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A finite-horizon condition-based maintenance policy for a two-unit system with dependent degradation processes

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ABSTRACT

This paper analyzes a condition-based maintenance (CBM) model for a system with two heterogeneous components in which degradation follows a bivariate gamma process. Unlike the traditional CBM formulation that assumes an infinite planning horizon, this paper evaluates the maintenance cost in a finite planning horizon, which is the practical case for most systems. In the proposed CBM policy, both components are periodically inspected and a preventive or corrective replacement might be carried out based on the state of degradation at inspection. The CBM model is formulated as a Markov decision process (MDP) and dynamic programming is used to compute the expected maintenance cost over a finite planning horizon.

The expected maintenance cost is minimized with respect to the preventive replacement thresholds for the two components. Unlike an infinite-horizon CBM problem, which leads to a stationary maintenance policy, the optimal policy in the finite-horizon case turns out to be non-stationary in the sense that the optimal actions vary at each inspection epoch. A numerical example is presented to illustrate the proposed model and investigate the influence of economic dependency and correlation between the degradation processes on the optimal maintenance policy. Numerical results show that a higher dependence between the degradation processes actually reduces the maintenance cost, while a higher economic dependence leads to higher preventive replacement thresholds.

KEY WORDS: Reliability; condition-based maintenance; Markov decision process; finite horizon; bivariate gamma process

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1 Introduction

With the fast development of modern industries, maintenance operations have played an increasingly important role in preventing system failures and improving operational safety. In recent years, advances in sensing and monitoring technologies have inspired rapid development of condition-based maintenance (CBM)(Elwany et al., 2011; Wu and Castro, 2020; Liu, Do, Iung and Xie, 2020; Omshi et al., 2020). CBM has shown its superiority over time-based maintenance from both the academic and practical points of view (Bei et al., 2019).

Several CBM strategies have been reported in the literature for single-unit systems. In general, maintenance strategies for a single-unit system cannot be directly applied to multi-unit systems due to various dependencies among components (economic, structural, and stochastic) (Dekker et al., 1997; Wang, 2002; Luo and Wu, 2018). Economic dependence indicates that simultaneous maintenance of multiple components can save cost compared with multiple individual maintenance actions, as the setup cost can be shared when the components are jointly maintained. Structural dependence occurs when the components structurally constitute a unified part, i.e., maintenance of one component requires the disassembly or replacement of other components. Stochastic dependence implies that failure or degradation of one component influences the state of degradation in other components (Olde Keizer et al., 2017).

Traditionally, group maintenance and opportunistic maintenance are two main approaches to dealing with economic dependence (Wildeman et al., 1997; Bouvard et al., 2011; Liu et al., 2017; Vu et al., 2018; Abbou and Makis, 2019). In the first approach, a common practice is to specify one or multiple maintenance thresholds in advance and then repair or replace the components whose degradation levels have exceeded the associated critical thresholds as found by an inspection (Van Horenbeek and Pintelon, 2013; Shafiee and Finkelstein, 2015; Olde Keizer et al., 2016; Chalabi et al., 2016; Verbert et al., 2017; Wu et al., 2017). As opposed to group maintenance under which the inspection interval and preventive maintenance thresholds are scheduled in advance, opportunistic maintenance does not require planned maintenance operations. In this approach, the maintenance action is initiated upon hitting a critical threshold (either at the system level or at the component level), and additional opportunistic thresholds are specified to determine the maintenance actions on other components (Tian and Liao, 2011; Huynh et al., 2015; Zhou et al., 2015; Zhang and Zeng, 2015; Liu et al., 2018).

Recently, several studies have been devoted to maintenance modelling and optimization with consideration of dependence among degradation processes. This type of dependence can occur in various scenarios, e.g., failure-induced, load-sharing, and common-mode (Olde Keizer et al., 2017). In the failure-induced scenario, failure of one component can cause damage to the remaining components (Rasmekomen and Parlikad, 2016; Olde Keizer et al., 2017; Berrade et al., 2018; Yuan et al., 2019; Liu, Zhao, Liu and Liu, 2020). Dao and Zuo (2016) proposed a selective maintenance model for a multi-unit system, where failure or degradation of one component increases the failure rates of other components. In the load-sharing scenario, multiple components share the total system load (Zhao et al., 2018; Xu et al., 2019). Failure or degradation of one component will increase the load on other components, resulting in an increased degradation level or failure rate. Shi and Zeng (2016) developed an opportunistic CBM policy considering the influence of one component's degradation level on the remaining useful life of other components. In the common-mode scenario, components follow similar deterioration or failure patterns when operating in a common environment. An increase in the degradation of one component is usually accompanied with a degradation increase of other components. Copula is a popular approach to model dependencies among components. Li et al. (2016) described the dependence of degradation processes via Lévy copulas, which are useful to model the dependence among degradation processes. In addition, dependence among degradation processes due to common environment can be characterized through a multivariate degradation process. Mercier and Pham (2012) adopted a bivariate non-decreasing Lévy process to model a two-unit system. The degree of dependency is described by the correlation coefficient of the process. Yet in that model, the components are assumed to be maintained simultaneously upon any component reaching a preventive replacement threshold. The authors further extended this work by considering separate maintenance (Mercier and Pham, 2014).

An extensive literature review shows that CBM for multi-unit systems has been largely limited to an infinite horizon scenario. A possible reason is that maintenance optimization in an infinite-horizon can be easily performed via an asymptotic approach based on the renewal reward theorem (de Jonge and Scarf, 2020). The asymptotic long-run cost rate converges to the ratio of the expected cost in a renewal cycle to the expected length of a renewal cycle, which is rather easy to obtain (Pandey and Van Der Weide, 2017). In reality, most equipment and facilities are designed to operate for a finite time period (Wang, Li and Xie, 2020; Wang, Zhao and Liu, 2020). Various factors contribute to the obsoleteness, such as demand change or incompatibility due to software/hardware upgrade. Several practical systems, such as piping systems, are usually evaluated in a finite time horizon. For example, Pandey et al. (2011) evaluated the life cycle maintenance cost of a piping system that is used for heat transport in nuclear power plants for a 30-year horizon. In this scenario, maintenance decisions based on the infinite-horizon assumption may provide a suboptimal solution.

This study investigate the CBM policy for a two-unit system in the finite-horizon setting. The system under investigation consists of two non-identical components, in which degradation processes are modeled as a bivariate gamma process. A component is said to be failed when its degradation level exceeds a specific failure threshold. Periodic inspection is carried out to detect and measure the degradation levels of the components. Each component will be preventively replaced if its degradation level exceeds a specific preventive maintenance threshold, and corrective replacement will be implemented if a component is found failed at inspection. We formulate the maintenance problem into a Markov decision process (MDP) framework and obtain the optimal maintenance policy by minimizing the expected maintenance cost. Specifically, we investigate the structure property of the optimal maintenance policy and obtain boundaries for various maintenance actions.

A backward dynamic programming algorithm is developed to compute the expected maintenance cost. Finally, we present a numerical example to illustrate the proposed maintenance model and investigate the influence of economic dependence and correlation between the degradation processes on the optimal maintenance policy.

Research contributions of this paper are summarized as follows:

- We develop a CBM model for a degrading system in a finite-time horizon.
- We consider a system with two non-identical dependent components and formulate the maintenance problem as a MDP.
- We investigate the structure of the optimal solution of maintenance policy and determine the boundaries for various maintenance actions.
- We investigate the influence of economic dependence and correlation between the degradation processes on the optimal maintenance policy.

The rest of the paper is organized as follows. Section 2 describes the bivariate gamma degradation processes and discusses the maintenance actions and costs. In Section 3, the maintenance problem is formulated as MDP and a solution algorithm is presented based on backward dynamic programming. Section 4 evaluates the maintenance cost under fixed preventive maintenance thresholds for comparison purposes. Section 5 presents an illustrative example to demonstrate the implementation of the proposed maintenance model. This section includes sensitivity analysis with respect to various model parameters. Finally, concluding remarks and future research directions are discussed in Section 6. All technical proofs are presented in the Appendix.

Nomenclature

$X_i(t)$	Degradation level of component i at time t
$X(t) = (X_1(t), X_2(t))$	Bivariate indicator denoting system degradation level at time t
$Ga(t; \alpha, \beta)$	Gamma process with shape parameter α and scale parameter β
ρ	correlation coefficient between the degradation level of the two components
$f_{X(t)}(x_1, x_2)$	Joint pdf of the system degradation level
$F_{X(t)}(x_1, x_2)$	Joint cdf of the system degradation level
L_i	Failure threshold of component i
P_i	Preventive maintenance threshold of component i
δ	Inspection interval
$c_i, c_{p,i}, c_{f,i}$	Cost of inspection, preventive maintenance for component i and corrective replacement for component i, respectively
c_s	Setup cost of maintenance actions

c_d	Downtime cost per unit time
γ	Discount rate
T_e	Length of the planning horizon
N	Total number of inspections
$V_{k\delta}(x_1, x_2)$	Value function, denoting minimum expected total discounted cost from period k to the terminal period (cost-to-go) for system at state (x_1, x_2)
$W_{k\delta}(x_1, x_2)$	Expected downtime cost between the kth and $(k+1)$th inspections for system at state (x_1, x_2)
T_k	Time interval between a system failure and the kth inspection
$d(T_k)$	Downtime cost between the kth and $(k+1)$th inspections
S_i	Discretized states of component i
M	Number of discretized states of the components

2 Problem statement

2.1 Bivariate gamma process of degradation

Degradation in a two-unit series system is modeled as a bivariate gamma process. In a univariate gamma process, degradation at time t, $Y(t)$, follows a gamma distribution with parameters (α, β), i.e., $Y(t) \sim Ga(t; \alpha, \beta)$, with the probability density function (pdf)

$$f_{\alpha t, \beta}(y) = \frac{\beta^{\alpha t}}{\Gamma(\alpha t)} e^{-\beta y} y^{\alpha t - 1}. \tag{1}$$

Degradation magnitudes in two components are denoted as $X(t) = (X_1(t), X_2(t))$. We construct the bivariate gamma process by trivariate reduction (Mercier and Pham, 2012). Let $\{Y_1(t)\}_{t \geq 0}$, $\{Y_2(t)\}_{t \geq 0}$, and $\{Y_u(t)\}_{t \geq 0}$ be three independent univariate gamma processes, with parameters (α_1, β), (α_2, β), and (α_u, β). Define $X_1(t) = Y_1(t) + Y_u(t)$, and $X_2(t) = Y_2(t) + Y_u(t)$. The process $\{X(t)\}_{t \geq 0} = \{(X_1(t), X_2(t))\}_{t \geq 0}$ is then a bivariate subordinator with gamma marginal processes and parameters (a_i, β), where $a_i = \alpha_i + \alpha_u$, $i = 1, 2$. In this way, the linear correlation coefficient between the two random variables $X_1(t)$ and $X_2(t)$ is

$$\rho = \frac{\alpha_u}{\sqrt{a_1 a_2}},$$

which is independent of t. Since the components follow gamma degradation processes, the degradation increment of each component, which is the increment of degradation level between any two time instants, must be nonnegative, the joint pdf of $X_1(t)$ and $X_2(t)$ is given by

$$f_{X(t)}(x_1, x_2) = \int_0^{x_m} f_{\alpha_1 t, \beta}(x_1 - u) f_{\alpha_2 t, \beta}(x_2 - u) f_{\alpha_u t, \beta}(u) du, \tag{2}$$

and the associated cumulative distribution function (cdf) is expressed as

$$F_{X(t)}(x_1, x_2) = \int_0^{x_m} F_{\alpha_1 t,\beta}(x_1 - u) F_{\alpha_2 t,\beta}(x_2 - u) f_{\alpha_u t,\beta}(u) du. \tag{3}$$

, where $x_m = \min\{x_1, x_2\}$.

2.2 Maintenance actions and costs

The system fails when degradation in any component exceeds a specific threshold. Periodic inspection is carried out to detect and measure the degradation level, with inspection interval δ and inspection cost c_i. We assume that the component failure mode is dormant, which can only be discovered at inspection. Upon inspection, a failed component will undergo corrective maintenance at a cost $c_{f,i}$ for component i. Meanwhile, the decision maker needs to determine whether to preventively maintain the other component. For each component, there are three possible maintenance actions: do nothing (DN), preventive maintenance (PM), and corrective maintenance (CM). We assume that for both PM and CM actions, a component will be replaced by a new identical one, i.e., it will be restored to the as-good-as-new state upon maintenance. The maintenance duration is assumed to be negligible. The PM cost of component i is denoted as $c_{p,i}$. In addition, a setup cost c_s is incurred whenever maintenance actions (either PM or CM) are implemented. The setup cost can be shared when multiple maintenance actions are carried out simultaneously. The unit costs associated with different maintenance actions are given as follows:

- c_i, if no replacement is implemented;
- $c_s + c_{p,i}$, if only component i is preventively replaced;
- $c_s + c_{p,1} + c_{p,2}$, if the two components are preventively replaced simultaneously;
- $c_s + c_{f,i}$, if only component i is correctively replaced;
- $c_s + c_{f,1} + c_{f,2}$, if the two components are correctively replaced simultaneously;
- $c_s + c_{p,i} + c_{f,j}$, if component i is preventively replaced and component j is correctively replaced.

If the system breaks down between two inspections, downtime cost will occur at c_d per unit time. It is assumed that the costs incurred at time t would be discounted to present value using discount factor, $e^{-\gamma t}$, for a fixed nonnegative rate γ ($\gamma \geq 0$), where $\gamma = 0$ implies no discount. It is further assumed that the system operates for a finite horizon T_e. With the inspection interval δ, the total number of inspections is given by $N = \lfloor T_e/\delta \rfloor$, where $\lfloor \cdot \rfloor$ is the floor operation. Since periodic inspection is performed on the system, for notational convenience, we denote $t_k = k\delta$, and t_k and $k\delta$ will be used interchangeably in the following context.

3 Formulation of the maintenance policy

In this section, we formulate the maintenance problem as a MDP model and investigate the structure property of the optimal maintenance policy, which shows that the optimal maintenance decision is a control-limit policy. In addition, we develop a backward dynamic programming algorithm to obtain the optimal maintenance policy and the associated cost.

3.1 MDP formulation

Let the value function, $V_{k\delta}(x_1, x_2)$, denote the minimum expected discounted cost from the current period k to the terminal period (cost-to-go) for the system at state (x_1, x_2). The optimality equation can be formulated as

$$V_{k\delta}(x_1,x_2) = \begin{cases} c_s + c_{f,1} + c_{f,2} + V_{k\delta}(0,0), \ x_1 > L_1 \ \& \ x_2 > L_2 \\ \min\{c_s + c_{f,1} + V_{k\delta}(0,x_2), c_s + c_{f,1} + c_{p,2} + V_{k\delta}(0,0)\}, x_1 > L_1 \ \& \ x_2 \le L_2 \\ \min\{c_s + c_{f,2} + V_{k\delta}(x_1,0), c_s + c_{p,1} + c_{f,2} + V_{k\delta}(0,0)\}, x_1 \le L_1 \ \& \ x_2 > L_2 \\ \min\left\{C_{0,k}, C_{1,k}, C_{2,k}, C_{12,k}\right\}, \text{otherwise} \end{cases} \tag{4}$$

$$C_{0,k} = e^{-\gamma\delta}(c_i + U_{k\delta}(x_1,x_2)) + W_{k\delta}(x_1,x_2), \tag{5}$$

$$C_{1,k} = c_s + c_{p,1} + V_{k\delta}(0,x_2), \tag{6}$$

$$C_{2,k} = c_s + c_{p,2} + V_{k\delta}(x_1,0), \tag{7}$$

$$C_{12,k} = c_s + c_{p,1} + c_{p,2} + V_{k\delta}(0,0), \tag{8}$$

where $U_{k\delta}(x_1,x_2)$ is the expected value function at the next inspection epoch and $W_{k\delta}(x_1,x_2)$ is the expected downtime cost within the kth and $(k+1)$th inspections. It follows that

$$U_{k\delta}(x_1,x_2) = E[V_{k+1}(X_{1,k+1}, X_{2,k+1})|X_{1,k} = x_1, X_{2,k} = x_2], \tag{9}$$

and

$$\begin{aligned} W_{k\delta}(x_1,x_2) &= E[d(T_k)|X_{1,k} = x_1, X_{2,k} = x_2] \\ &= E\left[\int_{T_k}^{\delta} c_d \cdot e^{-\gamma t} dt\right] \\ &= \int_0^{\delta} \frac{c_d(e^{-\gamma t} - e^{-\gamma\delta})}{\gamma} dF_{T_k}(t), \end{aligned} \tag{10}$$

where T_k is the time interval between a system failure and the kth inspection ($0 \leq T_k \leq \delta$), and $d(T_k)$ is the downtime cost within the kth and $(k+1)$th inspections. The cdf of T_k is given by

$$\begin{aligned} F_{T_k}(t) &= 1 - P(X_1(t) \leq L_1, X_2(t) \leq L_2 | X_{1,k} = x_1, X_{2,k} = x_2) \\ &= 1 - \int_0^{L_m - x_m} F_{\alpha_1 t, \beta}(L_1 - x_1 - u) F_{\alpha_2 t, \beta}(L_2 - x_2 - u) f_{\alpha_u t, \beta}(u) du, \end{aligned} \tag{11}$$

where $L_m - x_m = \min\{L_1 - x_1, L_2 - x_2\}$.

The explanation of the optimality equation (4) goes as follows. If the two components are found failed at the kth inspection, then CM will be implemented on both the components, and the system is recovered to the perfect condition. If one of the components fails, CM will be performed on the failed component and meanwhile the decision-maker needs to decide on whether to preventively maintain the other component. Otherwise, the decision-maker has four options: PM on both components, PM on component 1 only, PM on component 2 only, or do nothing till the next inspection, depending on which one is more cost-effective.

For each component, three maintenance actions (CM, PM, and DN) can be implemented. Therefore, there are totally nine maintenance actions that can be envisioned for the two-unit system, as illustrated in Fig. 1. In this figure, the two axes, $X_1(t_k)$ and $X_2(t_k)$, stand for the degradation levels of the two components at the kth inspection. The system state is divided into nine regions, each of which is associated with an optimal maintenance action. PM_{12} (resp. CM_{12}) represents PM (resp. CM) on both the two components, and PM_i (resp. CM_i) stands for PM (resp. CM) on component i. For each system state, there is an optimal maintenance action among the nine actions, which can be obtained from the value function. In the following, we will investigate the monotonicity property of the value function and the structure of the optimal maintenance policy.

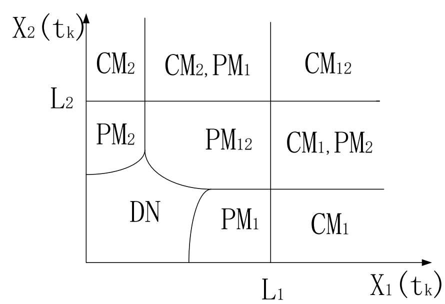

Figure 1: Illustrative regions of maintenance actions

Backward dynamic programming is employed to solve the problem. Usually, the terminal values can be determined based on the salvage value of the system. When the operating horizon T_e is large enough, arbitrarily setting the terminal values will not influence the optimal decision rules at the initial periods. Before we proceed to study the property of the value function, we first introduce the definition of stochastic dominance. A random variable X is stochastically less than a random variable Y, denoted by $X \prec_{st} Y$, if $P(X > t) \leq P(Y > t)$ for all $t \geq 0$. In addition, if $X \prec_{st} Y$, it holds that $E[f(X)] \geq E[f(Y)]$ for all non-increasing functions $f(\cdot)$. Stochastic dominance is used to establish the monotonic property of the value function on the component degradation level x_1 and x_2.

Proposition 1. *The value function, $V_{k\delta}(x_1, x_2)$, is nondecreasing in x_1 and x_2 for all $k = 0, 1, 2, ..., N$.*

The monotonicity of the value function leads to the optimal maintenance policy.

Proposition 2. *The optimal maintenance policy at inspection epoch t_k $(k = 0, 1, \ldots, N-1)$ is a two-dimensional control-limit policy. The control limits exhibit the following properties. There exists ζ_i $(i = 1, 2)$ and a function $h_i(\cdot)$, $x_i = \zeta_i$ is the boundary of PM_{12} and PM_{3-i} for $h_i(\zeta_i) \leq x_{3-i} < L_{3-i}$. For $0 \leq x_i < \zeta_i$, the optimal maintenance action is PM_{3-i} for $x_{3-i} > h_i(x_i)$ and DN for $x_{3-i} \leq h_i(x_i)$. There exists a line $x_2 = l(x_1)$ as the boundary of PM_{12} and DN for $\zeta_i \leq x_i < h_{3-i}(x_{3-i})$. In addition, $l(x_1)$ is nonincreasing in x_1.*

Proposition 2 presents the structure of the optimal maintenance policy in the case that the system is working at the inspection epoch. Fig. 2 illustrates the regions for PM and DN to better understand the optimal structure. As one can observe, the system state is divided into four regions, where each region corresponds to a maintenance action (DN, PM_1, PM_2, or PM_{12}).

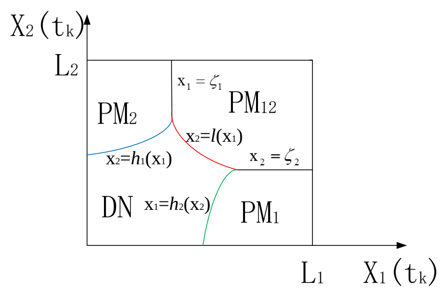

Figure 2: Illustration of the optimal two-dimensional control limit policy

Along with the discussion on the structure property of the optimal maintenance policy, another interesting finding on the region of system state in which PM_{12} is the optimal maintenance action, is discussed as follows.

Corollary 1. *Given $x_i = \zeta_i$ $(i = 1, 2)$ as the boundary of PM_{12} and PM_{3-i}, if PM_{12} is the optimal maintenance action upon system state (x_1, x_2), then $(x_1, x_2) \in [\zeta_1, L_1] \times [\zeta_2, L_2]$.*

The following corollary further shows the control limits when the degradation level at inspection exceeds the failure threshold.

Corollary 2. *If $x_i = \zeta_i$ is the boundary of PM_{12} and PM_{3-i}, it also serves as the boundary of CM_{3-i} and $\{CM_{3-i}, PM_i\}$ for $x_{3-i} \geq L_{3-i}$.*

Corollary 2 indicates that the control limits of preventive maintenance are also applicable to corrective maintenance. That is to say, the boundaries of corrective maintenance can be readily obtained when the boundaries of preventive maintenance are determined, which is useful for maintenance decision making. According to Corollary 2, the maintenance policy can be simplified by focusing on the optimal policy for preventive maintenance.

While a stationary maintenance policy was proposed by Sun et al. (2018) for a k-out-of-n: F system operating in an infinite horizon, the optimal policy in our study is non-stationary in such a way that the obtained control limits vary with the number of inspection k. This is due to the fact that the operating horizon is finite and the optimal maintenance decision varies with the remaining operating period. The optimal decision at the kth inspection, $\pi_k(x_1, x_2)$, can be obtained by solving the Bellman equation at the current system state (x_1, x_2). The optimal maintenance policy, $\Pi(x_1, x_2)$, is the set of optimal decision rules through the inspection epochs, i.e., $\Pi(x_1, x_2) = (\pi_1(x_1, x_2), \pi_2(x_1, x_2), ..., \pi_N(x_1, x_2))$.

3.2 Solution based on backward dynamic programming

Having discussed the value function and its properties, we proceed to devise an algorithm to compute the value function and the associate control limits at each inspection epoch. From the Bellman equation in Eq. (4), we can find that the total cost-to-go at period k depends only on the values at the next epoch and the expected downtime cost given the current system state. With this observation, we can employ backward dynamic programming for the calculation purpose (Puterman, 2014). Before implementing the backward dynamic programming algorithm, we first need to discretize the continuous degradation processes into a set of finite states. We discretize the degradation level of the two components into M states for some positive integer M. The degradation level of component i is evenly divided within the interval $[0, L_i]$, with the increment $\Lambda_i = L_i/M$. Then the component state is denoted as $S_i = \{\Lambda_i, 2\Lambda_i, ..., L_i - \Lambda_i, L_i\}$. Component i is said to stay in state $j\Lambda_i$ $(j \leq M)$ if the degradation level is within the interval $[(j-1)\Lambda_i, j\Lambda_i)$. Fig. 3 sketches how the discretization of component degradation process works.

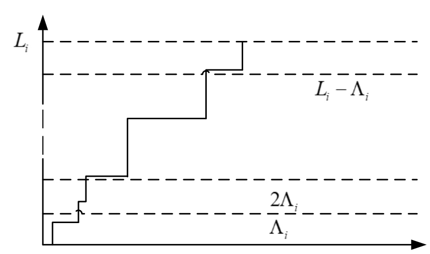

Figure 3: Discretization of the degradation process

The state space of the finite-horizon MDP is given by $\mathbf{S} = \{\mathbf{K} \times S_1 \times S_2\}$, where $\mathbf{K} = \{1, 2, ..., N\}$

is the set of decision epochs and "$\times$" denotes the Cartesian product of two sets. At the kth inspection, given that the system has not failed ($x_1 < L_1$ and $x_2 < L_2$), the transition probability of the system state at the next inspection is given as

$$\begin{aligned}&P(X_{1,k+1} = \omega_1, X_{2,k+1} = \omega_2 | X_{1,k} = x_1, X_{2,k} = x_2)\\&= \int_0^{\zeta_m} g_1(u) g_2(u) f_{\alpha_u t,\beta}(u) du,\end{aligned}$$

where $\zeta_m = \min\{\omega_1 - \Lambda_1 - x_1, \omega_2 - \Lambda_2 - x_2\}$, and $g_i(u) = F_{\alpha_i t,\beta}(\omega_i - x_i - u) - F_{\alpha_i t,\beta}(\omega_i - \Lambda_i - x_i - u)$, $i = 1, 2$.

Before implementing the dynamic programming algorithm, we first need to determine the boundary cost. Without loss of generality, we assume that the terminal cost is zero regardless of the degradation level, i.e., $V_{T_e}(x_1, x_2) = 0$, $\forall (x_1, x_2) \in \mathcal{R}_2^+$. Let T_m be the remaining time period beyond the last inspection, i.e., $T_m = T_e - N\delta$. If the operating horizon can be evenly divided by the inspection, i.e., $T_m = 0$, then we have $V_{N\delta}(x_1, x_2) = 0$. Otherwise, for $T_m > 0$, the expected downtime cost for the remaining period is given as

$$W_{T_m}(x_1, x_2) = \int_0^{T_m} \frac{c_d(e^{-\gamma t} - e^{-\gamma T_m})}{\gamma} dF_{T_N}(t).$$

At the Nth inspection, the value function $V_{N\delta}(x_1, x_2)$ can be evaluated as

$$V_{N\delta}(x_1, x_2) = \begin{cases} c_s + c_{f,1} + c_{f,2} + V_{N\delta}(0,0),\ x_1 > L_1 \ \&\ x_2 > L_2 \\ \min\{c_s + c_{f,1} + V_{N\delta}(0, x_2), c_s + c_{f,1} + c_{p,2} + V_{N\delta}(0,0)\}, x_1 > L_1 \ \&\ x_2 \leq L_2 \\ \min\{c_s + c_{f,2} + V_{N\delta}(x_1, 0), c_s + c_{p,1} + c_{f,2} + V_{N\delta}(0,0)\}, x_1 \leq L_1 \ \&\ x_2 > L_2 \\ \min\{C_{0,N}, C_{1,N}, C_{2,N}, C_{12,N}\}, \text{otherwise} \end{cases} \tag{12}$$

The terms $C_{1,N}, C_{2,N}, C_{12,N}$ can be readily obtained by Eqs. (6)-(8). However, the expected cost of DN, $C_{0,N}$, should be replaced by

$$C_{0,N} = e^{-\gamma\delta} c_i + W_{T_m}(x_1, x_2).$$

This is because the downtime cost is computed within the remaining period T_m and the terminal cost is set as 0 for all $(x_1, x_2) \in \mathcal{R}_2^+$. Detailed procedure of the backward dynamic programming algorithm is presented in Algorithm 1, which produces the optimal total discounted maintenance cost and the associated control limits.

Algorithm 1 Backward dynamic programming algorithm

Require: Parameters of the degradation processes; cost parameters; failure thresholds; discount factor and inspection interval.

Ensure: Value function $V_{k\delta}(x_1, x_2)$ and the associated control limits at each inspection epoch.

1: compute $P(X_{1,k+1}, X_{2,k+1}|X_{1,k} = x_1, X_{2,k} = x_2)$ and $W_{k\delta}(x_1, x_2)$ for all $x_1 \in [0, L_1]$ and $x_2 \in [0, L_2]$.
2: start at $k = N$ and initialize $V_{N\delta}$ according to Eq. (12).
3: **repeat**
4: replace k by $k-1$;
5: calculate $V_{k\delta}(0,0)$; $V_{k\delta}(0,0) = e^{-\gamma\delta}(c_i + U_{k\delta}(0,0)) + W_{k\delta}(0,0)$
6: let $x_1 = 0$;
7: **for all** x_2 **do**
8: calculate $V_{k\delta}(0, x_2)$.

$$V_{k\delta}(0,x_2) = \begin{cases} \min\{e^{-\gamma\delta}(c_i + U_{k\delta}(0,x_2)) + W_{k\delta}(0,x_2), & \\ c_s + c_{p,2} + V_{k\delta}(0,0)\}, & \text{if } x_2 \leq L_2 \\ c_s + c_{f,2} + V_{k\delta}(0,0), & \text{if } x_2 > L_2 \end{cases}$$

9: **end for**
10: let $x_2 = 0$;
11: **for all** x_1 **do**
12: calculate $V_{k\delta}(x_1, 0)$.

$$V_{k\delta}(x_1,0) = \begin{cases} \min\{e^{-\gamma\delta}(c_i + U_{k\delta}(x_1,0)) + W_{k\delta}(x_1,0), & \\ c_s + c_{p,1} + V_{k\delta}(0,0)\}, & \text{if } x_1 \leq L_1 \\ c_s + c_{f,1} + V_{k\delta}(0,0), & \text{if } x_1 > L_1 \end{cases}$$

13: **end for**
14: **for all** x_1 and x_2 **do**
15: calculate $V_{k\delta}(x_1, x_2)$ according to Eq. (4).
16: **end for**
17: **until** $k = 0$;
18: **return** $V_{k\delta}(x_1, x_2)$ and the optimal control limits.

At each inspection epoch, we first calculate the value function $V_{k\delta}(0,0)$ at perfect state (Step 5). It is expected that at the perfect state, one should do nothing and wait till the next inspection. Obviously, PM or CM cannot be optimal at state $(0,0)$. Then we proceed to calculate the value functions when only one component is perfect (Steps 8 and 12). This is because computation of the value function at an arbitrary system state requires values of $V_{k\delta}(x_1, 0)$ and $V_{k\delta}(0, x_2)$. With these values in hand, all the value functions can be obtained with the backward approach.

It is noteworthy that Algorithm 1 is designed for any given inspection interval δ. In practice, it is interesting to determine an optimal inspection interval so as to minimize the total maintenance cost. Recall that the number of inspections over the planning horizon is $N = \lfloor T_e/\delta \rfloor$. On the one hand, a large δ will increase the failure risk and expected downtime cost, which in turn results in a higher expected maintenance cost between two consecutive inspections. On the other hand, a longer inspection interval cuts down the number of inspections within the planning horizon and thus reduces the total inspection cost. This tradeoff implies that there exists an optimal inspection interval such that the total maintenance cost is minimized. The optimal inspection interval can be determined by

$$\delta^* = \arg\min_{\delta} V_{k\delta}(0,0), \text{ for } k = 0, \tag{13}$$

which is a one-dimensional optimization problem. For a given δ, we employ the backward dynamic programming to obtain the associated minimal maintenance cost. Each time when δ varies, we can have the corresponding optimal maintenance cost. The optimal inspection interval δ^* can be obtained by any derivative-free search approaches.

4 A simple case with fixed PM thresholds

Traditional CBM policies are usually performed by comparing the degradation levels with fixed PM thresholds. Although this simplified policy may be suboptimal, it is easy to implement in practical applications. The proposed MDP formulation can solve this problem.

Let P_i be the PM threshold for component i, $i = 1, 2$. The system states for different maintenance actions can be divided into nine regions, as shown in Fig. 4. Denote D_j $(j = 1, 2, ..., 9)$ as the regions of various maintenance actions. For fixed PM thresholds, the maintenance regions reduce to rectangular shapes instead of the general shapes in the optimal structure. This implies that the maintenance action on one component is independent of the degradation level of the other component.

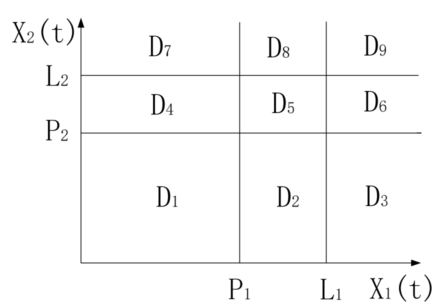

Figure 4: Illustration of the system state classification for given PM thresholds

Let $\Delta X_i(t_1, t_2) = X_i(t_2) - X_i(t_1)$ be the degradation increment of component i between t_1 and t_2 $(t_2 > t_1)$. In this study, the system is under periodic inspection with interval δ and the components follow gamma degradation processes. Since the gamma process is a stochastic process with independent, non-negative increments following gamma distribution (van Noortwijk, 2009), for notational simplicity, we can denote the degradation increment of component i within the interval δ as $\Delta X_i(\delta)$. Given the system state $X(t_k) = (x_1, x_2)$, $x_i < P_i$, at the kth inspection, the probability that the system state remains in domain D_1 at the next inspection is given by

$$\begin{aligned}
&P\{X(t_{k+1}) \in D_1 | X_{1,k} = x_1, X_{2,k} = x_2\} \\
&= P\{\Delta X_1(\delta) < P_1 - x_1, \Delta X_2(\delta) < P_2 - x_2\} \\
&= \int_0^{P_m - x_m} F_{\alpha_1 t,\beta}(P_1 - x_1 - u) F_{\alpha_2 t,\beta}(P_2 - x_2 - u) f_{\alpha_u t,\beta}(u) du \\
&= \int_0^{P_m - x_m} \prod_{i=1}^{2} \frac{\gamma(\alpha_i \delta, \beta(P_i - x_i - u))}{\Gamma(\alpha_i \delta)} \frac{\beta^{\alpha_u \delta}}{\Gamma(\alpha_u \delta)} e^{-\beta u} u^{\alpha_u \delta - 1} du.
\end{aligned} \tag{14}$$

The probability that the system state enters D_2 and D_3 can be computed as

$$\begin{aligned}
&P\{X(t_{k+1}) \in D_2 | X_{1,k} = x_1, X_{2,k} = x_2\} \\
&= P\{P_1 - x_1 < \Delta X_1(\delta) < L_1 - x_1, \Delta X_2(\delta) < P_2 - x_2\} \\
&= \int_0^{P_m - x_m} (F_{\alpha_1 t,\beta}(L_1 - x_1 - u) - F_{\alpha_1 t,\beta}(P_1 - x_1 - u)) F_{\alpha_2 t,\beta}(P_2 - x_2 - u) f_{\alpha_u t,\beta}(u) du,
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
&P\{X(t_{k+1}) \in D_3 | X_{1,k} = x_1, X_{2,k} = x_2\} \\
&= P\{\Delta X_1(\delta) > L_1 - x_1, \Delta X_2(\delta) < P_2 - x_2\} \\
&= \int_0^{P_m - x_m} (1 - F_{\alpha_1 t,\beta}(L_1 - x_1 - u)) F_{\alpha_2 t,\beta}(P_2 - x_2 - u) f_{\alpha_u t,\beta}(u) du,
\end{aligned} \tag{16}$$

where $P_m - x_m = \min\{P_1 - x_1, P_2 - x_2\}$. The probability that the system state falls into other regions at the next inspection can be obtained in a similar way. The value function in MDP is described as

$$V_{k\delta}(x_1, x_2) = \begin{cases}
c_s + c_{f,1} + c_{f,2} + V_{k\delta}(0, 0),\ X(t_k) \in D_9 \\
c_s + c_{p,1} + c_{f,2} + V_{k\delta}(0, 0), X(t_k) \in D_8 \\
c_s + c_{f,2} + V_{k\delta}(0, x_1), X(t_k) \in D_7 \\
c_s + c_{f,1} + c_{p,2} + V_{k\delta}(0, 0), X(t_k) \in D_6 \\
c_s + c_{p,1} + c_{p,2} + V_{k\delta}(0, 0), X(t_k) \in D_5 \\
c_s + c_{p,2} + V_{k\delta}(0, x_1), X(t_k) \in D_4 \\
c_s + c_{f,1} + V_{k\delta}(0, x_2), X(t_k) \in D_3 \\
c_s + c_{p,1} + V_{k\delta}(0, x_2), X(t_k) \in D_2 \\
e^{-\gamma \delta}(c_i + U_{k\delta}(x_1, x_2)) + W_{k\delta}(x_1, x_2), X(t_k) \in D_1
\end{cases} \tag{17}$$

The backward dynamic programming algorithm is employed to compute the maintenance cost using Eq. (17), which is detailed in Appendix B.

5 A practical example

In this section, a water supply pipe system is used as an example to illustrate the proposed maintenance model. The pipe system is used to transport water across areas, in which the pipes are subject to corrosion during usage. We focus on two adjacent pipes where the corrosion of the pipes are correlated because they operate in a similar soil environment. In the literature, gamma process has been widely used to describe the corrosion in pipes (Pandey et al., 2011; Ye et al., 2014). The corrosion in pipes follows a bivariate Gamma degradation process, with the parameters $(a_1, a_2, \beta, \rho) = (0.4, 0.5, 1, 0.6708)$. A pipe is assumed to fail when its degradation level exceeds its failure thresholds, $L_1 = 25$ and $L_2 = 15$. The system is subject to periodic inspection, at a cost $c_i = 3$. Corrective replacement is performed if a pipe is found failed upon inspection, and a preventive replacement is carried out if the degradation level of any pipe exceeds the PM threshold. Costs for preventive replacement and corrective replacement are given as $c_{p,1} = 40$, $c_{p,2} = 20$, and $c_{f,1} = 80$, $c_{f,2} = 40$. A setup cost $c_s = 30$ is incurred along with the maintenance actions on any component(s), which is close to the unit maintenance cost. The system failure mode is of latent nature, i.e., it can only be detected by an inspection. When the system is shut down, a downtime cost is incurred at $c_d = 100$ per unit time. The discount rate is set as $\gamma = 0.01$ per year. The system is designed to operate for a horizon of $T_e = 30$ years and will be discarded after the use period. Therefore, the terminal cost is set to 0 for all the system states, i.e., $V_{T_e}(x_1, x_2) = 0$, for all $x_1, x_2 \in \mathcal{R}_2^+$.

5.1 Optimal maintenance policy

To facilitate computation, we discretize the continuous degradation level of each component into $M = 20$ discrete states. For a fixed δ, the optimal maintenance policy can be obtained via Algorithm 1; the optimal inspection interval δ^* can be determined by a derivative-free search approach. Fig. 5 presents the expected total maintenance cost versus the inspection interval δ ranging from 1 to 10. The maintenance model leads to an optimal inspection interval of $\delta^* = 5$ and the associated optimal cost of 40.8. It is interesting to see that the expected maintenance cost in a finite horizon exhibits a unimodal trend in terms of the inspection interval.

At the optimal inspection interval $\delta^* = 5$, the total number of inspections is $T_e/\delta^* = 6$. The optimal control limits are shown in Figs. 6 and 7, where Fig. 6 presents the optimal PM thresholds for the first inspection and Fig. 7 shows the optimal PM thresholds for all the six inspection epochs. As stated in Proposition 2, the two-dimensional area is divided into four regions by the PM thresholds where each region corresponds to an optimal maintenance action. When the degradation levels of the two components are small enough, DN should be selected as the optimal maintenance action. When both components undergo large degradation, they should be preventively replaced simultaneously. Only one component will be replaced if one component has much higher degradation than the other one. Due to the finite nature of the operating horizon, the optimal control limits

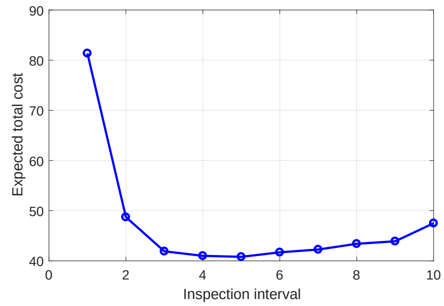

Figure 5: Expected total maintenance cost under different inspection intervals

vary with respect to the inspection interval. As shown in Fig. 7, the region for DN increases with the inspection index, while the opposite holds for simultaneous replacement (PM_{12}). This is due to the fact that the system can operate at a high risk of system failure when approaching to the terminal period. At the early stage of system operation, the maintenance policy is more conservative so as to reduce the risk of system failure.

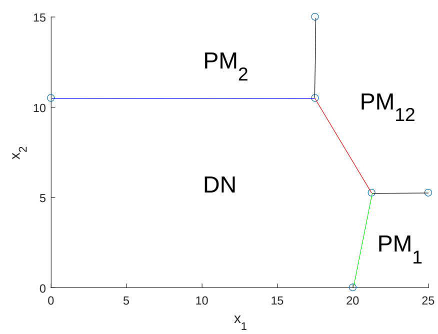

Figure 6: Optimal PM thresholds at the first inspection

In addition, Fig. 8 shows the variation of the expected total cost with respect to different initial degradation levels of the components. It is observed that the expected cost shows a nondecreasing trend with respect to either of the component degradation levels. This is due to the nondecreasing property of the value function, as stated in Proposition 1. The area where the expected cost remains constant with the component degradation level indicates preventive replacement of that component when its degradation level is within that area. This can be reflected by the value function in Eq. (4), where preventive replacement is carried out when the degradation level exceeds a certain threshold and restores the component to an as-good-as-new state.

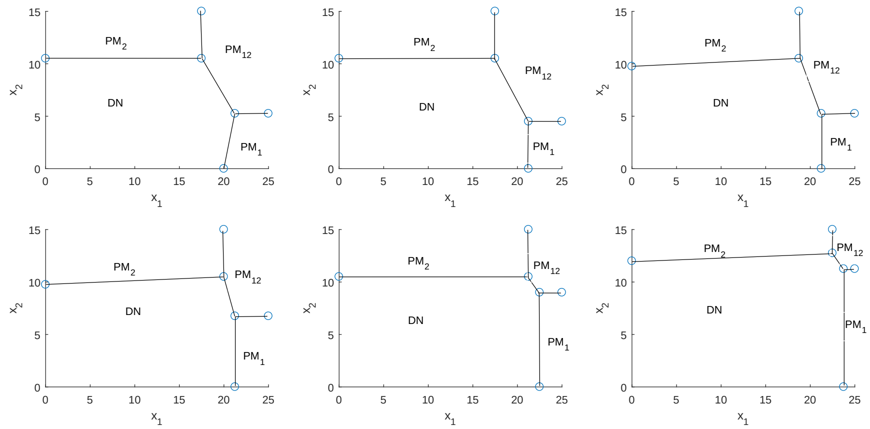

Figure 7: Optimal PM thresholds for the six inspection epochs

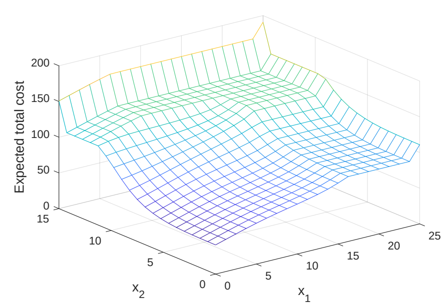

Figure 8: Expected total cost under different initial degradation levels

5.2 Sensitivity analysis

In this part, we investigate the effect of three parameters on the optimal maintenance policy—the correlation coefficient ρ, the setup cost c_s and the discount rate γ. In essence, the correlation coefficient ρ signifies the dependency level between the degradation processes, the setup cost c_s determines the economic dependence of the system, and the discount rate γ influences the importance of future cost.

Influence of correlation coefficient ρ. One contribution of this paper is to incorporate a realistic factor of degradation dependence between the components in maintenance decision making. Fig. 9 shows that the expected total cost decreases monotonically with ρ. This is because the components are in series and failure of any component can lead to system failure. When the two components have a lower correlation, the degradations of the two components are more likely to be random and the failure events more divergent. Hence, the system is more inclined to fail for a low correlation coefficient. On the other hand, the existence of economic dependence (common setup cost) contributes to the maintenance cost when the components are replaced individually. The two components are more likely to simultaneously reach the failure or PM thresholds and be replaced together with a higher correlation level between the degradation processes. Therefore, the common setup cost can be shared more frequently, which results in a lower maintenance cost. Fig. 10 further presents the optimal maintenance decisions at the first inspection for alternative values of the correlation coefficient ρ.

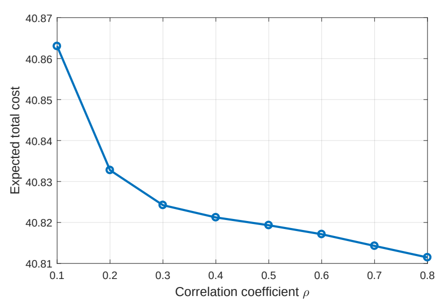

Figure 9: Expected total cost versus the correlation coefficient ρ

Influence of setup cost c_s. The setup cost reflects the economic dependence of the system. Fig. 11 shows monotonic increase of the expected cost in terms of the setup cost c_s. This is due to the monotonic nature of Eq. (4) with respect to the setup cost.

Fig. 12 shows how the optimal maintenance actions vary with respect to the setup cost ranging from $c_s = 0$ to $c_s = 80$. When the setup cost is reduced to 0, i.e., no economic dependence exists between the components, the regions for optimal maintenance policy are reduced to rectangular shapes rather than the irregular shapes that occur under finite setup cost. This indicates that the optimal maintenance decision on one component is independent from the condition of the other

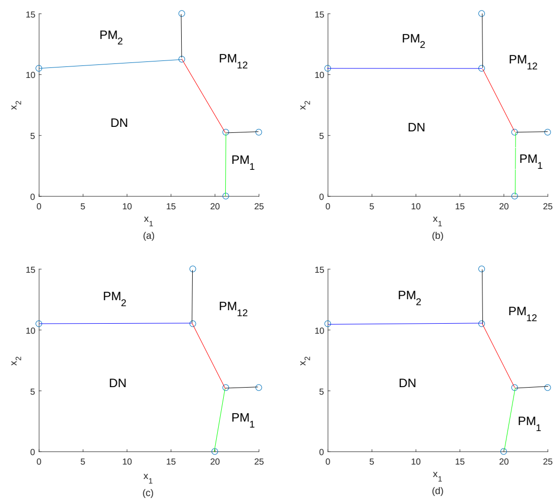

Figure 10: Optimal maintenance decisions at the first inspection with different correlation coefficients: (a)$\rho = 0.2$, (b)$\rho = 0.4$, (c)$\rho = 0.6$, (d)$\rho = 0.8$

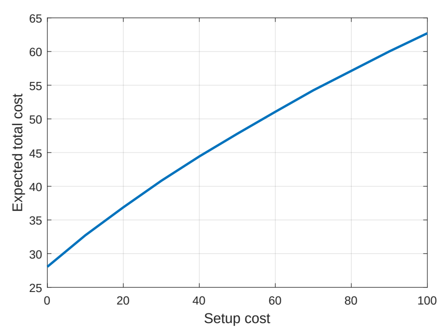

Figure 11: Expected total cost versus the setup cost c_s

component. On the other hand, the boundaries between PM_{12} and PM_1, and between PM_{12} and PM_2 (black lines) remain constant with different setup costs. This is due to the fact that setup cost is induced when any of the components is replaced, which makes no difference between separate replacement or simultaneous replacement. Another interesting finding is the region size of the four maintenance actions. The region for DN shows an increasing trend with the setup cost, while the region sizes of other maintenance actions (PM_1, PM_2 and PM_{12}) decrease. This is because the system is reluctant to be replaced at a high setup cost so as to reduce the maintenance cost. In other words, a high setup cost leads to higher PM thresholds for both the two components. As can be observed from Fig. 12, the boundary between PM_2 and DN shifts up and that between PM_1 and DN shifts right with the increase of setup cost, implying a less conservative policy.

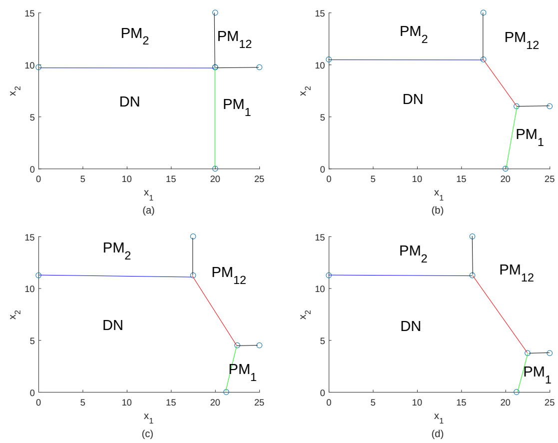

Figure 12: Optimal maintenance decisions at the first inspection with respect to setup cost: (a)$c_s = 0$, (b) $c_s = 20$, (c) $c_s = 40$, (d) $c_s = 60$

Influence of discount rate γ. The discount rate γ signifies the importance of future cost on the maintenance decisions. A small discount rate indicates that future cost is important, whereas a large value signifies that the present cost is far more important than the future cost. Fig. 13 shows that the expected total cost decreases monotonically with γ since a higher discount rate reduces future expenses.

Fig. 14 exhibits the optimal maintenance decisions at the first inspection with respect to the the discount rate γ. It can be observed that the boundaries for preventive replacement shift right or up, which indicates higher PM thresholds for both components and a more aggressive maintenance policy. In addition, the region size of DN increases with the discount rate while the region of PM_{12} shows an opposite trend, due to the fact that the discount rate evaluates the weight of future cost. A large γ implies that future costs will deplete fast and more effort should be devoted to the present condition rather than future degradations. In other words, for a large γ, the system is allowed to operate at a high failure risk since the failure cost occurring in future will add less value to the

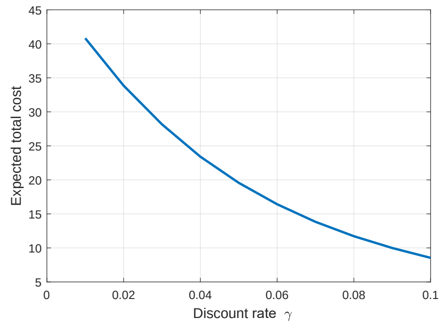

Figure 13: Expected total cost *vs* discount rate γ

current expense. Actually, the influence of the discount rate can be shown from the value function of Eq. (4), where the value for DN ($C_{0,k}$) decreases monotonically with γ. The optimal thresholds are achieved when the value for DN reaches the values for PM ($C_{1,k}$, $C_{2,k}$, and $C_{12,k}$), a higher threshold is therefore envisioned with a larger discount rate.

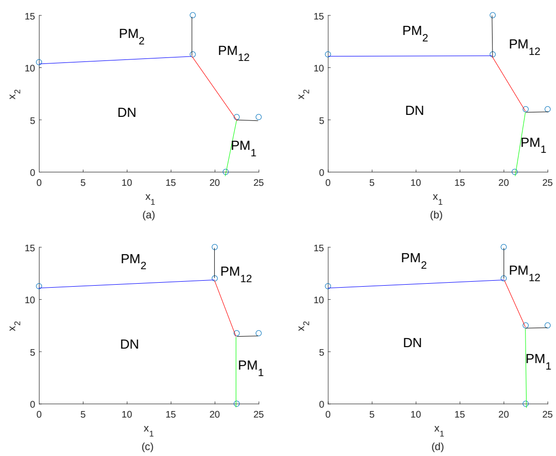

Figure 14: Optimal maintenance decisions at the first inspection with respect to discount rate: (a) $\gamma = 0.02$, (b) $\gamma = 0.04$, (c) $\gamma = 0.06$, (d) $\gamma = 0.08$

5.3 Cost evaluation under fixed PM thresholds

In practice, many engineering systems are subject to fixed PM thresholds due to industry standards or safety concerns. In this section, we consider the case in which the PM thresholds are exogenous. We will focus on the inspection interval and evaluate the maintenance cost for each inspection interval. The PM thresholds are set to $P_1 = 20$ and $P_2 = 12$, while the other parameters are identical to those before.

Fig. 15 presents the variation of the expected total cost with different inspection intervals under fixed PM thresholds. It can be observed that the minimum expected cost is achieved as 43.9 at the inspection interval $\delta^* = 3$. The maintenance cost curve exhibits a similar pattern to that under flexible PM thresholds. Nevertheless, the expected maintenance cost obtained with fixed PM thresholds is larger than that under flexible PM thresholds. This is because the fixed PM thresholds may not be the optimal policy. In addition, the optimal inspection interval is smaller than that under flexible PM thresholds.

We also compare the analytical results with Monte Carlo simulation in Fig. 15. One can observe that the analytical result matches the simulation for large inspection intervals while there exist some discrepancies for small inspection intervals. A possible reason is that the the numerical error is caused by discretization of the continuous degradation level in solving the value function. Another reason is the simulation process itself. In the current simulation, the failure time and system states are measured at integer points (e.g., 1, 2, 3,...). It neglects the case that the system may fail at non-integral epochs, in which the effect is more significant for a small inspection interval.

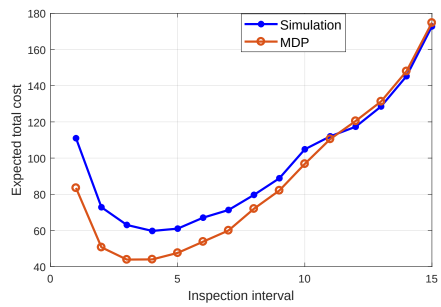

Figure 15: Expected total maintenance cost with different inspection intervals under fixed PM thresholds

With the fixed PM thresholds, we investigate the influence of correlation coefficient, setup cost and discount factor on the expected total cost. The results are shown in Fig. 16. As can be observed, the curves are quite similar to those under flexible PM thresholds, which indicates that the parameters have a similar impact on the expected maintenance cost. This is quite intuitive, as the value functions for fixed or flexible PM thresholds exhibit a similar structure.

Next, we investigate the optimal maintenance policy under the setting of an infinite horizon. As shown in Fig. 17, the optimal maintenance policy in an infinite horizon is a stationary two-dimensional control limit policy, which indicates that the maintenance boundaries remain constant at inspection epochs. The stationary infinite-horizon policy leads to an maintenance cost of 43.4, at an inspection interval $\delta^* = 5$. The suboptimality of the infinite-horizon policy can be observed when compared with the maintenance cost of the finite-horizon policy at 40.8.

In this study, we analytically present the optimal maintenance policy for a finite planning

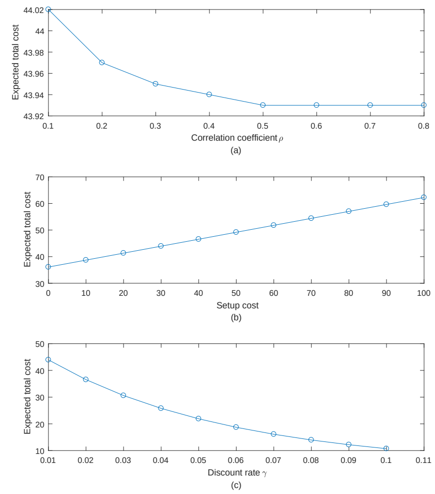

Figure 16: Parameter influence on the expected cost under fixed PM thresholds: (a) correlation coefficient ρ, (b) setup cost c_s, (c) discount rate γ

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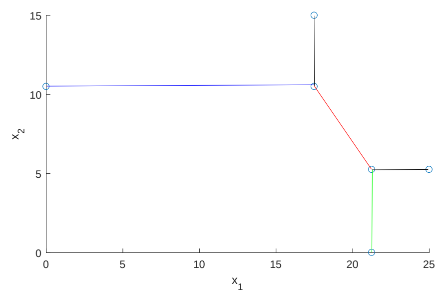

Figure 17: Optimal maintenance policy for an infinite horizon

horizon and show that the optimal policy is a dynamic control-limit policy. However, although the proposed policy is theoretically optimal, it may not be so easy to implement in practice compared with the stationary policy. One should take into consideration the cost due to the rearrangement of maintenance staff and reallocation of the resources when implementing the non-stationary policy. As can be observed, the difference between a dynamic and stationary policy depends largely on the length of the planning horizon. If the planning horizon is short, then the proposed dynamic policy is suggested. Yet if the planning horizon is relatively long but still finite, a hybrid policy that combines the two policies can be adopted, i.e., engineers are suggested to follow the stationary policy at the beginning of the horizon and switch to the dynamic policy when approaching to the end of the planning horizon.

6 Conclusions

This paper analyzes a condition-based maintenance policy for a two-unit series system within a finite-time horizon. The system under investigation consists of two heterogeneous components in which degradation processes follow a bivariate gamma process. The components are subject to periodic inspection and will be preventively replaced if their degradation levels exceed PM thresholds. The maintenance problem is formulated as a MDP model and the optimal maintenance policy is obtained by minimizing the expected total discounted cost over the horizon. The optimal maintenance decision turns out to be a two-dimensional control-limit policy. Different from an infinite-horizon case which permits a stationary optimal policy, the optimal maintenance policy for a finite horizon is non-stationary, which varies at each inspection epoch. For a relatively long but still finite planning horizon, a hybrid policy that combines the stationary and dynamic policies is suggested, i.e., engineers can follow the stationary policy at the beginning and switch to the dynamic policy when approaching to the end of the horizon. For engineering systems that may have fixed PM thresholds due to safety reasons or industrial standards, the proposed approach can also be employed to evaluate the optimal maintenance plan. In addition, the influence of stochastic and economic dependence is investigated through a numerical example. It shows that a higher dependence between the degradation processes reduces the maintenance cost while a higher economic dependence leads to higher PM thresholds.

Although the present work focuses on two-unit systems, it can be extended to multi-unit systems by generalizing the degradation process and Bellman equation. The control-limit policy applies to multi-unit systems as well. However, unlike a two-unit system whose optimal maintenance policy can be presented via a two-dimensional graph, maintenance decisions for multi-unit systems may lead to a multi-dimensional structure. The difficulty for this extension lies in the computational burden, where the size of transition probability and value functions grows dramatically with the number of components.

There are several issues and topics worth further exploration. In the current work, we assume that the inspection is perfect and reveals the degradation levels of both components simultaneously. In reality, the inspection may be comprised by measurement noise. In this case, some filtering approaches (e.g., particle filter, Kalman filter and its variants) can be employed to estimate the degradation level as a first step, followed by maintenance decision making. In addition, if the inspection cost is high, we may allow separate inspection on the components instead of the whole system. Moreover, several extensions on the maintenance model can be further investigated, such as imperfect repair, non-periodic inspection or optimization on availability instead of cost.

Acknowledgement

The authors would like to thank Prof Matthew Revie for providing suggestive comments that help improve the paper. The authors are also thankful to the Editor and the anonymous reviewers for their constructive comments.

Appendix

A Proofs

Proof of Proposition 1. Before proving the monotonicity property of the value function, we first need to show that the expected downtime cost $W_{k\delta}(x_1, x_2)$ is nondecreasing in x_1 and x_2. This property can be proved via stochastic dominance. Conditioned on the system state (x_1, x_2), the failure time T_k satisfies

$$P(T_k \le t | X_{1,k} = x_1, X_{2,k} = x_2) = P(X_1(t) > L_1 \cup X_2(t) > L_2 | X_{1,k} = x_1, X_{2,k} = x_2).$$

Let us first consider the degradation level of the component 1, $X_1(t)$. For $x_1^- < x_1^+$, it follows that

$$\begin{aligned} P(T_k \le t | X_{1,k} = x_1^-, X_{2,k} = x_2) &= P(X_1(t) > L_1 \cup X_2(t) > L_2 | X_{1,k} = x_1^-, X_{2,k} = x_2) \\ < P(X_1(t) > L_1 \cup X_2(t) > L_2 | X_{1,k} = x_1^+, X_{2,k} = x_2) &= P(T_k \le t | X_{1,k} = x_1^+, X_{2,k} = x_2). \end{aligned}$$

By the definition of stochastic order, we have the stochastic relationship of T_k conditioned on the component degradation level as $\langle T_k | X_{1,k} = x_1^-, X_{2,k} = x_2 \rangle \succ_{st} \langle T_k | X_{1,k} = x_1^+, X_{2,k} = x_2 \rangle$. On the other hand, conditioned on the system state $(X_{1,k} = x_1, X_{2,k} = x_2)$, we have the downtime cost (discounted at the kth inspection t_k) as

$$\begin{aligned} d(T_k) &= \int_{T_k}^{\delta} c_d \cdot e^{-\gamma(t-T_k)} dt \\ &= \frac{c_d(e^{-\gamma T_k} - e^{-\gamma\delta})}{\gamma}, \end{aligned}$$

which is decreasing in T_k. Therefore, we have

$$\begin{aligned} W_{k\delta}(x_1^-, x_2) &= E[d(T_k)|X_{1,k} = x_1^-, X_{2,k} = x_2] \\ &\le E[d(T_k)|X_{1,k} = x_1^+, X_{2,k} = x_2] = W_{k\delta}(x_1^+, x_2), \end{aligned}$$

which indicates that $W_{k\delta}(x_1, x_2)$ is nondecreasing in x_1. Similarly, we can prove that $W_{k\delta}(x_1, x_2)$ is nondecreasing in x_2.

We then prove the monotonicity property of the value function by mathematical induction. First, at the terminal period $k = N$, the value function is constant, which obviously satisfies the nondecreasing property. Assuming that the property holds at period $k+1$ ($0 \le k \le N-1$), we can show that this property also holds at period k.

Consider the case that no component fails at the kth inspection, *i.e.*, $x_1 < L_1$ & $x_2 < L_2$. We have

$$V_{k\delta}(x_1, x_2) = \min \left\{C_{0,k}, C_{1,k}, C_{2,k}, C_{12,k}\right\},$$

where

$$C_{0,k} = e^{-\gamma\delta}(c_i + U_{k\delta}(x_1, x_2)) + W_{k\delta}(x_1, x_2),$$

$$C_{1,k} = c_s + c_{p,1} + V_{k\delta}(0, x_2),$$

$$C_{2,k} = c_s + c_{p,2} + V_{k\delta}(x_1, 0),$$

$$C_{12,k} = c_s + c_{p,1} + c_{p,2} + V_{k\delta}(0, 0).$$

Recall that $U_{k\delta}(x_1, x_2)$ stands for the expected cost-to-go given the current system state, *i.e.*,

$$U_{k\delta}(x_1, x_2) = E[V_{k+1}(X_{1,k+1}, X_{2,k+1})|X_{1,k} = x_1, X_{2,k} = x_2].$$

Let us consider the effect of $X_{1,k} = x_1$ first. Since $V_{k+1}(X_{1,k+1}, X_{2,k+1})$ is nondecreasing with respect to $X_{1,k+1}$ and $X_{2,k+1}$, and $\langle X_{1,k+1}|X_{1,k} = x_1^-, X_{2,k} = x_2\rangle \prec \langle X_{1,k+1}|X_{1,k} = x_1^+, X_{2,k} = x_2\rangle$ for any $x_1^- < x_1^+$, we have

$$\begin{aligned} &U_{k\delta}(x_1^-, x_2) = E[V_{k+1}(X_{1,k+1}, X_{2,k+1})|X_{1,k} = x_1^-, X_{2,k} = x_2] \\ &\le E[V_{k+1}(X_{1,k+1}, X_{2,k+1})|X_{1,k} = x_1^+, X_{2,k} = x_2] = U_{k\delta}(x_1^+, x_2), \end{aligned}$$

which indicates that $U_{k\delta}(x_1, x_2)$ is nondecreasing in x_1. In a similar manner, we can prove that $U_{k\delta}(x_1, x_2)$ is nondecreasing in x_2. Combining the nondecreasing property of $W_{k\delta}(x_1, x_2)$ in x_1 and x_2, we can conclude that $C_{0,k}$ is nondecreasing in x_1 and x_2.

On the other hand, for $C_{1,k}$, obviously it is nondecreasing in x_1. In addition, we can easily show that $C_{1,k}$ is nondecreasing in x_2. For $V_{k\delta}(0,x_2)$, we have

$$V_{k\delta}(0,x_2) = \min\{e^{-\gamma\delta}(c_i + U_{k\delta}(0,x_2)) + W_{k\delta}(0,x_2), c_s + c_{p,2} + V_{k\delta}(0,0)\}.$$

Based on the previous discussion, we can have that the first term is nondecreasing in x_2. Obviously the second term is nondecreasing in x_2. Therefore, $V_{k\delta}(0,x_2)$ is nondecreasing in x_1 and x_2. We can conclude that $C_{1,k}$ is nondecreasing in x_1 and x_2. Similarly, we can prove that $C_{2,k}$ and $C_{12,k}$ are nondecreasing in x_1 and x_2. Since $C_{0,k}$, $C_{1,k}$, $C_{2,k}$ and $C_{12,k}$ are nondecreasing in x_1 and x_2, we can conclude that $V_{k\delta}(x_1,x_2)$ is nondecreasing in x_1 and x_2 for $x_1 < L_1$ & $x_2 < L_2$. With similar argument, we can prove that $V_{k\delta}(x_1,x_2)$ is nondecreasing in x_1 and x_2 for any $x_1 > 0$ and $x_2 > 0$. Hence, the induction hypothesis holds for all $k = 0,1,2,...,N$, which completes the proof. $\square$

Proof of Proposition 2. Let us first consider the boundary of PM_i and PM_{12}. For component 2, since the value function is nondecreasing in both x_1 and x_2, $C_{2,k} = c_s + c_{p,2} + V_{k\delta}(x_1,0)$ is nondecreasing in x_1 and constant in x_2, while $C_{12,k} = c_s + c_{p,1} + c_{p,2} + V_{k\delta}(0,0)$ is constant regardless the variation of x_1 and x_2. Therefore, there must exist ζ_1 that $C_{2,k} > C_{12,k}$ for $x_1 > \zeta_1$. Specifically, ζ_1 can be expressed as

$$\zeta_1 = \arg\max_{x_1}\{V_{k\delta}(x_1,0) \le c_{p,1} + V_{k\delta}(0,0)\}.$$

Via similar argument, we can have the line $x_2 = \zeta_2$ as the boundary of PM_1 and PM_{12}.

Then we focus on the boundary of PM_i and DN. Consider the maintenance action on component 1. For a given x_2, $C_{0,k} = e^{-\gamma\delta}(c_i + U_{k\delta}(x_1,x_2)) + W_{k\delta}(x_1,x_2)$ is nondecreasing in x_1 and $C_{1,k}$ is constant. There must exist a $h_2(x_2)$ that $C_{0,k} > C_{1,k}$ for $x_1 > h_2(x_2)$, which implies that the optimal maintenance action is PM_1 if $x_1 > h_2(x_2)$ and DN otherwise. $h_2(x_2)$ changes with the variation of x_2. The line $x_1 = h_2(x_2)$ constitutes the boundary of PM_1 and DN. Similarly, we can have $x_2 = h_1(x_1)$ as the boundary of PM_2 and DN.

The boundary of PM_{12} and DN can be obtained in a similar way. For a given x_1 ($L_1 > x_1 \ge \zeta_1$), since $C_{0,k}$ is nondecreasing in x_2 and $C_{12,k}$ is constant, there must exist a $l(x_1)$ ($L_2 > l(x_1) \ge \zeta_2$) that $C_{0,k} > C_{12,k}$ at the state (x_1,x_2) if $x_2 > l(x_1)$. The line $x_2 = l(x_1)$ can be obtained by varying x_1 within the interval $[\zeta_1, h_2(\zeta_2)]$.

In addition, since the value function is continuous on x_1 and x_2, it holds $C_{0,k} = C_{12,k}$ at the boundary line $x_2 = l(x_1)$. Suppose that $l(x_1)$ is nondecreasing in x_1. Without loss of generality, let $C_{0,k} = C_{12,k}$ at system state $(x_1^-, l(x_1^-))$. Then for system states $(x_1^+, l(x_1^+))$, $x_1^- < x_1^+$, we have $(C_{0,k}; (x_1^+, l(x_1^+))) > (C_{0,k}; (x_1^-, l(x_1^-))) = C_{12,k}$, which contradicts the argument that $(x_1^+, l(x_1^+))$ is along the boundary line $x_2 = l(x_1)$. Therefore, we can conclude that $l(x_1)$ is nonincreasing in x_1. $\square$

Proof of Corollary 1. We prove this corollary by contradiction. Suppose that $(x_1,x_2) \notin [\zeta_1, L_1] \times [\zeta_2, L_2]$. Without loss of generality, let $x_1 \in [0,\zeta_1)$. Based on the argument of Proposition 2, we have $C_{2,k} > C_{12,k}$ for $L_1 > x_1 \geq \zeta_1$ and $C_{2,k} < C_{12,k}$ for $0 \leq x_1 < \zeta_1$. If $x_1 \in [0, \zeta_1)$, then PM_2 is more cost-effective than PM_{12}, which contradicts the argument that PM_{12} is the optimal maintenance action. The argument also applies for $x_2 \in [0, \zeta_2)$. □

Proof of Corollary 2. If $x_1 = \zeta_1$ is the boundary of PM_{12} and PM_2, with the expression of $C_{2,k}$ and $C_{12,k}$, we have $V_{k\delta}(x_1, 0) \leq c_{p,1} + V_{k\delta}(0, 0)\}$ for $0 \leq x_1 \leq \zeta_1$ and $V_{k\delta}(x_1, 0) > c_{p,1} + V_{k\delta}(0, 0)\}$ for $L_1 > x_1 > \zeta_1$. For the case $x_1 \leq L_1$ & $x_2 > L_2$, by comparing the two terms in the value function, we can have $c_s + c_{f,2} + V_{k\delta}(x_1, 0) > c_s + c_{p,1} + c_{f,2} + V_{k\delta}(0, 0)$ for $L_1 > x_1 > \zeta_1$ and $c_s + c_{f,2} + V_{k\delta}(x_1, 0) \leq c_s + c_{p,1} + c_{f,2} + V_{k\delta}(0, 0)$ for $0 \leq x_1 \leq \zeta_1$, which indicates that the optimal maintenance action is CM_2 for $0 \leq x_1 \leq \zeta_1$ and $\{CM_2, PM_1\}$ for $L_1 > x_1 > \zeta_1$. The similar result can be obtained for component 2. □

B Backward dynamic programming algorithm under fixed PM thresholds

See Algorithm 2.

Algorithm 2 Backward dynamic programming algorithm under fixed PM thresholds

Require: Parameters of the degradation processes; cost parameters; failure and PM thresholds; discount factor and inspection interval.

Ensure: Value function $V_{k\delta}(x_1, x_2)$ at each inspection epoch.

1: compute $P(X_{1,k+1}, X_{2,k+1} | X_{1,k} = x_1, X_{2,k} = x_2)$ and $W_{k\delta}(x_1, x_2)$ for all $x_1 \in [0, L_1]$ and $x_2 \in [0, L_2]$.

2: start at $k = N$ and initialize $V_{N\delta}$ according to Eq. (12).

3: **repeat**

4: replace k by $k - 1$;

5: calculate $V_{k\delta}(0, 0)$. $V_{k\delta}(0, 0) = e^{-\gamma\delta}(c_i + U_{k\delta}(0, 0)) + W_{k\delta}(0, 0)$;

6: let $x_1 = 0$;

7: **for all** x_2 **do**

8: calculate $V_{k\delta}(0, x_2)$.

$$V_{k\delta}(0, x_2) = \begin{cases} e^{-\gamma\delta}(c_i + U_{k\delta}(0, x_2)) + W_{k\delta}(0, x_2), \text{ if } x_2 \leq P_2 \\ c_s + c_{p,2} + V_{k\delta}(0, 0), \text{ if } P_2 < x_2 \leq L_2 \\ c_s + c_{f,2} + V_{k\delta}(0, 0), \text{ if } x_2 > L_2 \end{cases}$$

9: **end for**

10: let $x_2 = 0$;

11: **for all** x_1 **do**

calculate $V_{k\delta}(x_1, 0)$.

$$V_{k\delta}(x_1, 0) = \begin{cases} e^{-\gamma\delta}(c_i + U_{k\delta}(x_1, 0)) + W_{k\delta}(x_1, 0), \text{if } x_1 \leq P_1 \\ c_s + c_{p,1} + V_{k\delta}(0, 0), \text{ if } P_1 < x_1 \leq L_1 \\ c_s + c_{f,1} + V_{k\delta}(0, 0), \text{ if } x_1 > L_1 \end{cases}$$

end for
for all x_1 and x_2 **do**
calculate $V_{k\delta}(x_1, x_2)$ according to Eq. (17).
end for
until $k = 0$;
return $V_{k\delta}(x_1, x_2)$ for all x_1, x_2 and $k \in \{0, 1, 2, ...N\}$.

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