Improving Inconsistency measured by the Geometric Consistency Index in the Analytic Hierarchy Process*

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Abstract

This paper presents a theoretical framework and a procedure for revising the judgements and improving the inconsistency of an Analytic Hierarchy Process (AHP) pairwise comparison matrix when the Row Geometric Mean is used as the prioritisation procedure and the Geometric Consistency Index (GCI) is the inconsistency measure. Inconsistency is improved by slightly modifying the judgements that further improve the GCI. Both the judgements and the derived priority vector will be close to the initial values. A simulation study is utilised to analyse the performance of the algorithm. The proposed framework allows the specification of the procedure to particular interests. It can also be used with inconsistency indices based on triads and as an intermediate step in the construction of consistency consensus matrices in AHP-group decision making. A numerical example illustrates the proposed procedure.

Keywords: Multiple criteria analysis, Analytic Hierarchy Process, Row Geometric Mean, Geometric Consistency Index, Inconsistency Improvement

1. Introduction

The Analytic Hierarchy Process (AHP) (Saaty, 1977, 1980) is one of the most widely employed multicriteria decision making techniques in practical situations (Ho, 2008; Ishizaka and Labib, 2011; Grzybowski, 2016). It is also one of the techniques that best responds to the challenges and needs of the Knowledge Society (Moreno-Jimenez and Vargas, 2018; Moreno-Jiménez et al., 2019).

AHP allows the decision maker some inconsistency when eliciting judgements and offers a procedure to assess the degree of internal coherence when incorporating their preferences into the model (judgements elicitation process). Given a pairwise comparison matrix, $A_{(nxn)} = (a_{ij})$ with $a_{ij}a_{ji} = 1$ and $a_{ij} > 0$, Saaty (1980) established that the matrix A is consistent if $a_{ij}a_{jk} = a_{ik}$, $\forall i, j, k = 1, \ldots, n$ (cardinal transitivity). As the measure of inconsistency, he proposed an index associated with the prioritisation procedure employed (eigenvector, EV), known as the Consistency Ratio (CR), and set the threshold of 10% ($CR \leq 0.1$) as the maximum level permitted for accepting the elicited judgements given by the decision maker.

*Funding: This research was supported by the Grupo Decisión Multicriterio Zaragoza research group (S35-17R, Government of Aragon) and FEDER funds.

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In recent years, the Row Geometric Mean (RGM) method (Saaty, 1980; Crawford and Williams, 1985), another of the oldest prioritisation methods in the AHP context, has become popular among the scientific community and is seen as an alternative to Saaty's initial proposal (EV). This is largely due to the RGM's psychological (Saaty, 1980; Brugha, 2000; Altuzarra et al., 2010), mathematical (Aguarón and Moreno-Jiménez, 2000; Escobar and Moreno-Jiménez, 2000; Aguarón et al., 2003, 2019; Brunelli, 2018) and statistical (Crawford and Williams, 1985; Altuzarra et al., 2007, 2010) properties. It is also one of the most widely employed, with numerous justifications in the scientific literature. Crawford and Williams (1985) justified it from the perspective of mathematical optimisation (minimisation of the log quadratic deviation between judgements and priority ratios) and statistical estimation (maximum likelihood estimation in multiplicative model with log-normal errors). Using a Bayesian approach, Altuzarra et al. (2007) proved that the RGM estimator is also the posterior median and mean of the priorities in these models when adopting a non-informative prior in the log-priorities and the consistency parameter of the model. Aguarón and Moreno-Jiménez (2010) provided a graph dominance justification that was similar to Saaty's for the EV method, but with the application of the geometric mean, instead of the arithmetic mean when evaluating the dominance of each element along all the walks. Lundy et al. (2017) discuss a spanning tree method and prove the mathematical equivalence of its preference vector to that of the RGM method. Csató (2018) proved that the ordering induced by RGM method is uniquely determined by three independent axioms: anonymity, responsiveness and aggregation invariance.

For the RGM method, Crawford and Williams (1985) proposed an unbiased estimator of the variance of log-errors as a measure of the inconsistency. Aguarón and Moreno-Jiménez (2003) referred to this measure as the Geometric Consistency Index (GCI) and established thresholds for the GCI with an interpretation analogous to the 10% of Saaty's CR. These values are: $GCI = 0.31$ for $n = 3$, $GCI = 0.35$ for $n = 4$ and $GCI = 0.37$ for $n > 4$.

When the inconsistency measures (CR or GCI) exceed the fixed thresholds (in some cases where little information is available, higher thresholds can be accepted), the decision maker must review their judgements until they reach the level of inconsistency required. If this is not achieved, the validity of the priority vector is not guaranteed, and the decision maker must seek additional knowledge to modify the judgements.

In general, and as suggested by Saaty, the review of judgements must be made personally by the decision maker, not automatically (Saaty, 2003). Nevertheless, some strategies for identifying the most inconsistent judgements and offering the decision maker procedures for improving inconsistency can be found in the scientific literature. Based on differentiation of the principal eigenvalue calculated by Harker (1987), Dadkhah and Zahedi (1993) suggested a method for improving inconsistency with the EV method. A number of authors subsequently put forward suggestions for improving inconsistency in AHP. An overview of these approaches can be found in Khatwani and Kar (2017) and Brunelli (2018). However, none of them include a procedure for improving inconsistency measured by the GCI.

Saaty (2003) suggested that the inconsistency should be improved by slightly modifying the judgements that further improve the inconsistency measure. Using analogous ideas, and assuming that input or typing errors made in the judgement elicitation process have been corrected, this paper presents a new procedure for revising judgements and improving the inconsistency measured by the GCI. In this way, and as Saaty recommended, both the judgements and the derived priority vector will be close to the initial values.

The work is structured as follows: Section 2 details the theoretical results necessary for the proposal; Section 3 describes the procedure for revising judgements and improving inconsistency, it goes on to offer an analysis of performance and presents some variants; Section 4 illustrates the proposed procedure by means of a numerical example; Section 5 highlights the most important conclusions of the study.

2. Theoretical Results

Let $A = (a_{ij})$ be a pairwise comparison matrix. The inconsistency measure proposed for the RGM method is the Geometric Consistency Index (Aguarón and Moreno-Jiménez, 2003):

$$GCI = \frac{2}{(n-1)(n-2)} \sum_{i<j} \log^2 e_{ij} \tag{1}$$

where $w = (w_i)$ is the priority vector obtained with the RGM method and $e_{ij} = a_{ij} w_j / w_i$ is the error obtained when the ratio of priorities ω_i/ω_j is approximated by a_{ij}.

Following a similar approach to that used by Dadkhah and Zahedi (1993) for the EV method, the derivatives of the GCI with respect to the judgements are calculated to identify which entries further improve inconsistency.

Theorem 1. *Given a pairwise comparison matrix, $A = (a_{ij})$ with $i, j = 1, \ldots, n$, the partial derivative of the Geometric Consistency Index with respect to a_{rs} is:*

$$\frac{\partial GCI}{\partial a_{rs}} = \frac{4}{(n-1)(n-2)} \frac{\log e_{rs}}{a_{rs}} \tag{2}$$

Proof. See Appendix A.

Theorem 2. *Given a pairwise comparison matrix, $A = (a_{ij})$ with $i, j = 1, \ldots, n$, the second order partial derivative of the Geometric Consistency Index with respect to a_{rs} is:*

$$\frac{\partial^2 GCI}{\partial a_{rs}^2} = \frac{4}{(n-1)(n-2)} \frac{1}{a_{rs}^2} \left(1 - \frac{2}{n} - \log e_{rs}\right) \tag{3}$$

Proof. Immediate from Theorem 1.

From Theorem 1, the judgements that further improve the GCI are those that maximise $|\frac{\log e_{rs}}{a_{rs}}|$. If $\log e_{rs} > 0$, the value a_{rs} should be decreased to reduce the GCI. Otherwise, the value of the judgement must be increased.

If we consider a judgement and its reciprocal (a_{rs} and a_{sr}), as $\log e_{rs} = -loge_{sr}$, the value $|\frac{\log e_{rs}}{a_{rs}}|$ will always be greater for the judgement that is smaller than the unit (a_{rs} or a_{sr}). In practice, taking into account the AHP comparison technique (the highest compared to the lowest), when variations are made in absolute terms, the evaluation of the previous expression ($|\frac{\log e_{rs}}{a_{rs}}|$) is suggested only for judgements $a_{rs} \geq 1$.

With the EV method (Dadkhah and Zahedi, 1993), the previous results consider absolute variations in the judgements of the matrix. However, increasing a unit in a small judgement (e.g. $a_{rs} = 2$) is not the same as in a large one (e.g. $a_{rs} = 8$). To deal with this problem, the consideration of relative changes in judgements is proposed, multiplying them by a factor $1 \pm \rho$, where ρ indicates the percentage of variation of the judgement; that is, absolute changes $\pm a_{rs}\rho$ are made.

This idea is formalised by means of the result proved by Aguarón et al. (2003); if judgement a_{rs} changes to a'_{rs}, the new value of the GCI is given by:

$$GCI(t_{rs}) = GCI_0 + \frac{2}{n(n-1)} \log^2 t_{rs} + \frac{4}{(n-1)(n-2)} \log t_{rs} \log e_{rs} \tag{4}$$

where $t_{rs} = a'_{rs}/a_{rs}$ and GCI_0 is the initial GCI of the matrix.

Theorem 3. *Given a pairwise comparison matrix, $A = (a_{ij})$ with $i, j = 1, \ldots, n$, the partial derivative of the Geometric Consistency Index with respect to the relative variation of a judgement ($t_{rs} = a'_{rs}/a_{rs}$) is:*

$$\frac{\partial GCI(t_{rs})}{\partial t_{rs}} = \frac{4}{n-1} \frac{1}{t_{rs}} \left(\frac{\log t_{rs}}{n} + \frac{\log e_{rs}}{n-2} \right) \tag{5}$$

Proof. Immediate from expression (4)

When considering small variations, the value of t_{rs} is moving around 1, so that the expression of the previous partial derivative is:

$$\left. \frac{\partial GCI(t_{rs})}{\partial t_{rs}} \right|_{t_{rs}=1} = \frac{4}{(n-1)(n-2)} \log e_{rs} \tag{6}$$

and, in contrast to absolute variations, the gradient associated with the variation of the judgement a_{rs} is given exclusively in terms of the error ($\log e_{rs}$). In the new proposal, this expression is used to select the judgements that must be considered for reducing the GCI.

Since the expression (4) is a parabolic function in $\log t_{rs}$ with a positive coefficient for $\log^2 t_{rs}$, the existence of a value that would provide the maximum reduction of the GCI is guaranteed (minimum of the parabola). This optimal value is then determined.

Corollary 1. *Given a pairwise comparison matrix, $A = (a_{ij})$ with $i, j = 1, \ldots, n$, the relative variation of judgement a_{rs} that produces the greatest decrease of the GCI is*

$$t^*_{rs} = a'_{rs}/a_{rs} = e_{rs}^{-n/(n-2)} \tag{7}$$

Proof. See Appendix A

By taking $t^*_{rs} = e_{rs}^{-n/(n-2)}$ the maximum decrease of the GCI is achieved. Substituting this value in (4), this variation is: $\Delta GCI = \frac{-2n}{(n-1)(n-2)^2} \log^2 e_{rs}$.

Remark 1. The judgement identified in (6) is the one that most rapidly decreases the value of the GCI; it is also the one that allows the greatest reduction in absolute terms.

Remark 2. The proposal for improving inconsistency measured by the GCI uses the RGM method as the prioritisation procedure. In addition to the relevance of the proposal for inconsistency measures defined in terms of the derived priority vector (particularly the RGM), it can also be used with inconsistency measures based on triads. Aguarón (2018, private document) proved that the inconsistency index (Triad Geometric Consistency Index, $TGCI$), defined for triads as

$$TGCI(A) = \frac{\sum_{i,j,k} \log^2 a_{ij} a_{jk} a_{ki}}{3n(n-1)(n-2)} \tag{8}$$

verifies that $TGCI(A) = GCI(A)$. To improve the inconsistency measured by the $TGCI$, the results described in this paper can be used. In addition, the similarities between the different measures identified by Brunelli (2018) can be used to transfer the results from this paper to other inconsistency indices, based on triads or the priority vectors. Moreover, in accordance with Saaty' suggestion for improving the inconsistency (slight modifications of the judgements), the inconsistency index used for triads ($TGCI$) considers an average of triads deviations from unity (not an extreme situation).

3. Procedure for improving the inconsistency

3.1. Algorithm

This section establishes a semi-automatic procedure for improving (reducing) the inconsistency of an AHP pairwise comparison matrix when the RGM is the prioritisation procedure and inconsistency is measured by the GCI. Based on the expressions derived from Theorem 3 and Corollary 1, the procedure has been designed to perform variations in the judgements in relative terms. Expression (6) is used to select the judgement that will be considered at each iteration and expression (7) provides the limit of the variation for this judgement. A modification beyond this value will produce an increase in the GCI.

In order to apply this semi-automatic procedure, the decision maker must previously indicate the maximum variation, in relative terms, that they would accept to modify the judgements. This value has been denominated as the *permissibility threshold* ρ. The algorithm is:

Algorithm for improving the GCI in terms of relative changes
Input: $A_{n\times n} = (a_{ij})$ a pairwise comparison matrix, ρ is the permissibility allowed in relative terms for the modification of judgements, $\overline{GCI}$ is the desired threshold for the GCI and J the set of indices corresponding to the judgements to be reviewed.
Output: The updated matrix (A'), where the new judgements a'_{rs} have been incorporated, and the associated value $GCI(A')$.

Step 0. Let $J = \{(r, s), \text{with } r < s\}$.
Step 1. Evaluate $\log e_{rs}$ for all $(r, s) \in J$.
Step 2. Choose the pair $(r', s') \in J$ for which $\log e_{r's'}$ has the largest absolute value.
Step 3. If $a_{r's'} > 1$ then let $(r, s) = (r', s')$. Otherwise, let $(r, s) = (s', r')$
Step 4. Modify a_{rs} considering the following value of the relative variation t_{rs} that will depend on the sign of $\log e_{rs}$

a. If $\log e_{rs} < 0$, use $t_{rs} = \min\left\{1+\rho, e_{rs}^{-n/(n-2)}\right\}$

b. If $\log e_{rs} > 0$, use $t_{rs} = \max\left\{\frac{1}{1+\rho}, e_{rs}^{-n/(n-2)}\right\}$

Update matrix A with new values $a'_{rs} = a_{rs}t_{rs}$ and $a'_{sr} = 1/a'_{rs}$.
Update $J = J \setminus (r', s')$.

Step 5. Calculate the new GCI. If the level of inconsistency has not reached the desired threshold ($GCI > \overline{GCI}$) and J is not empty, repeat steps 1 through 4. Otherwise, provide A' and $GCI(A')$.

If $\log e_{rs} < 0$ (Step 4a), it is necessary to increase a_{rs} ($t_{rs} \geq 1$) in order to reduce the GCI. In this case, the maximum relative increase delimited by the permissibility and the range of improvement is $t_{rs} = \min\left\{1+\rho, e_{rs}^{-n/(n-2)}\right\}$. If $\log e_{rs} > 0$ (Step 4b), the value of a_{rs} should be reduced ($t_{rs} \leq 1$). In this situation, the permissibility is incorporated as $\frac{1}{1+\rho}$ to keep the property of reciprocity.

When the judgements must belong to Saaty's fundamental scale (Saaty, 1980), the new value in Step 4 will be limited to the interval [1/9,9].

3.2. Performance of the algorithm

It can be verified that, by construction, the algorithm reduces the initial GCI, $GCI(A)$, whenever there are judgements that meet the required conditions. However, the algorithm does not guarantee the achievement of the desired threshold for the GCI ($\overline{GCI}$). Obviously, small values of permissibility will produce small modifications in the GCI.

A simulation study was undertaken, in which 100 000 matrices were generated for different combinations of n and $GCI(A)$, to determine the level of permissibility necessary to achieve the desired GCI in 99.5% (Table 1a) and 95% (Table 1b) of the situations. The desired values considered in this study were: $\overline{GCI} = 0.31$ for $n = 3$, $\overline{GCI} = 0.35$ for $n = 4$ and $\overline{GCI} = 0.37$ for $n > 4$.

Table 1: ρ values (%) that guarantee the success of the algorithm in 99.5% (a) and 95% (b) of situations

(a) 99.5% success

		$GCI(A)$				
		0.40	**0.45**	**0.50**	**0.60**	**0.75**
	3	5.0	7.3	9.6	13.9	20.0
	4	4.6	8.5	12.5	20.3	31.7
	5	3.4	7.8	12.5	22.1	37.1
n	**6**	3.6	8.4	13.3	23.6	39.6
	7	3.7	8.6	13.7	24.2	40.6
	8	3.8	8.7	13.8	24.3	40.8
	9	3.8	8.8	13.8	24.4	40.8

(b) 95% success

		$GCI(A)$				
		0.40	**0.45**	**0.50**	**0.60**	**0.75**
	3	4.9	7.3	9.5	13.8	20.0
	4	4.1	7.7	11.4	18.8	30.0
	5	3.1	7.1	11.3	19.7	32.6
n	**6**	3.3	7.6	12.1	21.3	35.3
	7	3.4	7.9	12.5	22.0	36.4
	8	3.4	8.0	12.7	22.3	37.0
	9	3.5	8.1	12.8	22.5	37.3

It can be seen that for an initial $GCI(A) = 0.40$, a $\rho = 5\%$ is sufficient to reach the acceptable $\overline{GCI}$ for $n = 3, \ldots, 9$ in 99.5% of situations; for an initial $GCI(A) = 0.40$, the necessary ρ is 8.8%. For 95% success, these values are $\rho = 4.9\%$ for $GCI(A) = 0.40$ and $\rho = 8.1\%$ for $GCI(A) = 0.45$. For the rest of initial $GCI(A)$, there are higher discrepancies in the necessary ρ for different values of n (see Table 1 to select the appropriate ρ). Note that the algorithm is able to provide acceptable levels of inconsistency if the permissibility of the decision maker is large enough.

To reach other desired thresholds of the GCI ($\overline{GCI}$), the necessary permissibility values can be calculated in a similar manner. This information may be used as an intermediate step in the construction of consistency consensus matrices in AHP-group decision making (Moreno-Jiménez et al., 2005, 2008; Escobar et al., 2015; Aguarón et al., 2016).

3.3. Variants of the algorithm

Expressions (6) and (7) establish (in relative terms) a framework for improving inconsistency measured by the GCI. The algorithm provides a realistic procedure for applying the above expressions in practical situations. However, the algorithm can be tailored to particular interests:

- To incorporate the personal intervention of the decision maker, the algorithm would not consider the permissibility value. The decision maker must provide a value of $t_{rs} \in \left[1, e_{rs}^{-n/(n-2)}\right]$ if $\log e_{rs} < 0$ (Step 4a) or a value of $t_{rs} \in \left[e_{rs}^{-n/(n-2)}, 1\right]$ if $\log e_{rs} > 0$ (Step 4b).

- The algorithm is designed to achieve an inconsistency below the desired threshold ($\overline{GCI}$), taking advantage of the improvements that allow the modification of the corresponding judgement. If the decision maker wants to exactly reach this desired threshold, it would be enough to solve the second degree equation set out in (4) once, for the first time in the algorithm, the modification of a judgement has been allowed to reach a $GCI(A') \leq \overline{GCI}$.

- The algorithm can be relaxed by eliminating permissibility ($\rho = \infty$) or allowing several modifications for each judgement (not updating J in Step 4). These variations allow the reduction of the number of iterations, smaller thresholds for the GCI or the minimisation of the number of modified judgements in order to achieve an acceptable inconsistency, as proposed by Bozóki et al. (2015). However, these actions could lead to changes in the judgements of remarkable intensity (Khatwani and Kar, 2017). In many situations, these changes provide values for judgements outside the priority stability intervals (Aguarón and Moreno-Jiménez, 2000) and cause important rank reversals.

- If the variations of the judgements are considered in absolute terms, as in Dadkhah and Zahedi (1993), the procedure can be adapted using expression (2) from Theorem 1 to select the judgements and then adjusting the remaining steps. If the procedure is to be semi-automatically applied, the decision-maker must provide, in advance, the maximum variation they would accept to modify the judgements in absolute terms.

Irrespective of the importance of the previous variants, it is worth mentioning that the proposed algorithm has been designed to achieve an acceptable inconsistency with slight variations of the judgements, as recommended by Saaty (2003).

4. Numerical Example

The procedure is illustrated with the same example used by Dadkhah and Zahedi (1993). The pairwise comparison matrix and the corresponding priority vector obtained with the RGM method are:

$$A = \begin{pmatrix} 1 & 5 & 6 & 7 \\ 1/5 & 1 & 4 & 6 \\ 1/6 & 1/4 & 1 & 4 \\ 1/7 & 1/6 & 1/4 & 1 \end{pmatrix} \quad w = (0.614, 0.239, 0.103, 0.045)$$

The associated value of the GCI is 0.504 which is greater than the threshold $\overline{GCI} = 0.35$ (for $n = 4$), so the matrix exceeds the tolerable values of inconsistency. The procedure is then applied with the aim of improving inconsistency, reducing the GCI to a value below 0.35.

In this illustrative example, it is assumed that the decision maker has established a *permissibility threshold* $\rho = 15\%$ (they would accept the modification of some judgements up to a 15% of their initial values), and, although it is not necessary in this example, the judgements belong to the interval $[1/9, 9]$.

Step 1. The matrices of the errors $E = (e_{ij})$ and of their corresponding logarithms $\log E = (\log e_{ij})$ are:

$$E = \begin{pmatrix} 1 & 1.944 & 1.007 & 0.511 \\ 0.514 & 1 & 1.727 & 1.126 \\ 0.993 & 0.579 & 1 & 1.739 \\ 1.958 & 0.888 & 0.575 & 1 \end{pmatrix} \log E = \begin{pmatrix} 0 & 0.665 & 0.007 & -0.672 \\ -0.665 & 0 & 0.546 & 0.119 \\ -0.007 & -0.546 & 0 & 0.553 \\ 0.672 & -0.119 & -0.553 & 0 \end{pmatrix}$$

Step 2. The maximum value of $\log e_{ij}$ in absolute terms is 0.672 which corresponds to the judgements (1,4) and (4,1) (as reciprocals, they correspond to the same judgement). These are the judgements that would most rapidly decrease the value of the GCI.

Step 3. The algorithm continues with element (1,4) as this is the judgement with a value greater than 1 ($a_{14} = 7 > 1$).

Step 4. Expression 4a is used, since $\log e_{14} = -0.672 < 0$. The value of a_{14} is modified by $t_{14} = \min\left\{1 + \rho, e_{14}^{-n/(n-2)}\right\} = \min\left\{1.15, 0.511^{-2}\right\} = 1.15$. The updated values of the judgements are $a'_{14} = a_{14}t_{14} = 7 \times 1.15 = 8.05$ and $a'_{41} = 1/8.05 = 0.124$.

Step 5. The GCI of the updated matrix is $GCI = 0.445$ which is still above the desired inconsistency threshold. As there are judgements that have not yet been considered, steps 1 to 4 of the procedure are repeated.

It was necessary to perform two more iterations of the algorithm to achieve a tolerable level of inconsistency (GCI below 0.35). Table 2 summarises the three iterations of the procedure. In the third iteration, the judgement that maximises $|\log e_{rs}|$ and, therefore, would most decrease inconsistency, is again $(1, 4)$, but the procedure does not select it since it has already been considered and modified in a previous iteration. The procedure continues with the following judgement that maximises $|\log e_{rs}|$ and has not been previously considered, $(3, 4)$. In this example, the modifications of the judgements in all the iterations have been determined by the permissibility (15%) and not by the value that provides the maximum possible reduction of inconsistency.

Table 2: Information on the 3 iterations of the procedure for $\rho = 15\%$

Iter#	GCI	(r, s)	a_{rs}	$\log a_{rs}$	$\uparrow\downarrow a_{rs}$	ρ limit	t^*_{rs}	t_{rs}	a'_{rs}	GCI'	$\nabla GCI(\%)$
1	0.504	(1,4)	7	-0.672	$\uparrow$	1.15	3.83	1.15	8.05	0.445	11.78
2	0.445	(1,2)	5	0.630	$\downarrow$	0.87	0.28	0.87	4.35	0.389	22.77
3	0.389	(3,4)	4	0.518	$\downarrow$	0.87	0.35	0.87	3.48	0.344	31.71

The final pairwise comparison matrix, A', its GCI, and the corresponding priority vector, w', are:

$$A' = \begin{pmatrix} 1 & \mathbf{4.35} & 6 & \mathbf{8.05} \\ \mathbf{0.23} & 1 & 4 & 6 \\ 1/6 & 1/4 & 1 & \mathbf{3.48} \\ \mathbf{0.12} & 1/6 & \mathbf{0.29} & 1 \end{pmatrix} \quad GCI(A') = 0.344 \quad w' = (0.611, 0.246, 0.099, 0.045)$$

The algorithm guarantees that the modifications made in some judgements to achieve an acceptable inconsistency ($GCI < 0.35$) do not exceed 15%. With just these small changes a significant improvement of inconsistency (a 31.71% reduction) is achieved in three iterations. In addition, the changes in the derived priorities are very small (see Table 5).

According to the simulation analysis performed, for $n = 4$ and a $GCI(A) = 0.50$, a value of ρ of 13% would be necessary (Table 1a) to guarantee the success of the algorithm with 99.5% probability. For the same GCI (0.50) and $n = 4$, $\rho = 12\%$ is enough to guarantee the success of the algorithm at 95% (Table 1b). For $n = 4$, and an initial GCI less than double what is allowed ($GCI = 0.75$), $\rho = 30\%$ is enough to achieve the success of the algorithm in 95% of cases.

Table 3 includes the results of the procedure (values of the judgements, the GCI, the reduction of GCI, the priorities obtained with the procedure and the number of iterations necessary to achieve an acceptable GCI) for three different values of the permissibility: $\rho = 10\%, 15\%$ and 20%. It can bee seen that the greater the permissibility, the greater the variation in the judgements, the smaller the number of modified judgements and the value of the GCI. It can also be observed that the priority vectors are close to the initial one (see Table 5 for details).

In the particular case of $\rho = 10\%$, although all six judgements were modified (only allowed once), it was not possible to achieve a matrix with acceptable inconsistency ($GCI < 0.35$). A permissibility of 11% would have been necessary to reach an acceptable level of inconsistency ($GCI = 0.349$) in 6 iterations. If repetition of judgements were allowed, a $GCI = 0.320$ would be reached in 5 iterations; two of them modifying judgement a_{12} with a cumulative relative change of 21%, and three modifying a_{14} with a cumulative relative change of 33.1%, both values are far from the initial ones.

Table 3: Results of the procedure for $\rho = 10\%, 15\%, 20\%$

	a_{12}	a_{13}	a_{14}	a_{23}	a_{24}	a_{34}	GCI	$\nabla GCI\%$	w_1	w_2	w_3	w_4	
Init. values	5.00	6.00	7.00	4.00	6.00	4.00	0.504	–	0.614	0.239	0.103	0.045	Iter.
$\rho = 10\%$	**4.55**	**5.92**	**7.70**	**3.64**	**5.45**	**3.64**	0.355	29.51	0.615	0.235	0.104	0.046	6
$\rho = 15\%$	**4.35**	6.00	**8.05**	4.00	6.00	**3.48**	0.344	31.71	0.611	0.246	0.099	0.045	3
$\rho = 20\%$	**4.17**	6.00	**8.40**	4.00	6.00	**3.33**	0.302	40.08	0.610	0.248	0.098	0.045	3

In the case of $\rho = 15\%$ (see Table 4), the changes in absolute terms between judgements a_{ij} and a'_{ij} are less than 1.05. On average, the change is 0.342 for each judgement. In relative terms, the maximum change in the judgements is 15% ($\rho = 15\%$) and the average is 6.12%. For priorities (see Table 5), the changes in absolute terms between w_i and w'_i are less than 0.007. On average, this change is 0.004 for each priority. In relative terms, the maximum change in the priorities is less than 4% (3.91%) and the average is less than 2% (1.98%).

In the case of continuing to apply the algorithm to the rest of the judgements (six iterations), an inconsistency of 0.295 would have been achieved, reducing the initial GCI by 41.43%. After the six iterations, the maximum and average relative differences for the judgements are 15% and 11.43%. For the priorities these values are 4.52% and 2.20%.

Tables 4 and 5 show the differences (maximum and average differences in absolute and relative terms) of the judgements and priorities with respect to the initial position for different levels of permissibility ($\rho = 10\%, 15\%$ and 20%).

Table 4: Differences between initial and final judgements for $\rho = 10\%, 15\%, 20\%$

	Max. Abs. Dif.	Avg. Abs. Dif.	Max. Rel. Dif. (%)	Avg. Rel. Dif. (%)
$\rho = 0.10$	0.700	0.419	10.00	7.96
$\rho = 0.15$	1.050	0.342	15.00	6.12
$\rho = 0.20$	1.400	0.483	20.00	8.89

Table 5: Differences between initial and final priority vectors for $\rho = 10\%, 15\%, 20\%$

	Max. Abs. Dif.	Avg. Abs. Dif.	Max. Rel. Dif. (%)	Avg. Rel. Dif. (%)
$\rho = 0.10$	0.004	0.002	3.06	1.52
$\rho = 0.15$	0.007	0.004	3.91	1.98
$\rho = 0.20$	0.010	0.005	5.08	2.59

where:

$$\text{Max. Abs. Dif.} = \max_i \left| x'_i - x_i \right| \qquad \text{Avg. Abs. Dif.} = \frac{1}{n} \sum_i \left| x'_i - x_i \right|$$

$$\text{Max. Rel. Dif.} = \max_i \left| \frac{x'_i - x_i}{x_i} \right| * 100 \qquad \text{Avg. Rel. Dif.} = \frac{1}{n} \sum_i \left| \frac{x'_i - x_i}{x_i} \right| * 100$$

With this example, Dadkhah and Zahedi (1993) obtained almost null inconsistency after ten modifications in judgements corresponding to five different entries. If the permissibility restriction is eliminated and judgements in the interval $[1/9, 9]$ are considered, our algorithm provides $GCI = 0$ in only three iterations by modifying the judgements $a'_{14} = 9$; $a'_{12} = 1.5$ and $a'_{34} = 1.5$.

The removal of the permissibility restriction in the algorithm would be in line with Bozóki et al. (2015), who suggested reducing inconsistency by modifying the minimum number of judgements. However, this does not follow Saaty's proposal (Saaty, 2003) as the modified judgements and priorities are clearly different from the initial values.

5. Conclusions

The two most common prioritisation methods in AHP are the EV and the RGM. Both have inconsistency measures (CR and GCI, respectively) based on the priority vector, but only the CR includes a procedure for improving the inconsistency. To solve this limitation in the RGM method, this paper has proposed a procedure for the improvement of inconsistency measured with the GCI. The proposal is similar to that put forward by Dadkhah and Zahedi (1993) for the CR.

Following Saaty's recommendation: *"to improve the validity of the priority vector, we must transform a given reciprocal judgement matrix to a near consistent matrix"*, and the procedure he suggested for improving inconsistency (Saaty, 2003): *"slightly modify judgements and the initial priorities"*, the proposal for the RGM method slightly modifies the judgements that improve the GCI faster and with greater intensity. The modification of judgements is always within the permissibility range set by the decision maker. It is understood, as Saaty argued, that the modification of the initial judgements must be made with the acceptance of the decision maker, and it must not be made automatically.

The proposal establishes a general framework for identifying the judgements that, in relative terms, should be modified for improving the GCI and determining the direction and intensity of the modification. A simulation study has been utilised to analyse the performance of the algorithm. The analysis provides the level of permissibility necessary to achieve the desired GCI in 95% and 99.5% of the situations for different values of n and GCI. It can be seen that the algorithm is able to provide acceptable levels of inconsistency if the permissibility of the decision maker is large enough. For example, for a $GCI = 0.60$, a 28% permissibility would be necessary to reach the acceptable inconsistency threshold with a probability of 99.5%.

The general framework is valid for the measurement scales that most commonly feature in the literature and can easily accommodate the interests and needs of decision makers. It is enough to adapt the framework (only the desired or maximum inconsistency threshold allowed and the permissibility associated with the relative changes of the judgements have been included as the initial parameters of the model) to the requirements of the decision maker. For example, if the decision maker wants to limit the modified values to an interval or select the judgements that provide a fixed level of the GCI with the minimum modification, this should be indicated in the Step 4 of the algorithm. If they want to improve inconsistency by modifying the lower number of judgements, permissibility should be eliminated. If the decision maker wants to work in absolute terms, then the procedure should refer to Theorems 1 and 2. These and other variants, including the extension of the results to fuzzy multiplicative preferences (Xu et al., 2019) and the improvement of inconsistency for its exploitation in group decision (Moreno-Jiménez et al., 2005, 2008; Escobar et al., 2015; Aguarón et al., 2016, 2019) will be the subject of a forthcoming paper.

In addition to the relevance of offering a procedure for improving inconsistency measured with the GCI (an issue that was absent from the scientific literature) and its adaptability to specific situations, the proposal put forward in this paper makes two specific contributions to the field: i) the judgement identified for improving the GCI is the one that most rapidly decreases the value of this index and allows the greatest reduction in absolute terms; ii) the results derived from Theorems 1 to 3 and Corollary 1 can be applied to inconsistency indices based on triads, in particular to the $TGCI$.

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Appendix A. Proofs of theorems

As a previous step to the proof of Theorem 1, Lemma 1 is included.

Lemma 1. *Given a pairwise comparison matrix, $A = (a_{ij})$ with $i, j = 1, \ldots, n$, the derivatives of the errors $e_{ij} = a_{ij}\omega_j/\omega_i$, where ω is the priority vector obtained with the RGM method are given by*

$$
\begin{aligned}
\frac{\partial e_{rs}}{\partial a_{rs}} &= \left(1 - \frac{2}{n}\right)\frac{e_{rs}}{a_{rs}} & \qquad \frac{\partial e_{sr}}{\partial a_{rs}} &= -\left(1 - \frac{2}{n}\right)\frac{e_{sr}}{a_{rs}} \\
\frac{\partial e_{rj}}{\partial a_{rs}} &= -\frac{1}{n}\frac{e_{rj}}{a_{rs}} \quad j \neq s & \qquad \frac{\partial e_{sj}}{\partial a_{rs}} &= \frac{1}{n}\frac{e_{sj}}{a_{rs}} \quad j \neq r \\
\frac{\partial e_{is}}{\partial a_{rs}} &= -\frac{1}{n}\frac{e_{is}}{a_{rs}} \quad i \neq r & \qquad \frac{\partial e_{ir}}{\partial a_{rs}} &= \frac{1}{n}\frac{e_{ir}}{a_{rs}} \quad i \neq s
\end{aligned}
$$

Proof. For error e_{rs} we have:

$$
e_{rs} = a_{rs}\frac{\omega_s}{\omega_r} = a_{rs}\left(\frac{a_{s1}\cdots a_{sr}\cdots a_{ss}\cdots a_{sn}}{a_{r1}\cdots a_{rr}\cdots a_{rs}\cdots a_{rn}}\right)^{1/n} = a_{rs}^{1-2/n}\left(\frac{a_{s1}\cdots \cancel{a}_{sr}\cdots a_{ss}\cdots a_{sn}}{a_{r1}\cdots a_{rr}\cdots \cancel{a}_{rs}\cdots a_{rn}}\right)^{1/n}
$$

And taking the derivative

$$
\frac{\partial e_{rs}}{\partial a_{rs}} = \left(1 - \frac{2}{n}\right)a_{rs}^{-2/n}\left(\frac{a_{s1}\cdots \cancel{a}_{sr}\cdots a_{ss}\cdots a_{sn}}{a_{r1}\cdots a_{rr}\cdots \cancel{a}_{rs}\cdots a_{rn}}\right)^{1/n} = \left(1 - \frac{2}{n}\right)\frac{\omega_s}{\omega_r} = \left(1 - \frac{2}{n}\right)\frac{e_{rs}}{a_{rs}}
$$

For error e_{sr} we use the relation $e_{sr} = 1/e_{rs}$:

$$
\frac{\partial e_{sr}}{\partial a_{rs}} = \frac{\partial e_{sr}}{\partial e_{rs}}\frac{\partial e_{rs}}{\partial a_{rs}} = \frac{-1}{e_{rs}^2}\left(1 - \frac{2}{n}\right)\frac{e_{rs}}{a_{rs}} = -\left(1 - \frac{2}{n}\right)\frac{e_{sr}}{a_{rs}}
$$

The term e_{rj} with $j \neq s$ can be expressed as:

$$
e_{rj} = a_{rj}\frac{\omega_j}{\omega_r} = a_{rj}\frac{\omega_j}{(a_{r1}\cdots a_{rs}\cdots a_{rn})^{1/n}} = a_{rj}a_{rs}^{-1/n}\frac{\omega_j}{\prod_{k\neq s} a_{rk}^{1/n}}
$$

And taking the derivative we have

$$
\frac{\partial e_{rj}}{\partial a_{rs}} = -\frac{1}{n}a_{rj}a_{rs}^{-1-1/n}\frac{\omega_j}{\prod_{k\neq s} a_{rk}^{1/n}} = -\frac{1}{n}\frac{a_{rj}}{a_{rs}}\frac{\omega_j}{\prod_k a_{rk}^{1/n}} = -\frac{1}{n}\frac{a_{rj}}{a_{rs}}\frac{\omega_j}{\omega_r} = -\frac{1}{n}\frac{e_{rj}}{a_{rs}}
$$

The rest of derivatives can be demonstrated analogously ■

Proof of Theorem 1 The *GCI* can be expressed as:

$$
GCI = \frac{1}{(n-1)(n-2)}\sum_{i\neq j}\log^2 e_{ij} \tag{A.1}
$$

The only terms of GCI (A.1) that depend on a_{rs} are those that are in rows r, s or columns r, s. Also, $\log^2 e_{ij} = \log^2 e_{ji}$ so we only consider two-time errors e_{rs}, e_{rj} with $j \neq s$ and e_{sj} with $j \neq r$:

$$\begin{aligned}
\frac{\partial GCI}{\partial a_{rs}} &= \frac{2}{(n-1)(n-2)} \frac{\partial}{\partial a_{rs}} \left(\log^2 e_{rs} + \sum_{j \neq r,s} \log^2 e_{rj} + \sum_{j \neq r,s} \log^2 e_{sj} \right) \\
&= \frac{2}{(n-1)(n-2)} \left[2 \log e_{rs} \frac{1}{e_{rs}} \left(1 - \frac{2}{n}\right) \frac{e_{rs}}{a_{rs}} + 2 \sum_{j \neq r,s} \log e_{rj} \frac{1}{e_{rj}} \frac{-1}{n} \frac{e_{rj}}{a_{rs}} \right. \\
&\quad \left. + 2 \sum_{j \neq r,s} \log e_{sj} \frac{1}{e_{sj}} \frac{1}{n} \frac{e_{sj}}{a_{rs}} \right] \\
&= \frac{4}{(n-1)(n-2)} \frac{1}{a_{rs}} \left[\left(1 - \frac{2}{n}\right) \log e_{rs} - \frac{1}{n} \sum_{j \neq r,s} \log e_{rj} + \frac{1}{n} \sum_{j \neq r,s} \log e_{sj} \right] \qquad \text{(A.2)}
\end{aligned}$$

It's known that $\sum_{j \neq i} \log e_{ij} = 0$ so

$$\sum_{j \neq r,s} \log e_{rj} = -\log e_{rs} \qquad \text{and} \qquad \sum_{j \neq r,s} \log e_{sj} = -\log e_{sr} = \log e_{rs} \qquad \text{(A.3)}$$

Substituting expressions (A.3) in (A.2):

$$\begin{aligned}
\frac{\partial GCI}{\partial a_{rs}} &= \frac{4}{(n-1)(n-2)} \frac{1}{a_{rs}} \left[\left(1 - \frac{2}{n}\right) \log e_{rs} + \frac{1}{n} \log e_{rs} + \frac{1}{n} \log e_{rs} \right] = \\
&= \frac{4}{(n-1)(n-2)} \frac{\log e_{rs}}{a_{rs}} \qquad \blacksquare
\end{aligned}$$

Proof of Corollary 1 From expression (5) we have:

$$\frac{\partial \Delta GCI}{\partial t_{rs}} = 0 \Rightarrow \left(\frac{\log t_{rs}}{n} + \frac{\log e_{rs}}{n-2} \right) = 0 \Rightarrow t^*_{rs} = e_{rs}^{-n/(n-2)}$$

This value (t^*_{rs}) corresponds to a minimum because the second derivative is positive since

$$\frac{\partial^2 GCI}{\partial t^2_{rs}} = \frac{4}{n(n-1)} \frac{1}{t^2_{rs}} \left[1 - \log t_{rs} - \frac{n}{n-2} \log e_{rs} \right]$$

and for $t^*_{rs} = e_{rs}^{-n/(n-2)}$ we have:

$$\left. \frac{\partial^2 GCI}{\partial t^2_{rs}} \right|_{t^*_{rs}} = \frac{4}{n(n-1)} e_{rs}^{2n/(n-2)} > 0 \qquad \blacksquare$$

This corollary can also be proved with Theorems 1 and 2.