced tree then (t, N) violates ET (as being a stronger axiom than WET). Therefore we have to prove that if (t, N) is an admissible competition such that t is balanced and $|N| > 2$, (that is, $h(t) > 1$) then (t, N) violates ET.

Let (t, N) be any admissible competition as above and consider R such that $1I2I \dots I(n-1)Pn$ holds. Take any probability matrix $\mathbf{p} \in \mathcal{P}_R$. Note that $p_{in} > 0.5$ and $p_{ij} = 0.5$ holds for all $i, j \in N \setminus \{n\}$ and $j \neq i$. Denote by t_ℓ and t_r the unique proper sub-trees of t such that $h(t_\ell) = h(t_r) = h(t) - 1$.

Take any $s \in \mathcal{S}^{t,N}$ and denote by N_ℓ and N_r the set of players seeded by s to the leaves of t_ℓ and t_r, respectively. Assume without loss of generality that $n \in N_\ell$.

For $k \in \{\ell, r\}$ let the notations $s_{|t_k}$ and $\mathbf{p}_{|k}$ stand for the restriction of s to t_k and of $\mathbf{p}$ to N_k, respectively. Notice then that for $k \in \{\ell, r\}$ and $i \in N_k$, $\varphi_i(t_k, s_{|t_k}, \mathbf{p}_{|k})$ is the probability that player i wins the sub-competition (t_k, N_k).

Claim: If (t_ℓ, N_ℓ) violates ET, then (t, N) violates ET as well.

Proof of the claim: Suppose that (t_ℓ, N_ℓ) violates ET, that is, there are $i, j \in N_\ell$ such that iIj and $\varphi_i(t_\ell, s_{|t_\ell}, \mathbf{p}_{|\ell}) \neq \varphi_j(t_\ell, s_{|t_\ell}, \mathbf{p}_{|\ell})$. Notice that

$$\varphi_i(t, s, \mathbf{p}) = \varphi_i(t_\ell, s_{|t_\ell}, \mathbf{p}_{|\ell}) \cdot \sum_{q \in N_r} p_{iq} \varphi_q(t_r, s_{|t_r}, \mathbf{p}_{|r})$$

and

$$\varphi_j(t, s, \mathbf{p}) = \varphi_j(t_\ell, s_{|t_\ell}, \mathbf{p}_{|\ell}) \cdot \sum_{q \in N_r} p_{jq} \varphi_q(t_r, s_{|t_r}, \mathbf{p}_{|r}).$$

Since $p_{ij} = 0.5$ implies by (2) that $p_{iq} = p_{jq}$ holds for all $q \in N_r$, we conclude from $\varphi_i(t_\ell, s_{|t_\ell}, \mathbf{p}_{|\ell}) \neq \varphi_j(t_\ell, s_{|t_\ell}, \mathbf{p}_{|\ell})$ that $\varphi_i(t, s, \mathbf{p}) \neq \varphi_j(t, s, \mathbf{p})$.

Consider now the unique sub-tree t^* of t_ℓ with $h(t^*) = 2$ and n being seeded by s to a leaf of t^*. Let $\{m_1, m_2, m_3\}$ be the set of the other three players seeded by s to the leaves of t^* and suppose that, without loss of generality, players m_3 and n play an initial match. For the probabilities of players m_2 and m_3 to win the sub-competition $(t^*, \{m_1, m_2, m_3, n\})$ we get

$$\varphi_{m_2}(t^*, s_{|t^*}, \mathbf{p}_{|\{m_1,m_2,m_3,n\}}) = p_{m_2,m_1} \cdot p_{m_2,m_3} \cdot p_{m_3,n} + p_{m_2,m_1} \cdot p_{m_2,n} \cdot p_{n,m_3}$$

and

$$\varphi_{m_3}(t^*, s_{|t^*}, \mathbf{p}_{|\{m_1,m_2,m_3,n\}}) = p_{m_3,n} \cdot p_{m_3,m_1} \cdot p_{m_1,m_2} + p_{m_3,n} \cdot p_{m_3,m_2} \cdot p_{m_2,m_1}.$$

Recall that $p_{m',n} > 0,5$ and $p_{m',nm''} = 0,5$ holds for all $m', m'' \in \{m_1, m_2, m_3\}$ and $m' \neq m''$. We conclude then that $\varphi_{m_3}(t^*, s_{|t^*}, \mathbf{p}_{|\{m_1,m_2,m_3,n\}}) > \varphi_{m_2}(t^*, s_{|t^*}, \mathbf{p}_{|\{m_1,m_2,m_3,n\}})$. Since $p_{m_2,m_3} = 0.5$, the sub-competition $(t^*, \{m_1, m_2, m_3, n\})$ violates ET.

Finally, by using the above claim and applying a consecutive reasoning as many times as necessary, we conclude that any admissible competition

(t, N) with t being a balanced tree and $h(t) > 2$ violates ET. ■