

New monotone measure-based integrals inspired by scientific impact problem

Michał Boczek^{a,*}, Anton Hovana^b, Ondrej Hutník^b, Marek Kaluszka^{a,†}

^a*Institute of Mathematics, Lodz University of Technology, 90-924 Lodz, Poland*

^b*Institute of Mathematics, Pavol Jozef Šafárik University in Košice, 040-01 Košice, Slovakia*

Abstract

In this paper, we define new functionals generalizing scientometric indices proposed by Mesiar and Gągolewski in 2016 to overcome some limitations of h -index. These functionals are integrals with respect to a monotone measure as well as aggregation functions under some mild conditions. We derive numerous properties of the new integrals and analyze subadditivity property in detail. We also give a partial solution to the problem posed by Mesiar and Stupňanová to find an algorithm for computing the pseudo-decomposition integral of n -th order based on operations $\oplus = +$ and $\odot = \wedge$, which will be useful in multi-criteria decision problems.

Keywords: scientometrics; Multiple criteria analysis; h -index; Aggregation function; Pseudo-decomposition integral.

1 Introduction

In order to compare the efficiency of work of two researchers, one must construct a rule that is the most objective and fair. It turns out that the task is very difficult. Currently, there are many scientometric indices known in the literature. Their calculations are based on two inputs: number of publications and number of citations of each publication (measuring the quality and importance of publications). Nowadays, the most popular scientometric index is h -index introduced in 2005 by Hirsch [23]. It is implemented in the largest scientific databases such as Scopus, or WoS. An axiomatic approach explaining the nature of h -index can be found in papers [8, 32, 33, 42], whereas its mathematical properties can be found in [17]. Torra and Narukawa [39] proved that h -index is the Sugeno integral

*Corresponding author

[†]*E-mail addresses:* michal.boczek.1@p.lodz.pl, ondrej.hutnik@upjs.sk, anton.hovana@student.upjs.sk,
kaluszka@p.lodz.pl

with respect to the counting measure. And because of specificity of the Sugeno integral, h -index has some drawbacks, see [31]. For example, let's consider two researchers possessing the same number of papers, say n , but each paper of the first one is cited n -times, and each paper of the second one is cited exactly $3n$ -times. Using the criterion of importance of author's publication and the number of quotations for each paper, one can see that the second researcher should have a higher scientometric index if both authors work in the same domain and have similar research experience measured by the years of work. However, h -index of both authors is the same and is equal to n . To overcome the above limitations, Mesiar and Gagolewski [28] have proposed two new indices based on the idea of h -index (for more details, see Section 5).

The present paper introduces and studies properties of two functionals: upper n -Sugeno integral and lower n -Sugeno integral of a measurable function with respect to a monotone measure. Both functionals are integrals in the spirit of definition introduced here. Integrals have many applications, especially in multicriteria decision theory, economy, optimalization or data classification [5, 10, 12, 14, 22, 23, 38, 39, 44]. To the best of our knowledge, there are only very few papers which describe the connection between integrals and scientometric indices, see [21, 39]. From those papers it follows that total number of citations is related to Choquet integral, h -index and Kosmulski $h(2)$ -index are related to Sugeno integral, Kosmulski MAXPROD index is related to Shilkret integral, etc. In this paper we further show that the upper and lower n -Sugeno integrals generalize scientometric indices introduced by Mesiar & Gagolewski and others, e.g. generalized Kosmulski index [13], iterated h -index [18], H_α and H^β indices [24].

The second main result of the work is a relation between the lower n -Sugeno integral and some special pseudo-decomposition integral introduced by Mesiar and Stupňanová in [30] (see Theorem 4.10 below). Their question from [30, Conclusion] motivated us to describe an algorithm for computation of the pseudo-decomposition integral.

Our paper is organized as follows. In Section 2 we provide basic notations and definitions we work with. In Sections 3 and 4 we introduce new concepts of upper and lower n -Sugeno integral, examine their basic properties and provide their equivalent forms. Section 5 includes applications of the obtained results mainly to aggregation and scientometrics. For better readability we postpone some technical proofs of our statements to Appendix.

2 Basic notations and preliminaries

Let (X, \mathcal{A}) be a measurable space, where \mathcal{A} is a σ -algebra of subsets of a non-empty set X . The class of all measurable functions $f: X \rightarrow Y$, where $Y = [0, \bar{y}]$ for $0 < \bar{y} \leq \infty$, is denoted by $\mathcal{F}_{(X,Y)}$. Usually,

we take $\bar{y} = 1$ or $\bar{y} = \infty$. A *monotone measure* on \mathcal{A} is a nondecreasing set function $\mu: \mathcal{A} \rightarrow [0, \infty]$, i.e., $\mu(A) \leq \mu(B)$ whenever $A \subset B$ with $\mu(\emptyset) = 0$ and $\mu(X) > 0$. The range of μ we write as $\mu(\mathcal{A})$. We denote by $\mathcal{M}_{(X, \mathcal{A})}$ the class of all monotone measures on (X, \mathcal{A}) . Given $f, g \in \mathcal{F}_{(X, Y)}$ and $\mu \in \mathcal{M}_{(X, \mathcal{A})}$, we say that g *dominates* f with respect to μ and write $f \leq_{\mu} g$ if $\mu(\{f \geq t\}) \leq \mu(\{g \geq t\})$ for all t , where $\{f \geq t\} = \{x \in X: f(x) \geq t\}$. Hereafter, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$. We say that a function $\circ: Y_1 \times Y_2 \rightarrow [0, \infty]$ is *nondecreasing* if $a_1 \circ a_2 \leq b_1 \circ b_2$ whenever $a_i \leq b_i$, where $a_i, b_i \in Y_i \subset [0, \infty]$ for $i = 1, 2$.

Sugeno integral of $f \in \mathcal{F}_{(X, Y)}$ with respect to $\mu \in \mathcal{M}_{(X, \mathcal{A})}$ [37, 41] is defined by

$$\text{Su}(\mu, f) := \sup_{t \in Y} \{t \wedge \mu(\{f \geq t\})\}. \quad (1)$$

To this day, many researchers introduced numerous generalizations of the Sugeno integral like generalized upper Sugeno integral, pseudo-decomposition integral or q-integral for $\bar{y} = \mu(X) = 1$, and studied their properties [6, 7, 14, 25, 30, 36].

To make our paper as self-contained as it gets, we give some properties of the Sugeno integral that we follow later. Hereafter, $c \searrow a$ and $c \nearrow a$ means that $c \rightarrow a$ for $c > a$ and $c < a$, respectively.

Lemma 2.1. *Let $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$. The Sugeno integral possesses the following properties:*

- (a) $\text{Su}(\mu, f) \in Y$,
- (b) $t > \mu(\{f \geq t\})$ for $t > \text{Su}(\mu, f)$ and $t < \mu(\{f \geq t\})$ for $t < \text{Su}(\mu, f)$,
- (c) $\text{Su}(\mu, f) = \lim_{t \nearrow \text{Su}(\mu, f)} (t \wedge \mu(\{f \geq t\}))$ if $\text{Su}(\mu, f) > 0$,
- (d) $\text{Su}(\mu, f) = \lim_{t \searrow \text{Su}(\mu, f)} (t \vee \mu(\{f > t\}))$ if $\text{Su}(\mu, f) < \bar{y}$,
- (e) $\text{Su}(\mu, f) = 0$ if and only if $\mu(\{f \geq t\}) = 0$ for all $t > 0$.

Proof. Properties (a) and (b) follow from (1), since $Y = [0, \bar{y}]$ (see also [41, Lemma 9.7]). Properties (c) and (d) follow from (b) as

$$\begin{aligned} \text{Su}(\mu, f) &= \lim_{t \nearrow \text{Su}(\mu, f)} t = \lim_{t \nearrow \text{Su}(\mu, f)} (t \wedge \mu(\{f \geq t\})), \\ \text{Su}(\mu, f) &= \lim_{t \searrow \text{Su}(\mu, f)} t = \lim_{t \searrow \text{Su}(\mu, f)} (t \vee \mu(\{f > t\})). \end{aligned}$$

To prove (e), by the definition of the Sugeno integral we have that

$$0 = \sup_{t \geq 0} \{t \wedge \mu(\{f \geq t\})\} = (0 \wedge \mu(X)) \vee \sup_{t > 0} \{t \wedge \mu(\{f \geq t\})\}.$$

Now it is evident that $\mu(\{f \geq t\}) = 0$ for all $t > 0$. □

We formulate properties which any integral should possess.

Definition 2.2. A functional $J: \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)} \rightarrow [0, \infty]$ is called an *integral* if

- (C₁) $J(\mu, f) \leq J(\mu, g)$ whenever $f \leq_\mu g$,
(C₂) $J(\mu, f) \leq J(\nu, f)$ whenever $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{A}$,
(C₃) $J(\mu, a\mathbb{1}_A) = s(a, \mu(A))$ for all $a \in Y$ and $A \in \mathcal{A}$, where $s: Y \times [0, \infty] \rightarrow Y$ is a nondecreasing function such that $s(a, 0) = s(0, b) = 0$ for any a, b .

For a fixed $\mu \in \mathcal{M}_{(X, \mathcal{A})}$, the property (C₃) is known in the literature as μ -generated property of the integral J , see [26, Definition 3.3]. Restriction to the class of monotone measures with $\mu(X) = 1$ and $Y = [0, 1]$ in Definition 2.2 is closely related to fuzzy integral introduced by Struk [35]. In fact, the condition (C₃) with $s(a, 1) = a = s(1, a)$ for any $a \in [0, 1]$ implies the conditions (2) and (3) in [35, Definition 1]. However, the assumption (1) from [35, Definition 1] is stronger than (C₁). Examples of integrals of nonnegative functions with respect to monotone measures include the Choquet integral [11], Sugeno integral or generalized upper Sugeno integral (see formula (2) in [5]) under some additional restrictions.

3 Upper n -Sugeno integral

In this section, we introduce a new type of integral with respect to a monotone measure. Our motivation for doing so comes from the lower 2- h -index defined by Mesiar and Gagolewski (see (19)).

We say that $\circ: Y \times Y \rightarrow Y$ is an *admissible fusion map* if it is nondecreasing and $0 \circ a \leq a$ for all $a \in Y$. The most important examples for $Y = [0, \infty]$ are: the standard addition, pseudo-addition [3], the standard product, minimum, maximum or means [2]. Moreover, for $Y = [0, 1]$ the examples are: boolean conjunctions such as semicopulas [1, 5, 15], copulas [16], t -norms, conjunctive aggregations [2] and fuzzy conjunctions [14], and other binary operations like uninorms, t -semiconorms or averaging aggregations [2].

Definition 3.1. Let $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and \circ be an admissible fusion map. For $n \geq 1$ the *upper n -Sugeno integral* is defined using the recurrence

$$\text{Su}_{n+1}^\circ(\mu, f) := \sup_{t \in Y} \{ (t \circ \text{Su}_n^\circ(\mu, f)) \wedge \mu(\{f \geq t\}) \}$$

with the initial condition $\text{Su}_1^\circ(\mu, f) := \text{Su}(\mu, f)$.

Lemma 2.1 (a) yields $\text{Su}_1^\circ(\mu, f) \in Y$. The induction implies that $\text{Su}_n^\circ(\mu, f) \in Y$ for all $n \geq 2$, so the functional $\text{Su}_n^\circ(\mu, f)$ in Definition 3.1 is well-defined. We show that the upper n -Sugeno integral is an integral in the sense of Definition 2.2.

Proposition 3.2. Let $n \geq 1$, $f, g \in \mathcal{F}_{(X, Y)}$ and $\mu, \nu \in \mathcal{M}_{(X, \mathcal{A})}$. Then

- (a) $\text{Su}_n^\circ(\mu, f) \leq \text{Su}_n^\circ(\mu, g)$ whenever $f \leq_\mu g$.

(b) $\text{Su}_n^\circ(\mu, f) \leq \text{Su}_n^\circ(\nu, f)$ whenever $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{A}$.

(c) $\text{Su}_n^\circ(\mu, a\mathbb{1}_A) = s_n(a, \mu(A))$ for $a \in Y$ and $A \in \mathcal{A}$, where $s_n: Y \times [0, \infty] \rightarrow Y$ is a nondecreasing function such that $s_n(a, 0) = s_n(0, b) = 0$ and $s_{n+1}(a, b) = (a \circ s_n(a, b)) \wedge b$ for all a, b .

Proof. Properties (a) and (b) follow immediately from Definition 3.1 and monotonicity of \circ . We shall prove by induction that (c) holds. Clearly, $\text{Su}(\mu, a\mathbb{1}_A) = a \wedge b$ with $b = \mu(A)$. If (c) is true for some $n \geq 1$, by monotonicity of function $t \mapsto t \circ b$ we have

$$\begin{aligned} \text{Su}_{n+1}^\circ(\mu, a\mathbb{1}_A) &= [(0 \circ \text{Su}_n^\circ(\mu, a\mathbb{1}_A)) \wedge \mu(X)] \vee \sup_{t \in (0, \bar{y}]} \{(t \circ \text{Su}_n^\circ(\mu, a\mathbb{1}_A)) \wedge \mu(\{a\mathbb{1}_A \geq t\})\} \\ &= (0 \circ s_n(a, b)) \vee [(a \circ s_n(a, b)) \wedge b] \\ &= (a \circ s_n(a, b)) \wedge b = s_{n+1}(a, b), \end{aligned}$$

as $0 \circ s_n(a, b) \leq s_n(a, b) = \text{Su}_n^\circ(\mu, a\mathbb{1}_A) \leq b \leq \mu(X)$ and $0 \circ s_n(a, b) \leq a \circ s_n(a, b)$. The induction hypothesis implies that s_{n+1} is nondecreasing and $s_{n+1}(a, 0) = s_{n+1}(0, b) = 0$, as desired. \square

Now, we provide other properties of the upper n -Sugeno integral. From now on, to shorten the notation, we write $\text{Su}_n^\circ(f)$ and $\text{Su}(f)$ instead of $\text{Su}_n^\circ(\mu, f)$ and $\text{Su}(\mu, f)$, respectively if there is no ambiguity.

Proposition 3.3. (a) Let $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$. If $\text{Su}(f) = 0$, then $\text{Su}_n^\circ(f) = 0$ for all n .

Moreover, if $\text{Su}_k^\circ(f) = 0$ for some $k > 1$ and $a \circ b > 0$ for all $a, b > 0$, then $\text{Su}_n^\circ(f) = 0$ for any n .

(b) $\text{Su}_n^\circ(af) \leq a \text{Su}_n^\circ(f)$ for some $a > 1$ and for all $(\mu, af) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and $n \geq 1$ provided that $(ax) \circ (ay) \leq a(x \circ y)$ for all $ax, ay \in Y$. Moreover, $\text{Su}_n^\circ(af) \geq a \text{Su}_n^\circ(f)$ for some $a \in (0, 1)$ and all $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ whenever $(ax) \circ (ay) \geq a(x \circ y)$ for all $x, y \in Y$.

(c) $\mu(A) = \text{Su}_n^\circ(\bar{y}\mathbb{1}_A)$ for any $A \in \mathcal{A}$ and $n \geq 1$ whenever $\mu(X) \leq \bar{y}$ and $\bar{y} \circ b \geq b$ for all $b \in Y$.

(d) (Idempotency) $\text{Su}_n^\circ(a\mathbb{1}_X) = a$ for all $a \in Y$ and $n \geq 1$ if and only if $\mu(X) \geq \bar{y}$ and $a \circ a = a$ for any $a \in Y$.

Proof. (a) If $\text{Su}(f) = 0$, then by Lemma 2.1 (e) we have $\mu(\{f \geq t\}) = 0$ for all $t > 0$. Hence

$$\text{Su}_2^\circ(f) = [(0 \circ 0) \wedge \mu(X)] \vee \sup_{t \in (0, \bar{y}]} \{(t \circ 0) \wedge 0\} = 0,$$

as $0 \circ 0 = 0$. Applying the induction, we will prove that $\text{Su}_n^\circ(f) = 0$ for all n .

Assume that $\text{Su}_k^\circ(f) = 0$ for some $k > 1$. Thus $t \circ \text{Su}_{k-1}^\circ(f) = 0$ for all $t > 0$ or $\mu(\{f \geq t\}) = 0$ for all $t > 0$. Suppose that $\text{Su}_{k-1}^\circ(f) > 0$. Then by the assumption on \circ , we have $\mu(\{f \geq t\}) = 0$ for all $t > 0$, so $\text{Su}(f) = 0$, which implies that $\text{Su}_{k-1}^\circ(f) = 0$, a contradiction. Therefore $\text{Su}_{k-1}^\circ(f) = 0$, which leads to $\text{Su}(f) = 0$, and so $\text{Su}_n^\circ(f) = 0$ for all n .

The proof of part (b) is again by induction on n . Clearly,

$$\begin{aligned} \text{Su}(af) &= \sup_{t \in Y} \{ (a(t/a)) \wedge \mu(\{f \geq t/a\}) \} \\ &\leq \sup_{s \leq \bar{y}/a} \{ (as) \wedge (a\mu(\{f \geq s\})) \} = a \text{Su}(f) \end{aligned}$$

for $a > 1$. Assume that the assertion holds for some $n \geq 1$. Then, by induction hypothesis,

$$\begin{aligned} \text{Su}_{n+1}^\circ(af) &= \sup_{s \leq \bar{y}/a} \{ ((as) \circ \text{Su}_n^\circ(af)) \wedge \mu(\{f \geq s\}) \} \\ &\leq \sup_{s \leq \bar{y}/a} \{ ((as) \circ (a\text{Su}_n^\circ(f))) \wedge (a\mu(\{f \geq s\})) \} \\ &\leq a \sup_{s \leq \bar{y}/a} \{ (s \circ \text{Su}_n^\circ(f)) \wedge \mu(\{f \geq s\}) \} = a \text{Su}_{n+1}^\circ(f). \end{aligned}$$

The proof for the case $0 < a < 1$ is analogous. To prove (c) and (d) one can use Proposition 3.2 (c). \square

Proposition 3.4. *The sequence $(\text{Su}_n^\circ(f))_{n \geq 1}$ is nondecreasing for all $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ if and only if $a \circ b \geq a \wedge b$ for all $a, b \in Y$.*

Proof. “ \Rightarrow ” By Proposition 3.2 (c) we get $\text{Su}_2^\circ(a\mathbb{1}_A) = (a \circ (a \wedge b)) \wedge b$, where $A \in \mathcal{A}$, $a \in Y$ and $b = \mu(A)$. Since $\text{Su}_2^\circ(a\mathbb{1}_A) \geq \text{Su}_1^\circ(a\mathbb{1}_A)$, we have $(a \circ (a \wedge b)) \wedge b \geq a \wedge b$ for all $a \in Y$, $b \in \mu(\mathcal{A}) \cap Y$ and any monotone measure μ . Hence, $a \circ b \geq (a \circ b) \wedge b \geq (a \circ (a \wedge b)) \wedge b \geq a \wedge b$ for all $a, b \in Y$.

“ \Leftarrow ” By assumption $\text{Su}_2^\circ(f) \geq \sup_{t \in Y} \{ (t \wedge \text{Su}(f)) \wedge \mu(\{f \geq t\}) \} = \text{Su}(f)$. Suppose that $\text{Su}_n^\circ(f) \geq \text{Su}_{n-1}^\circ(f)$ for some $n > 1$. By the monotonicity $t \mapsto a \circ t$ for all a and the induction hypothesis, we obtain

$$\begin{aligned} \text{Su}_{n+1}^\circ(f) &= \sup_{t \in Y} \{ (t \circ \text{Su}_n^\circ(f)) \wedge \mu(\{f \geq t\}) \} \\ &\geq \sup_{t \in Y} \{ (t \circ \text{Su}_{n-1}^\circ(f)) \wedge \mu(\{f \geq t\}) \} = \text{Su}_n^\circ(f), \end{aligned}$$

thus $(\text{Su}_n^\circ(f))_{n \geq 1}$ is a nondecreasing sequence. \square

Let $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$. Recall that

$$\text{Su}(f) = \inf_{t \in Y} \{ t \vee \mu(\{f > t\}) \} = \sup_{A \in \mathcal{A}} \{ \inf_{x \in A} f(x) \wedge \mu(A) \}, \quad (2)$$

see [5, 37, 41]. We present formulas for the upper n -Sugeno integral, which have the forms as in (2).

Theorem 3.5. *Let \circ be an admissible fusion map that is continuous in the first argument. Then for all $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and $n \geq 2$ we have*

$$\text{Su}_n^\circ(f) = \inf_{t \in Y} \{ (t \circ \text{Su}_{n-1}^\circ(f)) \vee \mu(\{f > t\}) \}.$$

Proof. The proof is given in Appendix. \square

Theorem 3.6. For all $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and $n \geq 2$ we have

$$\text{Su}_n^\circ(f) = \sup_{A \in \mathcal{A}} \left\{ \left(\inf_{x \in A} f(x) \circ \text{Su}_{n-1}^\circ(f) \right) \wedge \mu(A) \right\}.$$

Proof. Arguing as in the proof of Theorem 2.2 in [7], we shall show more than it is needed, i.e., if a map \circ is *nondecreasing in the first coordinate*, then

$$\sup_{t \in Y} \{ (t \circ a) \wedge \mu(\{f \geq t\}) \} = \sup_{A \in \mathcal{A}} \{ \left(\inf_{x \in A} f(x) \circ a \right) \wedge \mu(A) \} \quad (3)$$

for all $a \in Y$. In fact, let $t \in Y$ and $A_t = \{f \geq t\}$. Thus, $\inf_{x \in A_t} f(x) \geq t$ and

$$(t \circ a) \wedge \mu(\{f \geq t\}) \leq \left(\inf_{x \in A_t} f(x) \circ a \right) \wedge \mu(A_t) \leq \sup_{A \in \mathcal{A}} \{ \left(\inf_{x \in A} f(x) \circ a \right) \wedge \mu(A) \}.$$

Therefore, the left hand side in (3) is not greater than the right one. Let $A \in \mathcal{A}$ and $t_0 = \inf_{x \in A} f(x)$. Then $A \subset \{f \geq t_0\}$ and

$$\left(\inf_{x \in A} f(x) \circ a \right) \wedge \mu(A) \leq (t_0 \circ a) \wedge \mu(\{f \geq t_0\}) \leq \sup_{t \in Y} \{ (t \circ a) \wedge \mu(\{f \geq t\}) \}.$$

Thus, the left hand side in (3) is greater than or equal to the right one, which finishes the proof. \square

The most important property of an integral is subadditivity. The following concept was introduced in [5].

Definition 3.7. Let $\mu \in \mathcal{M}_{(X, \mathcal{A})}$. We say that $f, g \in \mathcal{F}_{(X, Y)}$ are μ -subadditive with respect to a map $\nabla: \mu(\mathcal{A}) \times \mu(\mathcal{A}) \rightarrow \mu(\mathcal{A})$ (μ - ∇ -subadditive for short) if

$$\mu(\{f > a\} \cup \{g > b\}) \leq \mu(\{f > a\}) \nabla \mu(\{g > b\})$$

for all $a, b \in Y$.

Note that all functions f, g are μ - ∇ -subadditive if $a \nabla b = (a + b) \wedge \mu(X)$ whenever the monotone measure μ is *subadditive*, that is, $\mu(A \cup B) \leq \mu(A) + \mu(B)$ for all $A, B \in \mathcal{A}$. Recall that $f, g \in \mathcal{F}_{(X, Y)}$ are *comonotone* if $(f(x) - f(y))(g(x) - g(y)) \geq 0$ for all $x, y \in X$. Equivalently, f and g are comonotone, if for any $t \in Y$ either $\{f > t\} \subset \{g > t\}$ or $\{g > t\} \subset \{f > t\}$. Thus, comonotone functions are μ - ∇ -subadditive for $\nabla \geq \vee$. For more examples of μ - ∇ -subadditive functions we refer to [5].

Theorem 3.8. Assume that $\nabla: \mu(\mathcal{A}) \times \mu(\mathcal{A}) \rightarrow \mu(\mathcal{A})$, $f, g \in \mathcal{F}_{(X, Y)}$ and

$$[(a + b) \wedge \bar{y}] \odot [(c + d) \wedge \bar{y}] \vee (\alpha \nabla \beta) \leq [(a \odot c) \vee \alpha] + [(b \odot d) \vee \beta] \quad (4)$$

for $a, b, c, d \in Y$ and $\alpha, \beta \in \mu(\mathcal{A})$ such that $a + b, c + d \in Y$ with $\odot \in \{\circ, \text{P}\}$, where the admissible fusion map \circ is continuous in the first argument and $x \text{P} y = x$ for any x, y . If f, g are μ - ∇ -subadditive and $f + g \in \mathcal{F}_{(X, Y)}$, then

$$\text{Su}_n^\circ(f + g) \leq \text{Su}_n^\circ(f) + \text{Su}_n^\circ(g) \quad (5)$$

for $n \geq 1$. Moreover, μ is a subadditive monotone measure whenever there is some $n \geq 1$ such that (5) holds for all $f, g \in \mathcal{F}_{(X,Y)}$ with $\mu(X) \leq \bar{y}$ and $\bar{y} \circ b \geq b$ for all $b \in Y$.

Proof. The proof is by induction on n . Let $n = 1$. Evidently, $\{f \leq a\} \cap \{g \leq b\} \subset \{f + g \leq a + b\}$. As $f(x) + g(x) \leq \bar{y}$ for any x and f, g are μ - ∇ -subadditive, we have

$$\begin{aligned} [(a + b) \wedge \bar{y}] \vee \mu(\{f + g > (a + b) \wedge \bar{y}\}) &\leq [(a + b) \wedge \bar{y}] \vee [\mu(\{f > a\}) \nabla \mu(\{g > b\})] \\ &\leq [a \vee \mu(\{f > a\})] + [b \vee \mu(\{g > b\})], \end{aligned}$$

where the last inequality holds by (4) with $x \odot y = x$. By (2) we get

$$\text{Su}(f + g) \leq [a \vee \mu(\{f > a\})] + [b \vee \mu(\{g > b\})].$$

Taking the lower bound for $a, b \in Y$ finishes the proof of (5) for $n = 1$.

Assume that (5) holds for some $n > 1$. By μ - ∇ -subadditivity and (4) with $\odot = \circ$, we get

$$\begin{aligned} &\left[[(a + b) \wedge \bar{y}] \circ [(\text{Su}_n^\circ(f) + \text{Su}_n^\circ(g)) \wedge \bar{y}] \right] \vee \mu(\{f + g > (a + b) \wedge \bar{y}\}) \\ &\leq \left[[(a + b) \wedge \bar{y}] \circ [(\text{Su}_n^\circ(f) + \text{Su}_n^\circ(g)) \wedge \bar{y}] \right] \vee [\mu(\{f > a\}) \nabla \mu(\{g > b\})] \\ &\leq [(a \circ \text{Su}_n^\circ(f)) \vee \mu(\{f > a\})] + [(b \circ \text{Su}_n^\circ(g)) \vee \mu(\{g > b\})]. \end{aligned}$$

By the induction hypothesis and $\text{Su}_n^\circ(f + g) \leq \bar{y}$, we have for all $a, b \in Y$

$$\begin{aligned} &[(a + b) \wedge \bar{y}] \circ \text{Su}_n^\circ(f + g) \vee \mu(\{f + g > (a + b) \wedge \bar{y}\}) \\ &\leq [(a \circ \text{Su}_n^\circ(f)) \vee \mu(\{f > a\})] + [(b \circ \text{Su}_n^\circ(g)) \vee \mu(\{g > b\})]. \end{aligned}$$

As a consequence of Theorem 3.5 we obtain

$$\text{Su}_{n+1}^\circ(f + g) \leq [(a \circ \text{Su}_n^\circ(f)) \vee \mu(\{f > a\})] + [(b \circ \text{Su}_n^\circ(g)) \vee \mu(\{g > b\})].$$

Taking the lower bound for a and then for b we finish the proof of (5).

Suppose that (5) is satisfied for some $n \geq 1$. Then Proposition 3.3 (c) yields $\text{Su}_k^\circ(\bar{y} \mathbb{1}_D) = \mu(D)$ for all D and k . Putting $f = \bar{y} \mathbb{1}_A$ and $g = \bar{y} \mathbb{1}_{B \setminus A}$ in (5), we obtain $\mu(A \cup B) \leq \mu(A) + \mu(B \setminus A) \leq \mu(A) + \mu(B)$. This completes the proof. \square

Corollary 3.9. *If $\mu \in \mathcal{M}_{(X,A)}$ is subadditive and $\mu(X) \leq \bar{y}$, then*

$$\text{Su}_n^+(f + g) \leq \text{Su}_n^+(f) + \text{Su}_n^+(g) \tag{6}$$

for any $n \geq 1$ and all $f, g \in \mathcal{F}_{(X,Y)}$ such that $f + g \in \mathcal{F}_{(X,Y)}$. Moreover, if there is n such that (6) holds for all $f, g \in \mathcal{F}_{(X,Y)}$, then μ is a subadditive monotone measure.

Proof. The assertion follows from Theorem 3.8 for $a \circ b = (a + b) \wedge \bar{y}$ and $a \nabla b = (a + b) \wedge \mu(X)$. \square

Corollary 3.10. *If $f, g \in \mathcal{F}_{(X,Y)}$ are comonotone functions and $\mu(\mathcal{A}) \subset Y$, then $\text{Su}_n^+(f + g) \leq \text{Su}_n^+(f) + \text{Su}_n^+(g)$ for all $n \geq 1$ and $f + g \in \mathcal{F}_{(X,Y)}$.*

Proof. Apply Theorem 3.8 for $a \circ b = (a + b) \wedge \bar{y}$ and $\nabla = \vee$. \square

Condition (4) is valid if $\nabla \leq +$ and $[(a + b) \wedge \bar{y}] \circ [(c + d) \wedge \bar{y}] \leq (a \circ c) + (b \circ d)$ for any $a, b, c, d \in Y$. Examples of maps \circ satisfying the last inequality are:

- (i) $a \circ b = \lambda a$ for $Y = [0, 1]$ and $\lambda \in (0, 1]$, or $Y = [0, \infty]$ and $\lambda > 0$,
- (ii) $a \circ b = a^\gamma$ with $\gamma \in (0, 1)$,
- (iii) $a \circ b = \lambda(a + b) + (1 - \lambda)(a \vee b)$ for $\lambda \in [0, 1]$ and $Y = [0, \infty]$,
- (iv) $a \circ b = a + b - ab$ for $Y = [0, 1]$.

As we have shown above, the upper n -Sugeno integral possesses several properties of the Sugeno integral, but not all. Hereafter, $a \mathbb{1}_X \vee f = (a \mathbb{1}_X) \vee f$ and $a \mathbb{1}_X \wedge f = (a \mathbb{1}_X) \wedge f$. We say that the integral J is *maxitive homogeneous* and *minitive homogeneous* if $J(\mu, a \mathbb{1}_X \vee f) = a \vee J(\mu, f)$ and $J(\mu, a \mathbb{1}_X \wedge f) = a \wedge J(\mu, f)$ for any a, μ, f , respectively. Next example demonstrates that it is not the case of upper and lower n -Sugeno integral.

Example 3.11. Let $f = 0.25 \mathbb{1}_A + 0.75 \mathbb{1}_B$, where $A \cap B = \emptyset$ and $A \cup B = X$. Assume that $\mu(X) = 1$ and $\mu(B) = 0.5$. It is clear that $(1/3) \mathbb{1}_X \wedge f = 0.25 \mathbb{1}_A + (1/3) \mathbb{1}_B$ and $(1/3) \mathbb{1}_X \vee f = (1/3) \mathbb{1}_A + 0.75 \mathbb{1}_B$. Thus,

$$\text{Su}_2^+(f) = 0.75, \quad \text{Su}_2^+((1/3) \mathbb{1}_X \wedge f) = 7/12, \quad \text{Su}_2^+((1/3) \mathbb{1}_X \vee f) = 5/6,$$

so $\text{Su}_2^+((1/3) \mathbb{1}_X \wedge f) > (1/3) \wedge \text{Su}_2^+(f)$ and $\text{Su}_2^+((1/3) \mathbb{1}_X \vee f) > (1/3) \vee \text{Su}_2^+(f)$.

Now we give one sufficient condition for minitive/maxitive homogeneity of the integral.

Proposition 3.12. *Let $n \geq 2$ and $(\mu, f) \in \mathcal{M}_{(X,A)} \times \mathcal{F}_{(X,Y)}$. Then*

- (a) $\text{Su}_n^\circ(a \mathbb{1}_X \wedge f) = a \wedge \text{Su}_n^\circ(f)$ for any $a \in Y$ if $(a \wedge b) \circ (a \wedge c) = a \wedge (b \circ c)$ for all $a, b, c \in Y$.
- (b) $\text{Su}_n^\circ(a \mathbb{1}_X \vee f) = a \vee \text{Su}_n^\circ(f)$ for any $a \in Y$ if $(a \vee b) \circ (a \vee c) = a \vee (b \circ c)$ for all $a, b, c \in Y$.

Proof. We show only (a) since the proof of (b) is analogous. Since $\text{Su}(a \mathbb{1}_X \wedge f) = a \wedge \text{Su}(f)$, we have from Theorem 3.6 that $\text{Su}_2^\circ(a \mathbb{1}_X \wedge f) = \sup_{A \in \mathcal{A}} \{[(a \wedge \inf_{x \in A} f(x)) \circ (a \wedge \text{Su}(f))]\} \wedge \mu(A) = a \wedge \text{Su}_2^\circ(f)$. The second induction step proceeds similarly. \square

The admissible fusion map $x \circ y = \varphi(x) \wedge \gamma(y)$ satisfies the condition of Proposition 3.12(a) if $\varphi, \gamma: Y \rightarrow Y$ are nondecreasing functions such that $\varphi(a) \wedge \gamma(a) = a$ for any $a \in Y$. The assumption of Proposition 3.12(b) holds if $x \circ y = \varphi(x) \vee \gamma(y)$, where the functions $\varphi, \gamma: Y \rightarrow Y$ are nondecreasing with $\varphi(a) \vee \gamma(a) = a$ for all $a \in Y$.

4 Lower n -Sugeno integral

This section is devoted to defining a new functional generalizing the upper 2- h -index (18) and its properties. We say that $\star: [0, \infty] \times Y \rightarrow [0, \infty]$ is a *link map* if it is nondecreasing and $0 \star a \leq a$ for all $a \in Y$. Clearly, the link map coincides with the admissible fusion map if and only if $Y = [0, \infty]$.

Definition 4.1. Let $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and \star be a link map. For $n \geq 1$ the *lower n -Sugeno integral* is defined by

$$\text{Su}_\star^{n+1}(\mu, f) := \sup_{t \in Y} \{t \wedge (\mu(\{f \geq t\}) \star \text{Su}_\star^n(\mu, f))\},$$

where $\text{Su}_\star^1(\mu, f) = \text{Su}(\mu, f)$.

It is clear that $\text{Su}_\star^n(\mu, f) \in Y$ for all n . The next proposition shows that the lower n -Sugeno integral satisfies all the properties in Definition 2.2.

Proposition 4.2. Let $n \geq 1$, $f, g \in \mathcal{F}_{(X, Y)}$ and $\mu, \nu \in \mathcal{M}_{(X, \mathcal{A})}$.

- (a) If $f \leq_\mu g$, then $\text{Su}_\star^n(\mu, f) \leq \text{Su}_\star^n(\mu, g)$.
- (b) If $\mu(A) \leq \nu(A)$ for all $A \in \mathcal{A}$, then $\text{Su}_\star^n(\mu, f) \leq \text{Su}_\star^n(\nu, f)$.
- (c) $\text{Su}_\star^n(\mu, a\mathbb{1}_A) = s_n(a, \mu(A))$ for $a \in Y$ and $A \in \mathcal{A}$, where $s_n: Y \times [0, \infty] \rightarrow Y$ is a nondecreasing function such that $s_n(a, 0) = s_n(0, b) = 0$ and $s_{n+1}(a, b) = a \wedge [b \star s_n(a, b)]$ for all a, b .

Proof. Parts (a) and (b) are immediate by induction.

(c) We have the following recurrence formula

$$\text{Su}_\star^{n+1}(\mu, a\mathbb{1}_A) = [a \wedge (\mu(A) \star \text{Su}_\star^n(\mu, a\mathbb{1}_A))] \vee [\bar{y} \wedge (0 \star \text{Su}_\star^n(\mu, a\mathbb{1}_A))],$$

where $a \in Y$. First we show that

$$\text{Su}_\star^n(\mu, a\mathbb{1}_A) \leq a \tag{7}$$

for all n and $a \in Y$. We use induction on n . In fact, $\text{Su}(a\mathbb{1}_A) = a \wedge \mu(A) \leq a$. Assume that (7) holds for some n . Since $0 \star a \leq a$, we have

$$\text{Su}_\star^{n+1}(\mu, a\mathbb{1}_A) \leq [a \wedge (\mu(A) \star a)] \vee [\bar{y} \wedge (0 \star a)] \leq a \vee a = a$$

and the proof of (7) is complete. From (7), we obtain $0 \star \text{Su}_\star^{n+1}(\mu, a\mathbb{1}_A) \leq 0 \star a \leq a \leq \bar{y}$ for all $a \in Y$. Hence,

$$\text{Su}_\star^{n+1}(\mu, a\mathbb{1}_A) = [a \wedge (\mu(A) \star \text{Su}_\star^n(\mu, a\mathbb{1}_A))] \vee [0 \star \text{Su}_\star^n(\mu, a\mathbb{1}_A)].$$

As $0 \star \text{Su}_\star^n(\mu, a\mathbb{1}_A) \leq 0 \star a \leq a$ and $0 \star \text{Su}_\star^n(\mu, a\mathbb{1}_A) \leq \mu(A) \star \text{Su}_\star^n(\mu, a\mathbb{1}_A)$, we get

$$\text{Su}_\star^{n+1}(\mu, a\mathbb{1}_A) = a \wedge [\mu(A) \star \text{Su}_\star^n(\mu, a\mathbb{1}_A)]. \tag{8}$$

Applying induction on n , we obtain the statement (c). \square

To shorten the notation, we write $\text{Su}_\star^n(f)$ instead of $\text{Su}_\star^n(\mu, f)$ if there is no ambiguity. Hereafter, for a link map \star and $\mu \in \mathcal{M}_{(X, \mathcal{A})}$ we use the convention $\mu_{k+1}^\star(A) := \mu(A) \star \mu_k^\star(A)$ for all $k \geq 1$ with $\mu_1^\star(A) = \mu(A)$ provided that $\mu_k^\star(X) \in Y$ for all k . It is evident that μ_n^\star is a monotone measure if $\mu_n^\star(X) > 0$. Some properties of the lower n -Sugeno integral are analogous to those of the upper n -Sugeno integral, which is shown in what follows.

Proposition 4.3. *Let $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and $A \in \mathcal{A}$.*

- (a) *If $\text{Su}(f) = 0$, then $\text{Su}_\star^n(f) = 0$ for all $n \geq 2$. If $\text{Su}_\star^k(f) = 0$ for some $k > 1$ and $a \star b > 0$ for all $a, b > 0$, then $\text{Su}_\star^n(f) = 0$ for all $n \geq 1$.*
- (b) $\text{Su}_\star^n(f) = \lim_{t \nearrow \text{Su}_\star^n(f)} (t \wedge (\mu(\{f \geq t\}) \star \text{Su}_\star^{n-1}(f)))$ if $\text{Su}_\star^{n-1}(f) > 0$.
- (c) $\text{Su}_\star^n(f) = \lim_{t \searrow \text{Su}_\star^n(f)} (t \vee (\mu(\{f > t\}) \star \text{Su}_\star^{n-1}(f)))$ if $\text{Su}_\star^n(f) < \bar{y}$.
- (d) *There is $a > 1$ such that $\text{Su}_\nu^n(ag) \leq a \text{Su}_\nu^n(g)$ for all $(\nu, ag) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and $n \geq 1$, if $x \star (ay) \leq a(x \star y)$ for all $x \in \nu(\mathcal{A})$ and $ay \in Y$. Moreover, $\text{Su}_\nu^n(ag) \geq a \text{Su}_\nu^n(g)$ for some $a \in (0, 1)$ and for all $(\nu, g) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and $n \geq 1$, if $x \star (ay) \geq a(x \star y)$ for all $x \in \nu(\mathcal{A})$ and $y \in Y$.*
- (e) $\text{Su}_\star^n(a \mathbb{1}_A) = \mu_n^\star(A)$ for $a \in [\max_{1 \leq k \leq n} \mu_k^\star(A), \bar{y}]$.
- (f) *(Idempotency) $\text{Su}_\star^n(a \mathbb{1}_X) = a$ for all $a \in Y$ and $n \geq 1$ if and only if $\mu(X) \geq \bar{y}$ and $\mu(X) \star a \geq a$ for all $a \in Y$.*

Proof. (a) The first assertion follows from Lemma 2.1 (e) and the induction, since $0 \star 0 = 0$. If $\text{Su}_\star^k(f) = 0$ for some $k > 1$, then repeating similar argument as in the proof of Proposition 3.3 (a), we get the assertion.

(b) and (c) By monotonicity of \star , we have $t > \mu(\{f \geq t\}) \star \text{Su}_\star^{n-1}(f)$ for $t > \text{Su}_\star^n(f)$ and $t < \mu(\{f \geq t\}) \star \text{Su}_\star^{n-1}(f)$ for $t < \text{Su}_\star^n(f)$. In consequence, both properties (b) and (c) hold. See the proof of Lemma 2.1 (c)-(d).

(d) The proof is similar to that of Proposition 3.3 (b).

(e) and (f) The proofs go by induction on n ; see (8). □

Proposition 4.4. *The sequence $(\text{Su}_\star^n(f))_{n=1}^\infty$ is nondecreasing for all $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ if and only if $a \star b \geq a \wedge b$ for all $a \in [0, \infty]$ and $b \in Y$.*

Proof. “ \Rightarrow ” By (8) for $n = 1$, we have $\text{Su}_\star^2(a \mathbb{1}_A) = a \wedge [b \star (a \wedge b)]$, where $A \in \mathcal{A}$, $a \in Y$ and $b = \mu(A)$. Since $\text{Su}_\star^1(a \mathbb{1}_A) \leq \text{Su}_\star^2(a \mathbb{1}_A)$, we obtain $a \wedge b \leq a \wedge [b \star (a \wedge b)] \leq b \star a$ for all $a \in Y$ and $b \in [0, \infty]$, as desired.

“ \Leftarrow ” The proof is similar to that of Proposition 3.4, so we omit it. □

Theorem 4.5. *For all $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and $n \geq 2$*

$$\text{Su}_\star^n(f) = \sup_{A \in \mathcal{A}} \left\{ \inf_{x \in A} f(x) \wedge (\mu(A) \star \text{Su}_\star^{n-1}(f)) \right\}.$$

Proof. Use the same arguments as in the proof of Theorem 3.6 for a nondecreasing map \star in the first coordinate. \square

The following extension of the first equality in (2) will be needed to prove the subadditivity property of the lower integral.

Theorem 4.6. *Assume that \star is a continuous link map in the first argument. Then for each $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and $n \geq 2$ we have*

$$\text{Su}_\star^n(f) = \inf_{t \in Y} \{t \vee (\mu(\{f > t\}) \star \text{Su}_\star^{n-1}(f))\}.$$

Proof. The proof is given in Appendix. \square

Next, we show that the lower n -Sugeno integral is also a subadditive functional under some extra assumptions.

Theorem 4.7. *Suppose that $\nabla: \mu(\mathcal{A}) \times \mu(\mathcal{A}) \rightarrow \mu(\mathcal{A})$, $f, g \in \mathcal{F}_{(X, Y)}$ and*

$$[(a + b) \wedge \bar{y}] \vee [(\alpha \nabla \beta) \circledast ((c + d) \wedge \bar{y})] \leq [a \vee (\alpha \circledast c)] + [b \vee (\beta \circledast d)] \quad (9)$$

for $a, b, c, d \in Y$ and $\alpha, \beta \in \mu(\mathcal{A})$ with $\circledast \in \{\star, P\}$, where \star is a continuous link map in the first argument and $xPy = x$ for any x, y . If f, g are μ - ∇ -subadditive and $f + g \in \mathcal{F}_{(X, Y)}$, then

$$\text{Su}_\star^n(f + g) \leq \text{Su}_\star^n(f) + \text{Su}_\star^n(g) \quad (10)$$

for all $n \geq 1$. Moreover, if (10) is valid for all $f, g \in \mathcal{F}_{(X, Y)}$ such that $f + g \in \mathcal{F}_{(X, Y)}$ and n such that $\mu_n^\star(X) > 0$, then the monotone measure μ_n^\star is subadditive.

Proof. We use the induction by n . The proof of subadditivity of the Sugeno integral (the case $n = 1$ and $\circledast = P$) can be found in the proof of Theorem 3.8. Assume that inequality (10) holds for some $n \geq 2$ and all μ - ∇ -subadditive functions f, g . Combining inductive hypothesis and (9) with $\circledast = \star$ yields

$$\begin{aligned} & [(a + b) \wedge \bar{y}] \vee [\mu(\{f + g > (a + b) \wedge \bar{y}\}) \star (\text{Su}_\star^n(f + g) \wedge \bar{y})] \\ & \leq [(a + b) \wedge \bar{y}] \vee [(\mu(\{f > a\}) \nabla \mu(\{g > b\})) \star ((\text{Su}_\star^n(f) + \text{Su}_\star^n(g)) \wedge \bar{y})] \\ & \leq [a \vee (\mu(\{f > a\}) \star \text{Su}_\star^n(f))] + [b \vee (\mu(\{g > b\}) \star \text{Su}_\star^n(g))] \end{aligned}$$

for all $a, b \in Y$. By Theorem 4.6, we get

$$\text{Su}_\star^{n+1}(f + g) \leq [a \vee (\mu(\{f > a\}) \star \text{Su}_\star^n(f))] + [b \vee (\mu(\{g > b\}) \star \text{Su}_\star^n(g))]$$

for any $a, b \in Y$. Taking infimum over a and then with b gives (10).

Put $f = a\mathbb{1}_A$ and $g = a\mathbb{1}_{B \setminus A}$ in (10), where $a \geq \max_{k \leq n} \mu_k^\star(A \cup B)$. Then from Proposition 4.3 (e) and by monotonicity of μ_n^\star , we get $\mu_n^\star(A \cup B) \leq \mu_n^\star(A) + \mu_n^\star(B \setminus A) \leq \mu_n^\star(A) + \mu_n^\star(B)$, as desired. \square

Example 4.8. There are many link maps \star with $Y = [0, \infty]$ such that subadditivity of μ_n^\star with arbitrary n implies subadditivity of μ . For instance, all idempotent operators (e.g. $a \star b = a^p b^{1-p}$ and $a \star b = p(a \wedge b) + (1-p)(a \vee b)$ with $p \in (0, 1)$) as well as mappings

$$\text{a) } a \star b = f(a)f(b), \quad \text{b) } a \star b = g(ab), \quad \text{c) } a \star b = h(a+b), \quad \text{d) } a \star b = (a^q + b^q)^{1/q},$$

where $q > 0$ and $f, g, h: [0, \infty] \rightarrow [0, \infty]$ are increasing superadditive functions¹ vanishing at 0, such that $h(x) \leq x$ for all x . In order to prove a) and b) one can use the inequality $a^n + b^n \leq (a+b)^n$.

Corollary 4.9. *If μ is a subadditive monotone measure, or $f, g \in \mathcal{F}_{(X, [0, \infty])}$ are comonotone functions, then*

$$\text{Su}_+^n(f+g) \leq \text{Su}_+^n(f) + \text{Su}_+^n(g) \quad (11)$$

for all $n \geq 1$. Moreover, if (11) is valid for some n and all $f, g \in \mathcal{F}_{(X, [0, \infty])}$, then μ is subadditive.

Proof. Put $Y = [0, \infty]$, $\star = +$ and $a \nabla b = (a+b) \wedge \mu(X)$ or $\nabla = \vee$ in Theorem 4.7 and use Example 4.8 (d) with $q = 1$. \square

Next, we give a partial solution to the problem posed in [30]. The question is how to compute the *pseudo-decomposition integral of n -th order* defined as

$$\text{I}_n^{\oplus, \odot}(\mu, f) = \sup \left\{ \bigoplus_{i=1}^n (a_i \odot \mu(A_i)) : \bigoplus_{i=1}^n a_i \mathbb{1}_{A_i} \leq f, a_i \in Y, A_1 \subset \dots \subset A_n \right\} \quad (12)$$

based on a pseudo-addition $\oplus: Y^2 \rightarrow Y$ and a \oplus -fitting pseudo-multiplication $\odot: Y \times [0, \mu(X)] \rightarrow Y$ (see [3, Definition 3.1 and 3.4]). The integral $\text{I}_n^{\oplus, \odot}$ is also called the *Benvenuti integral of n -th order*. Our aim is to compute the integral $\text{I}_n^{+, \wedge}(\mu, f)$. By the definition (12) we get

$$\begin{aligned} \text{I}_n^{+, \wedge}(\mu, f) &= \sup \left\{ \sum_{i=1}^n (a_i \wedge \mu(A_i)) : \sum_{i=1}^n a_i \mathbb{1}_{A_i} \leq f, A_1 \subset \dots \subset A_n \right\} \\ &= \sup \left\{ \sum_{i=1}^n (a_i \wedge \mu(A_i)) : \sum_{i=1}^n \left(\sum_{k=i}^n a_k \right) \mathbb{1}_{A_i \setminus A_{i-1}} \leq f, A_1 \subset \dots \subset A_n \right\} \\ &= \sup \left\{ \sum_{i=1}^n ((b_i - b_{i+1}) \wedge \mu(\{f \geq b_i\})) : 0 = b_{n+1} \leq b_n \leq \dots \leq b_1 \leq \bar{y} \right\} \end{aligned} \quad (13)$$

with $b_i = \sum_{k=i}^n a_k$ and $A_0 := \emptyset$, but computation of the integral from formula (13) is still a difficult task. However, there is a connection with the lower n -Sugeno integral.

Theorem 4.10. *For all $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and $n \geq 1$*

$$\text{I}_n^{+, \wedge}(\mu, f) = \text{Su}_+^n(\mu, f). \quad (14)$$

¹ A function f is superadditive if $f(a+b) \geq f(a) + f(b)$ for any a, b .

Proof. See Appendix. □

Combining the definition of $\text{Su}_+^n(\mu, f)$ with Theorem 4.10 gives the following simple recurrence scheme

$$I_n^{+, \wedge}(\mu, f) = \sup_{y \in Y} \{y \wedge (\mu(\{f \geq y\}) + I_{n-1}^{+, \wedge}(\mu, f))\}$$

with $I_1^{+, \wedge}(\mu, f) := \text{Su}(f)$.

Example 4.11. Let $X = [0, 1]$, $f(x) = x$ and $\mu(A) = (\lambda(A))^{1/2}$, where λ is the Lebesgue measure.

Then

$$I_n = \frac{2I_{n-1} - 1 + \sqrt{5 - 4I_{n-1}}}{2}, \quad n = 1, 2, \dots,$$

where $I_n := I_n^{+, \wedge}(\mu, f)$ and $I_0 := 0$. If $\mu(A) = (\lambda(A))^2$, then

$$I_n = \frac{3 - \sqrt{5 - 4I_{n-1}}}{2}, \quad n = 1, 2, \dots$$

The next result provides a connection between the lower n -Sugeno integral and the generalized Choquet integral introduced in [9] and deeply studied in [29].

Theorem 4.12. Let $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$ and $n \geq 2$. The lower n -Sugeno integral can be represented as

$$\text{Su}_+^n(f) = \inf \left\{ \sum_{i=1}^n ((b_i - b_{i+1}) \vee \mu(\{f > b_i\})) : 0 = b_{n+1} \leq b_n \leq \dots \leq b_1 \leq \bar{y} \right\}.$$

Proof. See Appendix. □

The lower 2-Sugeno integral is neither maxitive nor minitive homogeneous functional.

Example 4.13. Consider f as in the Example 3.11. Let $\mu(B) = 0.25$ and $\mu(X) = 1$. Then $\text{Su}_+^2(f) = 0.5$ and $\text{Su}_+^2((1/3)\mathbb{1}_X \vee f) = 7/12$, so $\text{Su}_+^2((1/3)\mathbb{1}_X \vee f) > (1/3) \vee \text{Su}_+^2(f)$.

Example 4.14. Let $f = 0.5\mathbb{1}_A$, $\mu(A) = 0.5$, $\mu(X) = 1$ and $\star = \cdot$. After simple calculations, we get $\text{Su}^2(f) = 0.25$ and $\text{Su}^2(0.1\mathbb{1}_X \wedge f) = 0.05$. Thus, $\text{Su}^2(0.1\mathbb{1}_X \wedge f) < 0.1 \wedge \text{Su}^2(f)$.

Now we give a sufficient condition for minitive/maxitive homogeneity.

Proposition 4.15. Suppose that $n \geq 2$ and $(\mu, f) \in \mathcal{M}_{(X, \mathcal{A})} \times \mathcal{F}_{(X, Y)}$.

- (a) $\text{Su}_\star^n(a\mathbb{1}_X \wedge f) = a \wedge \text{Su}_\star^n(f)$ for any $a \in Y$ if $a \wedge (b \star (a \wedge c)) = a \wedge (b \star c)$ for all $a, c \in Y$ and $b \in \mu(\mathcal{A})$.
- (b) $\text{Su}_\star^n(a\mathbb{1}_X \vee f) = a \vee \text{Su}_\star^n(f)$ for any $a \in Y$ if $\mu(X) \geq \bar{y}$ and $b \wedge c \leq b \star c \leq b \vee c$ for all $b \in [0, \infty]$ and $c \in Y$.

Proof. The proof of part (a) is similar to that of Proposition 3.12 (a) (applying Theorem 4.5), so we omit it.

Now we show (b) by induction. The proof for $n = 2$ will be omitted as it is quite similar to the proof of the second induction step. From Theorem 4.5 and the induction hypothesis, we have

$$\begin{aligned} \text{Su}_\star^n(a\mathbb{1}_X \vee f) &= \sup_A \left\{ (a \vee \inf_{x \in A} f(x)) \wedge [\mu(A) \star (a \vee \text{Su}_\star^{n-1}(f))] \right\} \\ &= \sup_A \left\{ (a \vee \inf_{x \in A} f(x)) \wedge [(\mu(A) \star a) \vee (\mu(A) \star \text{Su}_\star^{n-1}(f))] \right\} \\ &= \sup_A \left\{ [a \wedge (\mu(A) \star a)] \vee [a \wedge (\mu(A) \star \text{Su}_\star^{n-1}(f))] \right. \\ &\quad \left. \vee [\inf_{x \in A} f(x) \wedge (\mu(A) \star a)] \vee [\inf_{x \in A} f(x) \wedge (\mu(A) \star \text{Su}_\star^{n-1}(f))] \right\}, \end{aligned}$$

where we write \sup_A instead of $\sup_{A \in \mathcal{A}}$. Furthermore

$$\begin{aligned} \text{Su}_\star^n(a\mathbb{1}_X \vee f) &= [a \wedge (\mu(X) \star a)] \vee [a \wedge (\mu(X) \star \text{Su}_\star^{n-1}(f))] \vee \sup_A \left\{ \inf_{x \in A} f(x) \wedge (\mu(A) \star a) \right\} \\ &\quad \vee \sup_A \left\{ \inf_{x \in A} f(x) \wedge (\mu(A) \star \text{Su}_\star^{n-1}(f)) \right\}. \end{aligned}$$

By the assumption that $\mu(X) \star a \geq \mu(X) \wedge a = a$ and the fact that $a \wedge (\mu(X) \star \text{Su}_\star^{n-1}(f)) \leq a$, we have

$$\text{Su}_\star^n(a\mathbb{1}_X \vee f) = a \vee \sup_A \left\{ \inf_{x \in A} f(x) \wedge (\mu(A) \star a) \right\} \vee \text{Su}_\star^n(f). \quad (15)$$

Observe that

$$\begin{aligned} \sup_A \left\{ \inf_{x \in A} f(x) \wedge (\mu(A) \star a) \right\} &\leq \sup_A \left\{ \inf_{x \in A} f(x) \wedge (\mu(A) \vee a) \right\} \\ &= \sup_A \left\{ (\inf_{x \in A} f(x) \wedge \mu(A)) \vee (\inf_{x \in A} f(x) \wedge a) \right\} \\ &\leq \text{Su}(f) \vee a \leq \text{Su}_\star^n(f) \vee a, \end{aligned}$$

where the last inequality follows from Proposition 4.4. By (15), we obtain $\text{Su}_\star^n(a\mathbb{1}_X \vee f) = a \vee \text{Su}_\star^n(f)$, as desired. \square

The condition in (a) is satisfied if $x \star a \geq a$ for any x and a , e.g. $x \star a := (x^p + a^p)^{1/p}$ for $p > 0$. On the other hand, any OWA operator of the form $x \star y = p(x \wedge y) + (1-p)(x \vee y)$ satisfies the condition in (b) for $p \in [0, 1]$.

5 Applications

(A) Scientometric indices

We put $X = \mathbb{N}$, where $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of all positive integers, and $\mu: 2^{\mathbb{N}} \rightarrow [0, \infty]$ is the counting measure, i.e., $\mu(A) = \text{Card}(A)$ for any $A \in 2^{\mathbb{N}}$. A scholar with some publications is

formally described by an infinite vector $\mathbf{x} = (x_1, x_2, \dots)$, called a *scientific record*, where $x_i \in \mathbb{N}_0$ with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that $x_1 \geq x_2 \geq \dots$. The positive value of x_i gives the number of citations of i -th scholar publication, and the value $x_i = 0$ means either a paper with zero citations or a nonexisting paper. From now on we consider the scientific records with $x_1 \geq 1$. The h -index of \mathbf{x} is defined as follows [23, 28]

$$H(\mathbf{x}) = \max\{k: x_k \geq k\} = \max_k \{k \wedge x_k\}.$$

Note that there are several papers characterizing the Hirsch index via various axioms, e.g. [8, 32, 33, 42]. An interesting axiom $H(\mathbf{x}) = H(\mathbf{y}_{\mathbf{x}})$ is called the *symmetry* of the h -index, see [43, Proposition 3.1]. Here $\mathbf{y}_{\mathbf{x}} := (y_1, y_2, \dots)$ is called the *conjugate* of \mathbf{x} with $y_i = \sum_{k=1}^{\infty} \mathbb{1}_{\{x_k \geq i\}}$ providing the number of publications with at least i citations.

As it is well known, the Hirsch index has some drawbacks. In order to compensate some defects of h -index, many authors have introduced new scientometric indices that lead to better discrimination of some types of data than h -index (see [28, 31, 42]). Here we discuss a few of them and show that the upper/lower n -Sugeno integrals generalize some known scientometric indices. Firstly, recall that Narukawa and Torra [39] have shown that h -index is the Sugeno integral with respect to counting measure. In consequence, the upper/lower n -Sugeno integral generalizes h -index too.

(i) Generalized Kosmulski index There are several modifications of h -index based on the input k , e.g. $h_{\lambda}(\mathbf{x}) = \max\{k: x_k \geq \lambda k\}$ of Van Eck [40], $h(2)$ -index $H2(\mathbf{x}) = \max\{k: x_k \geq k^2\}$ of Kosmulski [27], or its extended version $\max\{k: x_k \geq k^m\}$ with $m = 3, 4, \dots$. In general, for any nondecreasing function $s: [0, \infty] \rightarrow [0, \infty]$ the *generalized Kosmulski index* is given by $K_s(\mathbf{x}) = \max\{k: x_k \geq s(k)\}$, see [13]².

Now we will show the connection between generalized Kosmulski index and upper/lower 2-Sugeno integral. For this purpose consider $\circ_s: [0, \infty]^2 \rightarrow [0, \infty]$ defined as $a \circ_s b = s(a)$ with $s: [0, \infty] \rightarrow [0, \infty]$ being a nondecreasing function such that $s(0) = 0$. Immediately, \circ_s is an admissible fusion as well as a link function with $Y = [0, \infty]$. Note that each scientific record \mathbf{x} uniquely determines a function $f: X \rightarrow \mathbb{N}_0$ as $x_i = f(i)$, and vice versa. Hence, the notation $Su_2^{\circ_s}(\mathbf{x})$ is justified and

$$Su_2^{\circ_s}(\mathbf{x}) = \max_k \{s(k) \wedge \mu(\{i: x_i \geq k\})\} = \max_j \{s(x_j) \wedge \mu(\{i: x_i \geq x_j\})\}.$$

Since μ is the counting measure, we get

$$Su_2^{\circ_s}(\mathbf{x}) = \max_j \{s(x_j) \wedge j\} = \max_j \{s(\mu(\{i: y_i \geq j\})) \wedge j\} = Su_{\circ_s}^2(\mathbf{y}_{\mathbf{x}}), \quad (16)$$

²In order to get an integer-valued index, in the original paper authors consider the function $s: \mathbb{N}_0 \rightarrow \mathbb{N}_0$ with $s(0) = 0$ and $s(k) \geq 1$ for each $k \in \mathbb{N}$.

where $\mathbf{y}_{\mathbf{x}}$ is the conjugate of \mathbf{x} . Moreover,

$$\begin{aligned} \text{Su}_{\circ_s}^2(\mathbf{x}) &= \max_k \{k \wedge s(\mu(\{i: x_i \geq k\}))\} = \max_k \{x_k \wedge s(\mu(\{i: x_i \geq x_k\}))\} \\ &= \max_k \{x_k \wedge s(k)\} = \max_k \{\mu(\{i: y_i \geq k\}) \wedge s(k)\} = \text{Su}_2^{\circ_s}(\mathbf{y}_{\mathbf{x}}). \end{aligned} \quad (17)$$

It is easy to see that $\text{Su}_{\circ_s}^2(\mathbf{x})$ and $\text{Su}_2^{\circ_s}(\mathbf{x})$ for $s(a) = a$ (under the convention $s(\infty) = \infty$) coincide with the h -index of \mathbf{x} . In consequence, this proves Proposition 31 from [43], i.e., the symmetry $H(\mathbf{x}) = H(\mathbf{y}_{\mathbf{x}})$ of h -index. However, the integrals $\text{Su}_2^{\circ_s}(\cdot)$ and $\text{Su}_{\circ_s}^2(\cdot)$ are not symmetric in general, i.e., the equalities $\text{Su}_2^{\circ_s}(\mathbf{x}) = \text{Su}_2^{\circ_s}(\mathbf{y}_{\mathbf{x}})$ and $\text{Su}_{\circ_s}^2(\mathbf{x}) = \text{Su}_{\circ_s}^2(\mathbf{y}_{\mathbf{x}})$ need not hold for each s and \mathbf{x} .

Example 5.1. For $s(a) = 2a$ and the scientific record $\mathbf{x} = (3, 0, \dots)$ with $\mathbf{y}_{\mathbf{x}} = (1, 1, 1, 0, \dots)$ we have $\text{Su}_2^{\circ_s}(\mathbf{x}) = 1 = \text{Su}_{\circ_s}^2(\mathbf{y}_{\mathbf{x}})$, but $\text{Su}_2^{\circ_s}(\mathbf{y}_{\mathbf{x}}) = 2 = \text{Su}_{\circ_s}^2(\mathbf{x})$. Note that for $s(a) = \lambda a$ with $\lambda > 0$ it follows from (17) that $\text{Su}_{\circ_s}^2(\mathbf{x}) = \max_k \{x_k \wedge (\lambda k)\}$. This index was introduced in [20, Definition 2].

Proposition 5.2. Let $\circ_s: [0, \infty]^2 \rightarrow [0, \infty]$ be such that $a \circ_s b = s(a)$ with $s: [0, \infty] \rightarrow [0, \infty]$ being an increasing and continuous function such that $s(0) = 0$. Then for each scientific record \mathbf{x} we have

$$\begin{aligned} (a) \quad K_s(\mathbf{x}) &= \text{Su}_2^{\circ_{[\hat{s}]}}(\mathbf{x}), \\ (b) \quad K_s(\mathbf{x}) &= \lfloor \text{Su}_2^{\circ_{\hat{s}}}(\mathbf{x}) \rfloor = \lfloor \hat{s}(\text{Su}_2^{\circ_s}(\mathbf{x})) \rfloor, \end{aligned}$$

where $\lfloor \cdot \rfloor$ is the floor function and $\hat{s} = s^{-1}$.

Proof. Observe that $K_s(\mathbf{x}) = \max\{k: s^{-1}(x_k) \geq k\} = \max\{k: \lfloor s^{-1}(x_k) \rfloor \geq k\} = \max_k \{k \wedge \lfloor s^{-1}(x_k) \rfloor\}$. From (16), we get $K_s(\mathbf{x}) = \text{Su}_2^{\circ_{[\hat{s}]}}(\mathbf{x})$, as $\circ_{[\hat{s}]}$ is an admissible fusion function. Next, note that $\text{Su}_2^{\circ_{[\hat{s}]}}(\mathbf{x}) = \lfloor \max_k \{k \wedge s^{-1}(x_k)\} \rfloor$, since $\lfloor k \wedge a \rfloor = k \wedge \lfloor a \rfloor$ for each $a \geq 0$ and $k \in \mathbb{N}$ and $\max_k g(z_k) = g(\max_k z_k)$ for any nondecreasing function g . Using (16) again, we obtain $K_s(\mathbf{x}) = \lfloor \text{Su}_2^{\circ_s}(\mathbf{x}) \rfloor$. Moreover, $K_s(\mathbf{x}) = \lfloor s^{-1}(\max_k \{s(k) \wedge x_k\}) \rfloor = \lfloor \hat{s}(\text{Su}_2^{\circ_s}(\mathbf{x})) \rfloor$, where the latter equality follows from (17). \square

All the above considerations are true also for the upper/lower n -Sugeno integral for any $n \geq 2$.

(ii) Upper and lower 2- h -indices We return back to the original indices our motivation comes from. Indeed, Mesiar and Gągolewski [28] introduced the upper 2- h -index and the lower 2- h -index of a scientific record \mathbf{x} as follows:

$$H_2^u(\mathbf{x}) = \max_k \{(k + H(\mathbf{x})) \wedge x_k\}, \quad (18)$$

$$H_2^l(\mathbf{x}) = H(\mathbf{x}) + \max_k \{(k - H(\mathbf{x}))_+ \wedge x_k\} = \max_k \{k \wedge (x_k + H(\mathbf{x}))\}, \quad (19)$$

where $a_+ = \max(a, 0)$. In other words, upper 2- h -index is h -index increased by the value of h -index calculated for the scientist's output after removing h citations from each work. On the other hand, lower 2- h -index is h -index increased by the value of h -index of a scientific record $\mathbf{x}|_{H(\mathbf{x})} = (x_{H(\mathbf{x})+1}, x_{H(\mathbf{x})+2}, \dots)$. The latter h -index of $\mathbf{x}|_{H(\mathbf{x})}$ corresponds to h -index of \mathbf{x} without publications in the Hirsch core, cf. [34].

Example 5.3. Let $\mathbf{x} = (6, 6, 4, 3, 1, 1, 1, 0, \dots)$. Clearly, $H(\mathbf{x}) = 3$, and $\mathbf{x}_{|H(\mathbf{x})} = (3, 1, 1, 1, 0, \dots)$ is the scientific record obtained from \mathbf{x} after removing the Hirsch core, i.e., the first three papers. Since $H(\mathbf{x}_{|H(\mathbf{x})}) = 1$, we have $H_2^l(\mathbf{x}) = 4$. Analogously, $H_2^u(\mathbf{x}) = 5$, see Fig. 1.

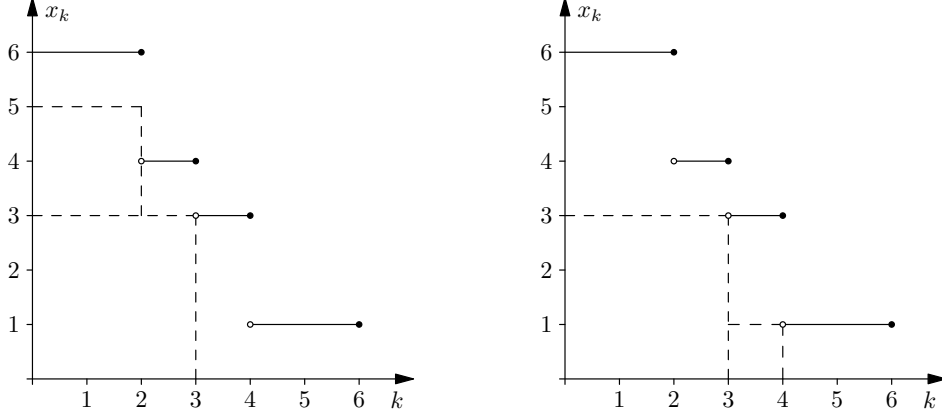


Figure 1: Illustration of formula (18) for $H_2^u(\mathbf{x}) = 5$ (left) and (19) for $H_2^l(\mathbf{x}) = 4$ (right).

Proposition 5.4. For each scientific record \mathbf{x} we have $Su_2^+(\mathbf{x}) = H_2^l(\mathbf{x}) = Su_+^2(\mathbf{y}_{\mathbf{x}})$ and $Su_+^2(\mathbf{x}) = H_2^u(\mathbf{x}) = Su_2^+(\mathbf{y}_{\mathbf{x}})$.

Proof. For proving the statements, the admissible fusion function and the link function is $\circ_s = +$. Repeating the considerations from (16) and (17) we finish the proof. \square

(iii) H_α and H^β -indices Next we show that the indices H_α and H^β recently introduced in [24] as

$$H_\alpha(\mathbf{x}) = \max_k \{ \lfloor (x_k/\alpha) \wedge \mu(\{i: x_i \geq x_k\}) \rfloor \} = \max_k \{ \lfloor (x_k/\alpha) \wedge k \rfloor \}, \quad \alpha > 0,$$

$$H^\beta(\mathbf{x}) = \lceil \max_k \{ x_k \wedge (\mu(\{i: x_i \geq x_k\})/\beta) \} \rceil = \lceil \max_k \{ x_k \wedge (k/\beta) \} \rceil, \quad \beta > 0,$$

are also a special case of upper/lower Sugeno integral. Here, $\lceil \cdot \rceil$ is the ceiling of a real number. Index H_α is able to compensate a lower number of citations and H^β compensates a lower number of papers.

Proposition 5.5. For each scientific record \mathbf{x} we have

- (i) $H_\alpha(\mathbf{x}) = Su_2^{\circ_s}(\mathbf{x})$ with $s(a) = \lfloor a/\alpha \rfloor$,
- (ii) $H^\beta(\mathbf{x}) = Su_{\circ_s}^2(\mathbf{x})$ with $s(a) = \lceil a/\beta \rceil$.

Proof. (i) Based on (16) we have $Su_2^{\circ_s}(\mathbf{x}) = \max_k \{ \lfloor x_k/\alpha \rfloor \wedge k \}$. To finish the proof one can use the fact that $\lfloor a \wedge k \rfloor = \lfloor a \rfloor \wedge k$ for each $a \geq 0$ and $k \in \mathbb{N}$.

(ii) Using (17) we get $Su_{\circ_s}^2(\mathbf{x}) = \max_k \{ x_k \wedge \lceil k/\beta \rceil \}$. Since $\lceil a \wedge k \rceil = \lceil a \rceil \wedge k$ for $a \geq 0$ and $k \in \mathbb{N}$, so $Su_{\circ_s}^2(\mathbf{x}) = \max_k \{ \lceil x_k \wedge (k/\beta) \rceil \}$. To get the statement, use $\max_k g(z_k) = g(\max_k z_k)$ for any nondecreasing function g . \square

Using the similar arguments as in the proof of Proposition 5.5 (ii) one can show that $H_\alpha(\mathbf{x}) = \lceil \text{Su}_2^{\circ s}(\mathbf{x}) \rceil$ with $s(a) = a/\alpha$ and $H^\beta(\mathbf{x}) = \lceil \text{Su}_{\circ s}^2(\mathbf{x}) \rceil$ with $s(a) = a/\beta$. Thence and from Proposition 5.2 (b) we conclude that H_α is a special case of the generalized Kosmulski index K_s with $s(k) = \alpha k$.

(iv) Iterated h -index In 2009 in García-Pérez [18, 19] considered a multidimensional h -index and showed that the additional components are useful to distinguish individuals with the same h -index. This approach has been studied further in [4] in order to provide its axiomatic characterization. Formally, the iterated h -index iH of a scientific record \mathbf{x} is a vector $iH(\mathbf{x}) = (iH_1(\mathbf{x}), iH_2(\mathbf{x}), \dots)$ with the components $iH_n(\mathbf{x})$ defined for each $n \in \mathbb{N}$ by

$$iH_n(\mathbf{x}) = \max_k \{k \wedge x_{iH_0(\mathbf{x})+\dots+iH_{n-1}(\mathbf{x})+k}\}$$

with $iH_0(\mathbf{x}) := 0$. Clearly, $iH_1(\mathbf{x}) = H(\mathbf{x})$ and $iH_1(\mathbf{x}) \geq iH_2(\mathbf{x}) \geq \dots$. Also, it is easy to see that $iH_n(\mathbf{x}) = \text{Su}_+^n(\mathbf{x}) - \text{Su}_+^{n-1}(\mathbf{x})$. Thus,

Proposition 5.6. *For each scientific record \mathbf{x} and each $n \in \mathbb{N}$ we have $\text{Su}_+^n(\mathbf{x}) = \sum_{k=1}^n iH_k(\mathbf{x})$.*

(v) p -index and c -index It follows from Proposition 5.6 that the functional defined by $\text{Su}_+^\infty(\mathbf{x}) := \sup_n \text{Su}_+^n(\mathbf{x})$ gives a number of publications with at least one citation. This index is known as the p -index (see [40, Definition 2.5]). On the other hand, the number $\text{Su}_\infty^+(\mathbf{x}) := \sup_n \text{Su}_n^+(\mathbf{x}) = x_1$ represents a number of citations of the most important paper and it is called the c -index [40, Definition 2.6], or the maximum-index [42, Definition 2.5]. The p - and c -indices measure almost completely opposite aspects of the performance of a researcher. The p -index can be seen as a measure of productivity with focusing on productivity (i.e., number of papers) and paying almost no attention to impact (i.e., number of times a paper has been cited). On the other hand, the c -index can be seen as a measure of impact with focusing on impact and paying no attention at all to productivity. For instance, it prefers a single highly cited paper over a large number of slightly lower cited papers. Finally, the s -index defined by $s(\mathbf{x}) = \sum_{i=1}^\infty x_i$ equals the total number of citations of all papers published by the scientist. Thus, the s -index takes into account all papers published by a scientist and not only the most cited paper (as in c -index).

(B) Aggregation functions

Nowadays, aggregation processes naturally appear in almost every discipline and importance of aggregation functions may be seen in various applications including data fusion, decision making, computer science, social choice, etc. We shall show here that both the upper n -Sugeno and the lower n -Sugeno

integrals are new aggregation functions under some mild assumptions on maps \circ and \star with a very natural max-min-type representations. Firstly we recall the definition of an aggregation function.

Definition 5.7. [2, Definition 1.5] A function $A: [0, \bar{y}]^m \rightarrow [0, \bar{y}]$ is said to be an *m-ary aggregation function*, if it is nondecreasing and it satisfies the boundary conditions $A(0, \dots, 0) = 0$ and $A(\bar{y}, \dots, \bar{y}) = \bar{y}$.

Put $X = \{1, 2, \dots, m\}$, $Y = [0, \bar{y}]$ and $\mu \in \mathcal{M}_{(X, \mathcal{A})}$ such that $\mu(X) \geq \bar{y}$. For $n \geq 2$ the upper n -Sugeno integral $\text{Su}_n^\circ(\mathbf{x})$ with $\mathbf{x} = (x_1, \dots, x_m)$, $x_i \in Y$, and an admissible fusion map \circ satisfying $\bar{y} \circ \bar{y} = \bar{y}$, is an m -ary aggregation function. Indeed, from Proposition 3.3 (d) we get $\text{Su}_n^\circ(\bar{y}\mathbb{1}_X) = \bar{y}$ and $\text{Su}_n^\circ(0\mathbb{1}_X) = 0$ for all n . Monotonicity follows from Proposition 3.2 (a). Additionally, the lower n -Sugeno integral $\text{Su}_\star^n(\mathbf{x})$ for $n \geq 2$ is also an aggregation function if the link map $\star: [0, \infty] \times Y \rightarrow [0, \infty]$ is such that $\star \geq \wedge$. In fact, the monotonicity follows from Proposition 4.2 (a), and by Proposition 4.3 (f) we have $\text{Su}_\star^n(\bar{y}\mathbb{1}_X) = \bar{y}$ and $\text{Su}_\star^n(0\mathbb{1}_X) = 0$. Moreover, for each $n \in \mathbb{N}$ we obtain

$$\text{Su}_{n+1}^\circ(\mathbf{x}) = \bigvee_{T \subset X} [((\bigwedge_{i \in T} x_i) \circ \text{Su}_n^\circ(\mathbf{x})) \wedge \mu(T)], \quad \text{Su}_\star^{n+1}(\mathbf{x}) = \bigvee_{T \subset X} [(\bigwedge_{i \in T} x_i) \wedge (\mu(T) \star \text{Su}_\star^n(\mathbf{x}))]$$

(see Theorem 3.6 and Theorem 4.5 for $\mathcal{A} = 2^X$) providing the weighted max-min-type representations of the two sequences of aggregation functions.

Conclusions

Generalizing the upper and lower 2- h -indices of Mesiar and Gągolewski [28] we have constructed upper and lower n -Sugeno integrals via iterating the Sugeno integral. These two classes of new functionals also include the generalized Kosmulski index [13] and H_α -index [24]. We have examined some of their universal mathematical properties that are useful in various fields such as scientometry, theory of integral and aggregation functions. Since there is only a few number of papers combining the above fields, the present paper stimulates a deeper study of the relationship between nonlinear functionals and scientometric indices. Thus, an applied research is supported by a theoretical research.

As a by-product, we have partially solved the question on computation of certain pseudo-decomposition integral providing the representation of Benvenuti integral of n -th order with respect to operations $\oplus = +$ and $\odot = \wedge$ as the lower n -Sugeno integral with respect to $+$. So, our approach provides a new way to look at pseudo-decomposition integrals and possibilities of their computation.

Acknowledgement

Authors would like to express their sincere thanks and gratitude to anonymous reviewers for their thoughtful suggestions toward the improvement of the paper. This work was supported by the Slovak Research and Development Agency under the contract No. APVV-16-0337. The work is also cofinanced by bilateral call Slovak-Poland grant scheme No. SK-PL-18-0032 together with the Polish National Agency for Academic Exchange under the contract No. PPN/BIL/2018/1/00049/U/00001.

Appendix

Proof of Theorems 3.5 and 4.6. The arguments are similar to those of [6, Theorem 2]. Put

$$S_{n+1} = \sup_{t \in Y} \{ (t \circ S_n) \wedge (\mu(\{f \geq t\}) \star S_n) \}, \quad Z_{n+1} = \inf_{t \in Y} \{ (t \circ Z_n) \vee (\mu(\{f > t\}) \star Z_n) \}$$

for all $n \geq 1$ with $S_1 = \text{Su}(f) = Z_1$, where

(A) the map \circ is given in Theorem 3.5 and $a \star b = a$, or

(B) the map \star is defined in Theorem 4.6 and $a \circ b = a$.

Assume that $S_1 > 0$ as if $S_1 = 0$, then $S_n = 0 = Z_n$ for all n (see Propositions 3.3 (a) and 4.3 (a)). By induction we show that $S_n = Z_n$ for all n . Clearly $S_1 = Z_1$. Suppose that $S_k = Z_k$ for all $k \leq n$. Set

$$I := \{t \in Y : \mu(\{f \geq t\}) \star S_n \geq t \circ S_n\}, \quad J := \{t \in Y : \mu(\{f > t\}) \star S_n \geq t \circ S_n\}.$$

Clearly, $J \subset I$ and $0 \in J$ in the case (B). In the case (A) we have $a \star b = a$, so we find that

$$\begin{aligned} 0 \circ S_n &\leq S_n = \inf_{t \in Y} \{ (t \circ S_{n-1}) \vee \mu(\{f > t\}) \} \\ &\leq (0 \circ S_{n-1}) \vee \mu(\{f > 0\}) \leq S_{n-1} \vee \mu(\{f > 0\}) \\ &\leq \dots \leq S_1 \vee \mu(\{f > 0\}) = \mu(\{f > 0\}), \end{aligned}$$

as $0 \circ y \leq y$ and $S_1 = \text{Su}(f) = \inf_{t \in Y} \{t \vee \mu(\{f > t\})\} \leq \mu(\{f > 0\})$. In consequence, $0 \in J$ in both cases. Since $t \mapsto t \circ S$ is nondecreasing and $t \mapsto \mu(\{f > t\}) \star S_n$ is nonincreasing, we have $I = [0, a]$ or $I = [0, a)$ and $J = [0, b]$ or $J = [0, b)$ with $b \leq a$. We need to show that $a = b$. Suppose that $b < a$. Hence by the definition of I and J , we have

$$\mu(\{f > t\}) \star S_n < t \circ S_n \leq \mu(\{f \geq t\}) \star S_n$$

for any $t \in (b, a)$. Let $b < d < c < a$. As $\{f \geq c\} \subset \{f > d\}$, we get

$$\mu(\{f \geq c\}) \star S_n \leq \mu(\{f > d\}) \star S_n < d \circ S_n \leq c \circ S_n \leq \mu(\{f \geq c\}) \star S_n,$$

a contradiction.

For each interval D , let $D^c = [0, \bar{y}] \setminus D$. By continuity of the maps $t \mapsto t \circ s$ and $t \mapsto t \star s$ we obtain

$$\begin{aligned} S_{n+1} &= \sup_{t \in Y} \{(t \circ S_n) \wedge (\mu(\{f \geq t\}) \star S_n)\} = \sup_{t \in I} \{t \circ S_n\} \vee \sup_{t \in I^c} \{\mu(\{f \geq t\}) \star S_n\} \\ &= \begin{cases} \bar{y} \circ S_n & \text{if } I = [0, \bar{y}], \\ (a \circ S_n) \vee (\mu(\{f \geq a^+\}) \star S_n) & \text{if } I = [0, a], a < \bar{y}, \\ (a \circ S_n) \vee (\mu(\{f \geq a\}) \star S_n) & \text{if } I = [0, a), a \leq \bar{y} \end{cases} \end{aligned}$$

with the convention that $\sup_{\emptyset} = 0$ and $\inf_{\emptyset} = \infty$. Observe that

- if $I = [0, a]$ for $a < \bar{y}$, then from the definition of I , $\mu(\{f \geq a^+\}) \star S_n \leq a \circ S_n$, so $S_{n+1} = a \circ S_n$,
- if $I = [0, a)$ for $a \leq \bar{y}$, we have $\mu(\{f \geq a\}) \star S_n < a \circ S_n$, so $S_{n+1} = a \circ S_n$.

This implies that $S_{n+1} = a \circ S_n$. Further, we have

$$\begin{aligned} Z_{n+1} &= \inf_{t \in Y} \{(t \circ S_n) \vee (\mu(\{f > t\}) \star S_n)\} = \inf_{t \in J} \{\mu(\{f > t\}) \star S_n\} \wedge \inf_{t \in J^c} \{t \circ S_n\} \\ &= \begin{cases} 0 \star S_n & \text{if } J = [0, \bar{y}], \\ (a \circ S_n) \wedge (\mu(\{f > a\}) \star S_n) & \text{if } J = [0, a], a < \bar{y}, \\ (a \circ S_n) \wedge (\mu(\{f > a^-\}) \star S_n) & \text{if } J = [0, a), a \leq \bar{y} \end{cases} \\ &= \begin{cases} 0 \star S_n & \text{if } J = [0, \bar{y}], \\ a \circ S_n & \text{if } J = [0, a], a < \bar{y} \text{ or } J = [0, a), a \leq \bar{y}, \end{cases} \end{aligned}$$

as

- if $J = [0, a]$ and $a < \bar{y}$, then $\mu(\{f > a\}) \star S_n \geq a \circ S_n$, so $Z_{n+1} = a \circ S_n$,
- if $J = [0, a)$ and $a \leq \bar{y}$, then $\mu(\{f > a^-\}) \star S_n \geq a \circ S_n$, and so $Z_{n+1} = a \circ S_n$.

Consequently, we need to show that $S_{n+1} = Z_{n+1}$ if $J = [0, \bar{y}] = I$. Indeed, we have $S_{n+1} = \bar{y} \circ S_n$ and $Z_{n+1} = 0 \star S_n$. Moreover,

$$0 \star S_n = \mu(\{f > \bar{y}\}) \star S_n \geq \bar{y} \circ S_n.$$

In the case (A), we have $0 = 0 \star S_n \geq \bar{y} \circ S_n \geq 0$, so $S_{n+1} = Z_{n+1}$. In the case (B), $S_{n+1} = \bar{y}$, $Z_{n+1} = 0 \star S_n$ and $0 \star S_n \geq \bar{y}$. As $S_n \geq 0 \star S_n$ and $S_n \leq \bar{y}$, we get $0 \star S_n = \bar{y}$, and so $S_{n+1} = Z_{n+1}$. The proof is complete. \square

Proof of Theorem 4.10. We begin with the formula (13). It is clear that $I_1^{+, \wedge}(\mu, f) = \text{Su}(f) = \text{Su}_+^1(f)$. From Lemma 2.1 (e) it follows that if $\text{Su}(f) = 0$, then $\mu(\{f \geq t\}) = 0$ for all $t > 0$. Hence by Proposition 4.3 (a), $I_n^{+, \wedge}(\mu, f) = 0 = \text{Su}_+^n(f)$ and the assertion holds for any n .

From now on let us assume $\text{Su}(f) > 0$. To get a better understanding, we first consider the case $n = 2$, that is, we show that $\text{Su}_+^2(f) = \sup_{b_2 \in Y} M_2(b_2)$, where

$$M_2(b_2) = (b_2 \wedge \mu(\{f \geq b_2\})) + \sup_{b_1 \geq b_2} \{(b_1 - b_2) \wedge \mu(\{f \geq b_1\})\}.$$

Clearly, $b_2 \wedge \mu(\{f \geq b_2\}) \leq \text{Su}(f)$, so $b_2 \wedge \mu(\{f \geq b_2\}) \leq b_2 \wedge \text{Su}(f)$ and

$$\begin{aligned} M_2(b_2) &\leq (b_2 \wedge \text{Su}(f)) + \sup_{b_1 \geq b_2 \wedge \text{Su}(f)} \{(b_1 - (b_2 \wedge \text{Su}(f))) \wedge \mu(\{f \geq b_1\})\} \\ &= \sup_{b_1 \geq b_2 \wedge \text{Su}(f)} \{b_1 \wedge (\mu(\{f \geq b_1\}) + (b_2 \wedge \text{Su}(f)))\} \\ &\leq \sup_{b_1 \geq b_2 \wedge \text{Su}(f)} \{b_1 \wedge (\mu(\{f \geq b_1\}) + \text{Su}(f))\} \\ &= \text{Su}(f) + \sup_{b_1 \geq b_2 \wedge \text{Su}(f)} \{(b_1 - \text{Su}(f)) \wedge \mu(\{f \geq b_1\})\}. \end{aligned}$$

Since $\sup_{b_1 \in [b_2 \wedge \text{Su}(f), \text{Su}(f)]} \{(b_1 - \text{Su}(f)) \wedge \mu(\{f \geq b_1\})\} = 0$, we have

$$M_2(b_2) \leq \sup_{b_1 \geq \text{Su}(f)} \{b_1 \wedge (\mu(\{f \geq b_1\}) + \text{Su}(f))\} = \text{Su}_+^2(f).$$

From the above it follows that

$$\sup_{b_2 \in Y} M_2(b_2) \leq \text{Su}_+^2(f). \quad (20)$$

Now we show that the reverse inequality holds in (20). Recall that $\text{Su}(f) > 0$. Evidently

$$\sup_{b_2 \in Y} M_2(b_2) \geq \lim_{b_2 \nearrow \text{Su}(f)} M_2(b_2).$$

By Lemma 2.1 (c), we get

$$\begin{aligned} \lim_{b_2 \nearrow \text{Su}(f)} M_2(b_2) &= \text{Su}(f) + \lim_{b_2 \nearrow \text{Su}(f)} \sup_{b_1 \geq b_2} \{(b_1 - b_2) \wedge \mu(\{f \geq b_1\})\} \\ &\geq \text{Su}(f) + \lim_{b_2 \nearrow \text{Su}(f)} \sup_{b_1 \geq b_2} \{(b_1 - \text{Su}(f)) \wedge \mu(\{f \geq b_1\})\} \\ &= \text{Su}(f) + \sup_{b_1 \geq \text{Su}(f)} \{(b_1 - \text{Su}(f)) \wedge \mu(\{f \geq b_1\})\}. \end{aligned}$$

Thus, $\sup_{a_2 \in Y} M_2(a_2) \geq \text{Su}_+^2(f)$, so there is the equality in (20), as claimed.

Now, we show that the assertion (14) holds for all $n > 2$. Observe that

$$\text{I}_n^{+, \wedge}(\mu, f) = \sup_{b_n \in Y} M_n(b_n, b_{n+1}),$$

where $b_{n+1} = 0$ and M_n is defined recursively using the formula

$$M_k(b_k, b_{k+1}) := [(b_k - b_{k+1}) \wedge \mu(\{f \geq b_k\})] + \sup_{b_{k-1} \geq b_k} M_{k-1}(b_{k-1}, b_k)$$

for $k = 2, \dots, n$ with the initial condition $M_1(b_1, b_2) = (b_1 - b_2) \wedge \mu(\{f \geq b_1\})$. Put $M_k^*(b_k) := \sup_{b_{k-1} \geq b_k} M_{k-1}(b_{k-1}, b_k)$ for $k \geq 2$. Mimicking the proof for $n = 2$, we obtain

$$\begin{aligned}
M_n(b_n, b_{n+1}) &\leq (b_n \wedge \text{Su}(f)) + \sup_{b_{n-1} \geq b_n} \{[(b_{n-1} - b_n) \wedge \mu(\{f \geq b_{n-1}\})] + M_{n-1}^*(b_{n-1})\} \\
&\leq (b_n \wedge \text{Su}(f)) \\
&\quad + \sup_{b_{n-1} \geq b_n \wedge \text{Su}(f)} \{[(b_{n-1} - (b_n \wedge \text{Su}(f))) \wedge \mu(\{f \geq b_{n-1}\})] + M_{n-1}^*(b_{n-1})\} \\
&= \sup_{b_{n-1} \geq b_n \wedge \text{Su}(f)} \{[b_{n-1} \wedge (\mu(\{f \geq b_{n-1}\}) + (b_n \wedge \text{Su}(f)))] + M_{n-1}^*(b_{n-1})\} \\
&\leq \sup_{b_{n-1} \geq b_n \wedge \text{Su}(f)} \{[b_{n-1} \wedge (\mu(\{f \geq b_{n-1}\}) + \text{Su}(f))] + M_{n-1}^*(b_{n-1})\} \\
&= \sup_{b_{n-1} \geq b_n \wedge \text{Su}(f)} N_{n-1}(b_{n-1}).
\end{aligned}$$

Here and subsequently,

$$N_k(b) := [b \wedge (\mu(\{f \geq b\}) + \text{Su}_+^{n-k}(f))] + M_k^*(b)$$

for all $k = 1, \dots, n-1$ with the convention $M_1^*(b) := 0$. By the very definition of $\text{Su}_+^2(f)$, we have $b \wedge (\mu(\{f \geq b\}) + \text{Su}(f)) \leq b \wedge \text{Su}_+^2(f)$ for $b \in Y$. Thus

$$\begin{aligned}
N_{n-1}(b_{n-1}) &\leq (b_{n-1} \wedge \text{Su}_+^2(f)) + \sup_{b_{n-2} \geq b_{n-1}} \{[(b_{n-2} - b_{n-1}) \wedge \mu(\{f \geq b_{n-2}\})] + M_{n-2}^*(b_{n-2})\} \\
&\leq (b_{n-1} \wedge \text{Su}_+^2(f)) \\
&\quad + \sup_{b_{n-2} \geq b_{n-1} \wedge \text{Su}_+^2(f)} \{[(b_{n-2} - (b_{n-1} \wedge \text{Su}_+^2(f))) \wedge \mu(\{f \geq b_{n-2}\})] + M_{n-2}^*(b_{n-2})\} \\
&= \sup_{b_{n-2} \geq b_{n-1} \wedge \text{Su}_+^2(f)} \{[b_{n-2} \wedge (\mu(\{f \geq b_{n-2}\}) + (b_{n-1} \wedge \text{Su}_+^2(f)))] + M_{n-2}^*(b_{n-2})\} \\
&\leq \sup_{b_{n-2} \geq b_{n-1} \wedge \text{Su}_+^2(f)} \{[b_{n-2} \wedge (\mu(\{f \geq b_{n-2}\}) + \text{Su}_+^2(f))] + M_{n-2}^*(b_{n-2})\} \\
&= \sup_{b_{n-2} \geq b_{n-1} \wedge \text{Su}_+^2(f)} N_{n-2}(b_{n-2})
\end{aligned}$$

for any $b_{n-1} \geq b_n \wedge \text{Su}(f)$. In the same manner we obtain for $k = 1, \dots, n-2$,

$$\sup_{b_{n-k} \geq b_{n-k+1} \wedge \text{Su}_+^k(f)} N_{n-k}(b_{n-k}) \leq \sup_{b_{n-k-1} \geq b_{n-k} \wedge \text{Su}_+^{k+1}(f)} N_{n-k-1}(b_{n-k-1}).$$

As a consequence, we get

$$M_n(b_n, b_{n+1}) \leq \sup_{b_{n-1} \geq b_n \wedge \text{Su}(f)} N_{n-1}(b_{n-1}) \leq \dots \leq \sup_{b_1 \geq b_2 \wedge \text{Su}_+^{n-1}(f)} N_1(b_1) = \text{Su}_+^n(f)$$

for all $b_n \in Y$. Therefore,

$$\text{I}_n^{+, \wedge}(\mu, f) \leq \text{Su}_+^n(f).$$

To finish the proof it is sufficient to show that $\sup_{b_n \in Y} M_n(b_n, b_{n+1}) \geq \text{Su}_+^n(f)$. Using Lemma 2.1 (c) and mimicking the proof for $n = 2$, we obtain

$$\begin{aligned}
\sup_{b_n \in Y} M_n(b_n, b_{n+1}) &\geq \lim_{b_n \nearrow \text{Su}(f)} M_n(b_n, b_{n+1}) \\
&= \text{Su}(f) + \lim_{b_n \nearrow \text{Su}(f)} \sup_{b_{n-1} \geq b_n} \{[(b_{n-1} - b_n) \wedge \mu(\{f \geq b_{n-1}\})] + M_{n-1}^*(b_{n-1})\} \\
&\geq \text{Su}(f) + \lim_{b_n \nearrow \text{Su}(f)} \sup_{b_{n-1} \geq b_n} \{[(b_{n-1} - \text{Su}(f))_+ \wedge \mu(\{f \geq b_{n-1}\})] + M_{n-1}^*(b_{n-1})\} \\
&= \text{Su}(f) + \sup_{b_{n-1} \geq \text{Su}(f)} \{[(b_{n-1} - \text{Su}(f)) \wedge \mu(\{f \geq b_{n-1}\})] + M_{n-1}^*(b_{n-1})\} \\
&= \sup_{b_{n-1} \geq \text{Su}(f)} N_{n-1}(b_{n-1}). \tag{21}
\end{aligned}$$

By Proposition 4.4, $\text{Su}_+^2(f) \geq \text{Su}(f) > 0$. Proposition 4.3 (b) and (21) implies

$$\begin{aligned}
\sup_{b_n \in Y} M_n(b_n, b_{n+1}) &\geq \lim_{b_{n-1} \nearrow \text{Su}_+^2(f)} \left([b_{n-1} \wedge (\mu(\{f \geq b_{n-1}\}) + \text{Su}(f))] + M_{n-1}^*(b_{n-1}) \right) \\
&= \text{Su}_+^2(f) + \lim_{b_{n-1} \nearrow \text{Su}_+^2(f)} \sup_{b_{n-2} \geq b_{n-1}} M_{n-2}(b_{n-2}, b_{n-1}). \tag{22}
\end{aligned}$$

Next, we get

$$\begin{aligned}
&\lim_{b_{n-1} \nearrow \text{Su}_+^2(f)} \sup_{b_{n-2} \geq b_{n-1}} M_{n-2}(b_{n-2}, b_{n-1}) \\
&\geq \lim_{b_{n-1} \nearrow \text{Su}_+^2(f)} \sup_{b_{n-2} \geq b_{n-1}} \{[(b_{n-2} - \text{Su}_+^2(f))_+ \wedge \mu(\{f \geq b_{n-2}\})] + M_{n-2}^*(b_{n-2})\} \\
&\geq \sup_{b_{n-2} \geq \text{Su}_+^2(f)} \{[(b_{n-2} - \text{Su}_+^2(f)) \wedge \mu(\{f \geq b_{n-2}\})] + M_{n-2}^*(b_{n-2})\}. \tag{23}
\end{aligned}$$

Thus from (22) and (23) we obtain

$$\sup_{b_n \in Y} M_n(b_n, b_{n+1}) \geq \sup_{b_{n-2} \geq \text{Su}_+^2(f)} N_{n-2}(b_{n-2}).$$

Repeating the same reasoning we get

$$\sup_{b_n \in Y} M_n(b_n, b_{n+1}) \geq \sup_{b_1 \geq \text{Su}_+^{n-1}(f)} N_1(b_1) = \text{Su}_+^n(f),$$

as required. The proof is complete. \square

Proof of Theorem 4.12. Let $n \geq 2$. We need to show that $\text{Su}_+^n(f) = L_n$, where

$$L_n = \inf \left\{ \sum_{i=1}^n ((b_i - b_{i+1}) \vee \mu(\{f > b_i\})) : 0 = b_{n+1} \leq b_n \leq \dots \leq b_1 \leq \bar{y} \right\}.$$

Clearly, $L_n = \inf_{b_n \in Y} M_n(b_n, b_{n+1})$, where M_n is defined recursively using the formula

$$M_k(b_k, b_{k+1}) := [(b_k - b_{k+1}) \vee \mu(\{f > b_k\})] + \inf_{b_{k-1} \geq b_k} M_{k-1}(b_{k-1}, b_k)$$

for any $k = 2, \dots, n$ with $M_1(b_1, b_2) := (b_1 - b_2) \vee \mu(\{f > b_1\})$. For simplicity, put $M_k^*(b_k) := \inf_{b_{k-1} \geq b_k} M_{k-1}(b_{k-1}, b_k)$ for $k = 2, \dots, n$. Since $b \vee \mu(\{f > b\}) \geq \text{Su}(f)$, we have $b \vee \mu(\{f > b\}) \geq b \vee \text{Su}(f)$ for all $b \in Y$. Thus,

$$\begin{aligned}
M_n(b_n, b_{n+1}) &\geq (b_n \vee \text{Su}(f)) + \inf_{b_{n-1} \geq b_n \wedge \text{Su}(f)} \{[(b_{n-1} - (b_n \vee \text{Su}(f))) \vee \mu(\{f > b_{n-1}\})] + M_{n-1}^*(b_{n-1})\} \\
&= \inf_{b_{n-1} \geq b_n \wedge \text{Su}(f)} \{[b_{n-1} \vee (\mu(\{f > b_{n-1}\}) + (b_n \vee \text{Su}(f)))] + M_{n-1}^*(b_{n-1})\} \\
&\geq \inf_{b_{n-1} \geq b_n \wedge \text{Su}(f)} \{[b_{n-1} \vee (\mu(\{f > b_{n-1}\}) + \text{Su}(f)))] + M_{n-1}^*(b_{n-1})\} \\
&= \inf_{b_{n-1} \geq b_n \wedge \text{Su}(f)} N_{n-1}(b_{n-1}), \tag{24}
\end{aligned}$$

where $N_k(b) := [b \vee (\mu(\{f > b\}) + \text{Su}_+^{n-k}(f))] + M_k^*(b)$ for $k = 1, \dots, n-1$ with $M_1^*(b_1) := 0$. Theorem 4.6 gives that $b \vee (\mu(\{f > b\}) + \text{Su}(f)) \geq b \vee \text{Su}_+^2(f)$ for all $b \in Y$. Thus,

$$\begin{aligned}
N_{n-1}(b_{n-1}) &\geq (b_{n-1} \vee \text{Su}_+^2(f)) + \inf_{b_{n-2} \geq b_{n-1}} \{[(b_{n-2} - b_{n-1}) \vee \mu(\{f > b_{n-2}\})] + M_{n-2}^*(b_{n-2})\} \\
&\geq (b_{n-1} \vee \text{Su}_+^2(f)) \\
&\quad + \inf_{b_{n-2} \geq b_{n-1} \wedge \text{Su}_+^2(f)} \{[(b_{n-2} - (b_{n-1} \vee \text{Su}_+^2(f))) \vee \mu(\{f > b_{n-2}\})] + M_{n-2}^*(b_{n-2})\} \\
&= \inf_{b_{n-2} \geq b_{n-1} \wedge \text{Su}_+^2(f)} \{[b_{n-2} \vee (\mu(\{f > b_{n-2}\}) + (b_{n-1} \vee \text{Su}_+^2(f)))] + M_{n-2}^*(b_{n-2})\} \\
&\geq \inf_{b_{n-2} \geq b_{n-1} \wedge \text{Su}_+^2(f)} \{[b_{n-2} \vee (\mu(\{f > b_{n-2}\}) + \text{Su}_+^2(f)))] + M_{n-2}^*(b_{n-2})\} \\
&= \inf_{b_{n-2} \geq b_{n-1} \wedge \text{Su}_+^2(f)} N_{n-2}(b_{n-2})
\end{aligned}$$

for any $b_{n-1} \geq b_n \wedge \text{Su}(f)$. Analogously, for $k = 1, \dots, n-2$ we get

$$\inf_{b_{n-k} \geq b_{n-k+1} \wedge \text{Su}_+^k(f)} N_{n-k}(b_{n-k}) \geq \inf_{b_{n-k-1} \geq b_{n-k} \wedge \text{Su}_+^{k+1}(f)} N_{n-k-1}(b_{n-k-1}), \tag{25}$$

where we use the fact that for all $b \in Y$ and $k = 2, \dots, n-1$

$$b \vee (\mu(\{f > b\}) + \text{Su}_+^{k-1}(f)) \geq b \vee \text{Su}_+^k(f)$$

(see Theorem 4.6). As a consequence of (25) and (24), we obtain

$$\begin{aligned}
M_n(b_n, b_{n+1}) &\geq \inf_{b_{n-1} \geq b_n \wedge \text{Su}(f)} N_{n-1}(b_{n-1}) \geq \dots \geq \inf_{b_1 \geq b_2 \wedge \text{Su}_+^{n-1}(f)} N_1(b_1) \\
&\geq \inf_{b_1 \in Y} N_1(b_1) = \text{Su}_+^n(f)
\end{aligned}$$

for all $b_n \in Y$. Therefore $L_n \geq \text{Su}_+^n(f)$. We show that $L_n \leq \text{Su}_+^n(f)$.

Let $\text{Su}_+^n(f) = \bar{y}$. Then

$$L_n \leq M_n(\bar{y}, b_{n+1}) = \bar{y} + M_n^*(\bar{y}) = \bar{y} = \text{Su}_+^n(f).$$

Assume that $\text{Su}_+^n(f) < \bar{y}$. Using Lemma 2.1 (d), we have

$$\begin{aligned}
L_n &\leq \lim_{b_n \searrow \text{Su}(f)} M_n(b_n, b_{n+1}) \\
&= \text{Su}(f) + \lim_{b_n \searrow \text{Su}(f)} \inf_{b_{n-1} \geq b_n} \{ [(b_{n-1} - b_n) \vee \mu(\{f > b_{n-1}\})] + M_{n-1}^*(b_{n-1}) \} \\
&\leq \text{Su}(f) + \lim_{b_n \searrow \text{Su}(f)} \inf_{b_{n-1} \geq b_n} \{ [(b_{n-1} - \text{Su}(f)) \vee \mu(\{f > b_{n-1}\})] + M_{n-1}^*(b_{n-1}) \} \\
&= \text{Su}(f) + \inf_{b_{n-1} \geq \text{Su}(f)} \{ [(b_{n-1} - \text{Su}(f)) \vee \mu(\{f > b_{n-1}\})] + M_{n-1}^*(b_{n-1}) \} \\
&= \inf_{b_{n-1} \geq \text{Su}(f)} N_{n-1}(b_{n-1}). \tag{26}
\end{aligned}$$

From (26), Propositions 4.4 and 4.3 (c) we get

$$\begin{aligned}
L_n &\leq \inf_{b_{n-1} \geq \text{Su}(f)} N_{n-1}(b_{n-1}) \leq \inf_{b_{n-1} \geq \text{Su}_+^2(f)} N_{n-1}(b_{n-1}) \\
&\leq \lim_{b_{n-1} \searrow \text{Su}_+^2(f)} ([b_{n-1} \vee (\mu(\{f > b_{n-1}\}) + \text{Su}(f))] + M_{n-1}^*(b_{n-1})) \\
&= \text{Su}_+^2(f) + \lim_{b_{n-1} \searrow \text{Su}_+^2(f)} M_{n-1}^*(b_{n-1}). \tag{27}
\end{aligned}$$

Next, we have

$$\begin{aligned}
\lim_{b_{n-1} \searrow \text{Su}_+^2(f)} M_{n-1}^*(b_{n-1}) &\leq \lim_{b_{n-1} \searrow \text{Su}_+^2(f)} \inf_{b_{n-2} \geq b_{n-1}} \{ [(b_{n-2} - \text{Su}_+^2(f)) \vee \mu(\{f > b_{n-2}\})] + M_{n-2}^*(b_{n-2}) \} \\
&\leq \inf_{b_{n-2} \geq \text{Su}_+^2(f)} \{ [(b_{n-2} - \text{Su}_+^2(f)) \vee \mu(\{f > b_{n-2}\})] + M_{n-2}^*(b_{n-2}) \}. \tag{28}
\end{aligned}$$

Thus, from (27) and (28) we obtain

$$L_n \leq \inf_{b_{n-1} \geq \text{Su}_+^2(f)} N_{n-2}(b_{n-2}).$$

Repeating the same reasoning we get

$$L_n \leq \inf_{b_1 \geq \text{Su}_+^{n-1}(f)} N_1(b_1) = \text{Su}_+^{n-1}(f) + \inf_{b \geq \text{Su}_+^{n-1}(f)} \{ (b - \text{Su}_+^{n-1}(f)) \vee \mu(\{f > b\}) \} = \text{Su}_+^n(f),$$

as required. \square

References

- [1] Bassan, B., & Spizzichino, F. (2005). Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. *Journal of Multivariate Analysis*, 93, 313–339.
- [2] Beliaikov, G., Bustince, H., & Calvo, T. (2016). *A practical guide to averaging functions*. (1st ed.). Heidelberg: Springer.

- [3] Benvenuti, P., Mesiar, R., & Vivona, D. (2002). Monotone set functions-based integrals. In E. Pap (Ed.), *Handbook of Measure Theory* (pp. 1329–1379). Amsterdam: Elsevier.
- [4] Béal, S., Ferrières, S., Rémila, E., & Solal, P. (2016). An axiomatization of the iterated h -index and applications to sport rankings. *Working papers*, 2016-11, CRESE.
- [5] Boczek, M., & Kaluszka, M. (2016). On the Minkowski-Hölder type inequalities for generalized Sugeno integrals with an application. *Kybernetika*, 52, 329–347.
- [6] Boczek, M., & Kaluszka, M. (2017). On conditions under which some generalized Sugeno integrals coincide: A solution to Dubois’ problem. *Fuzzy Sets and Systems*, 326, 81–88.
- [7] Borzová-Molnárová, J., Halčinová, L., & Hutník, O. (2015). The smallest semicopula-based universal integrals I: Properties and characterizations. *Fuzzy Sets and Systems*, 271, 1–17.
- [8] Brandão, L.C., & de Mello, J.C.C.B.S. (2019). A multi-criteria approach to the h -index. *European Journal of Operational Research*, 276, 357–363.
- [9] Bustince, H., Fernández, J., Kolesárová, A., & Mesiar, R. (2015). Directional monotonicity of fusion functions. *European Journal of Operational Research*, 244, 300–308.
- [10] Chateauneuf, A., Grabisch, M., & Rico, A. (2008). Modeling attitudes toward uncertainty through the use of the Sugeno integral. *Journal of Mathematical Economics*, 44, 1084–1099.
- [11] Choquet, G. (1954). Theory of capacities. *Annales de l’institut Fourier*, 5, 131–295.
- [12] Couceiro, M., Dubois, D., Prade, H., & Waldhauser, T. (2016). Decision-making with Sugeno integrals. *Order*, 33, 517–535.
- [13] Deineko, W.G., & Woeginger, G.J. (2009). A new family of scientific impact measures: The generalized Kosmulski-indices. *Scientometrics*, 80, 819–826.
- [14] Dubois, D., Prade, H., Rico, A., & Teheux, B. (2017). Generalized Qualitative Sugeno Integrals. *Information Sciences*, 415–416, 429–445.
- [15] Durante, F., & Sempi, C. (2005). Semicopulae. *Kybernetika*, 41, 315–328.
- [16] Durante, F., & Sempi, C. (2016). *Principles of copula theory*. (1st ed.). Boca Raton: CRC Press.
- [17] Franceschini, F., & Maisano, D.A. (2010). Analysis of the Hirsch index’s operational properties. *European Journal of Operational Research*, 203, 494–504.

- [18] García-Pérez, M. (2009). A multidimensional extension to Hirsch's h -index. *Scientometrics*, 81, 779–785.
- [19] García-Pérez, M. (2012). An extension of the h index that covers the tail and the top of the citation curve and allows ranking researchers with similar h . *Journal of Informetrics*, 6, 689–699.
- [20] Gągolewski, M., & Mesiar, R. (2012). Aggregating different paper quality measures with a generalized h -index. *Journal of Informetrics*, 6, 566–579.
- [21] Gągolewski, M., & Mesiar, R. (2014). Monotone measures and universal integrals in a uniform framework for the scientific impact assessment problem. *Information Sciences*, 263, 166–174.
- [22] Greco, S., Mesiar, R., Rindone, F., & Sipeky, L. (2016). Decomposition approaches to integration without a measure. *Fuzzy Sets and Systems*, 287, 37–47.
- [23] Hirsch, J.E. (2005). An index to quantify an individual's scientific research output. *Proceedings of the National Academy of Sciences of USA*, 102, 16569–16572.
- [24] Jin, L., Mesiar, R., Stupňanová, A.: Sugeno integrals, H_α and H^β indices: how to compare scientists from different academic areas. *IEEE Transactions on Fuzzy Systems*. DOI 10.1109/TFUZZ.2019.2914625.
- [25] Kaluszka, M., Okolewski, A., & Boczek, M. (2014). On Chebyshev type inequalities for generalized Sugeno integrals. *Fuzzy Sets and Systems*, 244, 51–62.
- [26] Kawabe, J. (2015). The bounded convergence in measure theorem for nonlinear integral functionals. *Fuzzy Sets and Systems*, 271, 31–42.
- [27] Kosmulski, M. (2006). A new Hirsch-index saves time and works equally well as the origin h -index. *ISSSI Newsletter*, 2, 4–6.
- [28] Mesiar, R., & Gągolewski, M. (2016). H-index and other Sugeno integrals: some defects and their compensation. *IEEE Transactions on Fuzzy Systems*, 24, 1668–1672.
- [29] Mesiar, R., Kolesárová, A., Bustince, H., Dimuro, G.P., & Bedregal, B.C. (2016). Fusion functions based discrete Choquet-like integrals. *European Journal of Operational Research*, 252, 601–609.
- [30] Mesiar, R., & Stupňanová, A. (2013). Decomposition integrals. *International Journal of Approximate Reasoning*, 54, 1252–1259.
- [31] Mingers, J., & Leydesdorff, J. (2015). A review of theory and practice in scientometrics. *European Journal of Operational Research*, 246, 1–19.

- [32] Miroiu, A. (2013). Axiomatizing the Hirsch index: Quantity and quality disjoined. *Journal of Informetrics*, 7, 10–15.
- [33] Quesada, A. (2011). Further characterizations of the Hirsch index. *Scientometrics*, 87, 107–114.
- [34] Rousseau, R. (2006). New developments related to the Hirsch index. *Science Focus*, 1, 23–25.
- [35] Struk, P. (2006). Extremal fuzzy integrals. *Soft Computing*, 10, 502–505.
- [36] Suárez García, F., & Gil Álvarez, P. (1986). Two families of fuzzy integrals. *Fuzzy Sets and Systems*, 18, 67–81.
- [37] Sugeno, M. (1974). Theory of Fuzzy Integrals and Its Applications. Tokyo Institute of Technology, Tokyo. Ph.D. thesis.
- [38] Torra, V., & Narukawa, Y. (2006). The interpretation of fuzzy integrals and their application to fuzzy systems. *International Journal of Approximate Reasoning*, 41, 43–58.
- [39] Torra, V., & Narukawa, Y. (2008). The h -index and the number of citations: Two fuzzy integrals. *IEEE Transactions on Fuzzy Systems*, 16, 795–797.
- [40] Van Eck, N.J., & Waltman, L. (2008). Generalizing the h - and g -indices. *Journal of Informetrics*, 2, 263–271.
- [41] Wang, Z., & Klir, G. (2009). *Generalized Measure Theory*. (1st ed.). New York: Springer.
- [42] Woeginger, G.J. (2008). An axiomatic characterization of the Hirsch-index. *Mathematical Social Sciences*, 56, 224–232.
- [43] Woeginger, G.J. (2008). A symmetry axiom for scientific impact indices. *Journal of Informetrics*, 2, 298–303.
- [44] Zhai, J., Zang, L., & Zhou, Z. (2018). Ensemble dropout extreme learning machine via fuzzy integral for data classification. *Neurocomputing*, 275, 1043–1052.