

A competitive search game with a moving target

Citation for published version (APA):

Duvocelle, B., Flesch, J., Staudigl, M., & Vermeulen, D. (2022). A competitive search game with a moving target. *European Journal of Operational Research*, 303(2), 945-957.
<https://doi.org/10.1016/j.ejor.2022.03.004>

Document status and date:

Published: 01/12/2022

DOI:

[10.1016/j.ejor.2022.03.004](https://doi.org/10.1016/j.ejor.2022.03.004)

Document Version:

Publisher's PDF, also known as Version of record

Document license:

Taverne

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.umlib.nl/taverne-license

Take down policy

If you believe that this document breaches copyright please contact us at:

repository@maastrichtuniversity.nl

providing details and we will investigate your claim.



Interfaces with Other Disciplines

A competitive search game with a moving target[☆]Benoit Duvocelle^{a,*}, János Flesch^b, Mathias Staudigl^c, Dries Vermeulen^b^a Toulouse School of Economics, Université Toulouse 1 Capitole, France^b Department of Quantitative Economics, Maastricht University, Netherlands^c Department of Data Science and Knowledge Engineering, Maastricht University, Netherlands

ARTICLE INFO

Article history:

Received 3 December 2020

Accepted 1 March 2022

Available online 7 March 2022

Keywords:

Game Theory

Search game

Two-player zero-sum game

Optimal strategies

Discrete time-varying Markov process

ABSTRACT

We introduce a discrete-time search game, in which two players compete to find an invisible object first. The object moves according to a time-varying Markov chain on finitely many states. The players are active in turns. At each period, the active player chooses a state. If the object is there then he finds the object and wins. Otherwise the object moves and the game enters the next period. We show that this game admits a value, and for any error-term $\varepsilon > 0$, each player has a pure (subgame-perfect) ε -optimal strategy. Interestingly, a 0-optimal strategy does not always exist. We derive results on the analytic and structural properties of the value and the ε -optimal strategies. We devote special attention to the important time-homogeneous case, where we show that (subgame-perfect) optimal strategies exist if the Markov chain is irreducible and aperiodic.

© 2022 Elsevier B.V. All rights reserved.

1. Introduction

The field of search problems is one of the original disciplines of Operations Research. In the basic settings, the searcher's goal is to find a hidden object, also called the target, either with maximal probability or as soon as possible. By now, the field of search problems has produced a wide range of models. The models in the literature differ from each other by the characteristics of the searchers and of the objects. Concerning objects, there might be one or several objects, mobile or not, and they might have no aim or their aim is to not be found. Concerning the searchers, there might be one or more. When there is only one searcher, the searcher faces an optimization problem. When there is more than one searcher, they might be cooperative or not. If the searchers cooperate, their aim is similar to the settings with one player: they might want to minimize the expected time of search, the worst case time, or

some search cost function. If the searchers do not cooperate, the problem becomes a search game with at least two strategic non-cooperative players, and hence game theoretic solution concepts and arguments will play an important role.

In almost all existing search games with more than one searcher, the searchers are assumed to cooperate in order to achieve a common goal. In real life it is often the case that the different searchers involved do not cooperate, for various reasons. For instance in nature, when several predators are looking for the same prey. Another relevant situation is when several companies have to dig to find a resource on a given surface (gold, coal, oil, lithium). The different companies do not have incentives to cooperate, and base their search on the choices that the companies have done before. It is clear in those examples that each searcher involved has to take into account the possible change of the object (a moving prey, new technologies to locate resources).

We introduce a competitive search game, played at discrete periods in \mathbb{N} . An object is moving according to a time-varying Markov chain on finitely many states. Two players compete to find the object first. They both know the Markov chain and the initial probability distribution of the object, but do not observe the current state of the object. Player 1 is active at odd periods, and player 2 is active at even periods. The active player chooses a state, and this choice is observed by the other player. If the object is in the chosen state, this player wins and the game ends. Otherwise, the object moves according to the Markov chain and the game continues at the next period. If the object is never found, the game lasts indefinitely. In that case, neither player wins.

[☆] We would like to thank Steve Alpern, Dietmar Berwanger, Michael Eichler, Rida Laraki, Thomas Lidbetter, Miquel Oliu-Barton, Jérôme Renault, Bernhard von Stengel, Xavier Venel, Yannick Viossat, Bruno Ziliotto, and the audiences of the game theory seminars in Paris and at the London School of Economics for their precious comments and referring us to related literature. We also thank the Associate Editor and three anonymous reviewers for reading the paper very carefully and providing many useful comments that helped us greatly improve the paper.

* Corresponding author.

E-mail addresses: benoit.duvocelle@hotmail.fr, benoit.duvocelle@ut-capitole.fr (B. Duvocelle), j.flesch@maastrichtuniversity.nl (J. Flesch), staudigl@maastrichtuniversity.nl (M. Staudigl), d.vermeulen@maastrichtuniversity.nl (D. Vermeulen).

When the active player chooses a state, he needs to take two opposing effects into account. First, if the object is at the chosen state, then he wins immediately. This aspect makes choosing states favorable where the object is located with a high probability. Second, if the object is not at the chosen state, then knowing this, the opponent gains information: the opponent can calculate the conditional probability distribution of the location of the object at the next period. This aspect makes choosing states favorable where, on condition that the object not being there, the induced conditional distribution at the next period disfavors the opponent. In particular, this conditional distribution should not be too informative, and for example it should not place too high a probability on a state. Clearly, in some cases there is no state that would be optimal for both scenarios at the same time, and hence the active player somehow needs to aggregate the two scenarios in order to make a choice.

Each player's goal is to maximize the probability to win the game, that is, to find the object first. In our model, we do not assume that the players take into account the period when the object is found. Of course, in most cases, maximizing the probability to win will entail at least partially that each player would prefer to find the object at earlier periods, thereby preventing the other player from finding the object. We refer to [Duvocelle, Flesch, Staudigl, & Vermeulen \(2020\)](#) for the finite horizon and on the discounted versions of the search game, where the period when the object is found also matters.

The two players have opposite interests, up to the event when the object is never found. More precisely, each player's preferred outcome is that he finds the object, but he is indifferent between the outcome that the other player finds the object and the outcome that the object is never found. As we will see, the possibility that neither player finds the object will only have minor role, and hence the two players have essentially opposite interests in the search game.

Main results. Our main results can be summarized as follows.

[1] We study the existence of (subgame-perfect) ε -equilibria. A strategy profile is called an ε -equilibrium, where $\varepsilon \geq 0$, if neither player can increase his winning probability by more than ε with a unilateral deviation. A subgame-perfect ε -equilibrium is a strategy profile that induces an ε -equilibrium in each subgame. We prove that each competitive search game admits a subgame-perfect ε -equilibrium in pure strategies, for all error-terms $\varepsilon > 0$ (cf. [Theorem 3](#)). The proof is based on topological properties of the game (cf. [Appendix Appendix A](#)). Interestingly, a 0-equilibrium does not always exist, not even in mixed strategies. This is demonstrated with an example (cf. [Example 2](#)). In the special case of time-homogeneous processes, if the Markov chain is irreducible and aperiodic, then there exists a subgame-perfect 0-equilibrium in pure strategies (cf. [Theorem 4](#)).

[2] We examine the properties of (subgame-perfect) ε -equilibria. We show that in each subgame-perfect ε -equilibrium where $\varepsilon > 0$ is small enough, the object is found with probability 1 (cf. [Proposition 15](#)), and that the set of ε -equilibrium payoffs converge to a singleton $(v_1, 1 - v_1)$, with $v_1 \in (0, 1]$ as ε vanishes (cf. [Proposition 9](#) and [Proposition 11](#)). This implies that the two players have essentially opposite interests, and we may consider v_1 to be the value of the game (cf. [Definition 10](#)) and the strategies of ε -equilibria as ε -optimal strategies (cf. [Definition 10](#) and [Proposition 11](#)).

[3] We investigate the properties of the value and the ε -optimal strategies. We show that the payoff functions have linear properties (cf. [Proposition 6](#)), which implies that the value is a Lipschitz continuous function with respect to the initial probability distribution of the location of the object (cf. [Theorem 8](#)). We also provide inequality properties of the value (cf. [Propositions 6](#) and [12](#), and [Corollary 13](#)).

[4] We present geometric properties of optimal actions in competitive search games. An action is said to be optimal if after choosing this action in period 1, player 1 can still guarantee the value, up to an arbitrary error-term. An optimal region of an action is the set of probability distributions for the initial location of the object where this action is optimal. We show that optimal regions are star-convex centered at the vertices of the unit simplex, and that the intersection of all the optimal regions is non-empty (cf. [Theorem 16](#)).

Related literature

In the literature, search games have been studied under many different assumptions. The models differ in various characteristics. For example, the number of searchers, the number of objects, the aim of the objects, and the search space. A discrete version of the model where the search space is a matrix is studied by [von Neumann \(1953\)](#). Several variations of this game were studied by [Neuts \(1963\)](#), [Efron \(1964\)](#), [Gittins & Roberts \(1979\)](#), [Roberts & Gittins \(1978\)](#), [Sakaguchi \(1977\)](#), [Subelman \(1981\)](#), [Berry & Mensch \(1986\)](#), and [Baston, Bostock, & Ruckle \(1990\)](#), among others.

The search game with an immobile hider was introduced by [Isaacs \(1965\)](#). [Beck & Newman \(1970\)](#) considered a search game with a hider hiding on a line according to some distribution and a searcher, starting from an origin and moving at fixed speed, who tries to find the hider as soon as possible. The continuous model was then generalized by [Gal \(1972\)](#), [Gal \(1974\)](#) and [Gal & Chazan \(1976\)](#) who extended the state space from a line to a surface.

There is a large literature on search games on graphs with an immobile hider. Among them, [Cao \(1995\)](#) and [Lidbetter \(2020\)](#) have studied a search games on trees. [Gal \(1979\)](#), [Reijnierse & Potters \(1993\)](#), [Dagan & Gal \(2008\)](#) and [Alpern, Baston, & Gal \(2008\)](#) examined search games on Eulerian networks. [Pavlović \(1995\)](#), [Gal \(2010\)](#), [Kikuta \(2004\)](#), [Alpern et al. \(2008\)](#), and [Alpern, Baston, & Gal \(2009\)](#) extended the analysis to more general networks. [Jotshi & Batta \(2008\)](#) proposed an algorithm to find a hider hidden uniformly at random on a network. More recently [Garrec & Scarsini \(2020\)](#) proposed a search game in a stochastic network. They proved that the value of such games always exists, and found upper and lower bounds of the value, and optimal strategies for certain types of games. Finally, [von Stengel & Werchner \(1997\)](#) proved that a particular search game played on a graph is NP-hard.

More relevantly to our paper, some authors dealt with discrete search problems with a moving object. [Pollock \(1970\)](#), [Schweitzer \(1971\)](#), [Dobbie \(1974\)](#) and [Kan \(1974\)](#) study the two-state problem. Assuming perfect detection, [Nakai \(1973\)](#) investigates the three-state problem. [Brown \(1980\)](#) considers the search for a target with Markov motion in discrete time and space using an exponential detection function. He provides a necessary and sufficient condition for an optimal search plan and an efficient iterative algorithm for generating optimal plans. [Washburn \(1983\)](#) studies a discrete effort analogue of [Brown \(1980\)](#), in which searchers decide the effort they want to invest in order to find the object at each location they visit. General necessary and sufficient conditions which extend Brown's results to an arbitrary stochastic process for any mixture of discrete and continuous time and space are given in [Stone \(1976\)](#). [Hohzaki & Iida \(2001\)](#) investigates a search problem for a moving target in which a searcher can anticipate the probabilities of routes selected by the targets but does not have any time information about when the target transits the route. [Zoroa, Fernández-Sáez, & Zoroa \(2011\)](#) study a pray-predator model in which the prey can move. They find optimal strategies for both the prey and the predator and compute the value of the game. [Abramovskaya, Fomin, Golovach, & Pilipczuk \(2016\)](#), [Angelopoulos & Lidbetter \(2020\)](#) and [Delavernhe, Jaillet, Rossi, & Sevaux \(2020\)](#) are recent papers which study search games with a mobile object. We should also mention the very recent PhD thesis

of Clarkson (2020), which contributes to the study of both search games with a mobile hider and to search games with an immobile object.

Most of the search games focus on the case of one searcher, or several cooperative searchers. Some problems with several cooperative searchers and one or several moving targets are mentioned in the book of Stone, Royset, & Washburn (2016).

To the best of our knowledge, only four models consider several non-cooperative searchers. Nakai (1986) investigates a non-zero-sum game in which two searchers compete with each other for quicker detection of an object hidden in one of n boxes, with exponential detection functions. Each player wishes to maximize the probability that he detects the object before the opponent detects it. The author shows the existence of an equilibrium point of the form of a solution of simultaneous differential equations, and gets explicit solution results showing that both players have the same equilibrium strategy even though the detection rates are different. In Nakai (1990), two searchers compete to find different objects before the other. Flesch, Karagözoğlu, & Perea (2009) investigate the problem in which an agent has to find an object that moves between two locations according to a discrete Markov process, with the additional costless option to wait instead of searching. They find a unique optimal strategy characterized by two thresholds and show that, in a clear contrast with our model, it can never be optimal to search the location with the lower probability of containing the object. They also analyze the case of multiple agents, where the agents not only compete against time but also against each other in finding the object. They find different kinds of subgame-perfect equilibria. Finally, in Alpern & Zeng (2021), the authors study several models of search games. Among them, one model considers a game played on three states with three players: two adversarial searchers and one hider. A searcher gets payoff 1 if he is the unique player to find the hider; 1/2 if both searchers find the hider at the same time and 0 if the other searcher finds the hider alone. If none of them found the hider, the game enters a new period. The payoff of the hider is the period at which he is found. The authors show that there exists a unique searcher-symmetric Nash equilibrium. Another related paper is Alpern (2021), where two competitive searchers wish to reach a target, with a known location, but with directions suggested by an unreliable GPS system (this game is a so-called Treasure Hunt game).

As in Nakai (1973), we investigate functional and structural properties of the objective function. Nakai proved that the function that allocates to a probability distribution the average number of looks before finding the object is continuous, concave and enjoy some linear properties. They also show that the optimal regions (see Section 5) are star-convex sets. These properties have also been studied in MacPhee & Jordan (1995) and in the PhD thesis of Jordan (1997).

For an introduction to search games, we recommend the books of Alpern & Gal (2006), Gal (1979), Gal (2010) and Garnaev (2012). We also refer to Gal (2013), and to the recent surveys Benkoski, Monticino, & Weisinger (1991) and Hohzaki (2016).

Structure of the paper. In Section 2, we present the model. In Section 3, we examine the existence of (subgame-perfect) ε -equilibrium, for $\varepsilon \geq 0$. In Section 4, we argue that the two players have essentially opposite interests, we define the value and the notion of ε -optimal strategies, we present some properties of the value of the game and we show the existence of ε -optimal strategies for both players, for all $\varepsilon > 0$. In Section 5, we define optimal actions and optimal regions of the game, and we give geometric properties of these sets. The conclusion is in Section 6.

2. The Model

The Game. We study a competitive search game G played by two players. Let $\mathbb{N} = \{1, 2, 3, \dots\}$. An object is moving according to a discrete-time Markov chain $(X_t)_{t \in \mathbb{N}}$ on a finite state space S . The set of probability distributions on S is denoted by $\Delta(S)$. The initial probability distribution of the object over the set S is given by $p \in \Delta(S)$, and the transition probabilities in period t are given by an $S \times S$ transition matrix $P_t = [P_t(i, j)]_{(i, j) \in S^2}$, where $P_t(i, j)$ is the probability for the object to move from state i to state j in period t .

At each period $t \in \mathbb{N}$, one of the players is active: At odd periods player 1 is the active player, and at even periods player 2 is the active player. The active player chooses a state $s_t \in S$, which we call the action in period t . If the object is at state $X_t = s_t$, then the active player finds the object and wins the game. Otherwise, the object moves according to the transition matrix P_t and the game enters period $t + 1$. We assume that each player observes the actions chosen by his opponent, and recalls perfectly all the actions chosen by both players from the beginning. The transition matrices $(P_t)_{t \in \mathbb{N}}$ and the initial distribution p are known to the players.

The aim of each player is to maximize the probability that he finds the object first.

Histories. A history in period $t \in \mathbb{N}$ is a sequence $h_t = (s_1, \dots, s_{t-1}) \in S^{t-1}$ of past actions with the property that it has a positive probability that the object is not found before period t if the players choose their actions according to h_t . By $H_t \subseteq S^{t-1}$ we denote the set of all histories in period t . Note that H_1 consists of the empty sequence. Let $\mathbb{N}^{\text{odd}} = \{1, 3, 5, \dots\}$ and $\mathbb{N}^{\text{even}} = \{2, 4, 6, \dots\}$. We denote by $H^{\text{odd}} = \bigcup_{t \in \mathbb{N}^{\text{odd}}} H_t$ the set of histories at odd periods, and by $H^{\text{even}} = \bigcup_{t \in \mathbb{N}^{\text{even}}} H_t$ the set of histories at even periods.

For a distribution $q \in \Delta(S)$ for the location of the object and a state $s \in S$ for which $q(s) < 1$, let $q^{-s} \in \Delta(S)$ denote the distribution q conditioned on the object not being in state s . That is, $q^{-s}(s) = 0$ and $q^{-s}(s') = \frac{q(s')}{1 - q(s)}$ for each state $s' \neq s$. With the help of these conditional distributions, the players can update the distribution for the current location of the object. Indeed, the initial distribution for the location of the object is p . If player 1 chooses state s_1 but he does not find the object there, then the players can update the distribution of the object in period 1 to p^{-s_1} . This implies that the distribution of the object in period 2 is $p_2 = p^{-s_1} P_1$, as the object moves once according to the transition matrix P_1 . The update procedure continues in a similar fashion and the distribution p_t for the location of the object in period t is calculated similarly, depending on the actions chosen in the earlier periods.

Strategies. The action sets for both players are $A_1 = A_2 = S$. A strategy $\sigma = (\sigma_t)_{t \in \mathbb{N}^{\text{odd}}}$ for player 1 is a sequence of functions $\sigma_t : H_t \rightarrow \Delta(S)$. The interpretation is that, at each period $t \in \mathbb{N}^{\text{odd}}$, given the history h_t , the strategy σ_t chooses to search state $s \in S$ with probability $\sigma_t(h_t)(s)$. Similarly, a strategy $\tau = (\tau_t)_{t \in \mathbb{N}^{\text{even}}}$ for player 2 is a sequence of functions $\tau_t : H_t \rightarrow \Delta(S)$.¹ We denote by Σ and \mathcal{T} the set of strategies for players 1 and 2, respectively. Note that $\Sigma = \prod_{h \in H^{\text{odd}}} \Delta(S)$ and $\mathcal{T} = \prod_{h \in H^{\text{even}}} \Delta(S)$. We say that a strategy σ for player 1 is pure if it uses no randomization: for each history $h \in H_t$ with t odd, $\sigma(h)$ places probability 1 on some action $s_h \in S$. Pure strategies are defined similarly for player 2.

We endow the strategy spaces Σ and \mathcal{T} with the topology of pointwise convergence. This is identical with the product topology on Σ and the product topology on \mathcal{T} . Under this topology, the

¹ A strategy (for either player) could also be defined as a function $\rho : \Delta(S) \rightarrow \Delta(S)$, with the interpretation that for each distribution $p \in \Delta(S)$ for the location of the object, the player should choose a state according to $\rho(p)$.

spaces Σ and \mathcal{T} are compact, and as H^{odd} and H^{even} are countable, Σ and \mathcal{T} are also metrizable.

Winning probabilities. We define the stopping time² of the game by $\Theta = \min\{t \in \mathbb{N} \mid s_t = X_t\}$. Consider a strategy profile (σ, τ) . The probability under (σ, τ) that player 1 wins is denoted by $u_1(\sigma, \tau) = \mathbb{P}_{\sigma, \tau}(\Theta \in \mathbb{N}^{\text{odd}})$, and that player 2 wins is denoted by $u_2(\sigma, \tau) = \mathbb{P}_{\sigma, \tau}(\Theta \in \mathbb{N}^{\text{even}})$. Note that $u_1(\sigma, \tau) + u_2(\sigma, \tau) = 1 - \mathbb{P}_{\sigma, \tau}(\Theta = \infty)$.

When we wish to emphasize the distribution $p \in \Delta(S)$ for the initial location of the object, we will write $u_1(\sigma, \tau)(p)$ and $u_2(\sigma, \tau)(p)$ for the winning probabilities of the players.

If the current history is h , we will use the notations $u_1(\sigma, \tau)(h)$ and $u_2(\sigma, \tau)(h)$ for the winning probabilities of the players conditioned on h being the current history.

Subgame-perfect ε -equilibrium. Let $\varepsilon \geq 0$ be an error-term. A strategy σ for player 1 is an ε -best response against strategy τ for player 2 if $u_1(\sigma, \tau) \geq u_1(\sigma', \tau) - \varepsilon$ for every strategy σ' of player 1. Similarly, a strategy τ for player 2 is an ε -best response against strategy σ for player 1 if $u_2(\sigma, \tau) \geq u_2(\sigma, \tau') - \varepsilon$ for every strategy τ' of player 2. A strategy profile (σ, τ) is called an ε -equilibrium if σ is an ε -best response against τ and τ is an ε -best response against σ . A subgame-perfect ε -equilibrium is a strategy profile (σ, τ) which is an ε -equilibrium in each subgame. That is, for each history h , for each $\sigma' \in \Sigma$, for each $\tau' \in \mathcal{T}$, $u_1(\sigma, \tau)(h) \geq u_1(\sigma', \tau)(h) - \varepsilon$ and $u_2(\sigma, \tau)(h) \geq u_2(\sigma, \tau')(h) - \varepsilon$.

An alternative interpretation of the game. We call the previous game Model [1]. Note that model [1] is a game in extensive form, where information is communicated to players over time. Now we present an alternative model of this game, which we call Model [2]. We stress that we only use Model [2] as an auxiliary model in our paper. We do this in order to apply existence results in the literature that are formulated for this type of models (cf. Theorem 3), namely models with perfect information.

Model [2]: Another way to describe our game is as follows. One could imagine that the game consists of two phases. In the first phase the players choose actions sequentially. More precisely, in the first phase, player 1 chooses an action at odd periods and player 2 chooses an action at even periods sequentially, just as before. This results in an infinite sequence of states (s_1, s_2, \dots) . The set of infinite histories is S^∞ . Every pure strategy profile (σ, τ) induces a unique infinite history $h_{\sigma, \tau}^\infty \in S^\infty$. In a second phase, players receive a payoff. Now, for $i = 1, 2$, consider the payoff function $f_i : S^\infty \rightarrow [0, 1]$ defined as follows. Consider an infinite history (s_1, s_2, \dots) . Take any pure strategy profile (σ, τ) such that $h_{\sigma, \tau}^\infty = (s_1, s_2, \dots)$ and define $f_i(s_1, s_2, \dots) = u_i(\sigma, \tau)$; note that this definition only depends on the realized history. The goal of each player is to maximize his payoff. Note that this is a game without an object.

We briefly argue that the above descriptions are equivalent. Indeed, for each pure strategy profile (σ, τ) , for each player $i = 1, 2$, we have $u_i(\sigma, \tau) = f_i(h_{\sigma, \tau}^\infty)$. Hence, (σ, τ) leads to the same payoff in both models.

3. Existence of equilibrium

In this section, we examine equilibria in competitive search games. In the first subsection, we show that there are search games for which there exist no 0-equilibrium, not even in mixed strategies. From a technical point of view, this is caused by discontinuity in the payoff functions of the players. In the second subsection, we focus on the notion of subgame-perfect ε -equilibrium, where $\varepsilon > 0$ is an error-term, and prove that each search game admits a subgame-perfect ε -equilibrium in pure strategies, for all

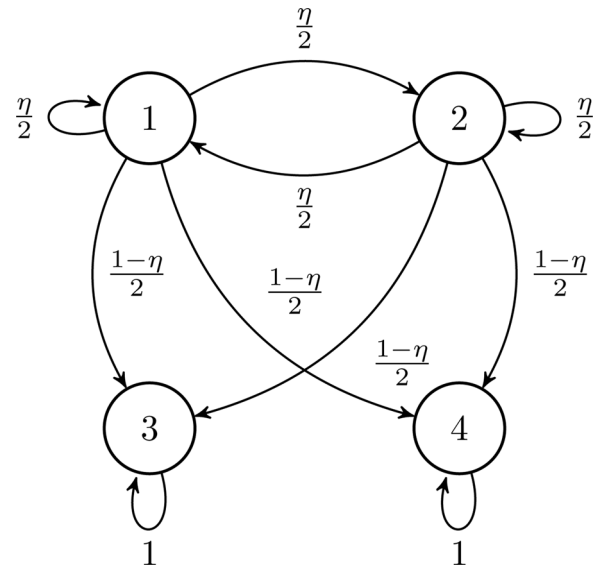


Fig. 1. A game without 0-equilibrium.

$\varepsilon > 0$. In the third subsection, we present sufficient conditions for the existence of a subgame-perfect 0-equilibrium in pure strategies.

3.1. Search games with no 0-equilibrium

Theorem 1. *There exist time-homogeneous competitive search games which admit no 0-equilibrium, not even in mixed strategies.*

We provide an example below of a time-homogeneous competitive search game which admits no 0-equilibrium, not even in mixed strategies (for another example, we refer to Duvoelle et al. (2020)). The main idea of this example is that during the game the active player is incentivised to choose a transient state of the Markov chain, as these are the only states where searching gives an advantage. However, if the players always choose a transient state, then they do not find the object with probability 1, and hence this cannot constitute a 0-equilibrium.

Example 2. Consider the game in Fig. 1. In this game, $\eta \in (0, \frac{1}{4})$ and the initial probability distribution is $p = (q, q, \frac{1}{2} - q, \frac{1}{2} - q)$, where $q \in (0, \frac{1}{4})$. Notice that states 1 and 2 have the same transition probabilities, and so do states 3 and 4. States 1 and 2 are transient, whereas states 3 and 4 are absorbing.

We show that this game admits no 0-equilibrium. The intuition for this claim is as follows. Consider period 1. Player 1 has a choice between a transient state (i.e., states 1 and 2) and an absorbing state (i.e., states 3 and 4).

Suppose first that player 1 chooses an absorbing state, say state 3. We will show (see Claim 1 below) that this, with optimal follow-up play, induces a winning probability of exactly $\frac{1}{2}$. Intuitively, if player 1 does not find the object in period 1, then it is very likely that the object is in state 4, and player 2 should respond by choosing state 4 in period 2. If player 2 does not find the object, then due to the transition probabilities, the object is very likely to be in state 3 in period 3. Indeed, we know that the object was not in state 3 in period 1 and not in state 4 in period 2, so the object has one more period to move from state 1 or state 2 to state 3 compared to state 4. Continuing this argument shows that the optimal continuation play consists of player 1 always choosing state 3 and player 2 always choosing state 4. As we will show, this gives a winning chance of exactly $\frac{1}{2}$ to each player.

² With the convention that $\min\{\emptyset\} = +\infty$.

Now suppose that player 1 chooses a transient state, say state 1. In addition, suppose that if period 3 is reached, then player 1 will choose a state with the highest probability of containing the object: state 3, unless player 2 chose state 3 in period 2, in which case player 1 chooses state 4. We show that this strategy guarantees a winning probability of strictly more than $\frac{1}{2}$ for player 1 (see Claim 2 below). Indeed, player 1 wins immediately with probability q , and as we show, the probability of winning in period 3 is strictly larger than $\frac{1}{2} - q$, so that the total probability that player 1 wins is strictly larger than $\frac{1}{2}$.

Based on the above discussion, player 1 should choose a transient state in period 1. If player 1 does not find the object, by an inductive argument, player 2 should also choose a transient state in period 2. Continuing this way, the players should choose transient states in all periods. Then, however, they do not find the object with probability 1, and hence this cannot constitute a 0-equilibrium (see Claim 3 below).

CLAIM 1: Let $\Sigma^{3,4}$ denote the set of strategies for player 1 that looks at state 3 or state 4 in period 1. Then,

$$\sup_{\sigma \in \Sigma^{3,4}} \inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau) = \frac{1}{2}.$$

PROOF OF CLAIM 1: Let $\tilde{\tau} = (\tilde{\tau}_t)_{t \in \mathbb{N}^{\text{even}}}$ be a pure strategy of player 2 such that, for all $t \in \mathbb{N}^{\text{even}}$, for all $h_t \in H_t$,

$$\tilde{\tau}_t(h_t) = \begin{cases} \text{state 3} & \text{if } h_t(t-1) = 4, \\ & \text{or if } t \geq 4 \text{ and } h_t(t-1) \in \{1, 2\} \\ & \text{and } h_t(t-2) = 4, \\ \text{state 4} & \text{otherwise,} \end{cases}$$

where $h_t(t-2)$ and $h_t(t-1)$ are the second-to-last and the last actions chosen under history h_t , respectively. Intuitively, $\tilde{\tau}$ always chooses a state that is most likely to contain the object, which is either state 3 or state 4.

Let σ^3 be the strategy of player 1 that always chooses state 3. Then, against σ^3 , the strategy $\tilde{\tau}$ always chooses state 4. It follows that $u_1(\sigma^3, \tilde{\tau}) = \frac{1}{2}$, as due to symmetry, the object will end up in state 3 with probability $\frac{1}{2}$ and in state 4 with probability $\frac{1}{2}$.

Now we show that the strategy σ^3 is a best response in $\Sigma^{3,4}$ against $\tilde{\tau}$, i.e., $u_1(\sigma, \tilde{\tau}) \leq \frac{1}{2}$ for each $\sigma \in \Sigma^{3,4}$. Assume that player 1 chooses state 3 in period 1; the argument is similar if player 1 chooses state 4 in period 1. Assume also that player 1 does not find the object in period 1. Then, by definition, $\tilde{\tau}$ looks at state 4 in period 2. If player 2 does not find the object in period 2, then the probability that the object is in state 3 is strictly more than $\frac{1}{2}$. Thus, if period 3 is reached, player 1 should look at state 3. Then, by definition, $\tilde{\tau}$ looks at state 4 in period 4. By repeating this argument, player 1 should always look at state 3 against $\tilde{\tau}$.

One can similarly check that $\tilde{\tau}$ minimizes player 2's winning chance against σ^3 , as after player 1 chooses state 3 it is best for player 2 to choose state 4. That is, $\frac{1}{2} \leq u_1(\sigma^3, \tau)$ for each strategy τ for player 1.

Hence,

$$\sup_{\sigma \in \Sigma^{3,4}} \inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau) \geq \inf_{\tau \in \mathcal{T}} u_1(\sigma^3, \tau) = u_1(\sigma^3, \tilde{\tau}) = \frac{1}{2}$$

and

$$\sup_{\sigma \in \Sigma^{3,4}} \inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau) \leq \sup_{\sigma \in \Sigma^{3,4}} u_1(\sigma, \tilde{\tau}) = u_1(\sigma^3, \tilde{\tau}) = \frac{1}{2},$$

which completes the proof of Claim 1. \square

Let $\sigma = (\sigma_t)_{t \in \mathbb{N}^{\text{odd}}}$ be any strategy of player 1 with the following properties: σ chooses state 1 in period 1, and in period 3 it chooses state 3 if player 2 did not choose state 3 in period 2, and it chooses state 4 otherwise. Intuitively, σ chooses one of the most likely states in period 3, which is either state 3 or state 4.

CLAIM 2: When player 1 uses σ he guarantees himself strictly more than $\frac{1}{2}$: $u_1(\sigma, \tau) > \frac{1}{2}$ for every τ .

PROOF OF CLAIM 2: It is sufficient to show that $u_1(\sigma, \tau) > \frac{1}{2}$ for an arbitrary pure strategy τ of player 2. Under σ , player 1 looks at state 1 in period 1 and finds the object with probability q in period 1. We only need to show that the probability under (σ, τ) that player 1 finds the object in period 3 is strictly more than $\frac{1}{2} - q$.

Suppose first that τ would choose state 1 or state 2 if period 2 is reached. A straightforward calculation shows that player 1 finds the object in period 3 with probability

$$\mathbb{P}_{\sigma, \tau}(\Theta = 3) = \frac{1}{2} - q + \frac{q \cdot (1 - \eta)}{2} + \frac{q \cdot \eta \cdot (1 - \eta)}{4},$$

which is strictly more than $\frac{1}{2} - q$.

Suppose now that τ would choose state 3 or state 4 if period 2 is reached. In this case, a straightforward calculation shows that player 1 finds the object in period 3 with probability

$$\mathbb{P}_{\sigma, \tau}(\Theta = 3) = \frac{1}{2} - q + \frac{q \cdot (1 - \eta)}{2} + \frac{q \cdot \eta \cdot (1 - \eta)}{2},$$

which is also strictly more than $\frac{1}{2} - q$. \square

CLAIM 3: There is no 0-equilibrium.

PROOF OF CLAIM 3: Assume by way of contradiction that there is a 0-equilibrium (σ', τ') . By Claim 1 and Claim 2, σ' chooses state 1 or state 2 with probability 1 in period 1. In both cases, in period 2 the current probability distribution for the location of the object is $(\frac{q \cdot \eta}{2(1-q)}, \frac{q \cdot \eta}{2(1-q)}, \frac{1}{2} - \frac{q \cdot \eta}{2(1-q)}, \frac{1}{2} - \frac{q \cdot \eta}{2(1-q)})$. Then, in period 2, the game is similar to the original one, with a parameter $q' = \frac{q \cdot \eta}{2(1-q)}$ instead of q , which still satisfies $q' \in (0, \frac{1}{4})$, and where the roles of the players are exchanged. Then, similarly, τ' chooses state 1 or state 2 with probability 1 in period 2. By following this process recursively, the players will choose states 1 and 2 with probability 1 forever. This leads to the payoff $\frac{4 \cdot q}{4 - \eta^2}$ for player 1. Then, player 1 has an incentive to deviate from σ' and to choose state 3 in period 1 to get a payoff of at least $\frac{1}{2} - q > \frac{4 \cdot q}{4 - \eta^2}$, a contradiction. \square

3.2. Existence of pure subgame-perfect ε -equilibrium

In this subsection we are interested in the existence of subgame-perfect ε -equilibrium, where $\varepsilon > 0$. In the next theorem, we show that all competitive search games admit a subgame-perfect ε -equilibrium in pure strategies, for each $\varepsilon > 0$. The proof relies on existence results for subgame-perfect ε -equilibria in games with bounded and lower semi-continuous payoff functions (see Flesch et al. (2010) or Flesch & Predtetchinski (2016)).

Theorem 3. Every competitive search game admits a pure subgame-perfect ε -equilibrium, for each $\varepsilon > 0$.

Proof. Consider Model [2] of a competitive search game, as described in Section 2. The payoffs are bounded and lower semi-continuous, in view of Proposition 18. Thus, by applying Theorem 2.3 of Flesch et al. (2010) (or by Theorem 4.1 of Flesch & Predtetchinski (2016)), the game admits a pure subgame-perfect ε -equilibrium for each $\varepsilon > 0$. \square

Revisiting Example 2. In view of Theorem 3, the game in Example 2 has a subgame-perfect ε -equilibrium in pure strategies, for each $\varepsilon > 0$. Let $\varepsilon > 0$ be arbitrary. Now we describe a subgame-perfect ε -equilibrium. The idea of the construction is that the players should choose state 1 or state 2 until the probability that the object is in one of these states is very low in comparison with ε , and after that they should choose the absorbing states. For the precise calculations we refer to Duvocelle et al. (2020). Let n be a large stage, i.e. $n \geq \frac{\ln(\frac{q}{\eta \varepsilon})}{\ln(\frac{2}{\eta})}$. Also, let (σ, τ) be a strategy profile that always chooses the most likely between state 3 or state 4. We define a strategy profile (σ^*, τ^*) as follows: The players choose state

1 until period n , and then they switch to (σ, τ) for the remaining game. If a player deviates to an absorbing state before period n , then they immediately switch to (σ, τ) . As one can show, under (σ^*, τ^*) , neither player can gain more than $\frac{q}{2} \cdot \left(\frac{q}{2}\right)^{n-1} \leq \varepsilon$ in any subgame by a unilateral deviation. Hence, (σ^*, τ^*) is a subgame-perfect ε -equilibrium.

3.3. Sufficient conditions for existence of pure subgame-perfect 0-equilibrium

In this subsection, we present sufficient conditions for the existence of a pure subgame-perfect 0-equilibrium. Consider a time-homogeneous competitive search game. In this case the transition matrix at each period is the same matrix P . A probability distribution $\pi \in \Delta(S)$ over the set S is called a *stationary distribution* for the transition matrix P if $\pi P = \pi$. It is known (see [Levin & Peres \(2017\)](#), Corollary 1.17 page 13 and Theorem 4.9 page 52) that if P is irreducible and aperiodic, i.e. for some $r \in \mathbb{N}$ all entries of the matrix P^r are strictly positive, then (i) there exists a unique stationary distribution $\pi \in \Delta(S)$, (ii) $\pi(s) > 0$ for all $s \in S$, and (iii) there exist constants $\beta \in (0, 1)$ and $c > 0$ such that for all distributions $p \in \Delta(S)$ for the initial location of the object and all periods $t \in \mathbb{N}$,

$$\|pP^t - \pi\|_{TV} \leq c \cdot \beta^t,$$

where $\|p - q\|_{TV} = \max_{A \subseteq S} \sum_{s \in A} (p(s) - q(s))$ is the total variation distance between $p, q \in \Delta(S)$. Note that for each $p, q \in \Delta(S)$, it holds that $\|p - q\|_{TV} = \frac{1}{2} \cdot \sum_{s \in S} |p(s) - q(s)|$ (see [Levin & Peres \(2017\)](#), pages 47–48).

Theorem 4. Consider a time-homogeneous competitive search game. Assume that the transition matrix P is irreducible and aperiodic. Then, no matter the initial probability distribution p , the object is found with probability 1 under every strategy profile (σ, τ) , i.e., $\mathbb{P}_{\sigma, \tau}(\Theta < \infty) = 1$. Hence, the payoff functions are continuous in this game, and there exists a subgame-perfect 0-equilibrium in pure strategies.

Proof. As mentioned above, the transition matrix P has a unique stationary distribution $\pi \in \Delta(S)$ and $\pi(s) > 0$ for all $s \in S$. Moreover, there exist constants $c > 0$ and $\beta \in (0, 1)$ such that $\|pP^t(s) - \pi(s)\| \leq c \cdot \beta^t$ for all $p \in \Delta(S)$, for all $s \in S$ and for all $t \in \mathbb{N}$. Hence, there exists $t^* \in \mathbb{N}$ with the following property: for all $p \in \Delta(S)$, for all $s \in S$, for all $t \geq t^*$, we have $(pP^{t-1})(s) > \frac{\delta}{2}$, where $\delta = \min_{s \in S} \pi(s)$. Without loss of generality we can assume that $t^* \geq 2$. \square

Let $\alpha = \frac{\delta}{4(t^*-1)}$. The proof is divided into four steps.

STEP 1: Let (σ, τ) be a pure strategy profile, and let $(s_t)_{t \in \mathbb{N}}$ denote the induced sequence of actions. We show that the object is found during the first t^* periods with probability at least α .

Proof. Recall that $p_t = (p_t(s))_{s \in S} \in \Delta(S)$ denotes the probability distribution of the location of the object in period t , for each $t \in \mathbb{N}$, conditional on it not being found through the history (s_1, \dots, s_{t-1}) .

If there is a period $t \leq t^*$ such that $p_t(s_t) \geq \alpha$, then under (σ, τ) , the object is found in period t with probability at least α , if it has not been found before. Hence, the claim of step 1 is true.

Therefore, it suffices to show that if at each period $t \leq t^* - 1$ we have $p_t(s_t) < \alpha$, then $p_{t^*}(s_{t^*}) \geq \alpha$. So assume that at each period $t \leq t^* - 1$ we have $p_t(s_t) < \alpha$. The idea of the calculation below is that, since the object is found with low probabilities at the first $t^* - 1$ periods, the probability distribution for the object in period t^* conditioned on it not being found during the first $t^* - 1$ periods is almost the same as the unconditional probability distribution. That is, p_{t^*} is close to pP^{t^*-1} , which is in turn close to the stationary distribution π .

Note that, if the players do not condition on the past, the probability distribution of the location of the object in period t^* is simply pP^{t^*-1} . Recall that, if $p(s) < 1$, then p^{-s} denotes the probability distribution p conditioned on the object not being in state s . We have

$$\begin{aligned} \|p_{t^*} - pP^{t^*-1}\|_{TV} &\leq \|p_{t^*} - p_{t^*-1}P\|_{TV} + \|p_{t^*-1}P - pP^{t^*-1}\|_{TV} \\ &= \|p_{t^*-1}^{-s_{t^*-1}}P - p_{t^*-1}P\|_{TV} + \|p_{t^*-1}P - pP^{t^*-1}\|_{TV} \\ &\leq \|p_{t^*-1}^{-s_{t^*-1}} - p_{t^*-1}\|_{TV} + \|p_{t^*-1} - pP^{t^*-2}\|_{TV} \\ &= p_{t^*-1}(s_{t^*-1}) + \|p_{t^*-1} - pP^{t^*-2}\|_{TV} \\ &< \alpha + \|p_{t^*-1} - pP^{t^*-2}\|_{TV} \\ &< \alpha \cdot (t^* - 1) + \|p_1 - pP^0\|_{TV} \\ &= \alpha \cdot (t^* - 1) \\ &= \frac{\delta}{4}. \end{aligned}$$

Here, in the first inequality we used the triangle inequality. In the first equality we used $p_{t^*} = p_{t^*-1}^{-s_{t^*-1}}P$, as p_{t^*} is the location of the object in period t^* conditioned on it not being found up to period $t^* - 1$ and $p_{t^*-1}^{-s_{t^*-1}}P$ expresses the distribution that arises when object moves once according to P after not being found up to period $t^* - 1$. The second inequality is true as $\|qP - q'P\|_{TV} \leq \|q - q'\|_{TV}$ for all $q, q' \in \Delta(S)$. The second equality follows from the above interpretation of $p_{t^*-1}^{-s_{t^*-1}}$ and the definition of the total variation norm. The third inequality is due to the assumption that at each period $t \leq t^* - 1$ we have $p_t(s_t) < \alpha$. The fourth inequality then follows by induction. The last two equalities are due to $p_1 = p$ and the choice of α . \square

Therefore,

$$p_{t^*}(s_{t^*}) \geq (pP^{t^*-1})(s_{t^*}) - \|p_{t^*} - pP^{t^*-1}\|_{TV} \geq \frac{\delta}{2} - \frac{\delta}{4} = \frac{\delta}{4} \geq \alpha.$$

This completes the proof of Step 1.

STEP 2: Consider any strategy profile (σ, τ) . We show that the object is found during the first t^* periods with probability at least α .

Proof. On the finite horizon t^* , each strategy can be equivalently represented as a mixed strategy, i.e. a probability distribution on the finite set of pure strategies on horizon t^* (see for example [Maschler, Solan, & Zamir \(2013\)](#)). Hence, Step 2 follows from Step 1. \square

STEP 3: Consider any strategy profile (σ, τ) . We show that the object is found with probability 1 under (σ, τ) . By [Proposition 18](#), this will imply that the payoff functions are continuous in this game.

Proof. By Step 2, the object is found during the first t^* periods with probability at least α . Since t^* and therefore α do not depend on the initial distribution of the object, if the object is not found in the first t^* periods, then it will be found between periods $t^* + 1$ and $2t^*$ with probability at least α . By repeating this argument, the object is found with probability 1 under (σ, τ) . \square

STEP 4: We show that there exists a subgame-perfect 0-equilibrium in pure strategies.

Proof. ³ In view of [Theorem 3](#), for each $n \in \mathbb{N}$, there exists a subgame-perfect $\frac{1}{n}$ -equilibrium (σ^n, τ^n) in pure strategies. Since the spaces of strategies Σ and \mathcal{T} are compact and metrizable, by

³ The claim of Step 4 follows from [Fudenberg & Levine \(1983\)](#), but for completeness, we give a proof based on [Theorem 3](#).

taking a subsequence if necessary, we can assume that the sequence $(\sigma^n, \tau^n)_{n \in \mathbb{N}}$ converges to a strategy profile (σ, τ) in pure strategies as $n \rightarrow \infty$. \square

For each $n \in \mathbb{N}$, each history h , each strategy σ' for player 1 and each strategy τ' for player 2, we have $u_1(\sigma^n, \tau^n)(h) \geq u_1(\sigma', \tau^n)(h) - \frac{1}{n}$ and $u_2(\sigma^n, \tau^n)(h) \geq u_2(\sigma^n, \tau')(h) - \frac{1}{n}$. Since by Step 3 the payoff functions u_1 and u_2 are continuous, by taking the limits as $n \rightarrow \infty$, we obtain for each h , each σ' and each τ' that $u_1(\sigma, \tau)(h) \geq u_1(\sigma', \tau)(h)$ and $u_2(\sigma, \tau)(h) \geq u_2(\sigma, \tau')(h)$. Hence, (σ, τ) is a subgame-perfect 0-equilibrium in pure strategies. \square

4. Existence and properties of the value and ε -optimal strategies

In competitive search games, the winning probabilities do not always add up to 1, as under certain strategy profiles it may have a positive probability that the object is never found. For instance, this is the case in the game in [Example 2](#), if the players always choose state 1. However, neither player is interested in the outcome when the object is never found. Therefore, as we argue in this section, in essence the players have opposite interest and the value is a natural solution concept for competitive search games.

We denote $v_1 = \sup_{\sigma \in \Sigma} \inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau)$ and $v_2 = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \Sigma} u_2(\sigma, \tau)$. Intuitively, a strategy σ guarantees for player 1 a winning probability of $\inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau)$, and therefore v_1 can be interpreted as the largest winning probability that player 1 can guarantee in the game. The interpretation of v_2 is similar.

More generally, when we wish to emphasize the initial distribution p of the object as a parameter, we will write $v_1(p) = \sup_{\sigma \in \Sigma} \inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau)(p)$ and $v_2(p) = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \Sigma} u_2(\sigma, \tau)(p)$.

Let Σ^s denote the set of those strategies for player 1 that choose state s in period 1. We define $v_1(p, s) = \sup_{\sigma \in \Sigma^s} \inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau)(p)$. Note that $v_1(p) = \max_{s \in S} v_1(p, s)$.

As discussed in [Section 2](#), each history h induces a conditional probability distribution for the location of the object, say p_h . We will use the notation $v_1(h) = v_1(p_h)$ and $v_2(h) = v_2(p_h)$.

Proposition 5. *We have*

$$v_1 = \inf_{\tau \in \mathcal{T}} \sup_{\sigma \in \Sigma} u_1(\sigma, \tau) \quad \text{and} \quad v_2 = \inf_{\sigma \in \Sigma} \sup_{\tau \in \mathcal{T}} u_2(\sigma, \tau).$$

Similar equalities hold for $v_1(p)$ and $v_2(p)$ for all $p \in \Delta(S)$.

Proof. We refer to the [Appendix Appendix B](#). \square

The next theorem discusses properties of the payoff functions and the functions $v_1(p)$ and $v_2(p)$ along line segments in $\Delta(S)$. For each state $s \in S$, let $e^s \in \Delta(S)$ denote the probability distribution that allocates probability 1 to state s and probability 0 to all other states.

Proposition 6. [1] *Let (σ, τ) be a strategy profile. Then, the payoff functions are linear in the initial probability distribution of the object: for every $p, q \in \Delta(S)$, for every $\lambda \in [0, 1]$, and for every player $i = 1, 2$:*

$$u_i(\sigma, \tau)(\lambda \cdot p + (1 - \lambda) \cdot q) = \lambda \cdot u_i(\sigma, \tau)(p) + (1 - \lambda) \cdot u_i(\sigma, \tau)(q). \quad (1)$$

[2] *For every $s \in S$, the map $p \mapsto v_1(p, s)$ is linear over every line passing through e^s : for every $p \in \Delta(S)$ and for every $\lambda \in [0, 1]$:*

$$v_1(\lambda \cdot e^s + (1 - \lambda) \cdot p, s) = \lambda + (1 - \lambda) \cdot v_1(p, s).$$

[3] *For every $p, q \in \Delta(S)$, for every $\lambda \in [0, 1]$, and for every player $i = 1, 2$:*

$$v_i(\lambda \cdot p + (1 - \lambda) \cdot q) \geq \lambda \cdot v_i(p).$$

Proof. We refer to the [Appendix Appendix B](#). \square

Remark 7. In part [2] of [Proposition 6](#), the linearity of $p \mapsto v_1(p, s)$ relies on the following fact: Let $p \in \Delta(S) \setminus \{e^s\}$, and let ℓ denote the line going through p and e^s . If by choosing state s player 1 does not find the object, then the conditional distribution of the location of the object, p^{-s} , stays on the line ℓ . Note, however, that the function $p \mapsto v_1(p, s)$ is generally non-linear. Indeed, consider a game that has 4 states, each of which is absorbing. For $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ and $q = (\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3})$, we have $v_1(p, 1) = \frac{2}{3}$ and $v_1(q, 1) = \frac{2}{3}$, but $v_1(\frac{1}{2} \cdot p + \frac{1}{2} \cdot q, 1) = \frac{1}{2}$. \diamond

We recall the definition of the total variation distance: for $p, q \in \Delta(S)$, the total variation distance between p and q is the non-negative number $\|p - q\|_{TV} = \max_{A \subset S} \sum_{s \in A} (p(s) - q(s))$.

Theorem 8. *For each player $i = 1, 2$, each strategy profile (σ, τ) , each state $s \in S$, the functions $p \mapsto u_i(\sigma, \tau)(p)$, $p \mapsto v_i(p)$, and $p \mapsto v_i(p, s)$ are 1-Lipschitz continuous with respect to the total variation distance.*

Proof. We only prove it for player 1. By part [1] of [Proposition 6](#), we have

$$u_1(\sigma, \tau)(p) = \sum_{s \in S} p(s) \cdot u_1(\sigma, \tau)(e^s),$$

$$u_1(\sigma, \tau)(q) = \sum_{s \in S} q(s) \cdot u_1(\sigma, \tau)(e^s).$$

Hence,

$$\begin{aligned} u_1(\sigma, \tau)(p) - u_1(\sigma, \tau)(q) &= \sum_{s \in S} [p(s) - q(s)] \cdot u_1(\sigma, \tau)(e^s) \\ &\leq \sum_{\substack{s \in S, \\ p(s) > q(s)}} [p(s) - q(s)] = \|p - q\|_{TV}, \end{aligned}$$

and similarly

$$u_1(\sigma, \tau)(q) - u_1(\sigma, \tau)(p) \leq \|p - q\|_{TV}.$$

Hence, $p \mapsto u_1(\sigma, \tau)(p)$ is 1-Lipschitz-continuous.

Taking the infimum over τ and the supremum over σ on both sides of the inequality $u_1(\sigma, \tau)(p) \leq u_1(\sigma, \tau)(q) + \|p - q\|_{TV}$ gives $v_1(p) \leq v_1(q) + \|p - q\|_{TV}$, which can be written $v_1(p) - v_1(q) \leq \|p - q\|_{TV}$. Similarly, $v_1(q) - v_1(p) \leq \|p - q\|_{TV}$. Hence, $p \mapsto v_1(p)$ is 1-Lipschitz-continuous too.

The proof for $p \mapsto v_1(p, s)$ is similar, but the supremum has to be taken over $\sigma \in \Sigma^s$. \square

Proposition 9. *We have $v_1 + v_2 = 1$, and in general, $v_1(p) + v_2(p) = 1$ for all $p \in \Delta(S)$.*

Proof. We refer to the [Appendix Appendix B](#). \square

Based on the [Proposition 9](#), in essence the players have opposite interests in competitive search games. This leads us to defining the value of competitive search games.

Definition 10. Consider a competitive search game. We call v_1 the value of the game. For each $\varepsilon \geq 0$, a strategy σ for player 1 is called ε -optimal, if $u_1(\sigma, \tau) \geq v_1 - \varepsilon$ for all strategies $\tau \in \mathcal{T}$. Similarly, a strategy τ for player 2 is called ε -optimal, if $u_2(\sigma, \tau) \geq v_2 - \varepsilon$ for all strategies $\sigma \in \Sigma$.

The following proposition relates the notions of profiles of ε -optimal strategies and ε -equilibrium, and shows that all ε -equilibria give almost the same payoffs for small ε .

Proposition 11. *Consider a competitive search game. Let $\varepsilon \geq 0$.*

[1] *If (σ, τ) is an ε -equilibrium, then σ and τ are ε -optimal strategies. Consequently, when $\varepsilon > 0$, each player has a pure ε -optimal strategy.*

[2] If σ and τ are ε -optimal strategies, then (σ, τ) is a 2ε -equilibrium.

[3] A strategy profile (σ, τ) is a 0-equilibrium if and only if σ and τ are 0-optimal strategies.

[4] If (σ, τ) is an ε -equilibrium then, under (σ, τ) , the object is found with probability at least $1 - \varepsilon \cdot |S|$. Moreover, $|u_1(\sigma, \tau) - v_1| \leq \varepsilon$ and $|u_2(\sigma, \tau) - v_2| \leq \varepsilon$.

Proof. We refer to the [Appendix Appendix B](#). \square

Proposition 12. Looking at a state in which the object is with zero probability is never better than looking anywhere else. That is, for all states $s, s' \in S$, for all $p \in \Delta(S)$, if $p(s') = 0$ then $v_1(p, s') \leq v_1(p, s)$.

Proof. Let \tilde{G} denote the game such that (i) the game is played at periods 2, 3, ...; (ii) the transition matrices of the Markov chain are P_2, P_3, \dots ; (iii) Player 2 is the active player in even periods, and player 1 is the active player in odd periods. For the game \tilde{G} , let \tilde{S} and \tilde{T} denote the sets of strategies, and $\tilde{u}_i(\sigma, \tau)(p)$ denote the expected winning chance of player $i = 1, 2$ under each strategy pair (σ, τ) and each initial distribution $p \in \Delta(S)$ for the object. Define for each player $i = 1, 2$

$$\tilde{v}_i(p) = \sup_{\sigma \in \tilde{S}} \inf_{\tau \in \tilde{T}} \tilde{u}_i(\sigma, \tau)(p).$$

Suppose that $p(s') = 0$ and let $s \in S$. We have

$$v_1(p, s) = p(s) + (1 - p(s)) \cdot \tilde{v}_1(p^{-s}P_1),$$

and because $p(s') = 0$, we have

$$v_1(p, s') = \tilde{v}_1(pP_1).$$

Note that $pP_1 = p(s) \cdot e^s P_1 + (1 - p(s)) \cdot p^{-s}P_1$. By part [3] of [Proposition 6](#) (applied to \tilde{G}),

$$\tilde{v}_2(pP_1) \geq (1 - p(s)) \cdot \tilde{v}_2(p^{-s}P_1).$$

Since by [Proposition 9](#) (applied to \tilde{G}) we have $\tilde{v}_1(q) + \tilde{v}_2(q) = 1$ for each $q \in \Delta(S)$, we derive

$$1 - \tilde{v}_1(pP_1) \geq (1 - p(s)) \cdot (1 - \tilde{v}_1(p^{-s}P_1)).$$

Rearranging gives

$$\tilde{v}_1(pP_1) \leq p(s) + (1 - p(s)) \cdot \tilde{v}_1(p^{-s}P_1).$$

This implies that $v_1(p, s) \geq v_1(p, s')$, as claimed. \square

Corollary 13. Consider a time-homogenous competitive search game. If p is a stationary distribution of P , then $v_1(p) \geq \frac{1}{2}$.

Proof. We refer to the [Appendix Appendix B](#). \square

Remark. We conjecture that if P is irreducible and aperiodic, then $v_1(p) > \frac{1}{2}$. The value $v_1(p)$ can be smaller than $\frac{1}{2}$ if p is not the stationary distribution. Indeed, the value is $\frac{1}{3}$ if the game has three states, initial probability distribution $p = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and transition matrix P such that at the second period the object is in state 1 with probability 1. \diamond

An ε -optimal strategy is a relevant solution concept, but it has the drawback that if the opponent makes a mistake, the continuation strategy does not have to be ε -optimal. A strategy σ for player 1 is called subgame ε -optimal if, in each subgame, the continuation strategy of σ is ε -optimal. More precisely, for each history $h \in H^{\text{odd}}$ and strategy $\tau \in \mathcal{T}$ for player 2

$$u_1(\sigma, \tau)(h) \geq v_1(h) - \varepsilon.$$

The definition of a subgame ε -optimal strategy for player 2 is similar. Note that each subgame ε -optimal strategy is ε -optimal.

Example 14. In this example, we show that there are ε -optimal strategies that are not subgame ε -optimal. The set of states is

$S = \{1, 2\}$, with each state being absorbing, and the initial probability distribution is $p = (1, 0)$. The value is $v_1 = 1$ and any optimal strategy of player 1 looks at state 1 in period 1. Hence, $v_2 = 0$ and all the strategies of player 2 are 0-optimal. In particular, it is optimal for player 2 to always choose state 2. Let τ denote this strategy.

Now suppose that player 1 makes a mistake and chooses state 2 in period 1. Then, the continuation strategy of τ from period 2 is not optimal. In fact, it would be the best for player 2 to choose state 1 in period 2 and win the game. \square

Proposition 15. Consider a competitive search game.

1. For every $\varepsilon > 0$, each player has a pure strategy which is subgame ε -optimal.
2. Let $\varepsilon \in (0, \frac{1}{|S|})$. If σ is a subgame ε -optimal strategy for player 1, then for every strategy τ of player 2, the object is found with probability 1 under the strategy profile (σ, τ) . A similar statement holds for player 2.

1.. Let $\varepsilon > 0$. By [Theorem 3](#), there exists a subgame perfect ε -equilibrium (σ, τ) in pure strategies. Now consider a subgame at a history h . Since the continuation strategies of σ and τ at h form an ε -equilibrium, it follows similarly to part [1] of [Proposition 11](#) that the continuation strategy of σ at h is ε -optimal in the subgame, and similarly the continuation strategy of τ at h is ε -optimal in the subgame. Hence, σ and τ are subgame ε -optimal.

[2] Let $\varepsilon \in (0, \frac{1}{|S|})$ and let σ be a subgame ε -optimal strategy. Consider a history h at an odd period. The strategy for player 1 which looks at a state with the highest probability guarantees $1/|S|$ in the subgame at h . So, $v_1(h) \geq 1/|S|$.

Now consider a strategy τ for player 2. Then, we have $u_1(\sigma, \tau)(h) \geq 1/|S| - \varepsilon > 0$. In particular, in the subgame at h , the object is found with probability at least $1/|S| - \varepsilon > 0$ under (σ, τ) . Since this holds for every history h at an odd period, by Lévy's zero-one law, the object is found with probability 1 under (σ, τ) . \square

5. Optimal actions

For the initial distribution $p \in \Delta(S)$, we call an action $s \in S$ optimal if it is optimal for player 1 to look at state s in period 1: $v_1(p, s) = v_1(p)$. For a given action $s \in S$, we denote by A_s the set of the initial distributions for which s is optimal: $A_s = \{p \in \Delta(S) \mid v_1(p, s) = v_1(p)\}$. We call A_s the optimality region of action s . Note that $\cup_{s \in S} A_s = \Delta(S)$.

Theorem 16. The optimality regions A_s have the following properties.

[1] If the initial probability distribution p is sufficiently close to e^s , for some state s , then choosing state s is the only optimal action. That is, the set $A_s \setminus \cup_{j \neq s} A_j$ is a neighborhood of e^s in $\Delta(S)$.

[2] For each subset $N \subseteq S$, the convex hull of the vertices e^s with $s \in N$ is included in the set $\cup_{s \in N} A_s$.

[3] There is an initial distribution at which choosing any state is optimal. That is, $\cap_{s \in S} A_s \neq \emptyset$.

[4] For all $s \in S$, the region A_s is star convex centered in e^s . That is, if $p \in A_s$ then the whole line segment between p and e^s is included in A_s .

Proof. [1] The statement follows from the facts that each $v_1(p, s)$ is continuous (cf. [Theorem 8](#)) in p and that $v_1(e^s, s) = 1$ and $v_1(e^s, j) < 1$ for all $j \neq s$.

[2] Let $p \in \text{conv}(\{e^s \mid s \in N\})$. Then $p(s) = 0$ for all $s \notin N$. By [Proposition 12](#), there is an optimal action $j \in N$, and hence $p \in \cup_{s \in N} A_s$.

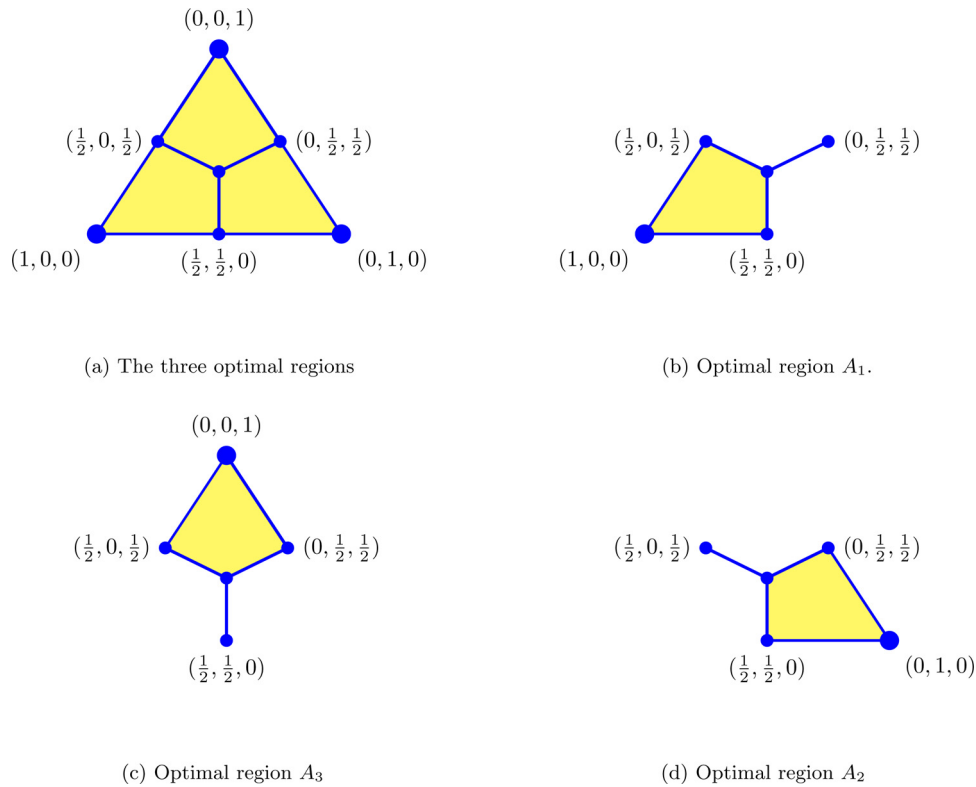


Fig. 2. Optimal regions when $P = I_3$.

[3] We will use the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem⁴, see [Knaster, Kuratowski, & Mazurkiewicz \(1929\)](#). By [Theorem 8](#), the function $p \mapsto v_1(p, s)$ is continuous for all $s \in S$. Thus, each region A_s is closed. From this fact and from [2], we can apply the KKM Theorem, and thus we can conclude that $\bigcap_{s \in S} A_s \neq \emptyset$.

[4] Let $s \in S$, let $p \in A_s$ and let $\lambda \in [0, 1]$. We want to show that $\lambda \cdot e^s + (1 - \lambda) \cdot p \in A_s$. By part [1] of [Proposition 6](#), for every strategy τ of player 2

$$\begin{aligned} & \sup_{\sigma \in \Sigma} u_1(\sigma, \tau)(\lambda \cdot e^s + (1 - \lambda) \cdot p) \\ &= \sup_{\sigma \in \Sigma} [\lambda \cdot u_1(\sigma, \tau)(e^s) + (1 - \lambda) \cdot u_1(\sigma, \tau)(p)] \\ &\leq \lambda \cdot \left[\sup_{\sigma \in \Sigma} u_1(\sigma, \tau)(e^s) \right] + (1 - \lambda) \cdot \left[\sup_{\sigma \in \Sigma} u_1(\sigma, \tau)(p) \right] \\ &= \lambda + (1 - \lambda) \cdot \left[\sup_{\sigma \in \Sigma} u_1(\sigma, \tau)(p) \right], \end{aligned}$$

where we used that $u_1(\sigma, \tau)(e^s) = 1$ for any strategy σ that looks at state s in period 1. Hence, by [Proposition 5](#)

$$\begin{aligned} v_1(\lambda \cdot e^s + (1 - \lambda) \cdot p) &= \inf_{\tau \in T} \sup_{\sigma \in \Sigma} u_1(\sigma, \tau)(\lambda \cdot e^s + (1 - \lambda) \cdot p) \\ &\leq \lambda + (1 - \lambda) \cdot \left[\inf_{\tau \in T} \sup_{\sigma \in \Sigma} u_1(\sigma, \tau)(p) \right] \\ &= \lambda + (1 - \lambda) \cdot v_1(p) \\ &= \lambda + (1 - \lambda) \cdot v_1(p, s). \end{aligned}$$

⁴ The KKM theorem states: Let $n \in \mathbb{N}$ be the cardinality of the set of states S , in other words $|S| = n$. Let Δ^n be the unit simplex in \mathbb{R}^n . A KKM covering is defined as a collection C_1, \dots, C_n of closed sets such that for any $N \subseteq \{1, \dots, n\}$, the convex hull of the vertices corresponding to N is covered by $\bigcup_{s \in N} C_s$. Then any KKM covering has a non-empty intersection, i.e.: $\bigcap_{s \in S} C_s \neq \emptyset$.

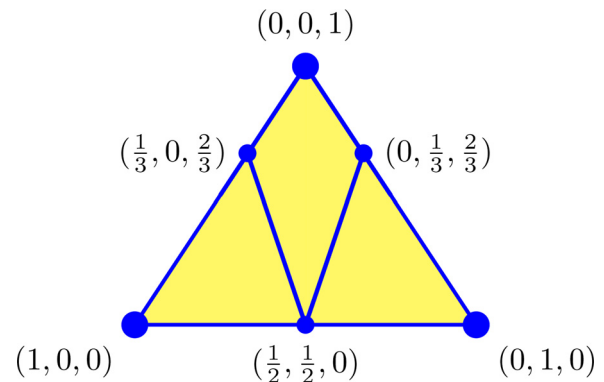


Fig. 3. Optimal regions when $P = Q$.

On the other hand, by part [2] of [Proposition 6](#), $v_1(\lambda \cdot e^s + (1 - \lambda) \cdot p, s) = \lambda + (1 - \lambda) \cdot v_1(p, s)$. Therefore, $v_1(\lambda \cdot e^s + (1 - \lambda) \cdot p) = v_1(\lambda \cdot e^s + (1 - \lambda) \cdot p, s)$. \square

Example 17. Consider the case in which the set of states is $S = \{1, 2, 3\}$. Let $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}$. The sets A_1 , A_2 and A_3 are represented in the time-homogeneous case where the transition matrix is the identity matrix in [Fig. 2](#), and the matrix Q in [Fig. 3](#).

Example 17 illustrates the statements of [Theorem 16](#). In particular here are some remarks.

(i) It makes intuitive sense that if the object is in a certain state with probability close to 1, then it is optimal to look at this state. Geometrically, this means that for all states $s \in S$, the set A_s contains a neighborhood of e^s in $\Delta(S)$.

(ii) Looking at a state s' such that $p(s') = 0$ can still be (weakly) optimal. For example, in Fig. 2 and Fig. 3 with initial probability distribution $p = (\frac{1}{2}, \frac{1}{2}, 0)$, looking at state 3 is just as good as looking at either state 1 or state 2. Fig. 3 shows that in general, if $N \subset S$, then $\cup_{s \in N} A_s$ does not necessarily include an open neighborhood of $\text{conv}\{e^s | s \in N\}$ in $\Delta(S)$.

(iii) Fig. 2 illustrates that the intersection of the regions $\cap_{s \in S} A_s$ can be more than a single point.

(iv) Fig. 2 illustrates the fact that the sets A_s are not always convex. We conjecture that their relative interior is convex.

6. Concluding remarks and future work

We introduced an infinite horizon search game, in which two players compete to find an object that moves according to a time-varying Markov chain. We proved that these games always admit a subgame-perfect ε -equilibrium in pure strategies, for all error-terms $\varepsilon > 0$, but not necessarily a 0-equilibrium. We showed that the ε -equilibrium payoffs converge to a vector $(v_1, 1 - v_1)$ as ε vanishes, and therefore we defined v_1 as the value of the game. We examined the analytical and structural properties of the solutions, and devoted attention to the important special case when the Markov chain is time-homogeneous, where stronger results hold.

We remark that, in these search games, the ε -optimal strategies are robust in the following sense: they are 2ε -optimal if the horizon of the game is finite but sufficiently long, and they are also 2ε -optimal in the discounted version of the game, provided that the discount factor is close to 1. For the precise statements and their proofs we refer to Duvocelle et al. (2020).

It would be interesting to generalize the results when the active player is chosen according to an arbitrary stochastic process. In the companion paper Duvocelle, Flesch, Shi, & Vermeulen (2021), we examine the variation in which the active player is chosen according to a fixed probability distribution at each period.

Another direction of research could be to investigate the link with influence games. Diffusion of opinions in a DeGroot-opinion model can be associated with a reverse Markov chain with particles moving in a network. In order to study influence games, Mandel & Venel (2020) introduced an auxiliary game based on this reverse Markov chain and where players try to catch more particles than their opponent making the problematic similar to a search game but with the introduction of several objects across time.

Also, one could introduce overlook probabilities to the model. In that case, even if the active player chooses the state that currently contains the object, there is a positive probability that the player fails to find it.

Declaration of Competing Interest

none

Appendix A. Topological properties of search games

Let X be a topological space. A function $f : X \rightarrow \mathbb{R}$ is called *lower semi-continuous* at $x \in X$ if, for every sequence $x^n \rightarrow x$, we have $\liminf_{n \rightarrow \infty} f(x^n) \geq f(x)$. A function $f : X \rightarrow \mathbb{R}$ is called *upper semi-continuous* at $x \in X$ if, for every sequence $x^n \rightarrow x$, we have $\limsup_{n \rightarrow \infty} f(x^n) \leq f(x)$. A function $f : X \rightarrow \mathbb{R}$ is called *continuous* at $x \in X$ if it is lower semi-continuous at x and upper semi-continuous at x .

A function $f : X \rightarrow \mathbb{R}$ is called *lower semi-continuous* (resp. *upper semi-continuous*, resp. *continuous*) if f is lower semi-continuous at all $x \in X$ (resp. upper semi-continuous at all $x \in X$, resp. continuous at all $x \in X$).

Proposition 18. Take a player $i \in \{1, 2\}$.

[1] The payoff function $u_i : \Sigma \times \mathcal{T} \rightarrow \mathbb{R}$ is lower semi-continuous.

[2] Assume that (σ, τ) is a strategy profile under which the object is found with probability 1, i.e., $\mathbb{P}_{\sigma, \tau}(\Theta < \infty) = 1$. Then, u_i is continuous at (σ, τ) .

Proof. [1] For each strategy profile $(\sigma, \tau) \in \Sigma \times \mathcal{T}$, for each period $n \in \mathbb{N}$, we denote by $u_i^{\leq n}(\sigma, \tau)$ the probability that player i finds the object during the first n periods under the strategy profile (σ, τ) . Note that $u_i^{\leq n}(\sigma, \tau)$ is non-decreasing in n and converges to $u_i(\sigma, \tau)$ as $n \rightarrow \infty$.

Let $(\sigma^k, \tau^k)_{k \in \mathbb{N}}$ be a sequence in $\Sigma \times \mathcal{T}$ converging to a strategy profile (σ, τ) . We have for each $n \in \mathbb{N}$

$$\begin{aligned} u_i^{\leq n}(\sigma, \tau) &= \lim_{k \rightarrow \infty} u_i^{\leq n}(\sigma^k, \tau^k) = \liminf_{k \rightarrow \infty} u_i^{\leq n}(\sigma^k, \tau^k) \\ &\leq \liminf_{k \rightarrow \infty} u_i(\sigma^k, \tau^k). \end{aligned}$$

Since $u_i^{\leq n}(\sigma, \tau)$ converges to $u_i(\sigma, \tau)$ as $n \rightarrow \infty$, we obtain

$$u_i(\sigma, \tau) \leq \liminf_{k \rightarrow \infty} u_i(\sigma^k, \tau^k),$$

which proves that u_i is lower semi-continuous.

[2] Assume that under the strategy profile (σ, τ) the object is found with probability 1. Thus, $u_1(\sigma, \tau) + u_2(\sigma, \tau) = 1$. Due to part 1, we only need to show that u_1 and u_2 are upper semi-continuous at (σ, τ) . We will prove it for u_1 ; the proof for u_2 is similar.

Let $(\sigma^k, \tau^k)_{k \in \mathbb{N}}$ be a sequence in $\Sigma \times \mathcal{T}$ converging to (σ, τ) . Then

$$\begin{aligned} \limsup_{k \rightarrow \infty} u_1(\sigma^k, \tau^k) &= 1 - \liminf_{k \rightarrow \infty} (1 - u_1(\sigma^k, \tau^k)) \\ &\leq 1 - \liminf_{k \rightarrow \infty} u_2(\sigma^k, \tau^k) \leq 1 - u_2(\sigma, \tau) = u_1(\sigma, \tau), \end{aligned}$$

where the first equality is a classic supinf equality applied to a limit, the first inequality comes from $u_1 + u_2 \leq 1$, the second inequality follows from part 1, and the second equality comes from the assumption we made on (σ, τ) . Hence, u_1 is upper semi-continuous at (σ, τ) , as desired. \square

Appendix B. Technical proofs

Proof of Proposition 5. In the expression $\inf_{\tau \in \mathcal{T}} \sup_{\sigma \in \Sigma} u_1(\sigma, \tau)$, player 1 is maximizing $u_1(\sigma, \tau)$ and player 2 is minimizing the same expression. Note that $(\sigma, \tau) \mapsto u_1(\sigma, \tau)$ is bounded, and by Proposition 18, it is lower semi-continuous, and hence Borel measurable. Thus, the equality $v_1 = \inf_{\tau \in \mathcal{T}} \sup_{\sigma \in \Sigma} u_1(\sigma, \tau)$ follows from Martin (1975). The equality $v_2 = \inf_{\sigma \in \Sigma} \sup_{\tau \in \mathcal{T}} u_2(\sigma, \tau)$ follows similarly. \square

Proof of Proposition 6. First we prove part [1]. Let (σ, τ) be a strategy profile, $p, q \in \Delta(S)$, $\lambda \in [0, 1]$ and $i \in \{1, 2\}$. The probability distribution $\lambda \cdot p + (1 - \lambda) \cdot q$ can be interpreted as follows: with probability λ the initial probability distribution is p and induces the expected payoff $u_i(\sigma, \tau)(p)$ for player i , and with probability $1 - \lambda$ the probability distribution is q and induces the expected payoff $u_i(\sigma, \tau)(q)$ for player i . Hence, the equality (1) holds.

Now we prove part [2]. Let $s \in S$, $p \in \Delta(S)$ and $\lambda \in [0, 1]$. Then, by part [1], we have for every $\sigma \in \Sigma^s$ and every $\tau \in \mathcal{T}$ that

$$\begin{aligned} u_1(\sigma, \tau)(\lambda \cdot e^s + (1 - \lambda) \cdot p) &= \lambda \cdot u_1(\sigma, \tau)(e^s) \\ &\quad + (1 - \lambda) \cdot u_1(\sigma, \tau)(p) = \lambda + (1 - \lambda) \cdot u_1(\sigma, \tau)(p). \end{aligned}$$

Hence, by taking the supremum over $\sigma \in \Sigma^s$ and the infimum over $\tau \in \mathcal{T}$

$$v_1(\lambda \cdot e^s + (1 - \lambda) \cdot p, s) = \lambda + (1 - \lambda) \cdot v_1(p, s).$$

Finally, we prove part [3]. Let $p, q \in \Delta(S)$, $\lambda \in [0, 1]$ and $i \in \{1, 2\}$. By part [1], for every strategy profile (σ, τ)

$$u_i(\sigma, \tau)(\lambda \cdot p + (1 - \lambda) \cdot q) = \lambda \cdot u_i(\sigma, \tau)(p) + (1 - \lambda) \cdot u_i(\sigma, \tau)(q) \geq \lambda \cdot u_i(\sigma, \tau)(p)$$

Hence, by taking the supremum over $\sigma \in \Sigma$ and the infimum over $\tau \in \mathcal{T}$, we obtain $v_i(\lambda \cdot p + (1 - \lambda) \cdot q) \geq \lambda \cdot v_i(p)$. \square

Proof of Proposition 9. We prove that $v_1 + v_2 = 1$.

Step 1. Let $\tilde{\Sigma}$ denote the set of strategies σ for player 1 such that for every strategy τ for player 2, the object is found with probability 1 under the strategy profile (σ, τ) , i.e. $\mathbb{P}_{\sigma, \tau}(\Theta < \infty) = 1$. Note that $\tilde{\Sigma}$ is nonempty; for example, $\tilde{\Sigma}$ contains the strategy for player 1 that always chooses a state according to the uniform distribution on S . We claim that

$$v_1 = \sup_{\sigma \in \tilde{\Sigma}} \inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau).$$

Proof of Step 1. Since $\tilde{\Sigma} \subseteq \Sigma$, we have $v_1 = \sup_{\sigma \in \Sigma} \inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau) \geq \sup_{\sigma \in \tilde{\Sigma}} \inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau)$.

Now we prove the opposite inequality:

$$\sup_{\sigma \in \tilde{\Sigma}} \inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau) \geq v_1. \quad (2)$$

Let $\varepsilon \in (0, \frac{1}{|S|})$. By [Mashiah-Yaakovi \(2015\)](#) (or alternatively, by the proof of Mertens and Neyman in [Mertens \(1990\)](#), or by [Flesch, Herings, Maes, & Predtetchinski \(2021\)](#)), there is a strategy σ for player 1 such that for every history h

$$\inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau)(h) \geq v_1(h) - \varepsilon. \quad (3)$$

We argue that $\sigma \in \tilde{\Sigma}$. So, take a strategy τ for player 2. Assume that h is a history at an odd period. In the subgame starting at h , player 1 can immediately win with probability $\frac{1}{|S|}$ if he chooses a state uniformly. So, $v_1(h) \geq \frac{1}{|S|}$, and hence $u_1(\sigma, \tau)(h) \geq \frac{1}{|S|} - \varepsilon > 0$. In particular, in the subgame at h , the object is found with probability at least $\frac{1}{|S|} - \varepsilon > 0$ under (σ, τ) . Since this holds for every history h at an odd period, by Lévy's zero-one law, the object is found with probability 1 under (σ, τ) . This proves that $\sigma \in \tilde{\Sigma}$, as desired.

Since $\sigma \in \tilde{\Sigma}$, by applying (3) to the empty history in period 1, we find $\inf_{\tau \in \mathcal{T}} u_1(\sigma, \tau) \geq v_1 - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, (2) follows.

Step 2. Define for each state $s \in S$

$$v_2(p, s) = \inf_{\sigma \in \Sigma^s} \sup_{\tau \in \mathcal{T}} u_2(\sigma, \tau)(p);$$

recall that Σ^s denotes the set of those strategies for player 1 that choose state s in period 1. We claim that for every $\delta > 0$ there is $w(\delta) > 0$ such that if $p \in \Delta(S)$ satisfies $p(s') \leq w(\delta)$ for some state $s' \in S$, then for all states $s \in S$ we have

$$v_2(p, s') + \delta \geq v_2(p, s). \quad (4)$$

Intuitively, if player 1 wishes to minimize player 2's winning chances, then choosing a state s' with a very low probability of containing the object cannot be much better than choosing any state s .

Proof of Step 2. Let \tilde{G} denote the game such that (i) the game is played at periods 2, 3, ...; (ii) the transition matrices of the Markov chain are P_2, P_3, \dots ; (iii) player 2 is the active player in even periods, and player 1 is the active player in odd periods. For the game \tilde{G} , let $\tilde{\Sigma}$ and $\tilde{\mathcal{T}}$ denote the sets of strategies, and $\tilde{u}_i(\sigma, \tau)(p)$ denote the expected winning chance of player $i = 1, 2$ under each strategy pair (σ, τ) and each initial distribution $p \in \Delta(S)$ for the object. Define for each player $i = 1, 2$

$$\tilde{v}_i(p) = \sup_{\sigma \in \tilde{\Sigma}} \inf_{\tau \in \tilde{\mathcal{T}}} \tilde{u}_i(\sigma, \tau)(p).$$

Note that $pP_1 = p(s) \cdot e^s P_1 + (1 - p(s)) \cdot p^{-s} P_1$. By part [3] of [Proposition 6](#) (applied to \tilde{G}),

$$\tilde{v}_2(pP_1) \geq (1 - p(s)) \cdot \tilde{v}_2(p^{-s} P_1). \quad (5)$$

Note that if player 1 chooses a state $s \in S$ then with probability $p(s)$ he finds the object immediately, and with probability $1 - p(s)$ he does not find it and then the distribution of the location of the object becomes $p^{-s} P_1$ in period 2, where player 2 is the active player. Therefore, $v_2(p, s) = (1 - p(s)) \cdot \tilde{v}_2(p^{-s} P_1)$. Thus, for all states $s, s' \in S$ we find

$$\begin{aligned} v_2(p, s) - v_2(p, s') &= (1 - p(s)) \cdot \tilde{v}_2(p^{-s} P_1) - (1 - p(s')) \cdot \tilde{v}_2(p^{-s'} P_1) \\ &= (1 - p(s)) \cdot \tilde{v}_2(p^{-s} P_1) - \tilde{v}_2(p^{-s'} P_1) + p(s') \cdot \tilde{v}_2(p^{-s'} P_1) \\ &\leq \tilde{v}_2(pP_1) - \tilde{v}_2(p^{-s'} P_1) + p(s') \\ &\leq \|(p - p^{-s'}) \cdot P_1\|_{TV} + p(s') \\ &\leq \|p - p^{-s'}\|_{TV} + p(s') \\ &= 2 \cdot p(s'), \end{aligned}$$

where in the first inequality we used (5) and $v_2(p^{-s'} P_1) \leq 1$, and in the second inequality we used [Theorem 8](#) (applied to the game \tilde{G}). Now the claim of Step 2 follows immediately.

Step 3. We claim that

$$v_2 = \inf_{\sigma \in \Sigma} \sup_{\tau \in \mathcal{T}} u_2(\sigma, \tau).$$

Proof of Step 3. Since $\tilde{\Sigma} \subseteq \Sigma$, we have $v_2 = \inf_{\sigma \in \Sigma} \sup_{\tau \in \mathcal{T}} u_2(\sigma, \tau) \leq \inf_{\sigma \in \tilde{\Sigma}} \sup_{\tau \in \mathcal{T}} u_2(\sigma, \tau)$.

Now we prove the opposite inequality: $v_2 \geq \inf_{\sigma \in \tilde{\Sigma}} \sup_{\tau \in \mathcal{T}} u_2(\sigma, \tau)$. Let $\varepsilon > 0$. By [Mashiah-Yaakovi \(2015\)](#) (or alternatively, by the proof of Mertens and Neyman in [Mertens \(1990\)](#), or by [Flesch et al. \(2021\)](#)), there is a pure strategy σ for player 1 such that for every history h

$$\sup_{\tau \in \mathcal{T}} u_2(\sigma, \tau)(h) \leq v_2(h) + \varepsilon.$$

Since σ is pure, $\sigma(h)$ places probability 1 on a state $s_h \in S$ for each history h .

For each $n \in \mathbb{N}$, let $w_n = w(\frac{\varepsilon}{2^n})$ where $w(\frac{\varepsilon}{2^n})$ is as in Step 3. Note that $w_n > 0$ for each n .

For each $n \in \mathbb{N}$, we define a strategy σ_n^* for player 1 as follows: start using σ , until a history h_1 occurs (if it occurs at all) such that s_{h_1} contains the object with probability at most w_1 . Then, at h_1 , choose a state which contains the object with the highest probability (which is at least $1/|S|$). Then, follow σ again, until a history h_2 occurs (if it occurs at all) such that s_{h_2} contains the object with probability at most w_2 . Then, at h_2 , choose a state which contains the object with the highest probability (which is at least $1/|S|$). Then, follow σ again, and so on, until a history h_n occurs (if it occurs at all) such that s_{h_n} contains the object with probability at most w_n . Then, at h_n , choose a state which contains the object with the highest probability (which is at least $1/|S|$), and then follow σ in the remaining game. Note that, along the play of the game, σ_n^* deviates from σ at most n times.

We define σ^* as the limit of the strategies σ_n^* when $n \rightarrow \infty$. This means that, along the play of the game, σ^* may deviate from σ at infinitely many histories. We argue that $\sigma^* \in \tilde{\Sigma}$ and

$$\sup_{\tau \in \mathcal{T}} u_2(\sigma^*, \tau) \leq v_2 + 4\varepsilon, \quad (6)$$

which will then complete the proof of Step 3.

We show first that $\sigma^* \in \tilde{\Sigma}$. Take any strategy τ for player 2, and consider the play according to (σ^*, τ) . By construction of σ^* : (i) whenever σ^* deviates from σ , the object is found immediately with probability at least $1/|S|$, (ii) if after a period t , σ^* never deviates from σ , then at every period $t' \geq t$, σ finds the object immediately with probability at least w_n for some fixed value of n . This implies that, under (σ^*, τ) , the object is found with probability 1. That is, $\sigma^* \in \tilde{\Sigma}$.

Now we show (6). Take any strategy τ for player 2. By the previous argument, for sufficiently large $n \in \mathbb{N}$, under (σ^*, τ) , the object is found with probability close to 1 within n periods. This

implies that σ^* deviates from σ at most n times, and hence, $u_2(\sigma^*, \tau)$ is close to $u_2(\sigma_n^*, \tau)$. Choose n so large that $u_2(\sigma^*, \tau) - \varepsilon \leq u_2(\sigma_n^*, \tau)$. By iteratively applying Step 3, we find that

$$u_2(\sigma^*, \tau) - \varepsilon \leq u_2(\sigma_n^*, \tau) \leq u_2(\sigma, \tau) + \varepsilon + \frac{\varepsilon}{2} + \dots + \frac{\varepsilon}{2^n} < u_2(\sigma, \tau) + 2\varepsilon \leq v_2 + 3\varepsilon.$$

Hence, (6) follows.

Step 4. We prove that $v_1 + v_2 = 1$. We have

$$\begin{aligned} v_1 &= \sup_{\sigma \in \tilde{\Sigma}} \inf_{\tau \in T} u_1(\sigma, \tau) = \sup_{\sigma \in \tilde{\Sigma}} \inf_{\tau \in T} [1 - u_2(\sigma, \tau)] \\ &= 1 - \inf_{\sigma \in \tilde{\Sigma}} \sup_{\tau \in T} u_2(\sigma, \tau) = 1 - v_2. \end{aligned}$$

Here, the first equality follows from Step 1, the second equality follows from the fact that if σ belongs to $\tilde{\Sigma}$ then the object is found with probability 1, and the last equality follows from Step 3. Hence, $v_1 + v_2 = 1$ as desired. \square

Proof of Proposition 11. Let $\varepsilon \geq 0$.

[1] Let (σ, τ) be an ε -equilibrium. Since $u_1(\sigma, \tau) \geq u_1(\sigma', \tau) - \varepsilon$ for all $\sigma' \in \Sigma$, we have $u_1(\sigma, \tau) \geq v_1 - \varepsilon$, which means that σ is ε -optimal. It follows similarly that τ is ε -optimal as well.

When $\varepsilon > 0$, by Theorem 3, there is an ε -equilibrium (σ, τ) in pure strategies. Hence, σ and τ are ε -optimal strategies for the players.

[2] Assume now that σ and τ are ε -optimal strategies for player 1 and player 2. Let $\sigma' \in \Sigma$. Then, $u_2(\sigma', \tau) \geq v_2 - \varepsilon$. By Proposition 9, we get that

$$u_1(\sigma', \tau) \leq 1 - u_2(\sigma', \tau) \leq 1 - (v_2 - \varepsilon) = v_1 + \varepsilon.$$

This implies that $u_1(\sigma, \tau) \geq v_1 - \varepsilon \geq u_1(\sigma', \tau) - 2\varepsilon$. Similarly, we obtain $u_2(\sigma, \tau) \geq u_2(\sigma, \tau') - 2\varepsilon$ for every $\tau' \in T$. So, (σ, τ) is a 2ε -equilibrium.

[3] This is a direct consequence of [1] and [2].

[4] Let (σ, τ) be an ε -equilibrium.

First we prove that under (σ, τ) , the object is found with probability at least $1 - \varepsilon \cdot |S|$. Suppose the opposite: it is found with probability $z := \mathbb{P}_{\sigma, \tau}(\theta < \infty) < 1 - \varepsilon \cdot |S|$. Let \tilde{H} denote those histories after which the object is found with probability at most $\gamma := \frac{1}{2}(\frac{1}{|S|} - \frac{\varepsilon}{1-\varepsilon})$:

$$\tilde{H} = \{h \in H : \mathbb{P}_{\sigma, \tau}(\theta < \infty | h) \leq \gamma\};$$

note that $\gamma > 0$. By Lévy's zero-one law, under (σ, τ) , it has probability at least $1 - z$ that a history in \tilde{H} arises during the play. Now consider the strategy σ' for player 1 that follows σ as long as no history arises in \tilde{H} , but as soon as this happens, σ' chooses each state with equal probability and plays arbitrarily afterwards. Then,

$$u_1(\sigma', \tau) - u_1(\sigma, \tau) \geq (1 - z) \cdot \left(\frac{1}{|S|} - \gamma \right) > \varepsilon.$$

This contradicts the fact that (σ, τ) is an ε -equilibrium.

Now prove the inequalities $|u_1(\sigma, \tau) - v_1| \leq \varepsilon$ and $|u_2(\sigma, \tau) - v_2| \leq \varepsilon$. By part [1], $u_1(\sigma, \tau) \geq v_1 - \varepsilon$ and similarly, $u_2(\sigma, \tau) \geq v_2 - \varepsilon$. Thus, by Proposition 9

$$u_1(\sigma, \tau) \leq 1 - u_2(\sigma, \tau) \leq 1 - (v_2 - \varepsilon) = v_1 + \varepsilon.$$

Similarly, $u_2(\sigma, \tau) \leq v_2 + \varepsilon$. These inequalities give $|u_1(\sigma, \tau) - v_1| \leq \varepsilon$ and $|u_2(\sigma, \tau) - v_2| \leq \varepsilon$. \square

Proof of Corollary 13. Assume first that there is a state $s \in S$ for which $p(s) = 0$. Then $p^{-s}P = pP = p$ and hence $v_1(p, s) = 1 - v_1(p)$. As $v_1(p) \geq v_1(p, s)$, we obtain $v_1(p) \geq 1 - v_1(p)$. Hence, $v_1(p) \geq \frac{1}{2}$.

Assume now that there is no state $s \in S$ for which $p(s) = 0$. The idea is that we add a new state that cannot be reached from the other states and where the object starts with probability 0. Therefore, adding this state does not change the value of the game. Also,

if player 1 chooses this state in period 1, then as p is a stationary distribution, the distribution of the object in period 2 will be also p . This allows us to use the previous case of the proof.

More precisely, consider the game G' that arises by adding a state w to G . More precisely, G' is the game with set of states $S' = S \cup \{w\}$, initial probability distribution p' such that $p'(s) = p(s)$ for each state $s \in S$ and $p'(w) = 0$, and transition matrix P' that has the same transition probabilities between states in S and makes w absorbing. Then, with probability 1, the object will never be in w . By Proposition 12, the players may ignore state w during the game. Then, p' is a stationary distribution of P' , and hence by the first part we find $v_1(p) = v'_1(p') \geq \frac{1}{2}$. \square

References

- Abramovskaya, T. V., Fomin, F. V., Golovach, P. A., & Pilipczuk, M. (2016). How to hunt an invisible rabbit on a graph. *European Journal of Combinatorics*, 52, 12–26.
- Alpern, S. (2021). The faulty GPS problem: Shortest time paths in networks with unreliable directions. Preprint arXiv:2111.09093, 1–32.
- Alpern, S., Baston, V., & Gal, S. (2008). Network search games with immobile hider, without a designated searcher starting point. *International Journal of Game Theory*, 37(2), 281–302.
- Alpern, S., Baston, V., & Gal, S. (2009). Searching symmetric networks with utilitarian-postman paths. *Networks: An International Journal*, 53(4), 392–402.
- Alpern, S., & Gal, S. (2006). *The Theory of Search Games and Rendezvous*: vol. 55. Springer Science & Business Media.
- Alpern, S., & Zeng, L. (2021). Social distancing, gathering, search games: Mobile agents on simple networks. Preprint arXiv:2111.05958, 1–34.
- Angelopoulos, S., & Lidbetter, T. (2020). Competitive search in a network. *European Journal of Operational Research*, 286, 781–790.
- Baston, V., Bostock, F., & Ruckle, W. H. (1990). The gold-mine game. *Journal of Optimization Theory and Applications*, 64(3), 641–650.
- Beck, A., & Newman, D. J. (1970). Yet more on the linear search problem. *Israel Journal of Mathematics*, 8(4), 419–429.
- Benkoski, S. J., Monticino, M. G., & Weisinger, J. R. (1991). A survey of the search theory literature. *Naval Research Logistics (NRL)*, 38(4), 469–494.
- Berry, D. A., & Mensch, R. F. (1986). Discrete search with directional information. *Operations Research*, 34(3), 470–477.
- Brown, S. S. (1980). Optimal search for a moving target in discrete time and space. *Operations Research*, 28(6), 1275–1289.
- Cao, B. (1995). Search-hide games on trees. *European Journal of Operational Research*, 80(1), 175–183.
- Clarkson, J. (2020). *Optimal search in discrete locations: extensions and new findings*. Lancaster University Ph.D. thesis.
- Dagan, A., & Gal, S. (2008). Network search games, with arbitrary searcher starting point. *Networks: An International Journal*, 52(3), 156–161.
- Delavernhe, F., Jaillet, P., Rossi, A., & Sevaux, M. (2020). Planning a multi-sensors search for a moving target considering traveling costs. *European Journal of Operational Research*, 292, 469–482.
- Dobbie, J. M. (1974). A two-cell model of search for a moving target. *Operations Research*, 22(1), 79–92.
- Duvoelle, B., Flesch, J., Shi, H. M., & Vermeulen, D. (2021). Search for a moving target in a competitive environment. *International Journal of Game Theory*, 1–11.
- Duvoelle, B., Flesch, J., Staudigl, M., & Vermeulen, D. (2020). A competitive search game with a moving target. Preprint arXiv:2008.12032, 1–34.
- Efron, B. (1964). Optimum evasion versus systematic search. *Journal of the Society for Industrial and Applied Mathematics*, 12(2), 450–457.
- Flesch, J., Herings, P. J.-J., Maes, J., & Predtchinski, A. (2021). Subgame maxmin strategies in zero-sum stochastic games with tolerance levels. *Dynamic Games and Applications*, 1–34.
- Flesch, J., Karagözoğlu, E., & Perea, A. (2009). Optimal search for a moving target with the option to wait. *Naval Research Logistics (NRL)*, 56(6), 526–539.
- Flesch, J., Kuipers, J., Mashiah-Yaakovi, A., Schoenmakers, G., Solan, E., & Vrieze, K. (2010). Perfect-information games with lower-semicontinuous payoffs. *Mathematics of Operations Research*, 35(4), 742–755.
- Flesch, J., & Predtchinski, A. (2016). Subgame-perfect epsilon-equilibria in perfect information games with common preferences at the limit. *Mathematics of Operations Research*, 41(4), 1208–1221.
- Fudenberg, D., & Levine, D. (1983). Subgame-perfect equilibria of finite-and infinite-horizon games. *Journal of Economic Theory*, 31(2), 251–268.
- Gal, S. (1972). A general search game. *Israel Journal of Mathematics*, 12(1), 32–45.
- Gal, S. (1974). Minimax solutions for linear search problems. *SIAM Journal on Applied Mathematics*, 27(1), 17–30.
- Gal, S. (1979). Search games with mobile and immobile hider. *SIAM Journal on Control and Optimization*, 17(1), 99–122.
- Gal, S. (2010). *Search games*. Wiley Encyclopedia of Operations Research and Management Science, Wiley Online Library.
- Gal, S. (2013). Search games: A review. In *Search theory* (pp. 3–15). Springer.
- Gal, S., & Chazan, D. (1976). On the optimality of the exponential functions for some minimax problems. *SIAM Journal on Applied Mathematics*, 30(2), 324–348.
- Garnaev, A. (2012). *Search games and other applications of game theory*: vol. 485. Springer Science & Business Media.

- Garrec, T., & Scarsini, M. (2020). Search for an immobile hider on a stochastic network. *European Journal of Operational Research*, 283(2), 783–794.
- Gittins, J., & Roberts, D. (1979). The search for an intelligent evader concealed in one of an arbitrary number of regions. *Naval Research Logistics Quarterly*, 26(4), 651–666.
- Hohzaki, R. (2016). Search games: Literature and survey. *Journal of the Operations Research Society of Japan*, 59(1), 1–34.
- Hohzaki, R., & Iida, K. (2001). Optimal ambushing search for a moving target. *European Journal of Operational Research*, 133(1), 120–129.
- Isaacs, R. (1965). *Differential games: A mathematical theory with applications to warfare and pursuit, control and optimization*. Control and Warfare Wiley, New York.
- Jordan, B. P. (1997). *On optimal search for a moving target*. Durham University Ph.D. thesis.
- Jotshi, A., & Batta, R. (2008). Search for an immobile entity on a network. *European Journal of Operational Research*, 191(2), 347–359.
- Kan, Y. (1974). A counterexample for an optimal search-and-stop model. *Operations Research*, 22(4), 889–892.
- Kikuta, K. (2004). A search game on a cyclic graph. *Naval Research Logistics (NRL)*, 51(7), 977–993.
- Knaster, B., Kuratowski, C., & Mazurkiewicz, S. (1929). Ein beweis des fixpunktsatzes für n -dimensionale simplexe. *Fundamenta Mathematicae*, 14(1), 132–137.
- Levin, D. A., & Peres, Y. (2017). *Markov chains and mixing times*. American Mathematical Society.
- Lidbetter, T. (2020). Search and rescue in the face of uncertain threats. *European Journal of Operational Research*, 285(3), 1153–1160.
- MacPhee, I. M., & Jordan, B. P. (1995). Optimal search for a moving target. *Probability in the Engineering and Informational Sciences*, 9(2), 159–182.
- Mandel, A., & Venel, X. (2020). Dynamic competition over social networks. *European Journal of Operational Research*, 280(2), 597–608.
- Martin, D. A. (1975). Borel determinacy. *Annals of Mathematics*, 102(2), 363–371.
- Maschler, M., Solan, E., & Zamir, S. (2013). *Game Theory*. Cambridge University Press, Cambridge.
- Mashiah-Yaakovi, A. (2015). Correlated equilibria in stochastic games with borel measurable payoffs. *Dynamic Games and Applications*, 5(1), 120–135.
- Mertens, J.-F. (1990). Repeated games. In *Game theory and applications* (pp. 77–130). Elsevier.
- Nakai, T. (1973). Model of search for a target moving among three boxes: Some special cases. *Journal of Operations Research Society of Japan*, 16, 151–162.
- Nakai, T. (1986). A search game with one object and two searchers. *Journal of Applied Probability*, 23(3), 696–707.
- Nakai, T. (1990). A preemptive detection game. *Journal of Information and Optimization Sciences*, 11(1), 1–15.
- von Neumann, J. (1953). A certain zero-sum two-person game equivalent to the optimal assignment problem. *Contributions to the Theory of Games*, 2(0), 5–12.
- Neuts, M. F. (1963). A multistage search game. *Journal of the Society for Industrial and Applied Mathematics*, 11(2), 502–507.
- Pavlović, L. (1995). A search game on the union of graphs with immobile hider. *Naval Research Logistics (NRL)*, 42(8), 1177–1189.
- Pollock, S. M. (1970). A simple model of search for a moving target. *Operations Research*, 18(5), 883–903.
- Reijnierse, J., & Potters, J. A. (1993). Search games with immobile hider. *International Journal of Game Theory*, 21(4), 385–394.
- Roberts, D. M., & Gittins, J. C. (1978). The search for an intelligent evader: Strategies for searcher and evader in the two-region problem. *Naval Research Logistics Quarterly*, 25(1), 95–106.
- Sakaguchi, M. (1977). A sequential allocation game for targets with varying values. *Journal of the Operations Research Society of Japan*, 20(3), 182–193.
- Schweitzer, P. J. (1971). Threshold probabilities when searching for a moving target. *Operations Research*, 19(3), 707–709.
- von Stengel, B., & Werchner, R. (1997). Complexity of searching an immobile hider in a graph. *Discrete Applied Mathematics*, 78(1–3), 235–249.
- Stone, L. D. (1976). *Theory of Optimal Search*: vol. 118. Elsevier.
- Stone, L. D., Royset, J. O., & Washburn, A. R. (2016). *Optimal Search for Moving Targets (International Series in Operations Research & Management Science 237)*. Cham, Switzerland: Springer.
- Subelman, E. J. (1981). A hide-search game. *Journal of Applied Probability*, 628–640.
- Washburn, A. R. (1983). Search for a moving target: The FAB algorithm. *Operations Research*, 31(4), 739–751.
- Zoroa, N., Fernández-Sáez, M. J., & Zoroa, P. (2011). A foraging problem: Sit-and-wait versus active predation. *European Journal of Operational Research*, 208(2), 131–141.