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Large Portfolio Losses in a Turbulent Market

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Abstract

Consider a large credit portfolio of defaultable obligors in a turbulent market. Accordingly, the credit quality process of each obligor is described by a stochastic differential equation consisting of a drift term reflecting the trend, an individual volatility term reflecting the idiosyncratic risk, and a common volatility term reflecting the systematic risk. Moreover, for each obligor a market beta is used to measure its loading on the systematic risk. The obligor defaults at the first passage time of the credit quality process. We approximate the portfolio loss as the portfolio size becomes large. For the usual case where the individual defaults do not become rare, we establish a limit theorem for the portfolio loss, while for the other case where the individual defaults become rare, which is due to portfolio effect, we establish an asymptotic estimate for its tail probability. Both results show that the portfolio loss is driven by the systematic risk, while this driving force is amplified by the market beta. As an application, we derive asymptotic estimates for the value at risk and expected shortfall of the portfolio loss. Moreover, we implement intensive numerical studies to examine the accuracy of the obtained approximations and conduct some sensitivity analysis.

Keywords: OR in banking, credit quality process, systematic risk, market beta, continuous Ocone martingale

1 Introduction

Default risk is manifested by losses arising from obligors' failure to fulfill their contractual obligations, where the general term obligor represents a person or an entity with contractual obligations or a defaultable asset. The severe and far-reaching impacts of the collapse of the financial system during the financial crisis of 2007–2009 have stimulated the need to effectively

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quantify and manage portfolio losses due to defaults. A precursor to a financial crisis is usually that markets become turbulent, generating large and random fluctuations. Kritzman and Li (2010) and Aghion et al. (2017) conduct empirical studies of turbulence in financial downturns.

In this paper, we consider a large credit portfolio of defaultable obligors in a turbulent market. In the current global economic environment, interest rates remain record low while in sharp contrast markets exhibit high volatilities. To address these new challenges, when introducing a credit quality process for each individual obligor, we focus primarily on the volatility process. Accordingly, the credit quality process of each obligor is described by a stochastic differential equation (SDE) consisting of a drift term reflecting the trend, an individual volatility term reflecting the idiosyncratic risk that affects this specific obligor only, and a common volatility term reflecting the systematic risk that is inherent in the entire market and not subject to diversification. Hence, the credit portfolio overall is dominated by the systematic risk. Moreover, a market beta is used to measure each individual obligor's loading on the systematic risk.

We describe the default of each obligor through the first passage time of its credit quality process, which is consistent with a majority of works in the credit risk literature; see e.g. Hilscher and Wilson (2017). The obligor's expected loss is determined by the exposure at default (EAD), the loss given default (LGD) as the percentage of the EAD, and the probability of default (PD).

We are concerned with the total amount of losses from defaults of a large credit portfolio of defaultable obligors, and we conduct an asymptotic study as the portfolio size tends to infinity. In the real world, large financial institutions like banks and credit card companies can easily have a portfolio consisting of tens of thousands of obligors. Assume that individual default thresholds are negative, determined by a negative deterministic function (called the representing default threshold) subject to individual variations. We consider two cases. In the first case, the representing default threshold does not vary with the portfolio size, which is to describe that the individual defaults, though still with different probabilities, do not become rare as the portfolio expands. In the second case, the representing default threshold diverges, which is to describe that the individual defaults become rare as the portfolio expands. While the former is often a standard assumption in the study of portfolio losses, the latter is more relevant for a large portfolio in which the rarity of defaults results from the portfolio effect, namely the decrease in overall risk due to the increase in the portfolio size. See Bassamboo et al. (2008) and Tang et al. (2019) for related discussions and see also Appendix B for two motivating examples illustrating that individual defaults can become rare under portfolio effect.

As our main results, for the former case we establish a limit theorem for the portfolio loss showing that the average loss converges weakly to a random variable expressed as a conditional expectation given the systematic risk, while for the latter case we establish an asymptotic estimate for the tail probability of the portfolio loss expressed as the tail of the systematic risk variable. Then we apply this asymptotic estimate to approximate the value at risk (VaR) and expected shortfall (ES) of the portfolio loss. Moreover, we implement intensive numerical studies to demonstrate that the approximations obtained by our results fit well in the tail area of the portfolio loss, and to test the sensitivity of the portfolio loss through its VaR and ES with respect to certain risk parameters.

The asymptotic study of portfolio losses has an immediate implication for economic capital assessment, in particular under prudent regulatory frameworks in the current catastrophic economic environment. For example, the Basel Committee on Banking Supervision (2019) stipulates that banks' default risk capital requirement must be based on a VaR model computed weekly in a one-year horizon at a 99.9% confidence level, and that banks' market risk capital requirement must be based on an ES model computed on a daily basis at a 97.5% confidence level according to the internal models approach. See Glasserman et al. (2007) for related discussions.

The topic of large portfolio losses has become increasingly interesting in the credit risk literature in recent years. So far the study of this topic has been developed along directions of both structural models and reduced-form models, and in each direction a great deal of attention has been paid to common factors (including in particular the systematic risk) among all obligors. Since the seminal work of Vasicek (1991), many researchers have carried out the study under static structural settings; see Lucas et al. (2001), Gordy (2003), Dembo et al. (2004), Schloegl and O'Kane (2005), Glasserman et al. (2007), Bassamboo et al. (2008), and Tang et al. (2019), among others. In parallel, there is also a large number of works under continuous-time settings. For example, Bush et al. (2011) consider a continuous-time structural model for a large portfolio of credit risky assets with emphasis on the correlation with a market factor, and investigate the large portfolio limit of this system through a stochastic partial differential equation. The work is further extended by Bujok and Reisinger (2012) who introduce a jump part to the asset value process. Since the establishment of a reduced-form model by Jarrow and Turnbull (1995), in the study of large portfolio losses, numerous variants of this model have been proposed to capture new features such as contagion, interaction, and self-excitation; see Giesecke and Weber (2004, 2006), Giesecke et al. (2013, 2015), Dai Pra et al. (2009), Dai Pra and Tolotti (2009), Hambly and Søjmark (2019), and Hambly and Kolliopoulos (2020), among others.

Methodologically, most of the above-mentioned works employ classical limit theorems including the law of large numbers (LLN), the central limit theorem (CLT), and the large deviation principle (LDP) to derive approximations for large portfolio losses. Computationally, as large portfolio losses are rare and, hence, difficult to observe under the naive Monte Carlo method, importance sampling becomes a commonly used alternative to increase the efficiency of simulation. See Glasserman and Li (2005), Glasserman et al. (2008), Bassamboo et al. (2008), Chan and Kroese (2010), Liu (2015), and Sirignano and Giesecke (2018), among others.

Our contribution to this literature is threefold. First, in terms of modeling, to reflect the turbulent market condition we model the credit quality process of each individual obligor by a SDE consisting of a drift term, an individual volatility term, and a common volatility term. We also allow for obligor-specific loadings on the systematic risk by using market betas. Second, methodologically, as we consider a large portfolio, those obligor-specific variables are randomized to a sequence of independent and identically distributed (i.i.d.) vectors. In other words, this randomization procedure identifies a continuum, which is well justified by the LLN, to underlie the large number of obligors of different risk types. Hence, the credit portfolio under our investigation can potentially be heterogeneous; see a detailed discussion in Subsection 2.2. Third, our main results are approximations to the portfolio loss, showing that the large portfolio loss is driven by the systematic risk while this driving force is amplified by the market beta. This aligns with an observation of Acharya et al. (2017) on the financial crisis of 2007–2009 that "financial institutions had levered up on similar large portfolios of securities and loans that faced little idiosyncratic risk, but large amounts of systematic risk."

The rest of the paper consists of five sections. Section 2 depicts a continuous-time structural model for the credit quality processes; Section 3 presents our main results; Section 4 implements intensive numerical studies; Section 5 concludes the paper with some remarks; finally, the Appendix collects the proofs of the three main results and constructs two motivating examples.

2 The credit risk model

2.1 Credit quality processes

Throughout the paper, we use $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ to denote a filtered probability space that accommodates all sources of randomness. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which represents the information available in the market, satisfies usual hypotheses, i.e., $\mathcal{F}_0$ contains all P null sets of $\mathcal{F}$ and $\{\mathcal{F}_t\}_{t\geq 0}$ is right-continuous.

Consider a credit portfolio of n defaultable obligors. For each obligor i, $i = 1, ..., n$, its credit quality process is described by a stochastic process $\{X_{i,t}\}_{t\geq 0}$, which starts with $X_{i,0} = 0$ and evolves according to the SDE

$$dX_{i,t} = \mu_{i,t}dt + \sigma_{i,t}dW_{i,t} + \beta_i\sigma_{0,t}dW_{0,t}, \qquad t \geq 0, \tag{2.1}$$

where $\{\mu_{i,t}\}_{t\geq 0}$ is a real-valued drift process, $\{W_{i,t}\}_{t\geq 0}$ and $\{W_{0,t}\}_{t\geq 0}$ are standard Brownian motions, $\{\sigma_{i,t}\}_{t\geq 0}$ and $\{\sigma_{0,t}\}_{t\geq 0}$ are nonnegative square-integrable volatility processes, all assumed to be $\{\mathcal{F}_t\}_{t\geq 0}$ adapted, and finally β_i is a nonnegative random variable, interpreted as market beta. Written in the form of stochastic integrals, the credit quality process defined by the SDE (2.1) equals

$$X_{i,t} = \int_0^t \mu_{i,s}ds + \int_0^t \sigma_{i,s}dW_{i,s} + \beta_i \int_0^t \sigma_{0,s}dW_{0,s} := M_{i,t} + I_{i,t} + \beta_i S_t. \tag{2.2}$$

As (2.2) shows, the volatility of each credit quality process consists of two parts, namely an individual part $\{I_{i,t}\}_{t\geq 0}$ reflecting the idiosyncratic risk that affects obligor i only and a common part $\{S_t\}_{t\geq 0}$ reflecting the systematic risk that is inherent in the entire market. The use of a market beta β_i allows each obligor i to be exposed to the systematic risk to a different extent. We will focus on the systematic part $\{S_t\}_{t\geq 0}$, which is a culprit behind the turbulent market. In particular, in our second main result we will assume ξ_T in (3.1), which integrates the systematic variance process, to be heavy tailed. In the current global economic environment, interest rate levels remain record low while in sharp contrast the markets exhibit high volatilities.

Our model (2.1) for a credit quality process slightly generalizes the one used by Hilscher and Wilson (2017) for the logarithms of unlevered asset values. See also Packham et al. (2013) and Cantia and Tunaru (2017), who model credit quality processes as similar stochastic integrals in the study of credit gap risk associated with many credit derivatives. All such models descend from the seminal structural model of default by Merton (1974). To see this clearly, like Hilscher and Wilson (2017) we may interpret $\{X_{i,t}\}_{t\geq 0}$ in (2.2) as the logarithm of an unlevered asset value. Broadly speaking, such continuous-time structural models incorporating various betas have been widely applied under the capital asset pricing model (CAPM) framework and the issue of model calibration has been extensively discussed; see e.g. Campbell et al. (2001) for an empirical study of the volatilities of common stocks at the market, industry, and firm levels.

Similarly to Tang et al. (2019), we assume that the individual default threshold of obligor i takes the form $\ell_i f_n$ for $i = 1, \ldots, n$, where f_n is a negative deterministic function, identical across the portfolio, representing the scale of the default thresholds, while each ℓ_i is a nonnegative random variable capturing an individual variation from the representing default threshold f_n. Alternatively, one may directly work on the individual default probabilities by assuming them to be exogenously given and then determining corresponding individual default thresholds. The two approaches are actually equivalent, but for our study of the portfolio loss under $n \to \infty$, it is more convenient to work on the representing default threshold f_n. Also note that the credit quality processes will be made identically distributed, and for this reason we must include individual variation factors ℓ_i in default thresholds to allow different individual default probabilities. We make the representing default threshold f_n depend on n so that the individual default probabilities also vary with the portfolio size n.

Then each obligor i defaults during a fixed time horizon $[0, T]$ for some $T > 0$ if the event

$$D_{i,T} = \left(\inf_{0 \le t \le T} X_{i,t} \le \ell_i f_n \right)$$

happens. In this way, obligor i's default indicator (DI) is $1_{D_{i,T}}$ and PD is equal to $P(D_{i,T})$. Note that these individual PDs are allowed to be different thanks to the use of individual variations.

2.2 The portfolio loss

Recall the credit portfolio of n defaultable obligors described above. For each obligor i, denote by θ_i its LGD, which can be decomposed into $\theta_i = \vartheta_i(1 - R_i)$ with ϑ_i denoting its EAD and R_i its recovery rate (RR). Collectively, the portfolio loss due to defaults by time T is given by

$$L_n = \sum_{i=1}^{n} \theta_i 1_{D_{i,T}}. \tag{2.3}$$

Our research target is the limiting behavior of L_n as the portfolio size n tends to ∞.

We make standing assumptions below:

Assumption 2.1 *Recall our model (2.1) for the credit quality processes:*

(a) The obligor-specific vectors $(\{M_{i,t}, I_{i,t}\}_{0 \le t \le T}, \theta_i, \beta_i, \ell_i)$, $i = 1, \ldots, n$, form a sequence of i.i.d. versions of a generic vector $(\{M_t, I_t\}_{0 \le t \le T}, \theta, \beta, \ell)$;

(b) The common part $\{S_t\}_{0 \le t \le T}$ is independent of the obligor-specific vectors;

(c) The three generic variables θ, β, and ℓ are nonnegative, but other than this they are completely general.

Some remarks follow. In Assumption 2.1(a), the generic obligor-specific vector process $\{M_t, I_t\}_{0 \le t \le T}$ consists of two stochastic integrals, $M_t = \int_0^t \mu_s ds$ and $I_t = \int_0^t \sigma_s dW_s$, $t \ge 0$, which are independent of the systematic process $S_t = \int_0^t \sigma_{0,s} dW_{0,s}$, $t \ge 0$. By Assumption 2.1(a,b), the credit quality processes $\{X_{i,t}\}_{0 \le t \le T}$, $i = 1, \ldots, n$, conditional on $\{S_t\}_{0 \le t \le T}$ are i.i.d., so that our model falls into the category of conditionally independent structural models, which is prevailing in credit risk modeling.

Assumption 2.1(a) ensures the homogeneity of the portfolio. Nevertheless, the general structure of the generic vector $(\{M_t, I_t\}_{0\le t\le T}, \theta, \beta, \ell)$ represents a continuum to underlie different obligors and, hence, it potentially allows for heterogeneity of the credit portfolio. We illustrate this in the following example; similar discussions can be found in Section 2 of Glasserman et al. (2007), Section 3 of Bassamboo et al. (2008), and Section 2 of Tang et al. (2019).

Example 2.1 Consider a finite partition of the index set as $\{1,\ldots,n\} = \bigcup_{j=1}^{k} N_j$ for a fixed finite positive integer k. Assume that each sub-portfolio approximately occupies a positive proportion of the whole portfolio; precisely,

$$\lim_{n\to\infty} \frac{|N_j|}{n} = q_j \in (0,1), \qquad j = 1,\ldots,k, \tag{2.4}$$

where each $|N_j|$ denotes the cardinality of the set N_j and the limits satisfy $\sum_{j=1}^{k} q_j = 1$. Further assume that within each sub-portfolio the triplets $(\theta_i, \beta_i, \ell_i)$, $i \in N_j$, are identical to a deterministic triplet, say, $(\theta_j^*, \beta_j^*, \ell_j^*)$. In this way, the portfolio is composed of a finitely many homogenous sub-portfolios but overall is heterogenous. On the other hand, (2.4) actually introduces a joint discrete distribution for (θ, β, ℓ),

$$P\left((\theta, \beta, \ell) = (\theta_j^*, \beta_j^*, \ell_j^*)\right) = q_j, \qquad j = 1,\ldots,k,$$

which stands for a continuum for the n obligors. In other words, as n becomes large, upon a randomization procedure, the whole portfolio may be thought of as composed of obligors with $(\theta_i, \beta_i, \ell_i)$, $i = 1,\ldots,n$, identical in distribution to (θ, β, ℓ) introduced above. This reduces to our situation under Assumption 2.1(a).

In the literature, for tractability both EAD and RR are often assumed to be independent random variables or even deterministic; see Glasserman et al. (2007), Liu (2015), and Cantia and Tunaru (2017), among many others. However, there is strong empirical evidence that EAD, LGD, and DI are dependent on each other. Actually, along this direction numerous models have been proposed to capture the interdependence between them; see e.g. Altman (2005), Frye and Jacobs (2012), and Jankowitsch et al. (2014). Our model well accommodates this feature by allowing complete flexibility for the interdependence between EAD, LGD, and DI. Furthermore, as in Assumption 2.1(c), we treat the LGD θ, the market beta β, and the variation factor ℓ as three general random variables of an arbitrary dependence structure.

The RR is the ratio of the recovery value to the EAD. Some works evaluate the EAD as the face value of the asset at default and the recovery value as the market value of the asset soon after default, while others evaluate the EAD as the outstanding value at default and the recovery value as the discounted value of all cash flows recovered. Based on the evaluation approaches to employ, there are mainly two types of RR: the market RR and the workout (also called ultimate) RR; see e.g. Calabrese and Zenga (2010) for a comprehensive review. RR can be larger than 100% due to differences in coupon rates (high) and prevailing interest rates (low), as pointed out by Schuermann (2004) and Mora (2012), or less than 0 if the sum of the discounted outgoing payments is higher than that of the incoming payments, as observed by Grunert and Weber (2009). Our work can be extended to allow RR to go above 100% or below 0 but this will make the presentation longer, which we skip here.

In this paper, we study the limiting behavior of the portfolio loss L_n in two cases. The first is a usual case in which the individual defaults do not become rare as the portfolio expands, which we realize by letting the representing default threshold f_n invariant with n. Our focus is on the second case in which the individual defaults become rare as the portfolio expands, which we realize by assuming $f_n \downarrow -\infty$ as $n \to \infty$. As our main contributions, for the first case we establish a limit theorem for L_n, and for the second case we establish an asymptotic estimate for the tail probability of L_n. Both results show that the portfolio loss L_n is driven by the systematic volatility process and amplified by the market beta.

We end this section with a remark. Our model (2.3) for the portfolio loss ignores the time effect of defaults. To address this, we need to define individual default times. For each obligor i, its default time is the first passage time $\tau_i(f_n) = \inf\{t \geq 0 | X_{i,t} \leq \ell_i f_n\}$. Then we can consider the present value of the portfolio loss

$$\tilde{L}_n = \sum_{i=1}^{n} \exp\left\{ -\int_0^{\tau_i(f_n)} r_t dt \right\} \theta_i 1_{(\tau_i(f_n) \leq T)},$$

where $\{r_t\}_{t \geq 0}$ is a risk-free interest rate process. The method developed in this paper does not help much on the study of the limit behavior of $\tilde{L}_n$ except for numerical solutions.

3 Main results

We study the limiting behavior of the portfolio loss (2.3) over a fixed time horizon $[0, T]$ for some $T > 0$. The integral

$$\xi_T = \int_0^T \sigma_{0,s}^2 ds, \tag{3.1}$$

which is finite almost surely, integrates the systematic variance process. We simply call ξ_T the systematic risk variable.

3.1 When the individual defaults do not become rare

In our first main result below, we consider a usual case in which the individual defaults, though still with different probabilities, do not become rare as the portfolio expands, which we realize by letting $f_n \equiv f < 0$.

Theorem 3.1 *Consider the credit risk model introduced in Section 2 in which the representing default threshold $f_n \equiv f < 0$ is fixed. If $0 < E[\theta] < \infty$, then we have the following convergence in distribution:*

$$\frac{L_n}{n} \xrightarrow{d} E\left[\theta 1_{\left(\inf_{0\le t\le T}(M_t+I_t+\beta S_t)\le \ell f\right)} \middle| \mathcal{F}_T^S\right] := \digamma^S, \qquad n \to \infty. \tag{3.2}$$

The random variable $\digamma^S$ above is a functional of $\{S_t\}_{0\le t\le T}$. The obtained convergence in distribution means that, for any fixed level $b \in \mathbb{R}$ (or, more precisely, for $0 < b < E[\theta]$) at which $\digamma^S$ is continuously distributed,[1] we have

$$\lim_{n\to\infty} P\left(\frac{L_n}{n} \le b\right) = P\left(\digamma^S \le b\right). \tag{3.3}$$

We learn from Theorem 3.1 that the limiting behavior of the portfolio loss is driven by the systematic volatility process $\{\sigma_{0,t}\}_{t\ge 0}$ via $S_t = \int_0^t \sigma_{0,s} dW_{0,s}$ for $0 \le t \le T$ while this driving force is amplified by the market beta β.

3.2 When the individual defaults become rare

Now we extend the study to a more important case in which the individual defaults become rare as the portfolio expands. As we target a large credit portfolio, the portfolio effect, namely the decrease in overall risk due to the portfolio size increase, may becomes considerable. To capture this, we follow Bassamboo et al. (2008) and Tang et al. (2019) to assume that the individual PD tends to 0 as $n \to \infty$, which we realize by assuming $f_n \downarrow -\infty$. In Appendix B we construct two motivating examples to illustrate that individual defaults can become rare under portfolio effect.

[1] Although we believe that for most practical cases this functional $\digamma^S$ should be continuously distributed over the range $[0, E\theta]$, unfortunately we have not been able to establish a general result to ensure this. Nevertheless, the convergence in distribution is often sufficient for applications.

To state our second main result, Theorem 3.2 below, we need to prepare some preliminaries. First, a positive measurable function h on $\mathbb{R}_+$ is said to be regularly varying at ∞ with index $\alpha \in \mathbb{R}$, written as $h \in \mathrm{RV}_\alpha$, if

$$\lim_{x\to\infty} \frac{h(xy)}{h(x)} = y^\alpha, \qquad y > 0.$$

When $\alpha = 0$, this defines a slowly varying function at ∞. See Bingham et al. (1987) and Resnick (1987) for textbook treatments of regular variation.

In condition (a) of Theorem 3.2, we assume that $\overline{F}_{\xi_T} \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$, which means that ξ_T has a power-like tail as

$$\overline{F}_{\xi_T}(x) = x^{-\alpha} l(x), \qquad x \geq 0,$$

for some slowly varying function l. Popular distributions with a power-like tail include Pareto, Student's t, Burr, Benktander Type I and II, Loggamma, and α–stable with $0 < \alpha < 2$; see Embrechts et al. (1997). This assumption describes the situation that the systematic risk inherent in the market becomes dominating and drives the credit quality of individual obligors to deteriorate, causing turbulence to the market. Such an assumption has been used by e.g. Mikosch and Rezapour (2013), who study a heavy-tailed stochastic volatility model and discover implications of a regularly varying stochastic volatility for exhibiting extremal clustering. See also Wang et al. (2011) for an empirical study of a heavy-tailed stochastic volatility model.

Next, Theorem 3.2 involves an auxiliary function

$$\psi(x) = E\left[\theta 1_{(\beta x \geq \ell)}\right], \qquad x \geq 0. \tag{3.4}$$

Due to the nonnegativity of (θ, β, ℓ) in Assumption 2.1(c), the function $\psi(x)$ is non-decreasing in x over $\mathbb{R}_+$ with its infimum $\psi_* := \inf_{x\geq 0} \psi(x) = E\left[\theta 1_{(\ell=0)}\right]$ and supremum $\psi^* := \sup_{x\geq 0} \psi(x) = E\left[\theta 1_{(\beta>0)\cup(\beta=\ell=0)}\right]$. If both β and ℓ are strictly positive, then the range (ψ_*, ψ^*) will be identical to $(0, E[\theta])$. Introduce the set

$$\Delta_\psi = \left\{x \in \mathbb{R}_+ : \psi_* < \psi(x) < \psi^*\right\}, \tag{3.5}$$

which may be empty if, for example, both β and ℓ are degenerate. To exclude such trivialities, assume that the set Δ_ψ is non-empty; subsequently, it must be a proper finite or infinite interval. The closure of the set Δ_ψ, denoted by $\mathrm{cl}\,(\Delta_\psi)$, forms the effective domain of the function ψ.

In condition (d) of Theorem 3.2, the function ψ is required to be continuous and strictly increasing over $\mathrm{cl}\,(\Delta_\psi)$. This is to ensure that for any $b \in (\psi_*, \psi^*)$ the ordinary inverse $\psi^{\leftarrow}(b)$ exists and falls into Δ_ψ. We remark that condition (d) is easily verifiable in concrete cases.

In particular, a sufficient condition is that the conditional distribution of $\frac{\beta}{\ell}$ given $(\ell > 0)$ is continuous and strictly increasing over its range.

Theorem 3.2 *Consider the credit risk model introduced in Section 2 in which the representing default threshold f_n varies such that $f_n \downarrow -\infty$ as $n \to \infty$. In addition to Assumption 2.1, assume that the systematic volatility process $\{\sigma_{0,t}\}_{t\geq 0}$ is independent of the corresponding Brownian motion $\{W_{0,t}\}_{t\geq 0}$.*[2] *Further assume the following:*

(a) $\overline{F}_{\xi_T} \in \mathrm{RV}_{-\alpha}$ *for some* $\alpha > 0$*;*

(b) $0 < E\left[\theta^2\right] < \infty$*;*

(c) $n\overline{F}_{\xi_T}\left(f_n^2\right) \to \infty$ *as* $n \to \infty$*;*

(d) the set Δ_ψ defined in (3.5) is non-empty and the function ψ defined in (3.4) is continuous and strictly increasing over $\mathrm{cl}\,(\Delta_\psi)$*.*

Then it holds for any fixed level $b \in (\psi_*, \psi^*) \subset (0, E[\theta])$ that

$$\lim_{n\to\infty} \frac{P\left(L_n > nb\right)}{\overline{F}_{\xi_T}(f_n^2)} = E\left[|\epsilon|^{2\alpha}\right] \left(\psi^{\leftarrow}(b)\right)^{-2\alpha}, \tag{3.6}$$

where ϵ is an independent standard normal random variable.

Theorem 3.2 gives an asymptotic estimate for the tail probability $P\left(L_n > nb\right)$, which lends us the same insight as Theorem 3.1. Precisely, the tail behavior of the portfolio loss L_n is driven by the systematic volatility process $\{\sigma_{0,t}\}_{t\geq 0}$ via $\xi_T = \int_0^T \sigma_{0,s}^2 ds$ while this driving force is amplified by the market beta β.

3.3 Approximations for the VaR and ES risk measures

We apply Theorem 3.2 to derive asymptotic estimates for the VaR and ES risk measures of the portfolio loss. These are two of the most important tail risk measures, widely used in the insurance and financial industries. Formally, for a general risk variable L its VaR and ES at level $0 < q < 1$ are given by, respectively,

$$\mathrm{VaR}_q(L) = \inf\{x \in \mathbb{R} | F_L(x) \geq q\} \qquad \text{and} \qquad \mathrm{ES}_q(L) = \frac{1}{1-q}\int_q^1 \mathrm{VaR}_p(L)dp.$$

Note that $\mathrm{ES}_q(L)$, as the arithmetic average of $\mathrm{VaR}_p(L)$ over $q < p < 1$, is greater (hence, more conservative) than $\mathrm{VaR}_p(L)$. In the literature, the ES risk measure has different names subject to some subtleties, such as tail value at risk (TVaR), conditional value at risk (CVaR),

[2]This is to ensure that the stochastic integral process $S_t = \int_0^t \sigma_{0,s} dW_{0,s}$, $t \geq 0$, is a continuous Ocone martingale, which is required in the proof.

conditional tail expectation (CTE), and tail conditional expectation (TCE). See Chapter 2 of McNeil et al. (2015) for more discussions on the two risk measures.

Consider the case with a representing default threshold varying with the portfolio size as specified in Theorem 3.2. In this case, to be consistent with the current prudent regulatory frameworks, we choose a high confidence level q_n. Precisely, assume that

$$\lim_{n\to\infty} \frac{1-q_n}{\overline{F}_{\xi_T}(f_n^2)} = cE\left[|\epsilon|^{2\alpha}\right] \tag{3.7}$$

for some constant $c > 0$, which makes Theorem 3.2 immediately applicable.

Theorem 3.3 *Under the conditions of Theorem 3.2, choose a confidence level $q_n \in (0,1)$ satisfying (3.7) for some constant $c > 0$. Then we have*

$$\lim_{n\to\infty} \frac{1}{n}\mathrm{VaR}_{q_n}(L_n) = \psi\left(c^{-\frac{1}{2\alpha}}\right) \tag{3.8}$$

and

$$\lim_{n\to\infty} \frac{1}{n}\mathrm{ES}_{q_n}(L_n) = \int_0^1 \psi\left((cx)^{-\frac{1}{2\alpha}}\right) dx, \tag{3.9}$$

where the function ψ is defined in (3.4).

4 Numerical studies

4.1 Model specifications

This subsection describes model setups for our numerical studies. Since the pioneering work of Barndorff-Nielsen and Shephard (2001), it has become popular to use a Lévy-driven Ornstein–Uhlenbeck (OU) process or one of its numerous extensions to model volatility changes underlying a financial asset. See Benth (2011), Muhle-Karbe et al. (2012), and Barndorff-Nielsen and Stelzer (2013), among others, who apply OU processes to model the continuous-time volatilities of financial asset values and point out that the OU-type models are capable of reproducing most of stylized features of volatilities observed from empirical studies such as jumps, clustering, and heavy tails. In particular, Packham et al. (2013) apply this idea to model the variance process of a credit quality process. Following this trend of research, in our numerical studies we model each of the systematic and idiosyncratic variance processes by an OU process.

Recall the SDE (2.1) for the credit quality process of each individual obligor. Denote by $V_{i,t} = \sigma_{i,t}^2$ and $V_{0,t} = \sigma_{0,t}^2$, $t \geq 0$, the idiosyncratic and systematic variance processes both of which are modeled by OU processes. We now focus on the systematic variance process. Assume that it starts with $V_{0,0} = v_{0,0} > 0$ and evolves according to

$$dV_{0,t} = -\gamma V_{0,t} dt + dJ_t, \qquad t \geq 0, \tag{4.1}$$

where $\gamma > 0$ is a constant, and $\{J_t\}_{t\geq 0}$ is a compound Poisson process of the standard form

$$J_t = \sum_{i=1}^{\Lambda_t} Y_i$$

with $\{\Lambda_t\}_{t\geq 0}$ a Poisson process with intensity $\lambda > 0$, and $\{Y_1, Y_2, \ldots\}$ a sequence of i.i.d. nonnegative random variables with a generic version Y independent of $\{\Lambda_t\}_{t\geq 0}$. This model is adopted from (26) of Barndorff-Nielsen and Shephard (2001) with the background driving Lévy process specified to be a compound Poisson process. The SDE (4.1) gives a closed-form expression for $V_{0,t}$ as

$$V_{0,t} = v_{0,0} e^{-\gamma t} + e^{-\gamma t} \int_0^t e^{\gamma s} dJ_s, \qquad t \geq 0,$$

which is thus a positive process. From now on, the time horizon is restricted to $[0, 1]$. In this way, the systematic risk variable $\xi = \xi_1$ defined in (3.1) assumes a closed-form expression

$$\begin{aligned} \xi &= \int_0^1 V_{0,t} dt \\ &= \frac{v_{0,0}}{\gamma}\left(1 - e^{-\gamma}\right) + \frac{1}{\gamma}\int_0^1 \left(1 - e^{-\gamma(1-t)}\right) dJ_t \\ &= \frac{v_{0,0}}{\gamma}\left(1 - e^{-\gamma}\right) + \frac{1}{\gamma}\sum_{i=1}^{\infty} Y_i \left(1 - e^{-\gamma(1-A_i)}\right) 1_{(A_i \leq 1)}, \end{aligned} \tag{4.2}$$

where each A_i is the ith arrival time of the Poisson process $\{\Lambda_t\}_{t\geq 0}$.

Assume that Y has a regularly varying tail, i.e. $\overline{F}_Y \in \mathrm{RV}_{-\alpha}$ for some $\alpha > 0$. By the one-dimensional version of Theorem 2.1 of Resnick and Willekens (1991), we have

$$\begin{aligned} \lim_{x\to\infty} \frac{\overline{F}_\xi(x)}{\overline{F}_Y(x)} &= \frac{1}{\gamma^\alpha}\sum_{i=1}^{\infty} E\left[\left(1 - e^{-\gamma(1-A_i)}\right)^\alpha 1_{(A_i\leq 1)}\right] \\ &= \frac{1}{\gamma^\alpha}\int_0^1 \left(1 - e^{-\gamma(1-t)}\right)^\alpha \sum_{i=1}^{\infty} P\left(A_i \in dt\right) \\ &= \frac{\lambda}{\gamma^\alpha}\int_0^1 \left(1 - e^{-\gamma(1-t)}\right)^\alpha dt \\ &= \frac{\lambda}{\gamma^\alpha}\int_0^1 \left(1 - e^{-\gamma t}\right)^\alpha dt, \end{aligned} \tag{4.3}$$

where the third step is due to $\sum_{i=1}^{\infty} P\left(A_i \leq t\right) = E[\Lambda_t] = \lambda t$. This leads to $\overline{F}_\xi \in \mathrm{RV}_{-\alpha}$ and thus Theorem 3.2 becomes applicable. By the way, the discussion above can be easily extended to the case that $\{J_t\}_{t\geq 0}$ is a subordinator with a regularly varying Lévy measure; actually, Corollary 2.1 of Hao and Tang (2012) readily gives a result regarding this.

Under the assumption that the systematic variance process $V_{0,t} = \sigma_{0,t}^2$ for $t \geq 0$ follows an OU process of form (4.1), we can simulate the systematic risk variable ξ based on its closed-form expression in (4.2). Indeed, a sample size $N_\xi = 2 \times 10^7$ of ξ only takes a few seconds to generate, and thus, the estimate of the tail probability of the portfolio loss given by Theorem 3.2 can be obtained in the blink of an eye. We assume that the idiosyncratic variance processes $V_{i,t} = \sigma_{i,t}^2$ for $t \geq 0$, $i = 1, \ldots, n$, are also OU of form (4.1) and independent of the systematic counterpart.

We adopt the well-known Vasicek model, which still falls into the OU family, to describe the individual drift processes. For each obligor i, we assume that its drift process $\{\mu_{i,t}\}_{t\geq 0}$ satisfies the SDE

$$d\mu_{i,t} = a(\bar{\mu} - \mu_{i,t})dt + \sigma^{(\mu)} dW_{i,t}^{(\mu)}, \qquad t \geq 0,$$

where a is the speed of reversion, $\bar{\mu}$ captures the long term mean, $\sigma^{(\mu)}$ measures the instantaneous volatility, and $\left\{W_{i,t}^{(\mu)}\right\}_{t\geq 0}$ is an independent standard Brownian motion. Given an initial value $\mu_{i,0} = \mu_0$, the process has an explicit expression as

$$\mu_{i,t} = \mu_0 e^{-at} + \bar{\mu}\left(1 - e^{-at}\right) + \sigma^{(\mu)} e^{-at} \int_0^t e^{as} dW_{i,s}^{(\mu)}, \qquad t \geq 0.$$

To generate samples of large portfolio losses, we need to simulate paths of dependent credit quality processes. This is done by applying the Euler–Maruyama method with a partition of the interval $[0, 1]$ into 10^3 equal subintervals. Direct simulation of the paths is time-consuming, while the asymptotic estimates obtained by Theorems 3.1–3.3 all allow a significant reduction in computation time, which demonstrates one of the strengths of our results.

We summarize the model specifications and simulation sizes in Table 4.1.

Table 4.1 is here.

4.2 Convergence in distribution and sensitivity analysis with respect to μ_0

Recall Theorem 3.1. Now assume that the LGD θ is independent of the systematic part $\mathcal{F}_T^S$ and the other idiosyncratic parts $(\{M_t, I_t\}_{0\leq t\leq T}, \beta, \ell)$. In Table 4.1, we set both the LGD θ and the market beta β to be exponentially distributed with mean 1 and set the individual variation factor $\ell = 1$. Then the convergence in (3.2) becomes

$$\frac{L_n}{n} \xrightarrow{d} P\left(\inf_{0\leq t\leq 1} (M_t + I_t + \beta S_t) \leq f \,\middle|\, \mathcal{F}_1^S\right). \tag{4.4}$$

In this subsection, we examine the convergence of (4.4) by drawing Q-Q plots of its both sides.

Individual drift processes	
$d\mu_{i,t} = a(\bar{\mu} - \mu_{i,t}) + \sigma^{(\mu)} dW_{i,t}^{(\mu)}$	$\bar{\mu} = \mu_0 \in [0, 0.05]$, $a = 0.15$, $\sigma^{(\mu)} = 0.015$
Idiosyncratic variance processes	
$dV_{i,t} = -\gamma_i V_{i,t} dt + dJ_{i,t}$	$v_{i,0} = 0.02$, $\gamma_i = 2$
Systematic variance process	
$dV_{0,t} = -\gamma V_{0,t} dt + dJ_t$	$v_{0,0} = 0.02$, $\gamma = 2$
Jump parts	
$\{J_{i,t}\}_{t\geq 0}$ compound Poisson	$\lambda_i = 1$, $\overline{F}_Y(y) = \left(\frac{1}{100y+1}\right)^{\alpha_i}$ for $y \geq 0$, $\alpha_i = 4$
$\{J_t\}_{t\geq 0}$ compound Poisson	$\lambda = 4$, $\overline{F}_Y(y) = \left(\frac{1}{100y+1}\right)^{\alpha}$ for $y \geq 0$, $\alpha = 1$
Other parameters	
LGD θ	exponential with mean 1
Time horizon	$T = 1$
Market beta β	exponential with mean 1
Individual variation ℓ	1
Simulation sizes	
Portfolio loss L_n	$N_L = 10^5$ or 5×10^6
Systematic risk variable ξ	$N_\xi = 2 \times 10^7$
The probability in (4.4)	$N_E = 2 \times 10^3$

Table 4.1: This table specifies the parameters of the credit quality processes (including the drift processes, the idiosyncratic and systematic volatility processes, the LGD, the market beta, and the individual variation factor), as well as the simulation sizes throughout the numerical studies.

For the left-hand side of (4.4), we use a double-layer Monte-Carlo simulation to draw a sample of size N_L from the portfolio loss L_n. In detail, we first generate a sample path of $\{S_t\}_{0\le t\le 1}$ and then generate a sample of size n from $(\{M_t, I_t\}_{0\le t\le 1}, \theta, \beta)$. These paths are jointly used to generate n credit quality processes, producing a portfolio of size n. Based on (2.3), we obtain a simulated value of L_n. By repeating this procedure N_L times, we obtain a desired sample from L_n.

To simulate the right-hand side of (4.4), we first generate a sample path of $\{S_t\}_{0\le t\le 1}$ and then for this path we generate a sample of size N_E from $(\{M_t, I_t\}_{0\le t\le 1}, \theta, \beta)$. These paths are jointly used to generate N_E credit quality processes, some of which lead to defaults. Then by calculating the percentage of this sample causing defaults we obtain an estimate of the right-hand side of (4.4). Note that the sample size N_E used in this step can be much smaller than the portfolio size n, showing that the computational cost for a large portfolio can be much reduced by the approximation. Also repeating N_L times, we end up with a sample of size N_L from the approximation.

In order to identify scenarios with reasonable PDs, we simulate the individual PD for various values of the threshold f and the initial value μ_0 of the drift processes. Table 4.2 shows that the individual PD drops to around 1% when the default threshold decreases to -1.6. Different values of μ_0 only cause small changes to the PD, especially when f is small. This is intuitively clear as in Table 4.2 the values of μ_0 are negligible compared to the values of f.

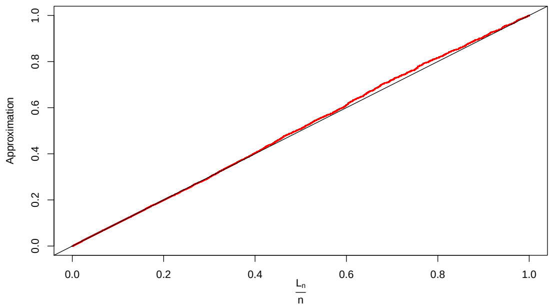
Table 4.2 is here.

μ_0 \ f	-0.4	-0.6	-0.8	-1.0	-1.2	-1.4	-1.6	-1.8	-2.0
0	0.1150	0.0630	0.0390	0.0260	0.0184	0.0137	0.0104	0.0081	0.0065
0.01	0.1125	0.0619	0.0384	0.0257	0.0181	0.0135	0.0103	0.0081	0.0064
0.02	0.1096	0.0607	0.0377	0.0252	0.0179	0.0131	0.0100	0.0079	0.0063
0.03	0.1068	0.0595	0.0371	0.0249	0.0177	0.0130	0.0099	0.0078	0.0063
0.04	0.1046	0.0584	0.0366	0.0245	0.0175	0.0128	0.0098	0.0077	0.0062
0.05	0.1020	0.0575	0.0360	0.0242	0.0172	0.0128	0.0097	0.0076	0.0062

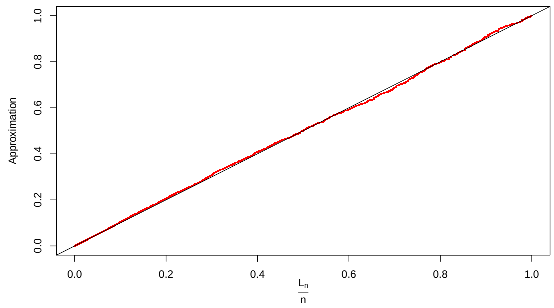
Table 4.2: This tabulates the simulated individual PDs for f varying between -0.4 and -2.0 with a stepsize 0.2 and for μ_0 varying between 0 and 0.05 with a stepsize 0.01. The simulation size for L_n is $N_L = 5 \times 10^6$.

From Table 4.2, we pick up four combinations for (f, μ_0), which are $(-0.6, 0.02)$, $(-1.0, 0.03)$, $(-1.6, 0)$, and $(-2.0, 0.05)$, corresponding to the individual PDs between 0.62% and 6.07%. The portfolio size n is chosen to be 2×10^4. Figure 4.1 gives the Q-Q plots corresponding to these four combinations. These Q-Q plots show that the approximation works nicely in mimicking the distribution of the portfolio loss.

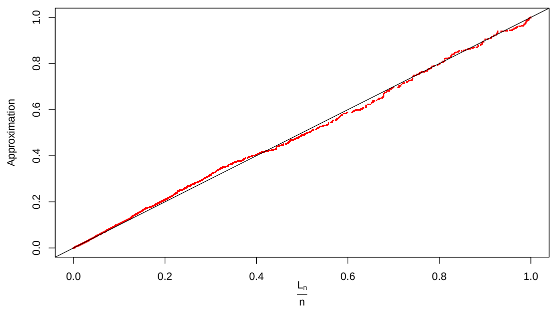
Figure 4.1 is here.



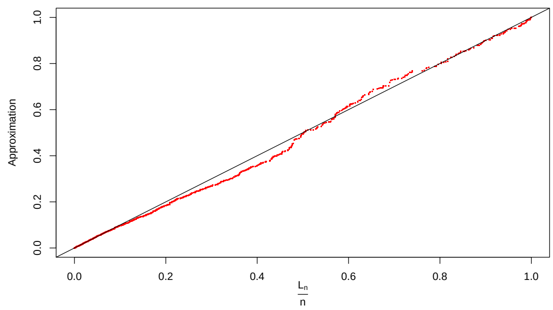
(a) Q-Q plot when $f = -0.6$ and $\mu_0 = 0.02$



(b) Q-Q plot when $f = -1.0$ and $\mu_0 = 0.03$



(c) Q-Q plot when $f = -1.6$ and $\mu_0 = 0$



(d) Q-Q plot when $f = -2.0$ and $\mu_0 = 0.05$

Figure 4.1: These Q-Q plots of $\frac{L_n}{n}$ and its approximation are used to examine the convergence of (4.4), for various combinations of the default threshold f and the initial value μ_0 of the drift processes. The simulation sizes for $\frac{L_n}{n}$ and its approximation are identically set to $N_L = 10^5$. The individual PDs are simulated to be 6.07%, 2.49%, 1.04%, and 0.62%, respectively.

Now we conduct a sensitivity analysis of the portfolio loss with respect to μ_0. We approximate the tail probability $P(L_n > nb)$ by simulating the right-hand side of (4.4) with a sample size N_L. While fixing the default threshold $f = -1.6$, we vary the values of μ_0 and b. The results are summarized in Table 4.3. We see that different values of μ_0 only cause small changes to the tail probability $P(L_n > nb)$, which can be explained in the same way as for Table 4.2.

Table 4.3 is here.

μ_0 \ b	0.2	0.3	0.4	0.5	0.6
0	(0.0125)	(0.0067)	(0.0038)	(0.0021)	(0.0011)
0.025	−3.34%	−3.32%	−3.49%	−4.03%	−4.62%
0.05	−4.32%	−4.18%	−4.30%	−4.82%	−5.61%

Table 4.3: This tabulates values of the approximated tail probability by (4.4) for $\mu_0 = 0$, 0.025, and 0.05, with b varying from 0.2 to 0.6 with a stepsize 0.1. The values in the parentheses are the approximated values of the tail probability when $\mu_0 = 0$. The values on the rows of $\mu_0 = 0.025$ and 0.05 represent the percentage changes from the values for $\mu_0 = 0$. The sample size for L_n is set to $N_L = 5 \times 10^6$.

4.3 An asymptotic estimate for the tail probabilities

Theorem 3.2 obtains an asymptotic estimate for the tail probability $P(L_n > nb)$ given by relation (3.6), which we now rewrite as

$$P(L_n > nb) \approx \overline{F}_\xi(f_n^2) E\left[|\epsilon|^{2\alpha}\right] (\psi^{\leftarrow}(b))^{-2\alpha}. \tag{4.5}$$

In this subsection, we check the accuracy of this asymptotic estimate. We will show that, for various values of the level b, the asymptotic estimate improves as the portfolio size n increases, as it should be.

We start from determining a proper divergence rate for f_n. Under the model specifications in Table 4.1, relation (4.3) shows that $\overline{F}_\xi$ and $\overline{F}_Y$ are asymptotically equivalent and, hence, $\overline{F}_\xi \in \mathrm{RV}_{-\alpha}$ with $\alpha = 1$. The condition $n\overline{F}_\xi\left(f_n^2\right) \to \infty$ roughly means that the divergence $f_n \downarrow -\infty$ is of order not higher than $n^{0.5}$. We choose $f_n = -0.15n^{0.25}$, so that the individual PD in a portfolio of size $n = 10^4$ is around 1%. We consider two representative portfolio sizes $n = 10^4$ and 2×10^4, and simulate corresponding individual PDs using a sample of size $N_L = 5 \times 10^6$. When the initial value μ_0 is 0, the individual PDs are 1.16% and 0.81%, respectively; when μ_0 is raised to 0.05, the individual PDs are slightly reduced to 1.11% and 0.78%, respectively.

We empirically estimate $P(L_n > nb)$ and $\overline{F}_\xi(f_n^2)$ by a sample of size $N_L = 5 \times 10^6$ from L_n and a sample of size N_ξ from ξ, respectively. As specified in Table 4.1, both the LGD θ and the market beta β follow the exponential distribution with mean 1, and the variation factor $\ell = 1$.

Then the auxiliary function (3.4) is simplified to

$$\psi(x) = E\left[\theta 1_{(\beta x \geq \ell)}\right] = e^{-\frac{1}{x}}.$$

Thus, $\psi^{\leftarrow}(b) = -\frac{1}{\log b}$ for $0 < b < 1$. Finally, we compute the ratio of both sides of (4.5), which should be close to 1 according to Theorem 3.2.

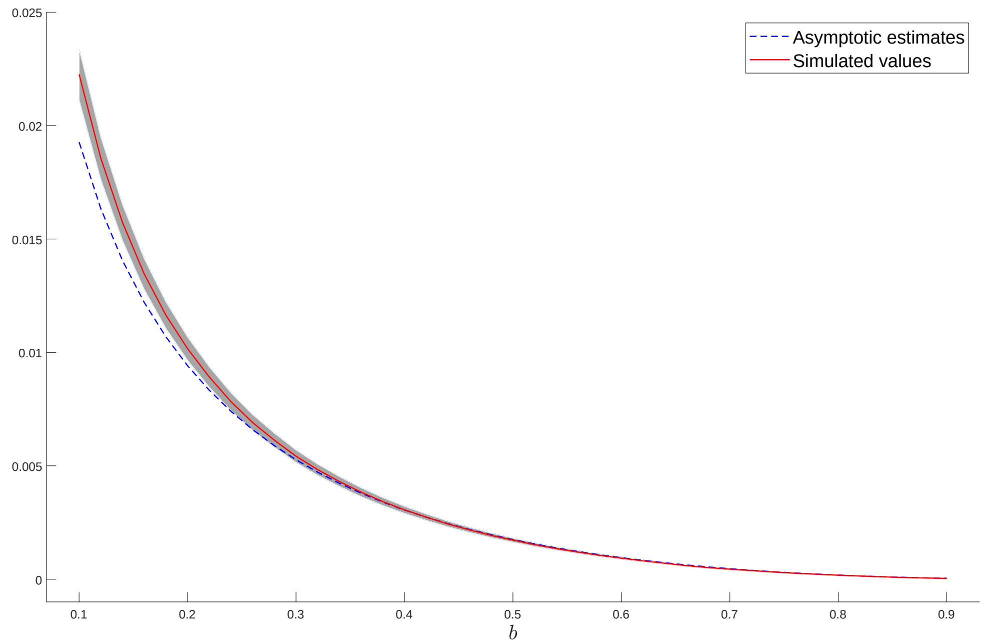
Figure 4.2(a) visualizes how much the estimated value given by (4.5) deviates from the simulated value of $P(L_n > nb)$ for a portfolio of size $n = 2 \times 10^4$ when $\mu_0 = 0$. The shaded region in this subfigure covers values within $1 \pm 5\%$ times of the simulated values of $P(L_n > nb)$. Figure 4.2(b), which plots the ratios of the simulated values to the asymptotically estimated values of $P(L_n > nb)$, supplements Figure 4.2(a) with ratios for more scenarios. Overall, the estimates corresponding to the larger portfolio size $n = 2 \times 10^4$ are more accurate. Figure 4.2(b) also shows that the ratio curves for $\mu_0 = 0.05$ are lower than the corresponding ones for $\mu_0 = 0$. The reason is that the raise of μ_0 from 0 to 0.05 represents an improvement in the credit quality, which decreases the simulated value of $P(L_n > nb)$ but does not influence the asymptotic estimate.

One may observe that, as the level b increases from 0.10 to 0.90, the performance of the asymptotic estimate first improves and then deteriorates. Our simulation study does not cover those values of b too close to 0 or $E[\theta] = 1$, for which the performance will become poor due to the rarity of either $(L_n \leq nb)$ for b close to 0 or $(L_n > nb)$ for b close to 1. When b is small, random perturbations in the idiosyncratic processes may have a non-negligible contribution to the tail probability, causing the asymptotic result to underestimate $P(L_n > nb)$. This effect is less significant when the portfolio size n is larger. When $b = 0.9$, the tail probability could be as low as 10^{-4}. In this case, the observed large fluctuations in the ratios are caused by simulation errors and can be offset by increasing the sample size N_L.

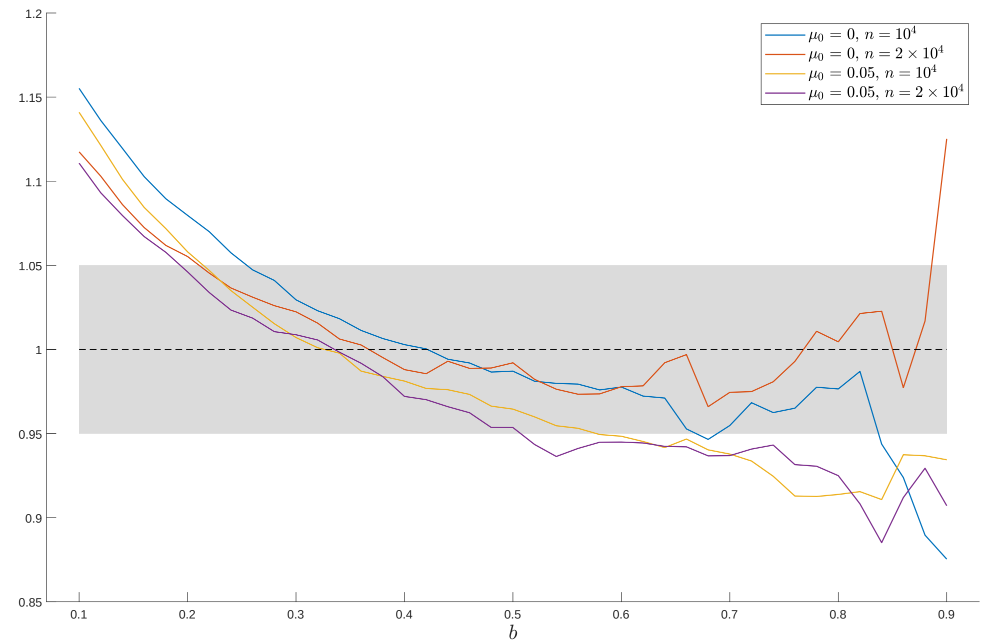
Figure 4.2 is here.

4.4 Asymptotic estimates for the VaR and ES

In this subsection, we check the accuracy of the asymptotic estimates for the VaR and ES of the portfolio loss obtained by Theorem 3.3. The VaR and ES of the portfolio loss are empirically estimated by a sample of size $N_L = 5 \times 10^6$ from L_n. We adopt the L-estimators introduced in Section 9.2.6 of McNeil et al. (2015). For a given level q, the L-estimator $\widehat{\mathrm{VaR}}_q(L_n)$ is given by the qth sample quantile, and the L-estimator $\widehat{\mathrm{ES}}_q(L_n)$ is the average of those sample points not less than $\widehat{\mathrm{VaR}}_q(L_n)$.



(a) Simulated and asymptotically estimated values of $P(L_n > nb)$



(b) Ratios of the simulated to asymptotically estimated values for different values of n and μ_0

Figure 4.2: These subfigures show the accuracy of the asymptotic estimate for $P(L_n > nb)$ by relation (4.5). The level b in both subfigures takes values between 0.1 and 0.9 with a stepsize of 0.02. The quantities $P(L_n > nb)$ and $\overline{F_\xi}(f_n^2)$ are simulated at the sizes $N_L = 5 \times 10^6$ and $N_\xi = 2 \times 10^7$, respectively. The first subfigure is plotted for a portfolio of size $n = 2 \times 10^4$, in which the drift processes have the same initial value $\mu_0 = 0$.

Now we work on the asymptotic estimates. Set $f_n = -0.15n^{0.25}$ as in Subsection 4.3. For a level q of practical interest, assume (3.7), that is,

$$\frac{1-q}{\overline{F}_{\xi}(f_n^2)} = cE\left[|\epsilon|^{2\alpha}\right].$$

Similarly to before, the tail probability $\overline{F}_{\xi}(f_n^2)$ is estimated by a sample of size N_{ξ} from ξ. The value of the constant c can thus be properly determined. Then the asymptotic estimates given by (3.8)–(3.9) are readily obtained numerically.

We consider three typical confidence levels $q = 99\%$, 99.5%, and 99.9% for three portfolio sizes $n = 5 \times 10^3$, 10^4, and 2×10^4. For two initial values $\mu_0 = 0$ and 0.05 of the drift processes, the simulation results are summarized in Tables 4.4–4.5. One can find that the asymptotic estimates given by Theorem 3.3 are very accurate, even for a smaller portfolio size like $n = 5 \times 10^3$. In view of this, when conducting the sensitivity analysis of the risk measures in the next subsection, we directly apply these asymptotic estimates.

Tables 4.4–4.5 are here.

		VaR			ES		
q		99%	99.5%	99.9%	99%	99.5%	99.9%
	simulated	0.3192	0.4384	0.6830	0.4777	0.5832	0.7773
$n = 5 \times 10^3$	estimated	0.3132	0.4401	0.6927	0.4797	0.5899	0.7859
	ratio	1.0192	0.9962	0.9859	0.9959	0.9887	0.9891
	simulated	0.2577	0.3760	0.6406	0.4207	0.5322	0.7458
$n = 10^4$	estimated	0.2494	0.3746	0.6446	0.4193	0.5344	0.7503
	ratio	1.0329	1.0036	0.9934	1.0034	0.9959	0.9940
	simulated	0.1990	0.3126	0.5884	0.3615	0.4752	0.7091
$n = 2 \times 10^4$	estimated	0.1904	0.3095	0.5918	0.3590	0.4762	0.7104
	ratio	1.0453	1.0101	0.9943	1.0070	0.9980	0.9981

Table 4.4: This tabulates the simulated values of $\frac{1}{n}\mathrm{VaR}_q(L_n)$ and $\frac{1}{n}\mathrm{ES}_q(L_n)$, their asymptotic estimates given by relations (3.8)–(3.9), as well as the ratios of the simulated to corresponding estimated values. We set the default threshold $f_n = -0.15n^{0.25}$, the initial value $\mu_0 = 0$, and the sample size $N_L = 5 \times 10^6$ for L_n.

		VaR			ES		
q		99%	99.5%	99.9%	99%	99.5%	99.9%
	simulated	0.3157	0.4356	0.6812	0.4751	0.5818	0.7776
$n = 5 \times 10^3$	estimated	0.3132	0.4401	0.6927	0.4797	0.5899	0.7859
	ratio	1.0079	0.9898	0.9833	0.9906	0.9864	0.9894
	simulated	0.2541	0.3723	0.6364	0.4170	0.5286	0.7430
$n = 10^4$	estimated	0.2494	0.3746	0.6446	0.4193	0.5344	0.7503
	ratio	1.0186	0.9939	0.9873	0.9946	0.9892	0.9902
	simulated	0.1977	0.3103	0.5833	0.3585	0.4712	0.7027
$n = 2 \times 10^4$	estimated	0.1904	0.3095	0.5918	0.3590	0.4762	0.7104
	ratio	1.0383	1.0028	0.9855	0.9987	0.9895	0.9891

Table 4.5: This tabulates the simulated values of $\frac{1}{n}\mathrm{VaR}_q(L_n)$ and $\frac{1}{n}\mathrm{ES}_q(L_n)$, their asymptotic estimates given by relations (3.8)–(3.9), as well as the ratios of the simulated to corresponding estimated values. The settings are the same as those for Table 4.4, except that $\mu_0 = 0.05$.

4.5 Sensitivity analysis with respect to α, β, and ℓ

Based on the asymptotic estimates by (3.8)–(3.9), we first conduct a sensitivity analysis of the VaR and ES of the portfolio loss with respect to the regular variation index α of the systematic risk variable ξ and the market beta β. In the baseline case, we have $\alpha = 1$ and β exponentially distributed with mean 1. Then we slightly alter α or the mean of β and monitor how much the estimated values of the VaR and ES change. As α decreases, the systematic risk variable ξ becomes more heavy tailed, and the risk measures are expected to increase. As the mean of β increases, the portfolio becomes more exposed to the systematic risk, and the risk measures are expected to increase as well. Tables 4.6–4.7 summarize the numerical results for a portfolio of size $n = 2 \times 10^4$, confirming our intuitive analysis. In addition, the tables show that the higher the confidence level q is, the more robust the risk measures become against the change in α or β. Moreover, they also show that the ES is more robust than the VaR. Last, we can see clearly that the regular variation index α of the systematic risk variable ξ plays a more important role than the market beta β in the large portfolio losses.

Tables 4.6–4.7 are here.

Parameter	% change	Approximated VaR		
		99%	99.5%	99.9%
α	+2%	−8.37%	−6.78%	−3.94%
	+1%	−4.54%	−3.63%	−2.06%
	($\alpha = 1$)	(0.1904)	(0.3095)	(0.5918)
	−1%	+4.66%	+3.69%	+2.05%
	−2%	+8.75%	+6.94%	+3.88%
$E[\beta]$	+2%	+3.31%	+2.33%	+1.03%
	+1%	+1.66%	+1.17%	+0.52%
	($E[\beta] = 1$)	(0.1904)	(0.3095)	(0.5918)
	−1%	−1.66%	−1.18%	−0.53%
	−2%	−3.33%	−2.37%	−1.06%

Table 4.6: This is a sensitivity test of the VaR based on Theorem 3.3 with respect to the regular variation index α and the mean of the market beta $E[\beta]$ changed by 1% or 2%.

A thorough investigation of the relative importance between α and β can be conducted. We skip it to keep the paper short but would like to refer the reader to a recent work by Rabitti and Borgonovo (2020).

We end our numerical studies with a sensitivity analysis of the VaR and ES of the portfolio loss with respect to the percentage of obligors of low credit quality. So far, the variation factor ℓ has been set to constant 1. In other words, all obligors in the portfolio have the same PD. To allow obligors of different credit quality, we may vary the distribution of ℓ to capture the heterogeneity. For this purpose, we consider a portfolio of size $n = 2 \times 10^4$, in which a proportion ρ of obligors have an identical default threshold $-0.075n^{0.25}$, while the remaining obligors have another identical default threshold $-0.15n^{0.25}$. To reflect this, we consider the representing default threshold to be $-0.15n^{0.25}$, meanwhile assume that the variation factor ℓ follows a two-point distribution,

$$P(\ell = 0.5) = \rho, \qquad P(\ell = 1) = 1 - \rho.$$

The auxiliary function ψ defined by (3.4) also changes accordingly. Clearly, the obligors with $\ell = 0.5$ are more likely to default. Thus, one expects that, as ρ varies from 0 to 1, the VaR and ES of the portfolio loss will both increase. Figure 4.3 plots the asymptotic estimates given by relations (3.8)–(3.9) with respect to ρ. This figure shows that the VaR and ES increase

Parameter	% change	Approximated ES		
		99%	99.5%	99.9%
	+2%	−5.66%	−4.68%	−2.78%
	+1%	−3.02%	−2.47%	−1.44%
α	($\alpha = 1$)	(0.3590)	(0.4762)	(0.7104)
	−1%	+3.05%	+2.48%	+1.42%
	−2%	+5.75%	+4.68%	+2.69%
	+2%	+1.86%	+1.38%	+0.66%
	+1%	+0.94%	+0.70%	+0.33%
$E[\beta]$	($E[\beta] = 1$)	(0.3590)	(0.4762)	(0.7104)
	−1%	−0.94%	−0.70%	−0.34%
	−2%	−1.90%	−1.42%	−0.68%

Table 4.7: This is a sensitivity test of the ES based on Theorem 3.3 with respect to the regular variation index α and the mean of the market beta $E[\beta]$ changed by 1% or 2%.

almost linearly with the proportion ρ of obligors with lower credit quality (or a higher default probability). It also shows that the more conservative the risk measure is, the less sensitive it is to a change in ρ.

Figure 4.3 is here.

5 Concluding remarks

We investigate the total amount of losses from defaults of a large credit portfolio in a turbulent market. The credit quality process of each obligor consists of a drift term reflecting the trend, an individual volatility term reflecting the idiosyncratic risk, and a common volatility term reflecting the systematic risk. Market betas are used to describe obligor-specific loadings on the systematic risk. As the portfolio expends, there are two cases. The first case is that the individual defaults do not become rare, and for this case we establish a limit theorem for the portfolio loss. The second case is that the individual defaults become rare, and for this case we establish an asymptotic estimate for the tail probability of the portfolio loss. Both results show that the portfolio loss is driven by the systematic risk, while this driving force is amplified by the market beta.

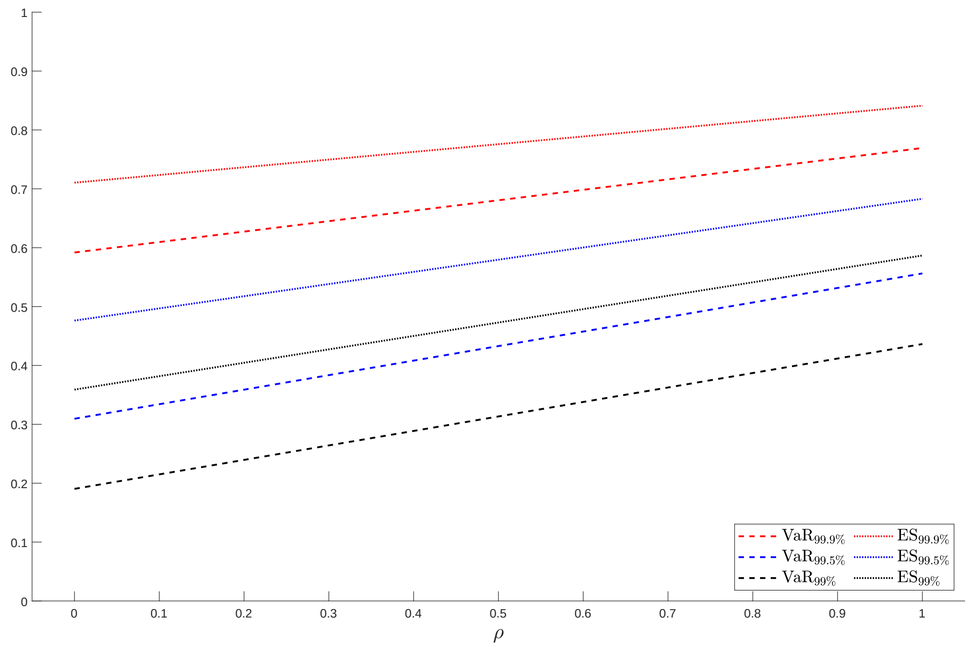


Figure 4.3: This figure plots the asymptotic estimates for the VaR and ES risk measures of $\frac{L_n}{n}$ by relations (3.8)–(3.9). The dashed lines correspond to the VaR and the dotted lines to the ES, with different colors representing different levels. This portfolio of size $n = 2 \times 10^4$ consists of obligors with different individual variation factors. For the representing threshold $f_n = -0.15n^{0.25}$, the obligors with $\ell = 0.5$ form a proportion ρ of the portfolio. The remaining obligors have $\ell = 1$.

Several extensions of our work are worthy of pursuit in the future. First, as seen from our numerical studies, the asymptotic estimate obtained by Theorem 3.2 becomes unstable when the level b is close to either the infimum ψ_* or the supremum ψ^* of the function ψ. Extra efforts need to be taken to deal with such cases. Moreover, when b is above ψ^*, the LLN approach does not work anymore. In a future work, we will give such challenging cases an in-depth investigation by employing CLT and LDP approaches instead. Second, our model describes a high volatility environment and our main discovery is that the portfolio loss is driven by the systematic risk contained in the volatility process. In doing so, however, the impacts of the idiosyncratic parts are integrated out or even cancelled out. It would be interesting to analyze the different roles of the individual drift processes, the idiosyncratic variance processes, and the systematic variance process in causing large portfolio losses. Third, it is desirable to introduce a jump component to the credit quality process (2.1) to reflect exogenous shocks to the market. In this situation, our method relying on a continuous Ocone martingale does not work anymore and we need to develop new methods for both theoretical and numerical studies. Finally and even more interestingly, the study of portfolio losses should be conducted under a chaotic market condition to better capture radical and irrational changes in the market, namely the butterfly effect, or more broadly, the systemic risk, of large credit portfolios. See Choi and Douady (2012) for an observation of chaos during the financial crisis of 2007–2009, and see Barnett and Serletis (2000) and Barnett et al. (2015) for literature reviews of the study of chaotic financial markets.

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Appendix A Proofs

Recall the credit quality process of each obligor i defined by the SDE (2.1) or its equivalent stochastic integral form (2.2). Then the portfolio loss (2.3) becomes

$$L_n = \sum_{i=1}^{n} \theta_i 1_{\left(\inf_{0\le t\le T}(M_{i,t}+I_{i,t}+\beta_i S_t)\le \ell_i f_n\right)}. \tag{A.1}$$

This expression serves as the starting point of the proofs of both Theorems 3.1 and 3.2.

Proof of Theorem 3.1

We start from (A.1) with $f_n \equiv f < 0$. Given $\mathcal{F}_T^S$, the σ-field generated by the common part $\{S_t\}_{0\le t\le T}$, the credit quality processes $\{X_{i,t}\}_{t\ge 0}$, $i = 1, \ldots, n$, are i.i.d., and so are their running minima. Conditioning all summands in (A.1) on $\mathcal{F}_T^S$ and applying the LLN, we obtain that, almost surely,

$$\frac{L_n}{n} \to E\left[\theta 1_{\left(\inf_{0\le t\le T}(M_t+I_t+\beta S_t)\le \ell f\right)} \middle| \mathcal{F}_T^S\right] = F^S, \qquad n \to \infty,$$

where $\{M_t\}_{0\le t\le T}$ and $\{I_t\}_{0\le t\le T}$ are two generic stochastic integral processes as specified in Assumption 2.1(a). Thus, for any $b\in\mathbb{R}$, any small $\varepsilon,\delta>0$, and all large n, say $n\ge N_0$, where N_0 is a random variable measurable to $\mathcal{F}_T^S$, it holds that

$$1_{(F^S\le b-\delta)}-\varepsilon\le P\left(\frac{L_n}{n}\le b\middle|\mathcal{F}_T^S\right)\le 1_{(F^S\le b+\delta)}+\varepsilon.$$

Thus, by Fatou's lemma,

$$\begin{aligned}\limsup_{n\to\infty}P\left(\frac{L_n}{n}\le b\right) &= \limsup_{n\to\infty}E\left[P\left(\frac{L_n}{n}\le b\middle|\mathcal{F}_T^S\right)\right]\\ &\le E\left[\limsup_{n\to\infty}P\left(\frac{L_n}{n}\le b\middle|\mathcal{F}_T^S\right)\right]\\ &\le P\left(F^S\le b+\delta\right)+\varepsilon\\ &\to P\left(F^S\le b\right),\qquad \delta,\varepsilon\downarrow 0.\end{aligned}$$

In the same way,

$$\liminf_{n\to\infty}P\left(\frac{L_n}{n}\le b\right)\ge P\left(F^S<b\right).$$

Thus, for $b\in\mathbb{R}$ at which the functional F^S is continuously distributed, a combination of the two results above gives (3.3). This concludes the claimed convergence in distribution.

Proof of Theorem 3.2

We still start from relation (A.1) and first derive an asymptotic lower bound for $P(L_n>nb)$. Note that

$$\inf_{0\le t\le T}(M_{i,t}+I_{i,t}+\beta_i S_t)\le\sup_{0\le t\le T}(M_{i,t}+I_{i,t})+\beta_i\inf_{0\le t\le T}S_t. \tag{A.2}$$

Denote by C_i the sup term on the right-hand side. For the inf term on the right-hand side, according to Proposition C.2 of Packham et al. (2013), the stochastic integral

$$S_t=\int_0^t\sigma_{0,s}dW_{0,s},\qquad t\ge 0,$$

due to the independence between $\{\sigma_{0,t}\}_{t\ge0}$ and $\{W_{0,t}\}_{t\ge0}$, is a continuous Ocone martingale; that is, it can be expressed as a time-changed Brownian motion

$$S_t=B_{[S,S]_t},\qquad t\ge 0, \tag{A.3}$$

for some Brownian motion $\{B_t\}_{t\ge0}$ independent of $[S,S]_t$, $t\ge0$, the quadratic variation process of S_t. Write

$$\xi_t=[S,S]_t=\int_0^t\sigma_{0,s}^2ds,\qquad 0\le t\le T,$$

which is consistent with the integral in (3.1). It follows from (A.3) that

$$\inf_{0\le t\le T} S_t \stackrel{d}{=} -|\epsilon|\sqrt{\xi_T}, \tag{A.4}$$

where ϵ is an independent standard normal random variable. Note that $\mathcal{F}_T^S$ is assumed to be independent of the other sources of randomness in the credit quality process, and so is $|\epsilon|\sqrt{\xi_T}$. Thus, by (A.4) we can simply replace $\inf_{0\le t\le T} S_t$ in (A.2) by $-|\epsilon|\sqrt{\xi_T}$, and then derive from (A.1) and (A.2) the following:

$$\begin{aligned} P\left(L_n > nb\right) &\ge P\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(C_i-\beta_i|\epsilon|\sqrt{\xi_T}\le \ell_i f_n\right)} > b\right) \\ &= \int_0^\infty P\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\frac{C_i}{f_n}+\beta_i x\ge \ell_i\right)} > b\right) P\left(\frac{|\epsilon|\sqrt{\xi_T}}{-f_n}\in dx\right). \end{aligned} \tag{A.5}$$

For $x\in\mathbb{R}_+$, write

$$\begin{aligned} \psi_n(x) &= E\left[\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\frac{C_i}{f_n}+\beta_i x\ge \ell_i\right)}\right] \\ &= E\left[\theta 1_{\left(\frac{C}{f_n}+\beta x\ge \ell\right)}\right] \\ &\to \psi(x), \qquad n\to\infty, \end{aligned} \tag{A.6}$$

where the limiting function ψ is defined in (3.4). By condition (d), ψ is continuous and strictly increasing over $\mathrm{cl}\left(\Delta_\psi\right)$, and hence it is continuous, non-decreasing, and bounded over $\mathbb{R}_+$. Subsequently, the convergence in (A.6) is uniform in $x\in\mathbb{R}_+$. In view of the convergence in (A.6), the integrand in (A.5) can be approximated by $1_{(\psi(x)>b)}$, and for this reason we introduce

$$d_n(x) = P\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\frac{C_i}{f_n}+\beta_i x\ge \ell_i\right)} > b\right) - 1_{(\psi(x)>b)}.$$

In terms of $d_n(x)$, we rewrite (A.5) as

$$P\left(\frac{L_n}{n} > b\right) \ge \int_0^\infty \left(1_{(\psi(x)>b)} + d_n(x)\right) P\left(\frac{|\epsilon|\sqrt{\xi_T}}{-f_n}\in dx\right) := K_1 + K_2. \tag{A.7}$$

By the condition $\overline{F}_{\xi_T}\in \mathrm{RV}_{-\alpha}$,

$$\begin{aligned} K_1 &= \int_0^\infty 1_{(\psi(x)>b)} P\left(\frac{|\epsilon|\sqrt{\xi_T}}{-f_n}\in dx\right) \\ &= P\left(\frac{|\epsilon|^2\xi_T}{\left(\psi^{\leftarrow}(b)\right)^2} > f_n^2\right) \\ &\sim E\left[|\epsilon|^{2\alpha}\right]\left(\psi^{\leftarrow}(b)\right)^{-2\alpha}\overline{F}_{\xi_T}\left(f_n^2\right), \end{aligned} \tag{A.8}$$

where the last step is due to Breiman's theorem; see Breiman (1965). For arbitrarily small $\delta>0$ such that $\psi_* < b-\delta < b+\delta < \psi^*$, further decompose K_2 into three parts as

$$K_2 = \left(\int_0^{\psi^{\leftarrow}(b-\delta)} + \int_{\psi^{\leftarrow}(b+\delta)}^{\infty} + \int_{\psi^{\leftarrow}(b-\delta)}^{\psi^{\leftarrow}(b+\delta)}\right) d_n(x) P\left(\frac{|\epsilon|\sqrt{\xi_T}}{-f_n}\in dx\right)$$

$$= \quad K_{21} + K_{22} + K_{23}. \tag{A.9}$$

Now we deal with K_{21}. By the uniformity of the convergence in (A.6), it holds for sufficiently large n, and uniformly for all $x \in (0, \psi^{\leftarrow}(b-\delta)]$ that

$$\psi_n(x) \le \psi(x) + \frac{\delta}{2} \le b - \frac{\delta}{2}.$$

By Chebyshev's inequality, the integrand $d_n(x)$ in K_{21} satisfies

$$\begin{aligned}
d_n(x) &= P\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\frac{C_i}{f_n}+\beta_i x\ge \ell_i\right)} > b\right) \\
&\le P\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\frac{C_i}{f_n}+\beta_i x\ge \ell_i\right)} - \psi_n(x) > \frac{\delta}{2}\right) \\
&\le \frac{4}{n\delta^2} var\left(\theta 1_{\left(\frac{C}{f_n}+\beta x\ge \ell\right)}\right) \\
&\le \frac{4E\left[\theta^2\right]}{n\delta^2}.
\end{aligned} \tag{A.10}$$

Under the condition that $n\overline{F}_{\xi_T}\left(f_n^2\right) \to \infty$, we have

$$0 \le K_{21} \le \frac{4E\left[\theta^2\right]}{n\delta^2} = o(1)\overline{F}_{\xi_T}\left(f_n^2\right). \tag{A.11}$$

A similar treatment can be applied to show that

$$K_{22} = o(1)\overline{F}_{\xi_T}\left(f_n^2\right). \tag{A.12}$$

Actually, by the uniformity of the convergence in (A.6) again, it holds for sufficiently large n, and uniformly for all $x \in (\psi^{\leftarrow}(b+\delta), \infty)$ that

$$\psi_n(x) \ge \psi(x) - \frac{\delta}{2} \ge b + \frac{\delta}{2}.$$

By Chebyshev's inequality, the integrand $d_n(x)$ in K_{22} satisfies

$$\begin{aligned}
|d_n(x)| &= \left|P\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\frac{C_i}{f_n}+\beta_i x\ge \ell_i\right)} > b\right) - 1\right| \\
&\le P\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\frac{C_i}{f_n}+\beta_i x\ge \ell_i\right)} - \psi_n(x) \le -\frac{\delta}{2}\right) \\
&\le \frac{4}{n\delta^2} var\left(\theta 1_{\left(\frac{C}{f_n}+\beta x\ge \ell\right)}\right) \\
&\le \frac{4E\left[\theta^2\right]}{n\delta^2}.
\end{aligned}$$

Then (A.12) follows similarly to (A.11). For K_{23}, noticing that $|d_n(x)| \le 1$, applying Breiman's theorem as in (A.8), we have

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\frac{|K_{23}|}{\overline{F}_{\xi_T}\left(f_n^2\right)}$$

$$
\begin{aligned}
&\leq \lim_{\delta\to 0}\limsup_{n\to\infty}\frac{P\left(\psi^{\leftarrow}(b-\delta)<\frac{|\epsilon|\sqrt{\xi_T}}{-f_n}\leq\psi^{\leftarrow}(b+\delta)\right)}{\overline{F}_{\xi_T}\left(f_n^2\right)} \\
&= E\left[|\epsilon|^{2\alpha}\right]\lim_{\delta\to 0}\left(\left(\psi^{\leftarrow}(b-\delta)\right)^{-2\alpha}-\left(\psi^{\leftarrow}(b+\delta)\right)^{-2\alpha}\right) \\
&= 0. \qquad (A.13)
\end{aligned}
$$

We conclude from (A.9)–(A.13) that

$$
K_2 = o(1)\overline{F}_{\xi_T}\left(f_n^2\right). \qquad (A.14)
$$

and subsequently conclude from (A.7)–(A.8) and (A.14) that

$$
\liminf_{n\to\infty}\frac{P\left(L_n>nb\right)}{\overline{F}_{\xi_T}(f_n^2)} \geq E\left[|\epsilon|^{2\alpha}\right]\left(\psi^{\leftarrow}(b)\right)^{-2\alpha}.
$$

The corresponding asymptotic upper bound can be derived by going exactly the same lines. Indeed, this becomes sufficiently clear after noticing that, similarly to (A.2),

$$
\inf_{0\leq t\leq T}\left(M_{i,t}+I_{i,t}+\beta_i S_t\right) \geq \inf_{0\leq t\leq T}\left(M_{i,t}+I_{i,t}\right)+\beta_i\inf_{0\leq t\leq T}S_t. \qquad (A.15)
$$

Denote by $\tilde{C}_i$ the first term on the right-hand side above. Correspondingly, we define

$$
\tilde{\psi}_n(x) = E\left[\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\frac{\tilde{C}_i}{f_n}+\beta_i x\geq\ell_i\right)}\right], \qquad (A.16)
$$

which also converges, uniformly over $\mathbb{R}_+$, to the limiting function ψ defined in (3.4).

Proof of Theorem 3.3

To prove relation (3.8), we start with

$$
P\left(L_n>\mathrm{VaR}_{q_n}\left(L_n\right)\right)\leq 1-q_n\leq P\left(L_n\geq\mathrm{VaR}_{q_n}\left(L_n\right)\right).
$$

Divide each side by $\overline{F}_{\xi_T}(f_n^2)$, take the limit $n\to\infty$, and apply relation (3.7), yielding

$$
\limsup_{n\to\infty}\frac{P\left(L_n>\mathrm{VaR}_{q_n}\left(L_n\right)\right)}{\overline{F}_{\xi_T}(f_n^2)}\leq cE\left[|\epsilon|^{2\alpha}\right]\leq\liminf_{n\to\infty}\frac{P\left(L_n\geq\mathrm{VaR}_{q_n}\left(L_n\right)\right)}{\overline{F}_{\xi_T}(f_n^2)}. \qquad (A.17)
$$

Relation (3.6) with b chosen such that $\left(\psi^{\leftarrow}(b)\right)^{-2\alpha}=c$ becomes

$$
\lim_{n\to\infty}\frac{P\left(L_n>nb\right)}{\overline{F}_{\xi_T}(f_n^2)} = cE\left[|\epsilon|^{2\alpha}\right]. \qquad (A.18)
$$

Comparing (A.17) with (A.18) and noticing the strict monotonicity of $\psi^{\leftarrow}$ over (ψ_*,ψ^*), we obtain

$$
\liminf_{n\to\infty}\frac{1}{n}\mathrm{VaR}_{q_n}\left(L_n\right)\geq b\geq\limsup_{n\to\infty}\frac{1}{n}\mathrm{VaR}_{q_n}\left(L_n\right).
$$

It follows that

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{VaR}_{q_n}(L_n)=b=\psi\left(c^{-\frac{1}{2\alpha}}\right),$$

giving relation (3.8).

To prove relation (3.9), we rewrite $\mathrm{ES}_{q_n}(L_n)$ as

$$\mathrm{ES}_{q_n}(L_n)=\mathrm{VaR}_{q_n}(L_n)+\frac{1}{1-q_n}\int_{\mathrm{VaR}_{q_n}(L_n)}^{\infty}\overline{F}_{L_n}(x)\,dx; \tag{A.19}$$

see Proposition 8.13 of McNeil et al. (2015). For any $\varepsilon>0$, we further split the integral term above into three parts as

$$\begin{aligned}\frac{1}{1-q_n}\int_{\mathrm{VaR}_{q_n}(L_n)}^{\infty}\overline{F}_{L_n}(x)\,dx &= n\left(\int_{\mathrm{VaR}_{q_n}\left(\frac{L_n}{n}\right)}^{E[\theta]-\varepsilon}+\int_{E[\theta]-\varepsilon}^{E[\theta]+\varepsilon}+\int_{E[\theta]+\varepsilon}^{\infty}\right)\frac{P\left(\frac{L_n}{n}>y\right)}{1-q_n}dy\\ &:= n(K_1+K_2+K_3).\end{aligned} \tag{A.20}$$

Note that in K_1,

$$\frac{P\left(\frac{L_n}{n}>y\right)}{1-q_n}\le\frac{P\left(\frac{L_n}{n}>\mathrm{VaR}_{q_n}\left(\frac{L_n}{n}\right)\right)}{1-q_n}\le 1.$$

Thus, applying the dominated convergence theorem we obtain

$$\begin{aligned}\lim_{n\to\infty}K_1 &= \int_0^{\infty}\lim_{n\to\infty}\frac{P\left(\frac{L_n}{n}>y\right)}{\overline{F}_{\xi_T}(f_n^2)}\cdot\frac{\overline{F}_{\xi_T}(f_n^2)}{1-q_n}\cdot 1_{\left(\mathrm{VaR}_{q_n}\left(\frac{L_n}{n}\right)<y\le E[\theta]-\varepsilon\right)}dy\\ &= \frac{1}{c}\int_{\psi\left(c^{-\frac{1}{2\alpha}}\right)}^{E[\theta]-\varepsilon}\left(\psi^{\leftarrow}(y)\right)^{-2\alpha}dy,\end{aligned} \tag{A.21}$$

where the second step applies both relations (3.7)–(3.8). For K_2, again by Theorem 3.2 and relation (3.7), we have

$$K_2\le\frac{P\left(\frac{L_n}{n}>E[\theta]-\varepsilon\right)}{\overline{F}_{\xi_T}(f_n^2)}\cdot\frac{\overline{F}_{\xi_T}(f_n^2)}{1-q_n}\cdot 2\varepsilon\to\frac{2\varepsilon}{c}\left(\psi^{\leftarrow}\left(E[\theta]-\varepsilon\right)\right)^{-2\alpha}. \tag{A.22}$$

As for K_3, recall the function $\tilde{\psi}_n(x)$ defined by (A.16), which clearly satisfies $\tilde{\psi}_n(x)\le E[\theta]$. For $y>E[\theta]+\varepsilon\ge\tilde{\psi}_n(x)+\varepsilon$, we derive from (A.1), (A.4), and (A.15) the following:

$$\begin{aligned}&P\left(\frac{L_n}{n}>y\right)\\ \le\ & P\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\tilde{C}_i-\beta_i|\epsilon|\sqrt{\xi_T}\le\ell_i f_n\right)}>y\right)\\ =\ & \int_0^{\infty}P\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i 1_{\left(\frac{\tilde{C}_i}{f_n}+\beta_i x\ge\ell_i\right)}-\tilde{\psi}_n(x)>y-\tilde{\psi}_n(x)\right)P\left(\frac{|\epsilon|\sqrt{\xi_T}}{-f_n}\in dx\right)\\ \le\ & \int_0^{\infty}\frac{E\left[\theta^2\right]}{n\left(y-\tilde{\psi}_n(x)\right)^2}P\left(\frac{|\epsilon|\sqrt{\xi_T}}{-f_n}\in dx\right)\\ \le\ & \frac{E\left[\theta^2\right]}{n(y-E[\theta])^2},\end{aligned}$$

where the third step applies Chebyshev's inequality as in (A.10). This leads to

$$K_3 \leq \frac{E\left[\theta^2\right]}{n\left(1-q_n\right)} \int_{E[\theta]+\varepsilon}^{\infty} \frac{dy}{\left(y-E[\theta]\right)^2} \rightarrow 0, \tag{A.23}$$

where the last step is due to relation (3.7) and the condition $n\overline{F}_{\xi_T}(f_n^2) \to \infty$. Plugging (A.21)–(A.23) into (A.20) and letting $\varepsilon \downarrow 0$, we obtain

$$\lim_{n\to\infty} \frac{1}{n\left(1-q_n\right)} \int_{\mathrm{VaR}_{q_n}(L_n)}^{\infty} \overline{F}_{L_n}(x)\, dx = \frac{1}{c} \int_{\psi\left(c^{-\frac{1}{2\alpha}}\right)}^{E[\theta]} \left(\psi^{\leftarrow}(y)\right)^{-2\alpha} dy.$$

Then plugging this and the estimate for $\mathrm{VaR}_{q_n}(L_n)$ given by (3.8) into (A.19), we finally obtain

$$\lim_{n\to\infty} \frac{1}{n} \mathrm{ES}_{q_n}(L_n) = \psi\left(c^{-\frac{1}{2\alpha}}\right) + \frac{1}{c} \int_{\psi\left(c^{-\frac{1}{2\alpha}}\right)}^{E[\theta]} \left(\psi^{\leftarrow}(y)\right)^{-2\alpha} dy = \int_0^1 \psi\left((cx)^{-\frac{1}{2\alpha}}\right) dx,$$

giving relation (3.9).

Appendix B Examples to illustrate portfolio effect

We show two self-contained examples to illustrate that individual defaults can become rare under portfolio effect. The first example looks at portfolio effect from the perspective of issuers, while the second example from the perspective of investors. These examples may be interesting in their own right.

Example B.1 Consider a bank who issues n defaultable products (bonds or loans), or an insurer who sells n insurance policies. For each product $i = 1, \ldots, n$, denote by θ_i the LGD representing the loss amount in the event of default, and denote by Z_i the default indicator with $P(Z_i = 1) = p \in (0,1)$. Assume that the LGDs $\theta_1, \ldots, \theta_n$ are i.i.d. and independent of the default indicators $Z_1, \ldots, Z_n$. Under the Basel Capital Accords, banks are allowed to follow the internal ratings-based (IRB) approach to calculating regulatory capital requirements. Suppose that each product is backed by a capital reserve equal to the VaR at a certain high level q. Specifically, let q satisfy $p \vee (1-p) < q < 1$ and be sufficiently close to 1 such that

$$\mathrm{VaR}_q(\theta Z) = \mathrm{VaR}_{\frac{q-(1-p)}{p}}(\theta) > E[\theta]. \tag{B.1}$$

If managed independently, the issuer will default on an individual product with probability

$$P\left(\theta Z > \mathrm{VaR}_q(\theta Z)\right) \leq P\left(\theta Z > E[\theta]\right),$$

which does not depend on the portfolio size n. If managed collectively, no individual product will default unless $\sum_{i=1}^{n} \theta_i Z_i > n\mathrm{VaR}_q(\theta Z)$. The latter, which represents the issuer's default on the whole portfolio, occurs with probability

$$P\left(\sum_{i=1}^{n}\theta_i Z_i > n\mathrm{VaR}_q(\theta Z)\right) \le P\left(\frac{1}{n}\sum_{i=1}^{n}\theta_i > \mathrm{VaR}_q(\theta Z)\right) \to 0, \qquad n\to\infty,$$

where the last step holds because $\frac{1}{n}\sum_{i=1}^{n}\theta_i$ converges almost surely to $E[\theta]$ by LLN while $\mathrm{VaR}_q(\theta Z) > E[\theta]$ by (B.1). This means that individual defaults will become rare under the portfolio effect. It is noteworthy that in this example no restriction is imposed on the dependence structure among the default indicators $Z_1, \ldots, Z_n$ and therefore we are allowed to incorporate certain common factors in modeling them.

Example B.2 Suppose a financial market composed of a variety of defaultable primitive assets. For each asset $i = 1, 2, \ldots$, an investment to it over a fixed time period either yields a return of rate r_i with probability $1 - p_i$ or incurs a loss of $100\varepsilon_i\%$ with probability p_i, for some $r_i > 0$, $\varepsilon_i \in (0, 1)$, and $p_i \in (0, 1)$. The latter corresponds to the default scenario of asset i, with a recovery rate $R_i = 1 - \varepsilon_i$. We introduce a Bernoulli random variable Z_i with $P(Z_i = 1) = p_i$ to describe the default indicator. Then one unit invested to asset i will accumulate to

$$G_i := (1 + r_i)\,1_{(Z_i=0)} + (1 - \varepsilon_i)\,1_{(Z_i=1)} = (1 + r_i) - (r_i + \varepsilon_i)\,Z_i. \tag{B.2}$$

For simplicity, assume that these assets yield the same expected accumulation factor larger than 1, that is, identically for all $i = 1, 2, \ldots$,

$$g := E\left[G_i\right] = (1 + r_i) - (r_i + \varepsilon_i)\,p_i > 1. \tag{B.3}$$

Consider an investor who invests in n such primitive assets in the market. In view of their identical expected accumulation factor, the investor naturally decides to minimize the variance of the investment portfolio. Denote by $\pi_i^{(n)}$ the percentage of the capital invested in asset i. By minimizing the variance of a unitized portfolio, recalling (B.2),

$$var\left(\sum_{i=1}^{n}\pi_i^{(n)}G_i\right) = var\left(\sum_{i=1}^{n}\pi_i^{(n)}\left(r_i + \varepsilon_i\right)Z_i\right),$$

the vector of optimal percentages is deduced to be

$$\boldsymbol{\pi}^{(n)}=\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^{\intercal}\Sigma^{-1}\mathbf{1}},$$

where Σ denotes the covariance matrix of the vector $\mathbf{G}$ with its (i, j) entry, by (B.3),

$$c_{ij} = (r_i + \varepsilon_i)\,(r_j + \varepsilon_j)\,Cov(Z_i, Z_j)$$

and Σ is assumed to be invertible. For the case with independent defaults, Σ is simplified to a diagonal matrix with its (i,i) entry $c_{ii} = (r_i + \varepsilon_i)^2 p_i(1-p_i)$, and thus each component of $\boldsymbol{\pi}^{(n)}$ is explicated to

$$\pi_i^{(n)} = \frac{\frac{1}{c_{ii}}}{\sum_{j=1}^{n} \frac{1}{c_{jj}}}, \qquad i = 1, \dots, n. \tag{B.4}$$

For other arguably more practical cases with correlated defaults, a similar analysis can still be conducted but will be more complicated, which we skip here.

Reasonably, an investor who holds a larger amount of capital will consider to extend his investment to more assets, for which case we use $n \to \infty$ to capture his increasing market capacity. We are interested in how much to be invested in those assets with significant default risk. To this end, for an arbitrarily fixed small number $\delta > 0$, define an index set $J_\delta^{(n)} = \{1 \le j \le n : p_j > \delta\}$, which collects such assets with significant default risk. Assume that the harmonic mean of p_1, ..., p_n tends to 0, namely,

$$H_n(p) = \frac{n}{\sum_{j=1}^{n} \frac{1}{p_j}} \to 0, \qquad n \to \infty, \tag{B.5}$$

which is interpreted as that, as the investor expands his investment portfolio, he will be prudent by considering some high-quality assets. This condition (B.5) is implied by, but is much weaker than, $p_n \to 0$. By (B.3), $(r_i + \varepsilon_i)(1-p_i) = g + \varepsilon_i - 1$, which lies between $g-1$ and 2. Thus, each $c_{ii} = (r_i + \varepsilon_i)^2 p_i(1-p_i)$ is bounded from both sides as

$$(g-1)^2 \frac{p_j}{1-p_j} \le c_{ii} \le 4 \frac{p_j}{1-p_j}. \tag{B.6}$$

Then by (B.4)–(B.6), the total percentage of the capital invested in those assets indexed by $J_\delta^{(n)}$ satisfies

$$\begin{aligned}
\sum_{j \in J_\delta^{(n)}} \pi_j^{(n)} &= \frac{\sum_{j \in J_\delta^{(n)}} \frac{1}{c_{jj}}}{\sum_{j=1}^{n} \frac{1}{c_{jj}}} \\
&\le \frac{4}{(g-1)^2} \frac{\sum_{j \in J_\delta^{(n)}} \frac{1-p_j}{p_j}}{\sum_{j=1}^{n} \frac{1-p_j}{p_j}} \\
&\le \frac{4}{(g-1)^2} \frac{n \frac{1-\delta}{\delta}}{\sum_{j=1}^{n} \frac{1}{p_j} - n} \to 0, \qquad \text{as } n \to \infty.
\end{aligned}$$

This means that collectively those assets with significant default risk can only occupy a negligible portion of this large investment portfolio, or, equivalently, the portfolio is overwhelmingly dominated by assets with low default risk.

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